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Quantitative coarse embeddings of quasi-Banach spaces into a Hilbert space

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QUANTITATIVE COARSE EMBEDDINGS OF QUASI-BANACH SPACES INTO A HILBERT SPACE

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ABSTRACT. We study how well a quasi-Banach space can be coarsely embedded into a Hilbert space. Given any quasi-Banach space X which coarsely embeds into a Hilbert space, we compute its Hilbert space compression exponent. We also show that the Hilbert space compression exponent of X is equal to the supremum of the amounts of snowflakings of X which admit a bi-Lipschitz embedding into a Hilbert space.

1. INTRODUCTION

Let (M, d_M) and (N, d_N) be metric spaces and let $T\colon M \to N$ be a mapping. Then T is called a *coarse embedding* if there are nondecreasing functions $\rho_1, \rho_2\colon [0,\infty) \to [0,\infty)$ such that $\lim_{t\to\infty} \rho_1(t) = \infty$ and

$$\rho_1(d_M(x,y)) \le d_N(T(x),T(y)) \le \rho_2(d_M(x,y)) \text{ for all } x,y \in M.$$

We say that M *coarsely embeds* into N if there is a coarse embedding of M into N. The reader should be warned that what we call a coarse embedding is called a uniform embedding by some authors. We use the term coarse embedding because in the nonlinear geometry of Banach spaces the term uniform embedding is used for a uniformly continuous injective mapping whose inverse is also uniformly continuous.

Randrianarivony [Ra, Theorem 1] gave a characterization of those quasi-Banach spaces which coarsely embed into a Hilbert space. More precisely, she proved that a quasi-Banach space coarsely embeds into a Hilbert space if and only if it is linearly isomorphic to a subspace of $L_0(\mu)$ for some probability space $(\Omega, \mathcal{B}, \mu)$ ($L_0(\mu)$ is the space of all equivalence classes of real measurable functions on $(\Omega, \mathcal{B}, \mu)$ with the topology of convergence in probability). In this note, we are interested in how well a quasi-Banach space can be coarsely embedded into a Hilbert space. To measure it, we will use the following notion introduced by Guentner and Kaminker [GK, Definition 2.2].

Suppose again that (M, d_M) and (N, d_N) are metric spaces, with M unbounded. Recall that a mapping $T\colon M \to N$ is *large-scale Lipschitz* if there is $A > 0$ and $B \ge 0$ such that $d_N(T(x),T(y)) \le A d_M(x,y) + B$ for all $x, y \in M$. The *compression exponent* of M in N, denoted by $\alpha_N(M)$, is defined to be the supremum of all $\alpha \ge 0$ for which there is a large-scale Lipschitz mapping $T\colon M \to N$ and constants $C, t > 0$ such that $d_N(T(x),T(y)) \ge C d_M(x,y)^\alpha$ if $d_M(x,y) \ge t$ (with the understanding that $\alpha_N(M) = 0$ if there is no such α). It is clear that $\alpha_N(M) \le 1$ (since M is unbounded) and that if $\alpha_N(M) > 0$, then M coarsely embeds into N. The closer $\alpha_N(M)$ is to one, the "better" we can coarsely embed M into N. The *Hilbert space compression exponent* of M, denoted by $\alpha(M)$, is the supremum of all $\alpha \ge 0$ for which there is a Hilbert space H, a large-scale Lipschitz mapping $T\colon M \to H$ and constants $C, t > 0$ such that $\|T(x) - T(y)\|_H \ge C d_M(x,y)^\alpha$ if $d_M(x,y) \ge t$.

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Equivalently,

$$\alpha(M) = \sup_{H \text{ is a Hilbert space}} \alpha_H(M).$$

Analogous remarks to those on $\alpha_N(M)$ apply to $\alpha(M)$ as well.

Our method of establishing a lower estimate for the Hilbert space compression exponent of a quasi-Banach space actually gives a stronger information. We will use one more type of parameter which will capture this additional information.

Let (M, d_M) and (N, d_N) be metric spaces. Recall that a mapping $T\colon M \to N$ is called a *bi-Lipschitz embedding* if there are constants $A, B > 0$ such that

$$Ad_M(x,y) \leq d_N(T(x), T(y)) \leq Bd_M(x,y) \text{ for all } x, y \in M. \tag{1}$$

Recall also that if $0 < \alpha < 1$, then d_M^α is also a metric on M (the space (M, d_M^α) is sometimes called the *α-snowflaked version* of (M, d_M)). We denote by $s_N(M)$ the supremum of all $0 < \alpha \leq 1$ for which the space (M, d_M^α) admits a bi-Lipschitz embedding into (N, d_N). Let further $s(M)$ be the supremum of all $0 < \alpha \leq 1$ for which the space (M, d_M^α) admits a bi-Lipschitz embedding into a Hilbert space. It is clear that if M is unbounded, then $0 \leq s_N(M) \leq \alpha_N(M) \leq 1$ and $0 \leq s(M) \leq \alpha(M) \leq 1$. The parameter $s_N(M)$ was introduced and studied by Albiac and Baudier [AB] in the case when M and N were ℓ_p-spaces.

We use symbols $\alpha_N(M)$, $\alpha(M)$, $s_N(M)$ and $s(M)$ when the metrics on M and N are clear from the context, otherwise we write for example $\alpha_N(M, d_M)$.

