Solvability of some problems in fluid mechanics: Is weak really weak?

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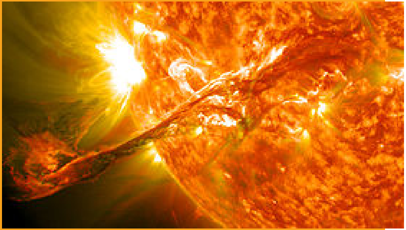
Turbulence or viscosity?

honey



Sun

Eulerian description of motion

Leonhard Paul Euler [1707-1783]

Physical space

- time $t \in [0, \infty)$
- position $\mathbf{x} \in \Omega \subset R^3$

Phenomenological static variable

- mass density $\varrho = \varrho(t, x)$

Bulk motion

- macroscopic velocity $\mathbf{u} = \mathbf{u}(t, x)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t,\mathbf{x}) = \mathbf{u}\Big(t, \mathbf{X}(t,\mathbf{x})\Big), \;\; \mathbf{X}(0,\mathbf{x}) = \mathbf{x}$$

Euler system (compressible, inviscid)

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance - Newton's Second Law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0$$

Solvability - classical way

Jacques Hadamard, [1865 - 1963]

Existence

Given problem is solvable for any choice of (admissible) data

Uniqueness

Solutions are uniquely determined by the data

Stability

Solutions depend continuously on the data

Solvability - modern way

Jacques-Louis Lions, [1928 - 2001]

Approximations

Given problem admits an approximation scheme that is solvable analytically and, possibly, numerically

Uniform bounds

Approximate solutions possesses uniform bounds depending solely on the data

Stability

The family of approximate solutions admits a limit representing a (generalized) solution of the given problem

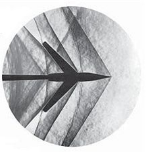
Singularities in nonlinear models

Blow-up singularities - concentrations

Solutions become large (infinite) in a finite time.
There is too much energy pumped in the system

Shock waves - oscillations

Shocks are singularities in "derivatives".
Originally smooth solutions become discontinuous in a finite time

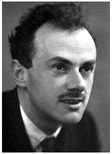


Weak vs. strong

- *Pointwise* (ideal) values of functions are replaced by their *integral averages*. This idea is close to the physical concept of *measurement*
- Derivatives in the equations replaced by integrals:

$$\frac{\partial u}{\partial x} \approx \varphi \mapsto -\int u \partial_x \varphi, \ \varphi \text{ a smooth } \textit{test} \text{ function}$$

Dirac distribution: $\delta_0 : \ \varphi \mapsto \varphi(0)$



Paul Adrien Maurice Dirac [1902-1984]

Field equations - classical vs. weak formulation

Mass conservation - integral formulation

$$\int_B \varrho(t_2,\cdot)\ \mathrm{d}x - \int_B \varrho(t_1,\cdot)\ \mathrm{d}x = -\int_{t_1}^{t_2}\int_{\partial B} \varrho\mathbf{u}\cdot\mathbf{n}\ \mathrm{dS}_x$$

Mass conservation - Equation of continuity

$$\partial_t\varrho + \mathrm{div}_x(\varrho\mathbf{u}) = 0$$

Mass conservation - weak formulation

$$\int\int \varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\ \mathrm{d}x\mathrm{d}t = 0 \text{ for } \textit{any} \text{ smooth } \varphi$$

Existence

Global-in-time solutions (in general) do not exist. Weak solutions may exist but may not be uniquely determined by the initial data.

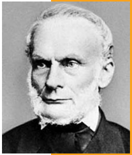
Mechanical energy

$$E = \frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho),\ \ P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z$$

Admissibility criteria - mechanical energy dissipation

$$\partial_t E + \mathrm{div}_x (E\mathbf{u} + p(\varrho)\mathbf{u}) \boxed{\leq} 0$$

Do some solutions lose energy?



bf Rudolph Clausius, [1822–1888]

First and Second law of thermodynamics

Die Energie der Welt ist constant; Die Entropie der Welt strebt einem Maximum zu

Mechnical energy balance for compressible fluid

$$\textbf{classical:}\ \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}|\mathbf{u}|^2 + P(\varrho)\ \mathrm{d}x = 0$$

$$\textbf{weak:}\ \frac{\mathrm{d}}{\mathrm{d}t}\int \frac{1}{2}|\mathbf{u}|^2 + P(\varrho)\ \mathrm{d}x \boxed{\leq} 0$$

Bad or good news for compressible Euler?

Camillo DeLellis [*1976]

Existence

Good news: There exists a global-in-time weak solution of compressible Euler system for "any" initial data.
Bad news: There are infinitely many...

Dissipative solutions

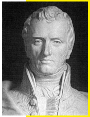
Good news: Most of the "wild" solutions produce energy.
Bad news: There is a vast class of data for which there exist infinitely many admissible (dissipative) weak solutions. Specifically, for any initial distribution of the density, there is a velocity field...

László Székelyhidi [*1977]

Viscosity solutions or maximal dissipation?

The "correct" solutions are obtained as limits of the viscous system

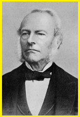
Navier-Stokes system (compressible, viscous)



Claude Louis Marie Henri Navier [1785-1836]

Mass conservation

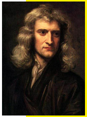
$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$



George Gabriel Stokes [1819-1903]

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \boxed{\mathrm{div}_x \mathbb{S}}$$

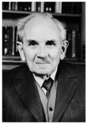


Isaac Newton [1643-1727]

Newton's rheological law

$$\mathbb{S} = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

State of the art for viscous fluids


Jean Leray - Royal society (1991)

Jean Leray [1906-1998] Global existence of weak solutions for the incompressible Navier-Stokes system (3D)

Olga Aleksandrovna Ladyzhenskaya [1922-2004] Global existence of classical solutions for the incompressible 2D Navier-Stokes system

Pierre-Louis Lions[*1956] Global existence of weak solutions for the compressible barotropic Navier-Stokes system (2,3D)

and many, many others...

Some good news to finish...

Numerical solution for the compressible Navier-Stokes system

Numerical schemes based on a combination of finite volumes - finite elements schemes (Karlsen, Karper, Gallouet et al.)

Synergy analysis-numerics

- Certain numerical schemes converge to *weak* solutions
- Convergence is unconditional and even error estimates are available of the limit solution is smooth
- Bounded weak solutions *are* smooth
- Bounded solutions of the numerical scheme converge (with error estimates) to the smooth solution