Scales in Mathematical Fluid Dynamics

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(all pictures in the text thanks to *wikipedia*)

Motto

Johann von Neumann [1903-1957]

In mathematics you don't understand things. You just get used to them.

Fluids in the real world

- wheather prediction
- ships, planes, cars, trains
- astrophysics, gaseous stars
- rivers, floods, oceans, tsunami waves
- human body, blood motion

Mathematical issues

- Modeling
- Analysis of models, well-posedness, stability, determinism (?)
- Numerical analysis and implementations, computations

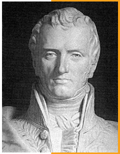
Millennium problems (?)

Clay Mathematics Institute, Providence, RI

- Birch and Swinnerton-Dyer Conjecture
- Hodge Conjecture
- Navier-Stokes Equation
- P vs NP Problem
- Poincaré Conjecture
- Riemann Hypothesis
- Yang-Mills and Mass Gap

Navier-Stokes system

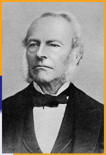
- $\mathbf{u} = \mathbf{u}(t,x)$ .... velocity of an incompressible viscous fluid
- $\Pi = \Pi(t,x)$ .... pressure



Claude Louis Marie Henri Navier [1785-1836]

Incompressibility constraint

$$\mathrm{div}_x \mathbf{u} = 0$$



George Gabriel Stokes [1819-1903]

Momentum balance

$$\partial_t \mathbf{u} + \mathrm{div}_x (\mathbf{u} \otimes \mathbf{u}) + \nabla_x \Pi = \Delta \mathbf{u}$$

Mathematical modeling of fluids in motion

Molecular dynamics

Fluids understood as huge families of individual particles (atoms, molecules)

Kinetic models

Large ensembles of particles in *random* motion, description in terms of averages

Continuum fluid mechanics

Phenomenological theory based on observable quantities - mass density, temperature, velocity field

Models of turbulence

Essentially based on classical continuum mechanics but description in terms of averaged quantities

Good models?

Stephen William Hawking [*1942]

A model is a good model if it:

- Is elegant
- Contains few arbitrary or adjustable elements
- Agrees with and explains all existing observation
- Makes detailed predictions about future observations that disprove or falsify the model if they are not borne out

Linear vs. nonlinear models

Linear equations

- Solutions built up from elementary functions - modes
- Solvability by means of the symbolic calculus - Laplace and Fourier transform
- Limited applicability

Nonlinear equations

- Explicit solutions known only exceptionally: solitons, simple shock waves
- Possible singularities created by nonlinearity - blow up and/or shocks
- Almost all genuine models are nonlinear

Solvability - classical sense

Jacques Hadamard, [1865 - 1963]

- **Existence.** Given problem is solvable for any choice of (admissible) data
- **Uniqueness.** Solutions are uniquely determined by the data
- **Stability.** Solutions depend continuously on the data

Solvability - modern way

Jacques-Louis Lions,
[1928 - 2001]

- **Approximations.** Given problem admits an approximation scheme that is solvable analytically and, possibly, numerically
- **Uniform bounds.** Approximate solutions possesses uniform bounds depending solely on the data
- **Stability.** The family of approximate solutions admits a limit representing a (generalized) solution of the given problem

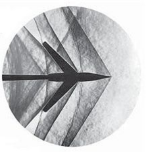
Singularities in nonlinear models

Blow-up singularities - concentrations

Solutions become large (infinite) in a finite time.
There is too much energy pumped in the system

Shock waves - oscillations

Shocks are singularities in "derivatives".
Originally smooth solutions become discontinuous in a finite time

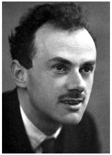


Weak vs. strong

- *Pointwise* (ideal) values of functions are replaced by their *integral averages*. This idea is close to the physical concept of *measurement*
- Derivatives in the equations replaced by integrals:

$$\frac{\partial u}{\partial x} \approx \varphi \mapsto -\int u \partial_x \varphi, \ \varphi \text{ a smooth } \textit{test} \text{ function}$$

Dirac distribution: $\delta_0 : \ \varphi \mapsto \varphi(0)$



Paul Adrien Maurice Dirac [1902-1984]

Field equations - classical vs. weak formulation

- $\mathbf{u} = \mathbf{u}(t,x)$ .... velocity field
- $\varrho = \varrho(t,x)$ ....mass density

Mass conservation

$$\int_B \varrho(t_2,\cdot)\ \mathrm{d}x - \int_B \varrho(t_1,\cdot)\ \mathrm{d}x = -\int_{t_1}^{t_2}\int_{\partial B} \varrho\mathbf{u}\cdot\mathbf{n}\ \mathrm{dS}_x$$

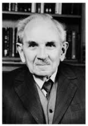
Equation of continuity

$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0$$

Weak formulation

$$\int\int \varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\ \mathrm{d}x\mathrm{d}t = 0 \text{ for } \textit{any} \text{ smooth } \varphi$$

State of the art



Jean Leray - Royal society (1991)

Jean Leray [1906-1998] Global existence of weak solutions for the incompressible Navier-Stokes system (3D)

Olga Aleksandrovna Ladyzhenskaya [1922-2004] Global existence of classical solutions for the incompressible 2D Navier-Stokes system

Pierre-Louis Lions[*1956] Global existence of weak solutions for the compressible barotropic Navier-Stokes system (2,3D)

and many, many others...