The values of $s(X)$ and $\alpha(X)$ are known if X is a space ℓ_p or $L_p(0,1)$ for $0 < p < \infty$. Let us recall the results. Recall first that if $0 < p < 1$, then the canonical metric on ℓ_p is defined by $d_p(x,y) = \sum_{i=1}^\infty |x_i - y_i|^p$, where $x = (x_i), y = (y_i)$, and similarly the canonical metric on $L_p(0,1)$ is defined by $d_p(f,g) = \int_0^1 |f(t) - g(t)|^p \mathrm{d}t$. Baudier [Ba, Corollaries 2.23 and 2.19] proved that if $0 < p < q < \infty$ and $q \geq 1$, then

$$s_{\ell_q}(\ell_p) = \alpha_{\ell_q}(\ell_p) = \frac{\max\{p,1\}}{q} \tag{2}$$

(the case $q = 1$ was already proved in [Al, Proposition 4.1(ii)]). It follows that if $0 < p \leq 2$, then

$$s(\ell_p) = \alpha(\ell_p) = \frac{\max\{p,1\}}{2}. \tag{3}$$

If $p > 2$, then ℓ_p does not coarsely embed into a Hilbert space (this was first proved in [JR]), hence $s(\ell_p) = \alpha(\ell_p) = 0$.

It also follows from [Ba, after Corollary 2.19], [MN, Remark 5.10] and [Al, Proposition 6.5] that if $0 < p \leq 2$, $q \geq 1$ and $p < q$, then

$$s_{L_q(0,1)}(L_p(0,1)) = \alpha_{L_q(0,1)}(L_p(0,1)) = \frac{\max\{p,1\}}{\min\{q,2\}}.$$

Hence if $0 < p \leq 2$, then

$$s(L_p(0,1)) = \alpha(L_p(0,1)) = \frac{\max\{p,1\}}{2}. \tag{4}$$

If $p > 2$, then $s(L_p(0,1)) = \alpha(L_p(0,1)) = 0$ since $L_p(0,1)$ does not coarsely embed into a Hilbert space (because it contains an isometric copy of ℓ_p).

Let us mention that unlike the case of the spaces ℓ_p described in (2), the precise values of $s_{L_q(0,1)}(L_p(0,1))$ and $\alpha_{L_q(0,1)}(L_p(0,1))$ are not known if $2 < p < q$. However, some estimates are known. If $2 < p < q$, a construction due to Mendel and Naor [MN, Remark 5.10] shows that $\alpha_{L_q(0,1)}(L_p(0,1)) \geq s_{L_q(0,1)}(L_p(0,1)) \geq \frac{p}{q}$, and Naor and Schechtman [NS] recently proved that $s_{L_q(0,1)}(L_p(0,1)) < 1$.

In this note, we compute the values of $s(X)$ and $\alpha(X)$ for any quasi-Banach space X which coarsely embeds into a Hilbert space. A few remarks are in order.

If X is a Banach space with a norm $\|.\|$, then the canonical metric on X is given by $(x,y) \mapsto \|x-y\|$ and there is no problem with the definition of $s(X)$ and $\alpha(X)$. However, if X is a general quasi-Banach space, we cannot speak about some canonical metric on X. The usual way how to introduce a metric on X is to use a theorem of Aoki [Ao] and Rolewicz [Ro] (see also [BL, Proposition H.2]), which says that there is $0 < r \leq 1$ and an equivalent quasi-norm $\|.\|$ on X which is r-*subadditive*, that is, $\|x+y\|^r \leq \|x\|^r + \|y\|^r$ for all $x, y \in X$. Then $(x,y) \mapsto \|x-y\|^r$ is an invariant metric on X, which induces the same topology on X as the original quasi-norm. Of course, there are many such metrics on X and $s(X)$ and $\alpha(X)$ depend on the metric. (On the other hand, it is clear that the coarse embeddability of X into a Hilbert space does not depend on the choice of the above described metric. When we say that X coarsely embeds into a Hilbert space, it is understood that it is with respect to any such metric on X.) So, if X is a quasi-Banach space which coarsely embeds into a Hilbert space, we compute $s(X)$ and $\alpha(X)$ with respect to any such metric on X. The result is stated in Theorem 3.1. If X does not coarsely embed into a Hilbert space, then, of course, $s(X) = \alpha(X) = 0$ with respect to any such metric on X. The corresponding results for the spaces ℓ_p and $L_p(0,1)$, $0 < p < \infty$, mentioned above are a particular case of this since the canonical metrics on ℓ_p and $L_p(0,1)$ for any $0 < p < \infty$ are of the form described above.

2. Preliminaries

The notation and terminology is standard, as may be found for example in [BL]. All vector spaces throughout the paper are supposed to be over the real field. Recall that if $(\Omega, \mathcal{B}, \mu)$ is a measure space, where μ is a nonnegative measure, and $0 < p < \infty$, then $L_p(\mu)$ is the (quasi-)Banach space of all equivalence classes of real measurable functions f on $(\Omega, \mathcal{B}, \mu)$ for which $\|f\|_p = \left(\int |f|^p \mathrm{d}\mu\right)^{\frac{1}{p}} < \infty$. If $1 \leq p < \infty$, then $\|.\|_p$ is a norm on $L_p(\mu)$, whereas if $0 < p < 1$, it is only a quasi-norm (except in the trivial cases when $L_p(\mu)$ is zero or one-dimensional). If $0 < p < 1$, then the canonical metric on $L_p(\mu)$ is given by $d_p(f,g) = \|f-g\|_p^p = \int |f-g|^p \mathrm{d}\mu$. If $1 \leq p < \infty$, then the canonical metric on $L_p(\mu)$ is given by the norm (as on any Banach space), and we denote it by d_p as well, so $d_p(f,g) = \|f-g\|_p$. If not stated otherwise, all metric properties of the space $L_p(\mu)$ for any $0 < p < \infty$ are regarded with respect to the metric d_p. Special cases like $L_p(0,1)$, ℓ_p and $\ell_p^n, n \in \mathbb{N}$, are defined in a standard way.

