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GENERALIZED DARCY–OSEEN RESOLVENT PROBLEM

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Abstract. In this paper we study the well-posedness of a coupled Darcy-Oseen resolvent problem, describing the fluid flow between free fluid domains and porous media separated by a semipermeable membrane. The influence of osmotic effects, induced by the presence of a semipermeable membrane, on the flow velocity is reflected in the transmission conditions on the surface between the free-fluid domain and the porous medium. To prove the existence of a weak solution of the generalized Darcy-Oseen resolvent system we consider two auxiliary problem: a mixed Navier-Dirichlet problem for the generalized Oseen resolvent system and Robin problem for an elliptic equation related to the general Darcy equations.

1. Introduction

The transport of macromolecules from a free fluid domain to a porous medium separated by a semipermeable membrane is an important biological question as well as an interesting and challenging problem for the mathematical modelling and analysis. As examples for selective transport of macromolecules between free fluid and porous medium, we can consider transport of macromolecules by water in plant tissues or by blood in arteries. Water flow in plant tissues takes place in two different physical domains separated by semipermeable membranes, denoted as symplast and apoplast [5]. The apoplast is composed of cell walls and intercellular spaces, while the symplast is constituted by cell insides, which can be connected by plasmodesmata. The complex microstructure of the apoplast, composed of polymers and microfibrils, can be represented as a porous medium and the water flow in the cell walls and intercellular space is modeled by Darcy's law. The Stokes and/or Brinkmann equations can be used to define the flow velocity in the cell inside (cytoplasm) and plasmodesmata. Similar situation is found in models of early atherosclerotic lesions [8] or low-density lipoproteins transfer through arterial walls, e.g. [19, 20, 25]. Here, the blood velocity in artery lumen is described by Stokes or Navier-Stokes equations, while arterial walls are modelled as porous media and Darcy equations are considered.

The aim of this article is to study the well-posedness of a general coupled free-fluid and porous-medium model for transport processes in biological tissues. The main difference of our problem to coupled Stokes–Darcy, Brinkman-Darcy, and Navier-Stokes–Darcy problems studied before (see e.g. [13, 14, 15, 24] and references therein) is that the free fluid and the porous media domains do not interact directly. A semipermeable membrane separates there two domains and controls actively and passively the fluxes of the water (or blood) and the solutes [5, 8]. Thus the appropriate transmission conditions on the boundary between free-fluid domain and porous media need to be considered to represented the regulation of the water (or blood) flow from the cell inside (or artery lumen) into the intercellular space (or arterial walls), [5, 8].

In contrast to previous work on transport of macromolecules from artery lumen to arterial walls [8], where the dependence of fluid flow across the membrane on the oncotic pressure difference was neglected (to simplify the analysis), we consider the influence of the osmotic effect on the flow velocity. This is reflected in the transmission conditions on the boundary between cell insides and cell walls, comprising the normal component of the Darcy velocity $\mathbf{v}^D \cdot \mathbf{n}$ and a given function $\mathbf{g} \cdot \mathbf{n}$. The function $\mathbf{g} \cdot \mathbf{n}$ corresponds to the difference between the solute concentrations in the cell insides and the intercellular space (or in the artery lumen and arterial walls), respectively, see e.g. [5, 8]. Our model includes also the situation when only a part of the boundary between the cell insides and the intercellular space (cell walls) is semipermeable.

The analysis presented here is a generalization of the results obtained in [18], where a coupled Stokes–Darcy system with a constant permeability in the Darcy equations was analysed. For the flow velocity in the domain with free fluid we consider generalized Oseen resolvent equations, while the fluid flow in the porous medium is modelled by the Darcy equations with a general space-dependent permeability tensor. Althrough, there are many results on existence of solutions of Oseen system [2, 3, 4, 10, 22, 23], coupled generalized Darcy–Oseen problem was not considered before.