What may go wrong...

What is not (?) in classical models

- the fluid velocity may become large or even infinite
- infinite speed of propagation
- "incompressibility" and the non-local character of the pressure in the incompressible models

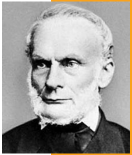
Mathematical problems

- Gap between the existence and uniqueness theory - weak solutions exist globally in time but are not (known to be) unique; strong (classical) solutions (are known to) exist only locally in time
- Possibility of blow-up or concentrations of solutions at some points
- Possibility of fast oscillations, shock waves (?)

Way out?

- Better (more accurate) models
- Better mathematics
- Both?

Do some solutions lose energy?



Rudolph Clausius, [1822–1888]

First and Second law of thermodynamics

Die Energie der Welt ist constant; Die Entropie der Welt strebt einem Maximum zu

Kinetic energy balance for a viscous incompressible fluid

classical: $$\frac{\mathrm{d}}{\mathrm{d}t} \int \frac{1}{2} |\mathbf{u}|^2 \ \mathrm{d}x = -\nu \int |\nabla_x \mathbf{u}|^2$$

weak: $$\frac{\mathrm{d}}{\mathrm{d}t} \int \frac{1}{2} |\mathbf{u}|^2 \ \mathrm{d}x \boxed{\leq} - \nu \int |\nabla_x \mathbf{u}|^2$$

Complete fluid systems

State variables

Mass density

$$\varrho = \varrho(t, x)$$

Absolute temperature

$$\vartheta = \vartheta(t, x)$$

Velocity field

$$\mathbf{u} = \mathbf{u}(t, x)$$

Thermodynamic functions

Pressure

$$p = p(\varrho, \vartheta)$$

Internal energy

$$e = e(\varrho, \vartheta)$$

Entropy

$$s = s(\varrho, \vartheta)$$

Transport

Viscous stress

$$\mathbb{S} = \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$$

Heat flux

$$\mathbf{q} = \mathbf{q}(\vartheta, \nabla_x \vartheta)$$

Field equations

Total energy conservation

$$\frac{\mathrm{d}}{\mathrm{d}t} \int \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e(\varrho, \vartheta) \right) \ \mathrm{d}x = 0$$

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$$

Entropy production

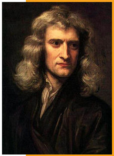
$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho s \mathbf{u}) + \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right) \boxed{\geq} \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \mathbf{u} - \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Second law

Joseph Fourier [1768-1830]

Fourier's law

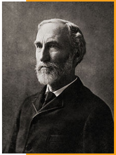
$$\mathbf{q} = -\kappa(\vartheta)\nabla_x \vartheta$$



Isaac Newton [1643-1727]

Newton's rheological law

$$\mathbb{S} = \mu(\vartheta)\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\right) + \eta(\vartheta)\mathrm{div}_x \mathbf{u}\mathbb{I}$$



Willard Gibbs
[1839-1903]

Gibbs' relation:

$$\vartheta Ds(\varrho, \vartheta) = De(\varrho, \vartheta) + p(\varrho, \vartheta) D \left(\frac{1}{\varrho} \right)$$

Thermodynamics stability:

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \ \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Boundary conditions

Impermeability

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

No-slip

$$\mathbf{u}_{\tan}|_{\partial\Omega} = 0$$

No-stick

$$[\mathbb{S} \cdot \mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

Navier's slip

$$[\mathbb{S} \cdot \mathbf{n}]_{\tan} + \beta[\mathbf{u}]_{\tan} = 0$$

Thermal insulation

$$\mathbf{q} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Mathematics of complete system

- Weak solutions exist globally in time for any physically admissible data
- Strong solutions exist locally in time
- **Weak-strong uniqueness.** A weak solution coincides with the strong solution emanating from the same initial data as long as the latter exists. Strong solutions are unique in the class of weak solutions
- **Long-time stability.** Any weak solution stabilizes to an equilibrium state for large time
- **Conditional regularity.** Any weak solution with a bounded velocity gradient is regular (strong)

However...