Let X be a quasi-Banach space (for a brief overview of quasi-Banach spaces see for example [BL, Appendix H]). As we have already mentioned, by the theorem of Aoki and Rolewicz, there is $0 < r \leq 1$ and an equivalent quasi-norm $\|.\|$ on X which is r-*subadditive*, that is, $\|x+y\|^r \leq \|x\|^r + \|y\|^r$ for all $x, y \in X$. In particular, $(x,y) \mapsto \|x-y\|^r$ is an invariant metric on X, which we denote by $d_{\|.\|,r}$ and which induces the same topology on X as the original quasi-norm. Let $0 < r \leq 1$. An r-subadditive quasi-norm on X is called an r-*norm* (so a 1-norm is just a norm). If there is an equivalent r-norm on X, then we say that X is r-*normable* (and instead of 1-normable we just say *normable*). We denote by M_X the set of all $0 < r \leq 1$ for which X is r-normable. Furthermore, we define $r_X = \sup M_X$. By the theorem of Aoki and Rolewicz, we have $M_X \neq \emptyset$ and hence $r_X > 0$. It is clear that M_X is either the interval $(0, r_X]$ or $(0, r_X)$.

For example, if X is a Banach space, then clearly $M_X = (0,1]$ and $r_X = 1$. Let $0 < p < 1$ and consider a space $L_p(\mu)$ for some nonnegative measure μ. Then $\|.\|_p$ is a p-norm on $L_p(\mu)$ and the canonical metric d_p on $L_p(\mu)$ is the metric $d_{\|.\|_p,p}$. If $L_p(\mu)$ is in addition infinite-dimensional, then it is not hard to prove that $M_{L_p(\mu)} = (0,p]$, and hence $r_{L_p(\mu)} = p$.

As we have said, if X is a quasi-Banach space which coarsely embeds into a Hilbert space, then our goal is to compute $s(X, d_{\|.\|,r})$ and $\alpha(X, d_{\|.\|,r})$ for any $r \in M_X$ and any equivalent r-norm $\|.\|$ on X. To state (and prove) the result, we will need the notion of type of a quasi-Banach space and some of its properties.

A quasi-Banach space X, equipped with a quasi-norm $\|.\|$, is said to have *type* p, where $0 < p \leq 2$, if there is a constant $C > 0$ such that for every $n \in \mathbb{N}$ and every $x_1, \dots, x_n \in X$ we have

$$\mathbb{E} \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\|^p \leq C^p \sum_{i=1}^{n} \|x_i\|^p,$$

where $\mathbb{E}$ denotes the expectation with respect to a uniform choice of signs $(\varepsilon_1, \dots, \varepsilon_n) \in \{-1, 1\}^n$. Note that if $|||.|||$ is a quasi-norm on X equivalent to $\|.\|$, then $(X, |||.|||)$ has type p if and only if $(X, \|.\|)$ has type p. We define

$$p_X = \sup\{0 < p \leq 2 : X \text{ has type } p\}.$$

The quantities p_X and r_X are related as follows.

Lemma 2.1. *Let X be a quasi-Banach space. Then $r_X = \min\{p_X, 1\}$.*

Proof. If $r \in M_X$, then it is clear that X has type r. Hence $r_X \leq p_X$ and since $r_X \leq 1$, we obtain $r_X \leq \min\{p_X, 1\}$.

Let us show that $r_X \geq \min\{p_X, 1\}$. If $p_X > 1$, then, by [Ka2, Theorem 2.1(2)], X is normable, and therefore $r_X = 1 = \min\{p_X, 1\}$. If $p_X \leq 1$, then, by [Ka2, Theorem 2.1(1)], $r_X \geq p_X = \min\{p_X, 1\}$. □

In particular, it follows from Lemma 2.1 that if X is a quasi-Banach space, then $p_X > 0$ (since $r_X > 0$). Let us mention that we will not actually need the full strength of Lemma 2.1, but only the trivial inequality $r_X \leq p_X$.

We will also use the following result. For Banach spaces it is the classical theorem of Maurey and Pisier [MP] (see also [MS, 13.2. Theorem]). The generalization to quasi-Banach spaces presented here was proved by Kalton [Ka1]. Recall that if X and Y are quasi-Banach spaces and $T\colon X \to Y$ is a linear mapping, then one defines $\|T\| = \sup\{\|T(x)\| : \|x\| \leq 1\}$. A quasi-Banach space Y is said to be *finitely representable* in a quasi-Banach space X if for every $\varepsilon > 0$ and every finite-dimensional subspace E of Y there is a subspace F of X with $\dim F = \dim E$ and a linear isomorphism $T\colon E \to F$ such that $\|T\| \cdot \|T^{-1}\| \leq 1 + \varepsilon$.

Theorem 2.2 (Kalton)**.** *Let X be an infinite-dimensional quasi-Banach space equipped with an r-norm, where $0 < r \leq 1$. Then ℓ_{p_X} is finitely representable in X.*

The above theorem follows from [Ka1, Theorem 4.6]. Let us mention that [Ka1, Theorem 4.6] is stated for the so-called *convexity type* $p(X)$ of X instead of for our p_X. However, it is not difficult to prove using the results of [Ka1] that $p(X) = p_X$.

