Weak solutions to problems in fluid mechanics

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Abstract formulation

Variable coefficients "Euler system"

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{H}[\mathbf{v}]) \odot (\mathbf{v} + \mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} + \mathbb{M}[\mathbf{v}] \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0,$$

$$\mathbf{v} \odot \mathbf{w} \equiv \mathbf{v} \otimes \mathbf{w} - \frac{1}{2} \mathbf{v} \cdot \mathbf{w} \mathbb{I}$$

Kinetic energy

$$\frac{1}{2} \frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} = E[\mathbf{v}]$$

Data

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Example: Savage-Hutter model for avalanches

Unknowns

flow height .... $h = h(t, x)$
depth-averaged velocity .... $\mathbf{u} = \mathbf{u}(t, x)$

$$\partial_t h + \mathrm{div}_x(h\mathbf{u}) = 0$$

$$\partial_t(h\mathbf{u}) + \mathrm{div}_x(h\mathbf{u} \otimes \mathbf{u}) + \nabla_x(ah^2) = h\left(-\gamma\frac{\mathbf{u}}{|\mathbf{u}|} + \mathbf{f}\right)$$

Periodic boundary conditions

$$\Omega = \left([0,1]|_{\{0,1\}}\right)^2$$

Transformation - Step I

Helmholtz decomposition

$$h\mathbf{u} = \mathbf{v} + \mathbf{V} + \nabla_x \Psi$$

where

$$\mathrm{div}_x \mathbf{v} = 0, \ \int_\Omega \Psi \ \mathrm{d}x = 0, \ \int_\Omega \mathbf{v} \ \mathrm{d}x = 0, \ \mathbf{V} \in R^2$$

Fixing h and the potential Ψ

$$\partial_t h + \Delta \Psi = 0$$

$$h(0,\cdot) = h_0, \ -\partial_t h(0,\cdot) = \Delta \Psi_0$$

Problem I

Equation

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{V} + \nabla_x \Psi) \otimes (\mathbf{v} + \mathbf{V} + \nabla_x \Psi)}{h} + \left(a h^2 + \partial_t \Psi \right) \mathbb{I} \right)$$

$$+ \partial_t \mathbf{V}$$

$$= h \left(-\gamma \frac{\mathbf{v} + \mathbf{V} + \nabla_x \Psi}{|\mathbf{v} + \mathbf{V} + \nabla_x \Psi|} + \mathbf{f} \right),$$

Constraints and initial conditions

$$\mathrm{div}_x \mathbf{v} = 0, \ \int_\Omega \mathbf{v}(t, \cdot) \ \mathrm{d}x = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{V}(0) = \mathbf{V}_0$$

Transformation - Step II

Prescribing the kinetic energy

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}+\nabla_x\Psi|^2}{h} = E \equiv \Lambda(t) - ah^2 - \partial_t\Psi$$

Problem II

$$\partial_t \mathbf{v} + \partial_t \mathbf{V}$$

$$+\mathrm{div}_x\left(\frac{(\mathbf{v}+\mathbf{V}+\nabla_x\Psi)\otimes(\mathbf{v}+\mathbf{V}+\nabla_x\Psi)}{h} - \frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}+\nabla_x\Psi|^2}{h}\mathbb{I}\right)$$

$$= -\gamma\left(\frac{h}{2E}\right)^{1/2}(\mathbf{v}+\mathbf{V}+\nabla_x\Psi) + h\mathbf{f}$$

Transformation - Step III

Determining function V

$$\partial_t \mathbf{V} - \left[\frac{1}{|\Omega|} \int_\Omega \gamma \left(\frac{h}{2E} \right)^{1/2} \mathrm{d}x \right] \mathbf{V}$$

$$= \frac{1}{|\Omega|} \int_\Omega \left[\gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \nabla_x \Psi) + h\mathbf{f} \right] \mathrm{d}x, \;\; \mathbf{V}(0) = \mathbf{V}_0$$

Problem III

Equation

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi) \odot (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi)}{h} \right)$$

$$= -\gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi)$$

$$+ \frac{1}{|\Omega|} \int_\Omega \gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi) \ \mathrm{d}x + h\mathbf{f} - \frac{1}{|\Omega|} \int_\Omega h\mathbf{f} \ \mathrm{d}x$$

$$\mathbf{v} \odot \mathbf{w} = \mathbf{v} \otimes \mathbf{w} - \frac{1}{2} \mathbf{v} \cdot \mathbf{w} \mathbb{I}$$