Sir Winston Churchill, [1874–1965]

However beautiful the strategy, you should occasionally look at the results

Scaled Navier-Stokes-Fourier system

Mass conservation

$$[\mathrm{Sr}]\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

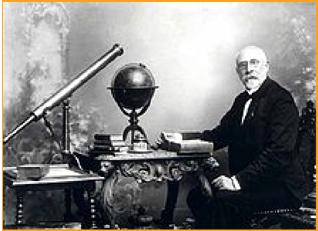
$$[\mathrm{Sr}]\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \left[\frac{1}{\mathrm{Ma}^2}\right] \nabla_x p(\varrho, \vartheta) = \left[\frac{1}{\mathrm{Re}}\right] \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$$

Entropy balance

$$[\mathrm{Sr}]\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \left[\frac{1}{\mathrm{Pe}}\right] \mathrm{div}_x \left(\frac{\mathbf{q}}{\vartheta}\right) = \sigma$$

$$\sigma = \frac{1}{\vartheta}\left(\left[\frac{\mathrm{Ma}^2}{\mathrm{Re}}\right] \mathbb{S} : \nabla_x \mathbf{u} - \left[\frac{1}{\mathrm{Pe}}\right] \frac{\mathbf{q} \cdot \nabla_x \vartheta}{\vartheta}\right)$$

Characteristic numbers - Strouhal number



Čeněk Strouhal
[1850-1922]

Strouhal number

$$[\mathrm{Sr}] = \frac{\mathrm{length}_{\mathrm{char}}}{\mathrm{time}_{\mathrm{char}}\mathrm{velocity}_{\mathrm{char}}}$$

Scaling by means of Strouhal number is used in the study of the long-time behavior of the fluid system, where the characteristic time is large

Mach number

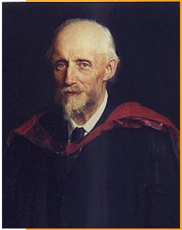
Ernst Mach [1838-1916]

Mach number

$$[\mathrm{Ma}] = \frac{\mathrm{velocity}_{\mathrm{char}}}{\sqrt{\mathrm{pressure}_{\mathrm{char}}/\mathrm{density}_{\mathrm{char}}}}$$

Mach number is the ratio of the characteristic speed to the speed of sound in the fluid. Low Mach number limit, where, formally, the speed of sound is becoming infinite, characterizes incompressibility

Reynolds number



Osborne Reynolds
[1842-1912]

Reynolds number

$$[\mathrm{Re}] = \frac{\mathrm{density}_{\mathrm{char}}\mathrm{velocity}_{\mathrm{char}}\mathrm{length}_{\mathrm{char}}}{\mathrm{viscosity}_{\mathrm{char}}}$$

High Reynolds number is attributed to turbulent flows, where the viscosity of the fluid is negligible

Péclet number

Jean Claude Eugène Péclet [1793-1857]

Péclet number

$$[\mathrm{Pe}] = \frac{\mathrm{pressure}_{\mathrm{char}} \mathrm{velocity}_{\mathrm{char}} \mathrm{length}_{\mathrm{char}}}{\mathrm{heat\ conductivity}_{\mathrm{char}} \mathrm{temperature}_{\mathrm{char}}}$$

High Péclet number corresponds to low heat conductivity of the fluid that may be attributed to turbulent flows

Inviscid incompressible limit

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2} \nabla_x p(\varrho, \vartheta)} = \boxed{\varepsilon^a} \mathrm{div}_x \mathbb{S}(\vartheta, \nabla_x \mathbf{u})$$

Entropy production

$$\partial_t(\varrho s(\varrho, \vartheta)) + \mathrm{div}_x(\varrho s(\varrho, \vartheta)\mathbf{u}) + \boxed{\varepsilon^b} \mathrm{div}_x \left(\frac{\mathbf{q}(\vartheta, \nabla_x \vartheta)}{\vartheta} \right)$$

$$= \frac{1}{\vartheta} \left(\boxed{\varepsilon^{2+a}} \mathbb{S}(\vartheta, \nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \boxed{\varepsilon^b} \frac{\mathbf{q}(\vartheta, \nabla_x \vartheta) \cdot \nabla_x \vartheta}{\vartheta} \right)$$

Boundary conditions and total energy conservation

Navier's slip

$$\mathbf{u}\cdot\mathbf{n}|_{\partial\Omega}=0,\ \ \varepsilon^c[\mathbb{S}(\vartheta,\nabla_x\mathbf{u})\mathbf{n}]_{\rm tan}+\beta(\vartheta)\mathbf{u}|_{\partial\Omega}=0,\ \ c,\beta>0$$

Energy insulation

$$\mathbf{q}(\vartheta,\nabla_x\vartheta)\cdot\mathbf{n}|_{\partial\Omega}=-\beta(\vartheta)\varepsilon^d|\mathbf{u}|^2|_{\partial\Omega},\ \ d=2+a-c-b$$

Total mass and energy conservation

$$\frac{\rm d}{{\rm d}t}\int_\Omega\varrho\ {\rm d}x=0,\ \ \frac{\rm d}{{\rm d}t}\int_\Omega\left(\varepsilon^2\varrho|\mathbf{u}|^2+\varrho e(\varrho,\vartheta)\right)\ {\rm d}x=0$$

Target (limit) system

Leonhard Paul Euler
[1707-1783]

Incompressible Euler system

$$\mathrm{div}_x \mathbf{v} = 0$$

$$\partial_t \mathbf{v} + \mathrm{div}_x (\mathbf{v} \otimes \mathbf{v}) + \nabla_x \Pi = 0$$

$$\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega} = 0$$

Transport equation for temperature deviation

$$\partial_t \mathcal{T} + \mathbf{v} \cdot \nabla_x \mathcal{T} = 0$$