**Shape sensitivity analysis of time-dependent flows of incompressible non-Newtonian fluids**

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# 1 Introduction

We consider the time-dependent flow of an incompressible fluid in a bounded domain $\Omega := B \setminus S$ in $\mathbb{R}^2$, where $B$ is a container and $S$ is an obstacle, described by the system of equations:

$$\begin{aligned}
\partial_t \mathbf{v} + \operatorname{div}(\mathbf{v} \otimes \mathbf{v}) - \operatorname{div} \mathbb{S}(\mathbb{D}\mathbf{v}) + \nabla p + \mathbb{C}\mathbf{v} &= \mathbf{f} && \text{in } Q,\\
\operatorname{div} \mathbf{v} &= 0 && \text{in } Q,\\
\mathbf{v} &= 0 && \text{on } \Sigma,\\
\mathbf{v}(0,\cdot) &= \mathbf{v}_0 && \text{in } \Omega.
\end{aligned} \tag{$P(\Omega)$}$$

Here $Q := (0,T) \times \Omega$, $\Sigma := (0,T) \times \partial\Omega$, where $(0,T)$ is a time interval of arbitrary length, $\mathbf{v}$, $p$, $\mathbb{C}$, $\mathbf{f}$ stands for the velocity, the pressure, the constant skew-symmetric Coriolis tensor and the body force, respectively. The traceless part $\mathbb{S}$ of the Cauchy stress can depend on the symmetric part $\mathbb{D}\mathbf{v}$ of the velocity gradient; for simplicity we assume the power-law model

$$\mathbb{S}(\mathbb{D}\mathbf{v}) = (1 + |\mathbb{D}\mathbf{v}|^2)^{r-2}\mathbb{D}\mathbf{v},\ r \in [2,4). \tag{1}$$

The Coriolis term $\mathbb{C}\mathbf{v}$ appears e.g. when the change of variables is performed in order to take into account the flight scenario of the obstacle in the fluid or gas.

We investigate the differentiability of the drag functional

$$J(\Omega) := \int_0^T \int_{\partial S} (\mathbb{S}(\mathbb{D}\mathbf{v}) - p\mathbb{I})\mathbf{n} \cdot \mathbf{d}, \tag{2}$$

where $\mathbf{d}$ is a given constant unit vector, with respect to the variations of the shape of the obstacle.

Let $\mathbf{T} \in \mathcal{C}^2(\mathbb{R}^2, \mathbb{R}^2)$ be a vector field vanishing in the vicinity of $\partial B$ and define the mapping

$$\mathbf{y}(\mathbf{x}) = \mathbf{x} + \varepsilon \mathbf{T}(\mathbf{x}).$$

For small $\varepsilon > 0$ the mapping $\mathbf{x} \mapsto \mathbf{y}(\mathbf{x})$ takes diffeomorphically the region $\Omega$ onto $\Omega_\varepsilon = B \setminus S_\varepsilon$ where $S_\varepsilon = \mathbf{y}(S)$. Denoting $Q_\varepsilon := (0,T) \times \Omega_\varepsilon$, $\Sigma_\varepsilon := (0,T) \times \partial\Omega_\varepsilon$, we consider the counterpart of problem $(P(\Omega))$ in $Q_\varepsilon$, with the data $\mathbf{f}_{|Q_\varepsilon}$ and $\mathbf{v}_{0|\Omega_\varepsilon}$. The new problem will be denoted by $(P(\Omega_\varepsilon))$ and its solution by $(\bar{\mathbf{v}}_\varepsilon, \bar{p}_\varepsilon)$.

The shape derivative $\mathbf{v}'$ and the material derivative $\dot{\mathbf{v}}$ are formally introduced by

$$\mathbf{v}' := \lim_{\varepsilon \to 0} \frac{\bar{\mathbf{v}}_\varepsilon - \mathbf{v}}{\varepsilon}, \quad \dot{\mathbf{v}} := \lim_{\varepsilon \to 0} \frac{\bar{\mathbf{v}}_\varepsilon \circ \mathbf{y} - \mathbf{v}}{\varepsilon}.$$

In order to obtain the boudary representation of the shape gradient it is convenient to introduce the adjoints through the linearized system:

*Find the couple* $(\mathbf{w}, s)$ *such that*

$$\begin{aligned} -\partial_t \mathbf{w} - 2(\mathbb{D}\mathbf{w})\mathbf{v} - \operatorname{div}\left[\mathbb{S}'(\mathbb{D}\mathbf{v})^\top \mathbb{D}\mathbf{w}\right] + \nabla s - \mathbb{C}\mathbf{w} &= \mathbf{0} && \text{in } Q, \\ \operatorname{div}\mathbf{w} &= 0 && \text{in } Q, \\ \mathbf{w} &= \mathbf{d} && \text{on } \Sigma, \qquad (P_{\mathrm{adj}}(\Omega)) \\ \mathbf{w}(T,\cdot) &= \mathbf{0} && \text{in } \Omega. \end{aligned}$$

It is possible to show under certain assumptions on $\Omega$, $\mathbf{f}$ and $\mathbf{v}_0$ that there is a unique weak solution of $(P(\Omega))$ which satisfies:

$$\mathbf{v} \in L^\infty(0,T;\mathbf{W}^{2,q}(\Omega)), \quad \nabla\mathbf{v} \in \mathcal{C}^{0,\alpha}(\overline{Q}), \tag{3}$$

for some $q > 2$ and $\alpha > 0$, see [1] for further details.

# 2 Main results

The first step is the existence of the material derivative.

**Theorem 1.** *Let* (3) *be satisfied. Then*

*(i) the material and shape derivatives of* $\mathbf{v}$ *exist and depend contiuously on* $\|\mathbf{T}\|_{\mathcal{C}^2}$*;*

*(ii) the adjoint problem* $(P_{\mathrm{adj}}(\Omega))$ *has a unique weak solution;*

*(iii) the shape gradient of* $J$ *exists and is given by*

$$dJ(\Omega;\mathbf{T}) = -\lim_{\delta\searrow 0}\int_0^{T-\delta}\int_{\partial S}\left[\left(\mathbb{S}'(\mathbb{D}\mathbf{v})^\top\mathbb{D}\mathbf{w} - s\mathbb{I}\right) : \frac{\partial\mathbf{v}}{\partial\mathbf{n}}\otimes\mathbf{n} + \mathbf{f}\cdot\mathbf{d}\right]\mathbf{T}\cdot\mathbf{n}. \tag{4}$$

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# Reference

[1] J. Sokołowski and J. Stebel. Shape sensitivity analysis of time-dependent flows of incompressible non-Newtonian fluids. *Control and Cybernetics*. Submitted.