3. Main result

Theorem 3.1. *Let X be a quasi-Banach space which coarsely embeds into a Hilbert space. Then for every $r \in M_X$ and every equivalent r-norm $\|.\|$ on X we have*

$$s(X, d_{\|.\|,r}) = \alpha(X, d_{\|.\|,r}) = \min\left\{\frac{p_X}{2r}, 1\right\}.$$

Before we turn to the proof of the above theorem, let us make a few remarks. First, note that Theorem 3.1 yields in particular that if X is a Banach space which coarsely embeds into a Hilbert space, then

$$s(X) = \alpha(X) = \frac{p_X}{2}. \tag{5}$$

As we have said before, (3) and (4) follow from Theorem 3.1. Indeed, let $0 < p \le 2$ and consider an infinite-dimensional space $L_p(\mu)$ for some nonnegative measure μ. Then $L_p(\mu)$ coarsely embeds into a Hilbert space (see [No, Proposition 4.1] or Lemma 3.2 bellow). If $1 \le p \le 2$, then we can use (5) and obtain

$$s(L_p(\mu)) = \alpha(L_p(\mu)) = \frac{p_{L_p(\mu)}}{2} = \frac{p}{2}.$$

If $0 < p < 1$, then Theorem 3.1 yields

$$s(L_p(\mu)) = \alpha(L_p(\mu)) = s(L_p(\mu), d_{\|.\|_p,p}) = \alpha(L_p(\mu), d_{\|.\|_p,p}) = \min\left\{\frac{p_{L_p(\mu)}}{2p}, 1\right\} = \frac{1}{2}.$$

In particular, this gives (3) and (4).

Let X be a quasi-Banach space which coarsely embeds into a Hilbert space, let $r \in M_X$ and let $\|.\|$ be an equivalent r-norm on X. By Theorem 3.1 and Lemma 2.1 we have

$$\alpha(X, d_{\|.\|,r}) = \min\left\{\frac{p_X}{2r}, 1\right\} \ge \min\left\{\frac{p_X}{2r_X}, 1\right\} \ge \frac{1}{2},$$

and this estimate is of course sharp ($\alpha(\ell_1) = \frac{1}{2}$). This is not true for general metric spaces. For example, Arzhantseva, Druţu and Sapir [ADS, Theorem 1.5] proved that for every $\alpha \in [0, 1]$ there is a finitely generated group, equipped with a word length metric, that coarsely embeds into a Hilbert space and whose Hilbert space compression exponent is equal to α.

Note also that in Theorem 3.1 we cannot omit the assumption that X coarsely embeds into a Hilbert space. Indeed, if X is a quasi-Banach space which does not coarsely embed into a Hilbert space, $r \in M_X$ and $\|.\|$ is an equivalent r-norm on X, then $s(X, d_{\|.\|,r}) = \alpha(X, d_{\|.\|,r}) = 0 < \min\left\{\frac{p_X}{2r}, 1\right\}$, since $p_X > 0$.

Let us now prove Theorem 3.1. Let us first consider the inequality $s(X, d_{\|.\|,r}) \ge \min\left\{\frac{p_X}{2r}, 1\right\}$. Our method of proof is a quantification of Randrianarivony's proof that if X is a quasi-Banach space which is linearly isomorphic to a subspace of $L_0(\mu)$ for some probability space $(\Omega, \mathcal{B}, \mu)$, then X coarsely embeds into a Hilbert space [Ra, Proof of Theorem 1]. We will use the following well-known fact.

Lemma 3.2. *Let $0 < p \le 2$ and let $(\Omega, \mathcal{B}, \mu)$ be a measure space, where μ is a nonnegative measure. Then there is a Hilbert space H and a mapping $S\colon L_p(\mu) \to H$ such that $\|S(x) - S(y)\|_H = \|x - y\|_p^{\frac{p}{2}}$ for all $x, y \in L_p(\mu)$.*

Proof. The function $\|.\|_p^p$ on $L_p(\mu)$ is negative definite by [BL, p. 186, Examples. (iii)] (for a survey on negative definite kernels and functions see [BL, Chapter 8]) and $\|0\|_p^p = 0$, and therefore, by [BL, Proposition 8.5(ii)], there is a Hilbert space H and a mapping $S\colon L_p(\mu) \to H$ such that $\|x - y\|_p^p = \|S(x) - S(y)\|_H^2$ for all $x, y \in L_p(\mu)$. Let us mention that the proof of [BL, Proposition 8.5(ii)] actually gives a complex Hilbert space H, but it is easy to see that there is a real Hilbert space H with the desired properties. □

Proof of $s(X, d_{\|.\|,r}) \ge \min\left\{\frac{p_X}{2r}, 1\right\}$ in Theorem 3.1. Let $r \in M_X$ and let $\|.\|$ be an equivalent r-norm on X.

Since X coarsely embeds into a Hilbert space, [Ra, Theorem 1] implies that there is a probability space $(\Omega, \mathcal{B}, \mu)$ such that X is linearly isomorphic to a subspace of $L_0(\mu)$. By [BL, Theorem 8.15], then, the space X is linearly isomorphic to a subspace of $L_p(\mu)$ for every $0 < p < p_X$.