To prove the existence of a weak solution of the generalized Darcy–Oseen resolvent problem we consider two auxiliary problem: a mixed Navier–Dirichlet problem for the generalized Oseen resolvent system and a Robin problem for an elliptic equation related to the general Darcy equations. The Riesz representation theorem for a continuous linear functional on the Hilbert space $H^1(\Omega)$ and the fact that $H^{-1/2}(\partial\Omega)$ is a closed subspace of the dual space $[H^1(\Omega)]'$ are used to prove the existence of a weak solution of the Robin type problem. To show the existence of a solution of a mixed Navier–Dirichlet problem for the generalized Oseen resolvent system we combine the existence results for a mixed Navier–Dirichlet problem for Stokes system, obtained in [18], and a compact perturbation argument.

Notice that considering some of the parameters in the generalized Oseen resolvent equations to be zero we recover Stokes, Oseen or Brinkmann system. Brinkman equations describe flows through some types of porous media (i.e. fibrous porous media for swarms of particles of low concentration) [6, 7, 21]. The Brinkman system is also an extension of Darcy's law when boundary layer regions near the soil phase of a porous media cannot be neglected [9] and can be derived from the Stokes or Navier–Stokes system with slip boundary conditions at the surfaces of solid structures [1]. Brinkman system can also be viewed as an approximation of the Navier–Stokes equations at low Reynolds numbers, see e.g. [15]. The Oseen problem is a popular linearization of the Navier-Stokes equations [2, 11]. Thus, modelling the fluid flow in the symplast of plant tissues by generalized Oseen resolved equations allow us to consider the situation when the water flow inside plant cells is modelled by Stokes equations and Brinkmann equations describe the water velocity in plasmodesmata.

The paper is organized as follows. In Section 2 we formulate mathematical model for transport processes in a domain composed of free fluid domain and a porous medium separated by a semipermeable membrane. In Sections 3 and 4 two auxiliary problems: the mixed Navier–Dirichlet problem for an Oseen resolvent system and the Robin type problem are analyzed. The main result of the paper, i.e. the existence and uniqueness (up to a constant for pressures) of a weak solution of the generalized Darcy–Oseen resolvent problem, is proved in Section 5.

2. Formulation of the mathematical model

Let $\Omega \subset \mathbb{R}^3$ be a bounded domain, i.e. a bounded open connected set, with Lipschitz boundary $\partial\Omega$, and suppose that Ω_S is a nonempty subdomain of Ω with Lipschitz boundary $\partial\Omega_S$ such that $\Omega_S \neq \Omega$. Then $\Omega_D := \Omega \setminus \overline{\Omega_S}$ is a bounded open set, not necessarily connected. We suppose that Ω_D has a Lipschitz boundary, too. Note that $\partial\Omega_S \cap \partial\Omega_D \cap \Omega$ is always nonempty, and it is locally the graph of a Lipschitz function. Let Γ denote a nonempty closed subset of $\partial\Omega_S \cap \partial\Omega_D$. Then Γ might reach the boundary $\partial\Omega$ or not. If G is a component of Ω_D, we suppose that $\Gamma \cap \partial G$ has positive surface measure.

In Ω we consider the following coupled boundary value transmission problem:

$$-\eta\Delta\mathbf{v}^S + \lambda\mathbf{v}^S + \mathbf{k}\cdot\nabla\mathbf{v}^S + \nabla p^S = 0, \qquad \operatorname{div}\mathbf{v}^S = 0 \qquad \text{in } \Omega_S, \tag{1}$$

$$\mathbf{v}^D + K\nabla p^D = 0, \qquad \operatorname{div}\mathbf{v}^D = 0 \qquad \text{in } \Omega_D, \tag{2}$$

$$\mathbf{v}^S = \mathbf{f} \qquad \text{on } \partial\Omega_S \setminus \Gamma, \tag{3}$$

$$\mathbf{v}^D\cdot\mathbf{n} = h \qquad \text{on } \partial\Omega_D \setminus \Gamma, \tag{4}$$

$$\mathbf{v}^D\cdot\mathbf{n} - \mathbf{v}^S\cdot\mathbf{n} = h, \qquad \mathbf{v}^S_\tau = \mathbf{f}_\tau \qquad \text{on } \Gamma, \tag{5}$$

$$\left[-T_\eta(\mathbf{v}^S,p^S)\,\mathbf{n} - \alpha\mathbf{v}^S + \frac{1}{2}(\mathbf{k}\cdot\mathbf{n})\mathbf{v}^S\right]\cdot\mathbf{n} = \beta p^D + \gamma\mathbf{v}^D\cdot\mathbf{n} - \mathbf{g}\cdot\mathbf{n} \quad \text{on } \Gamma. \tag{6}$$