Transformation - Step IV

Solving elliptic problem

$$\mathrm{div}_x \mathbb{M} \equiv \mathrm{div}_x \left(\nabla_x \mathbf{m} + \nabla_x^t \mathbf{m} - \mathrm{div}_x \mathbf{m} \mathbb{I} \right)$$

$$= -\gamma \left(\frac{h}{2E} \right)^{1/2} \left(\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi \right)$$

$$+ \frac{1}{|\Omega|} \int_\Omega \gamma \left(\frac{h}{2E} \right)^{1/2} \left(\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi \right) \, \mathrm{d}x + h\mathbf{f} - \frac{1}{|\Omega|} \int_\Omega h\mathbf{f} \, \mathrm{d}x,$$

$$\int_\Omega \mathbb{M}(t, \cdot) \, \mathrm{d}x = 0 \text{ for any } t \in [0, T].$$

Abstract formulation, revisited

Variable coefficients "Euler system"

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{H}[\mathbf{v}]) \odot (\mathbf{v} + \mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} + \mathbb{M}[\mathbf{v}] \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0,$$

Kinetic energy

$$\frac{1}{2} \frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} = E[\mathbf{v}]$$

Data

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Abstract operators

Boundedness

b maps bounded sets in $L^\infty((0,T)\times\Omega;R^N)$ on bounded sets in $C_b(Q,R^M)$

Continuity

$$b[\mathbf{v}_n] \to b[\mathbf{v}] \text{ in } C_b(Q;R^M) \text{ (uniformly for } (t,x)\in Q \text{)}$$

whenever

$$\mathbf{v}_n \to \mathbf{v} \text{ in } C_{\rm weak}([0,T];L^2(\Omega;R^N))$$

Causality

$\mathbf{v}(t,\cdot)=\mathbf{w}(t,\cdot)$ for $0\le t\le\tau\le T$ implies $b[\mathbf{v}]=b[\mathbf{w}]$ in $[(0,\tau]\times\Omega]$

Subsolutions

Field equations, differential constraints

$$\partial_t \mathbf{v} + \mathrm{div}_x \mathbb{F} = 0,\ \mathrm{div}_x \mathbf{v} = 0$$

$$\mathbf{v}(0,\cdot) = \mathbf{v}_0,\ \mathbf{v}(T,\cdot) = \mathbf{v}_T$$

Non-linear constraint

$$\mathbf{v} \in C(Q; R^N),\ \mathbb{F} \in C(Q; R^{N\times N}_{\mathrm{sym},0}),$$

$$\frac{N}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{H}[\mathbf{v}])\otimes(\mathbf{v}+\mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} - \mathbb{F} + \mathbb{M}[\mathbf{v}]\right] < E[\mathbf{v}]$$

Subsolution relaxation

Algebraic inequality

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} \leq \frac{N}{2}\lambda_{\max}\left[\frac{(\mathbf{v}+\mathbf{H}[\mathbf{v}])\otimes(\mathbf{v}+\mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} - \mathbb{F} + \mathbb{M}[\mathbf{v}]\right]$$

$$< E[\mathbf{v}]$$

Solutions

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} = E[\mathbf{v}]$$

$$\Rightarrow$$

$$\mathbb{F} = \frac{(\mathbf{v}+\mathbf{H}[\mathbf{v}])\odot(\mathbf{v}+\mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} + \mathbb{M}[\mathbf{v}]$$

Oscillatory lemma

A variable coefficients variant of DeLellis-Székelyhidi's result

Hypotheses:

$$U \subset R \times R^N, \; N = 2, 3 \text{ bounded open set}$$

$$\tilde{\mathbf{h}} \in C(U; R^N), \; \tilde{\mathbb{H}} \in C(U; R^{N \times N}_{\mathrm{sym},0}), \; \tilde{e}, \; \tilde{r} \in C(U), \; \tilde{r} > 0, \; \tilde{e} \leq \overline{e} \text{ in } U$$

$$\frac{N}{2} \lambda_{\max} \left[\frac{\tilde{\mathbf{h}} \otimes \tilde{\mathbf{h}}}{\tilde{r}} - \tilde{\mathbb{H}} \right] < \tilde{e} \text{ in } U.$$