Let p be such that $0 < p < p_X$ and let $\varphi\colon X \to L_p(\mu)$ be an isomorphism into. Then there are $A, B > 0$ such that

$$A\|x\| \le \|\varphi(x)\|_p \le B\|x\| \text{ for every } x \in X.$$

By Lemma 3.2, there is a Hilbert space H and a mapping $S\colon L_p(\mu) \to H$ such that

$$\|S(x) - S(y)\|_H = \|x - y\|_p^{\frac{p}{2}} \text{ for all } x, y \in L_p(\mu).$$

Let $T = S \circ \varphi$. Then T maps X into H and for all $x, y \in X$ we have

$$A^{\frac{p}{2}}(\|x-y\|^r)^{\frac{p}{2r}} \leq \|T(x) - T(y)\|_H \leq B^{\frac{p}{2}}(\|x-y\|^r)^{\frac{p}{2r}}.$$

Hence if p is such that $\frac{p}{2r} \leq 1$, then T is a bi-Lipschitz embedding of $(X, d_{\|.\|,r}^{\frac{p}{2r}})$ into H. It follows that $s(X, d_{\|.\|,r}) \geq \min\left\{\frac{p_X}{2r}, 1\right\}$. □

Remark 3.3. The above proof actually shows that if $r \in M_X$ and $\|.\|$ is an equivalent r-norm on X, then for every $\alpha > 0$ such that $\alpha < \frac{p_X}{2r}$ and $\alpha \leq 1$ the space $(X, d_{\|.\|,r}^{\alpha})$ admits a bi-Lipschitz embedding into a Hilbert space.

Since the inequality $s(X, d_{\|.\|,r}) \leq \alpha(X, d_{\|.\|,r})$ in Theorem 3.1 is trivial, to complete the proof of Theorem 3.1 it only remains to prove the inequality $\alpha(X, d_{\|.\|,r}) \leq \min\left\{\frac{p_X}{2r}, 1\right\}$.

First, let us recall several useful notions. Let (M, d_M) and (N, d_N) be metric spaces and let $T\colon M \to N$ be a mapping. The *Lipschitz constant* of T is defined by

$$\operatorname{Lip}(T) = \sup_{x,y \in M, x \neq y} \frac{d_N(T(x), T(y))}{d_M(x,y)}.$$

If $T\colon M \to N$ is injective, then the *distortion* of T is defined by

$$\operatorname{distortion}(T) = \operatorname{Lip}(T) \cdot \operatorname{Lip}(T^{-1}),$$

where T^{-1} is regarded as a mapping on $T(M)$. Let us mention that if $\operatorname{distortion}(T) < \infty$, then T is a bi-Lipschitz embedding and $\operatorname{distortion}(T) = \inf \frac{B}{A}$, where the infimum is taken over all constants $A, B > 0$ for which (1) holds. The *distortion* of M in N is defined by

$$c_N(M) = \inf_{T\colon M \to N \text{ injective}} \operatorname{distortion}(T).$$

A metric space (M, d_M) is called *d-discrete*, where $d > 0$, if $d_M(x,y) \geq d$ for all $x, y \in M, x \neq y$. The *diameter* of M is defined by $\operatorname{diam}(M) = \sup_{x,y \in M} d_M(x,y)$.

We will use the following modification of a lemma of Austin [Au, Lemma 3.1], which in its original form was used for estimating from above the compression exponents in L_p-spaces of certain groups. A version of Austin's lemma was also used by Baudier [Ba, proof of Corollary 2.22] to show that if $0 < p \leq 1 \leq q < \infty$, then $\alpha_{L_q}(\ell_p) \leq \frac{1}{\min\{q,2\}}$.

Lemma 3.4. *Let X be a quasi-Banach space, $r \in M_X$ and $\|.\|$ be an equivalent r-norm on X. Let Y be a Banach space. Suppose further that (M_n, δ_n), $n \in \mathbb{N}$, are finite d-discrete metric spaces, where $d > 0$, such that*

- *$\operatorname{diam}(M_n) \to \infty$,*
- *there is $\gamma \in (0,1]$ and $A, B > 0$ such that for each $n \in \mathbb{N}$ there is a mapping $f_n\colon M_n \to X$ satisfying*

$$A\delta_n(x,y)^\gamma \leq \|f_n(x) - f_n(y)\|^r \leq B\delta_n(x,y) \text{ for all } x, y \in M_n,$$

- *there is $\eta \in (0,1]$ and $K > 0$ such that $c_Y(M_n) \geq K\operatorname{diam}(M_n)^\eta$ for every $n \in \mathbb{N}$.*

Then $\alpha_Y(X, d_{\|.\|,r}) \leq \frac{1-\eta}{\gamma}$.

Proof. If $\alpha_Y(X, d_{\|.\|,r}) = 0$, then the result is trivial, so suppose that $\alpha_Y(X, d_{\|.\|,r}) > 0$. Let $\alpha \in (0, \alpha_Y(X, d_{\|.\|,r})]$ be such that there is a large-scale Lipschitz mapping

$T\colon (X, d_{\|.\|,r}) \to Y$ and constants $C, t > 0$ such that $\|T(x)-T(y)\|_Y \geq C(\|x-y\|^r)^\alpha$ if $\|x-y\|^r \geq t$. Then for some $D > 0$ we have

$$C(\|x-y\|^r)^\alpha \leq \|T(x)-T(y)\|_Y \leq D\|x-y\|^r \text{ if } \|x-y\|^r \geq t.$$

By rescaling if necessary, we may clearly suppose that $t \leq Ad^\gamma$.