Here $\mathbf{k}\in\mathbb{R}^3$, and $\eta,\beta>0$, $\alpha,\gamma\geq 0$ are constants, $K=(K_{ij})$ is a symmetric 3×3–matrix function with entries $K_{ij}\in L^\infty(\Omega_D)$, and $\lambda=(\lambda_{ij})$ is a measurable 3×3–matrix function with entries $\lambda_{ij}\in L^\infty(\Omega_D)$. Notice that $\mathbf{k}\cdot\nabla\mathbf{v} = (\mathbf{k}\cdot\nabla v_1, \mathbf{k}\cdot\nabla v_2, \mathbf{k}\cdot\nabla v_3)$.

The vector $\mathbf{v}^D = (v^D_1, v^D_2, v^D_3)$ denotes the generalized Darcy velocity vector in Ω_D, and $\mathbf{v}^S = (v^S_1, v^S_2, v^S_3)$ represents the generalized Oseen resolvent velocity in Ω_S. In the transmission condition (6),

$$T_\eta(\mathbf{v}^{\mathbf{S}},p^S) = 2\eta\,\mathbf{D}\mathbf{v}^S - p^S I$$

means the stress tensor, where

$$\mathbf{D}\mathbf{v} = \frac{1}{2}\left[\nabla\mathbf{v} + (\nabla\mathbf{v})^T\right]$$

is the symmetric 3×3 - gradient of $\mathbf{v}$ and I the 3×3 - unity matrix. We denote by $\mathbf{n}^S$ and by $\mathbf{n}^D$ the exterior unit normal vectors of Ω_S and Ω_D, respectively. By $\mathbf{n}$ we mean $\mathbf{n}=\mathbf{n}^S$ on $\partial\Omega_S$ and $\mathbf{n}=-\mathbf{n}^D$ on $\partial\Omega_D$. Moreover, we use $\mathbf{v}_\mathbf{n} = (\mathbf{v}\cdot\mathbf{n})\mathbf{n}$ for the normal part of $\mathbf{v}$, and $\mathbf{v}_\tau = \mathbf{v}-\mathbf{v}_\mathbf{n}$ for the tangential part of $\mathbf{v}$. Finally, we suppose that the matrix function K satisfies a uniform ellipticity condition, i.e. there exists a constant $a>0$ such that for a.a. $\mathbf{x}\in\Omega_D$ we have

$$\sum_{i,j=1}^{3} K_{ij}(\mathbf{x})\,\xi_i\xi_j \geq a|\xi|^2 \qquad \forall\,\xi=(\xi_1,\xi_2,\xi_3)\in\mathbb{R}^3, \tag{7}$$

and that the matrix λ is nonnegative, i.e. for a.a. $\mathbf{x} \in \Omega_S$ we have

$$\sum_{i,j=1}^{3} \lambda_{ij}(\mathbf{x})\, \xi_i \xi_j \geq 0 \qquad \forall\, \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3. \tag{8}$$

The equation (2) represents a generalization of the standard Darcy equations where $K = I$. For $\lambda = 0$ and $k = 0$ the system (1) coincides with the Stokes system, for $\lambda = 0$ and $k \neq 0$ with the Oseen system, and for $\lambda = c\,I$ with $c > 0$ and $k \neq 0$ with the Oseen resolvent system. Finally, if $k = 0$ and $\xi \cdot (\lambda(\mathbf{x})\,\xi) > 0$ for a.a. $\mathbf{x} \in \Omega_S$ and $0 \neq \xi \in \mathbb{R}^3$, then (1) is called the Brinkman system.