Conclusion:

$$\mathbf{w}_n \in C_c^\infty(U; R^N),\ \mathbb{G}_n \in C_c^\infty(U; R^{N\times N}_{\mathrm{sym},0}),\ n = 0, 1, \dots$$

$$\partial_t \mathbf{w}_n + \mathrm{div}_x \mathbb{G}_n = 0,\ \mathrm{div}_x \mathbf{w}_n = 0 \text{ in } R \times R^N,$$

$$\frac{N}{2}\lambda_{\max}\left[\frac{(\tilde{\mathbf{h}} + \mathbf{w}_n)\otimes(\tilde{\mathbf{h}} + \mathbf{w}_n)}{\tilde{r}} - (\tilde{\mathbb{H}} + \mathbb{G}_n)\right] < \tilde{e} \text{ in } U,$$

$$\mathbf{w}_n \to 0 \text{ weakly in } L^2(U; R^N)$$

$$\liminf_{n\to\infty} \int_U \frac{|\mathbf{w}_n|^2}{\tilde{r}}\ \mathrm{d}x\mathrm{d}t \geq \Lambda(\overline{e}) \int_U \left(\tilde{e} - \frac{1}{2}\frac{|\tilde{\mathbf{h}}|^2}{\tilde{r}}\right)^2\ \mathrm{d}x\mathrm{d}t$$

Basic ideas of proof

Localization

Localizing the result of DeLellis and Széhelyhidi to "small" cubes by means of scaling arguments

Linearization

Replacing all continuous functions by their means on any of the "small" cubes

Eliminating singular sets

Applying Whitney's decomposition lemma to the non-singular sets (e.g. out of the vacuum $\{h = 0\}$)

Energy and other coefficients depending on solutions

Applying compactness of the abstract operators in C

Results

Result (A)

The set of subsolutions is non-empty $\Rightarrow$ there exists infinitely many weak solutions of the problem with the same initial data

Initial energy jump

$$\frac{1}{2}\frac{|\mathbf{v}_0 + \mathbf{H}[\mathbf{v}_0]|^2}{h[\mathbf{v}_0]} \boxed{<} \liminf_{t\to 0} \frac{1}{2}\frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]}$$

Result (B)

The set of subsolutions is non-empty $\Rightarrow$ there exists a dense set of times such that the values $\mathbf{v}(t)$ give rise to non-empty subsolution set with

$$\frac{1}{2}\frac{|\mathbf{v}_0 + \mathbf{H}[\mathbf{v}_0]|^2}{h[\mathbf{v}_0]} \boxed{=} \liminf_{t\to 0} \frac{1}{2}\frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]}$$

Application to Savage-Hutter model

Theorem [with A.Swierczewska-Gwiazda, P. Gwiazda]

(i) Let the initial data

$$h_0 \in C^2(\Omega),\ \mathbf{u}_0 \in C^2(\Omega; R^2), h_0 > 0 \text{ in } \Omega$$

be given, and let $\mathbf{f}$ and a be smooth.
Then the Savage-Hutter system admits infinitely many weak solutions in $(0, T) \times \Omega$.

(ii) Let $T > 0$ and

$$h_0 \in C^2(\Omega),\ h_0 > 0$$

be given.
Then there exists

$$\mathbf{u}_0 \in L^\infty(\Omega; R^2)$$

such that the Savage-Hutter system admits infinitely many weak solutions in $(0, T) \times \Omega$ satisfying the energy inequality.

Example II, Euler-Fourier system

Joint work with E.Chiodaroli and O.Kreml

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x (\varrho \vartheta) = 0$$

Internal energy balance

$$\frac{3}{2} \Big[\partial_t (\varrho \vartheta) + \mathrm{div}_x (\varrho \vartheta \mathbf{u}) \Big] - \Delta \vartheta = - \varrho \vartheta \mathrm{div}_x \mathbf{u}$$

Example III, Euler-Korteweg-Poisson system

Joint work with D.Donatelli and P.Marcati

Mass conservation - equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equations - Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho)$$

$$= \boxed{\varrho \nabla_x \left(K(\varrho)\Delta_x \varrho + \frac{1}{2} K'(\varrho)|\nabla_x \varrho|^2 \right)} - \varrho \mathbf{u} + \varrho \nabla_x V$$

Poisson equation

$$\Delta_x V = \varrho - \overline{\varrho}$$

Example IV, Euler-Cahn-Hilliard system

Model by Lowengrub and Truskinovsky

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p_0(\varrho, c) = \mathrm{div}_x \left(\varrho \nabla_x c \otimes \nabla_x c - \frac{\varrho}{2} |\nabla_x c|^2 \mathbb{I} \right)$$

Cahn-Hilliard equation

$$\partial_t(\varrho c) + \mathrm{div}_x(\varrho c \mathbf{u}) = \Delta \left(\mu_0(\varrho, c) - \frac{1}{\varrho} \mathrm{div}_x \left(\varrho \nabla_x c \right) \right)$$