Let $n \in \mathbb{N}$. Let us estimate from above the distortion of $T \circ f_n\colon M_n \to Y$. If $x, y \in M_n, x \neq y$, then

$$\|f_n(x) - f_n(y)\|^r \geq A\delta_n(x,y)^\gamma \geq Ad^\gamma \geq t,$$

hence

$$C(\|f_n(x)-f_n(y)\|^r)^\alpha \leq \|T \circ f_n(x) - T \circ f_n(y)\|_Y \leq D\|f_n(x) - f_n(y)\|^r,$$

and therefore

$$CA^\alpha \delta_n(x,y)^{\gamma\alpha} \leq \|T \circ f_n(x) - T \circ f_n(y)\|_Y \leq DB\delta_n(x,y)$$

(in particular, $T \circ f_n$ is injective). Consequently,

$$\begin{aligned}
&\operatorname{distortion}(T\circ f_n) = \operatorname{Lip}(T \circ f_n) \cdot \operatorname{Lip}\left((T\circ f_n)^{-1}\right) \\
&= \max_{x,y\in M_n, x\neq y} \frac{\|T\circ f_n(x) - T\circ f_n(y)\|_Y}{\delta_n(x,y)} \cdot \max_{x,y\in M_n, x\neq y} \frac{\delta_n(x,y)}{\|T\circ f_n(x) - T \circ f_n(y)\|_Y} \\
&\leq \frac{BD}{A^\alpha C} \max_{x,y\in M_n, x\neq y} \delta_n(x,y)^{1-\gamma\alpha} \\
&= \frac{BD}{A^\alpha C} \operatorname{diam}(M_n)^{1-\gamma\alpha}.
\end{aligned}$$

Hence

$$c_Y(M_n) \leq \frac{BD}{A^\alpha C} \operatorname{diam}(M_n)^{1-\gamma\alpha}$$

and from the assumption that $c_Y(M_n) \geq K\operatorname{diam}(M_n)^\eta$ it follows that

$$\operatorname{diam}(M_n)^\eta \leq \frac{BD}{A^\alpha CK} \operatorname{diam}(M_n)^{1-\gamma\alpha}.$$

Since $\operatorname{diam}(M_n) \to \infty$, we obtain $\eta \leq 1 - \gamma\alpha$, and therefore $\alpha \leq \frac{1-\eta}{\gamma}$. Hence $\alpha_Y(X, d_{\|.\|,r}) \leq \frac{1-\eta}{\gamma}$. □

Proof of $\alpha(X, d_{\|.\|,r}) \leq \min\left\{\frac{p_X}{2r}, 1\right\}$ in Theorem 3.1. If the space X is finite-dimensional, then the statement is trivial. So suppose that X is infinite-dimensional, and let $r \in M_X$ and $\|.\|$ be an equivalent r-norm on X. To obtain the upper estimate for $\alpha(X, d_{\|.\|,r})$, we will use Lemma 3.4. The role of the metric spaces (M_n, δ_n) in Lemma 3.4 will be played by the following sequence of metric spaces. For $n \in \mathbb{N}$, let $H_n = \{0,1\}^n$ (the so-called *Hamming cube*), equipped with the ℓ_1 metric d_1 (i.e. the metric inherited from ℓ_1^n when considering H_n as a subset of ℓ_1^n). In other words, the distance between two sequences from H_n is equal to the number of places where they differ (this is also called the *Hamming distance*). Then (H_n, d_1) is finite, 1-discrete and $\operatorname{diam}(H_n, d_1) = n$.

Let us first construct appropriate embeddings of the Hamming cubes H_n into X. Let $n \in \mathbb{N}$. By Theorem 2.2, there is a linear mapping $S_n\colon \ell_{p_X}^n \to X$ such that

$$\|x\|_{p_X} \leq \|S_n(x)\| \leq 2\|x\|_{p_X} \text{ for every } x \in \ell_{p_X}^n.$$

Define a mapping $\varphi_n\colon H_n \to \ell_{p_X}^n$ by $(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n)$. Then for $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in H_n$ we have

$$\|\varphi_n(x) - \varphi_n(y)\|_{p_X} = \left(\sum_{i=1}^n |x_i - y_i|^{p_X}\right)^{\frac{1}{p_X}} = \left(\sum_{i=1}^n |x_i - y_i|\right)^{\frac{1}{p_X}} = d_1(x,y)^{\frac{1}{p_X}},$$

where the second equality follows from the fact that $|x_i - y_i| \in \{0,1\}$ for every i. Let $f_n = S_n \circ \varphi_n \colon H_n \to X$. If $x, y \in H_n$, then

$$d_1(x,y)^{\frac{r}{p_X}} \leq \|f_n(x) - f_n(y)\|^r \leq 2^r d_1(x,y)^{\frac{r}{p_X}} \leq 2^r d_1(x,y),$$

where the last inequality holds since $d_1(x,y)$ is either zero or greater or equal to one and $\frac{r}{p_X} \leq 1$ by Lemma 2.1.

Now, let H be an infinite-dimensional Hilbert space. It follows from the work of Enflo [En] (see also [Ma, 15.4.1 Theorem]) that $c_H(H_n, d_1) = \sqrt{n} = \operatorname{diam}(H_n, d_1)^{\frac{1}{2}}$ for every $n \in \mathbb{N}$. We apply Lemma 3.4 and obtain

$$\alpha_H(X, d_{\|.\|,r}) \leq \frac{1 - \frac{1}{2}}{\frac{r}{p_X}} = \frac{p_X}{2r}.$$

Hence $\alpha(X, d_{\|.\|,r}) \leq \frac{p_X}{2r}$, and since $\alpha(X, d_{\|.\|,r}) \leq 1$, we have $\alpha(X, d_{\|.\|,r}) \leq \min\left\{\frac{p_X}{2r}, 1\right\}$. □

Note that the above proof of the inequality $\alpha(X, d_{\|.\|,r}) \leq \min\left\{\frac{p_X}{2r}, 1\right\}$ in Theorem 3.1 does not use the assumption that the space X coarsely embeds into a Hilbert space.

Let us conclude with several remarks.

Remark 3.5. The inequality $s(X, d_{\|.\|,r}) \leq \min\left\{\frac{p_X}{2r}, 1\right\}$ in Theorem 3.1 can easily be proved using the notion of Enflo type.

Recall that a metric space (M, d_M) has *Enflo type* p, where $1 \leq p < \infty$, if there is a constant $C > 0$ such that for every $n \in \mathbb{N}$ and every $f \colon \{-1,1\}^n \to M$ we have

$$\mathbb{E}\, d_M(f(\varepsilon), f(-\varepsilon))^p \leq C^p \sum_{i=1}^{n} \mathbb{E}\, d_M(f(\varepsilon), f(\varepsilon_1, \dots, \varepsilon_{i-1}, -\varepsilon_i, \varepsilon_{i+1}, \dots, \varepsilon_n))^p, \tag{6}$$

where $\mathbb{E}$ denotes the expectation with respect to a uniform choice of signs $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{-1,1\}^n$. We set

$$\text{E-type}(M) = \sup\{1 \leq p < \infty : M \text{ has Enflo type } p\}$$

(note that this is a supremum of a nonempty set since M always has Enflo type 1 by the triangle inequality).

Now, let X be a quasi-Banach space, $r \in M_X$ and $\|.\|$ be an equivalent r-norm on X. It is easy to prove that then

$$\text{E-type}(X, d_{\|.\|,r}) \leq \frac{p_X}{r}.$$

Suppose that $\alpha \in (0,1]$ is such that $(X, d_{\|.\|,r}^{\alpha})$ admits a bi-Lipschitz embedding into a Hilbert space H. It is well known that E-type$(H) = 2$ (this can be proved following the ideas from [En]). Using [AB, Proposition 2.3] we obtain

$$\frac{\text{E-type}(X, d_{\|.\|,r})}{\alpha} \geq \text{E-type}(H) = 2,$$

hence

$$\alpha \leq \frac{\text{E-type}(X, d_{\|.\|,r})}{2} \leq \frac{p_X}{2r}.$$

Therefore $s(X, d_{\|.\|,r}) \leq \min\left\{\frac{p_X}{2r}, 1\right\}$.

Note that as in the proof of the inequality $\alpha(X, d_{\|.\|,r}) \leq \min\left\{\frac{p_X}{2r}, 1\right\}$ in Theorem 3.1 we did not use the assumption that the space X coarsely embeds into a Hilbert space.

Remark 3.6. The choice of the ℓ_1 metric on the Hamming cubes H_n in the proof of the inequality $\alpha(X, d_{\|.\|,r}) \leq \min\left\{\frac{p_X}{2r}, 1\right\}$ in Theorem 3.1 for X infinite-dimensional was not essential. Given $r \in M_X$ and an equivalent r-norm $\|.\|$ on X, we can actually use the ℓ_p metric d_p on H_n for any $p \in [1, 2)$ such that $p \leq \frac{p_X}{r}$ (note that $\frac{p_X}{r} \geq 1$ by Lemma 2.1 and that we do not need to consider the ℓ_p metrics for $0 < p < 1$ since they are all equal to the ℓ_1 metric on H_n). Indeed, take such a p. Then (H_n, d_p) is 1-discrete and $\operatorname{diam}(H_n, d_p) = n^{\frac{1}{p}}$ for every $n \in \mathbb{N}$. Following the same lines as above, we construct for every $n \in \mathbb{N}$ a mapping $f_n\colon H_n \to X$ such that for all $x, y \in H_n$ we have

$$d_p(x, y)^{\frac{pr}{p_X}} \leq \|f_n(x) - f_n(y)\|^r \leq 2^r d_p(x, y)^{\frac{pr}{p_X}} \leq 2^r d_p(x, y),$$

where the last inequality holds since $d_p(x, y)$ is either zero or greater or equal to one and $\frac{pr}{p_X} \leq 1$ by our assumption on p. If H is an infinite-dimensional Hilbert space, then $c_H(H_n, d_p) = \operatorname{diam}(H_n, d_p)^{1-\frac{p}{2}}$ for every $n \in \mathbb{N}$ (this may be proved following the same lines as in [Ma, 15.4.1 Theorem]). Lemma 3.4 then yields

$$\alpha_H(X, d_{\|.\|,r}) \leq \frac{1 - (1 - \frac{p}{2})}{\frac{pr}{p_X}} = \frac{p_X}{2r}$$

and we again conclude that $\alpha(X, d_{\|.\|,r}) \leq \min\left\{\frac{p_X}{2r}, 1\right\}$.

Besides taking $p = 1$, another natural choice would be to take $p = \max\{p_X, 1\}$ if $p_X < 2$. If $p_X = 2$, then we have trivially $\alpha(X, d_{\|.\|,r}) \leq 1 = \min\left\{\frac{p_X}{2r}, 1\right\}$.

Remark 3.7. If $p_X > 1$, we can give an alternative proof of the inequality $\alpha(X, d_{\|.\|,r}) \leq \min\left\{\frac{p_X}{2r}, 1\right\}$ in Theorem 3.1 by reducing it to the case of ℓ_p-spaces, which is already known from [Ba]. Suppose that X is an infinite-dimensional quasi-Banach space with $p_X > 1$ which coarsely embeds into a Hilbert space. By [Ka2, Theorem 2.1(2)], X is normable, so we can assume that X is a Banach space.

Let us first estimate $\alpha(X)$ (that is, the Hilbert space compression exponent of X with respect to the canonical metric on X given by the norm). It is easy to see that there is an infinite-dimensional separable closed subspace Y of X such that $p_Y = p_X$. Clearly, the space Y coarsely embeds into a Hilbert space. By [Ra, Theorem 1], there is a probability space $(\Omega, \mathcal{B}, \mu)$ such that Y is linearly isomorphic to a subspace of $L_0(\mu)$. Since $p_Y > 1$, [BL, Theorem 8.15] implies that Y is isomorphic to a subspace of $L_1(\mu)$. Since Y is separable, [Wo, III.A.2] implies that there is a separable $L_1(\mu')$ for some nonnegative measure μ' such that Y is isomorphic to a subspace of $L_1(\mu')$. It follows from the isomorphic classification of separable L_1-spaces [Wo, III.A.1] that Y is isomorphic to a subspace of $L_1(0, 1)$. By a theorem of Guerre and Levy [GL, Théorème 1], there is a subspace of Y isomorphic to ℓ_{p_Y}. Hence, by (3),

$$\alpha(X) \leq \alpha(\ell_{p_Y}) = \frac{p_Y}{2} = \frac{p_X}{2}.$$

Now, let $r \in M_X = (0, 1]$ and let $\|.\|$ be an equivalent r-norm on X. It follows easily from the definition that $\alpha(X, d_{\|.\|,r}) \leq \frac{1}{r}\alpha(X)$, and therefore $\alpha(X, d_{\|.\|,r}) \leq \frac{1}{r}\frac{p_X}{2}$. Hence $\alpha(X, d_{\|.\|,r}) \leq \min\left\{\frac{p_X}{2r}, 1\right\}$.

Remark 3.8. The proof of the inequality $\alpha(X, d_{\|.\|,r}) \leq \min\left\{\frac{p_X}{2r}, 1\right\}$ in Theorem 3.1 can be generalized to give an upper estimate for compression exponents of quasi-Banach spaces in general Banach spaces.

First, suppose that a metric space (M, d_M) has Enflo type $p \in [1, \infty)$ with a constant $C > 0$ (see Remark 3.5 for the definition). Let $n \in \mathbb{N}$ and consider the ℓ_1 metric d_1 on $\{-1, 1\}^n$. Let $f\colon \{-1, 1\}^n \to M$ be injective. Using the estimate

$$\frac{1}{\operatorname{Lip}(f^{-1})} d_1(\varepsilon, \varepsilon') \leq d_M(f(\varepsilon), f(\varepsilon')) \leq \operatorname{Lip}(f) d_1(\varepsilon, \varepsilon') \text{ for all } \varepsilon, \varepsilon' \in \{-1, 1\}^n,$$

we obtain easily from (6) that

$$\mathrm{distortion}(f)=\mathrm{Lip}(f)\cdot\mathrm{Lip}(f^{-1})\geq\frac{1}{C}n^{1-\frac{1}{p}}.$$

Hence (recall that $H_n=\{0,1\}^n$)

$$c_M(H_n,d_1)=c_M(\{-1,1\}^n,d_1)\geq\frac{1}{C}n^{1-\frac{1}{p}}.$$

Now, let X be a quasi-Banach space, $r\in M_X$, $\|.\|$ be an equivalent r-norm on X, and let Y be a Banach space. Let us show that then $\alpha_Y(X,d_{\|.\|,r})\leq\min\{\frac{p_X}{rp_Y},1\}$. If X is finite-dimensional, then the statement is trivial. So suppose that X is infinite-dimensional. If $p_Y=1$, then, since $r\leq p_X$ by Lemma 2.1, we have trivially $\alpha_Y(X,d_{\|.\|,r})\leq 1=\min\{\frac{p_X}{rp_Y},1\}$. So suppose that $p_Y>1$. If Y has type $p>1$, then, by a theorem of Pisier [Pi, Theorem 7.5], it has Enflo type q for every $1\leq q<p$. So if $p\in(1,p_Y)$, then Y has Enflo type p (say with a constant C), and therefore $c_Y(H_n,d_1)\geq\frac{1}{C}n^{1-\frac{1}{p}}=\frac{1}{C}\mathrm{diam}(H_n,d_1)^{1-\frac{1}{p}}$ for every $n\in\mathbb{N}$. Using the same method as in the proof of Theorem 3.1, we obtain

$$\alpha_Y(X,d_{\|.\|,r})\leq\frac{1-(1-\frac{1}{p})}{\frac{r}{p_X}}=\frac{p_X}{rp}.$$

Hence $\alpha_Y(X,d_{\|.\|,r})\leq\min\{\frac{p_X}{rp_Y},1\}$.

To illustrate this result and its limitations, let $0<p<q<\infty$ and $q\geq 1$. As mentioned in (2), we then have $\alpha_{\ell_q}(\ell_p)=\frac{\max\{p,1\}}{q}$. Our result above gives the estimate

$$\alpha_{\ell_q}(\ell_p)\leq\frac{\max\{\min\{p,2\},1\}}{\min\{q,2\}},$$

which is clearly an equality if in addition $q\leq 2$, but not if $q>2$.

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