In the next we suppose that there exists some $\mathbf{\Theta} \in [H^1(\Omega_S)]^3$ with $\mathbf{\Theta} = 0$ on $\partial\Omega \setminus \Gamma$, with $\mathbf{\Theta}_\tau = 0$ on Γ, and with

$$\int_\Gamma \mathbf{\Theta} \cdot \mathbf{n}\; \mathrm{d}\sigma_{\mathbf{y}} = 1\ . \tag{9}$$

Note that the latter condition is certainly satisfied if the surface Γ contains a non-trivial sufficiently smooth part (of class C^2).

Notice that the situations when $\partial\Omega_S \setminus \Gamma = \emptyset$ or $\partial\Omega_D \setminus \Gamma = \emptyset$ are included in the analysis presented here. If $\overline{\Omega_S} \subset \Omega$ and $\partial\Omega_S = \Gamma$, then the condition (3) disappears. As examples we can consider a domain $\Omega = \{\boldsymbol{x} \in \mathbb{R}^3;\ |x_j| < 2,\ j = 1,2,3\}$, $\Omega_S = \{\boldsymbol{x} \in \mathbb{R}^3;\ |\boldsymbol{x}| < 1\}$, $\Omega_D = \Omega \setminus \Omega_S$, and $\Gamma = \{\boldsymbol{x} \in \mathbb{R}^3;\ |\boldsymbol{x}| = 1\}$.) In the situation when $\overline{\Omega_D} \subset \Omega$ and $\partial\Omega_D = \Gamma$ the boundary condition (4) disappears. For example, if $\Omega = \{\boldsymbol{x} \in \mathbb{R}^3;\ |x_j| < 2,\ j = 1,2,3\}$, $\Omega_D = \{\boldsymbol{x} \in \mathbb{R}^3;\ |\boldsymbol{x}| < 1\}$, and $\Gamma = \{\boldsymbol{x} \in \mathbb{R}^3;\ |\boldsymbol{x}| = 1\}$. Such structure is characteristic for plant cells that are connected with each other via plasmodesmata. Then the domain Ω_S represents cell inside and plasmodesmata and the domain Ω_D describes a cell wall and intercellular space.

The interface Γ might reach the boundary, e.g. $\Omega = \{\boldsymbol{x} \in \mathbb{R}^3;\ |x_j| < 3\}$, $\Omega_S = \{\boldsymbol{x} \in \Omega;\ x_1 < 0\}$, $\Omega_D = \{\boldsymbol{x} \in \Omega;\ 0 < x_1\}$, and $\Gamma = \{\boldsymbol{x} \in \mathbb{R}^3;\ x_1 = 0,\ |x_2| \leq 3,\ |x_3| \leq 3\}$, or might not reach the boundary, e.g. $\Omega = \{\boldsymbol{x} \in \mathbb{R}^3;\ 1 < |\boldsymbol{x}| < 3\}$, $\Omega_S = \{\boldsymbol{x} \in \mathbb{R}^3;\ 2 < |\boldsymbol{x}| < 3\}$, $\Omega_D = \{\boldsymbol{x} \in \mathbb{R}^3; 1 < |\boldsymbol{x}| < 2\}$, $\Gamma = \{\boldsymbol{x} \in \mathbb{R}^3;\ |\boldsymbol{x}| = 2\}$, $\partial\Omega_D \setminus \Gamma = \{\boldsymbol{x} \in \mathbb{R}^3;\ |\boldsymbol{x}| = 1\}$, $\partial\Omega_S \setminus \Gamma = \{\boldsymbol{x} \in \mathbb{R}^3;\ |\boldsymbol{x}| = 3\}$.

3. A Mixed Navier-Dirichlet problem for the Oseen resolvent system

To prove the main result of the paper, i.e. the existence of a unique (up to a constant for pressures) weak solution of the model (1)–(6), we shall use two auxiliary problems.

As a first auxiliary problem, in this section we consider a mixed Navier–Dirichlet problem for the generalized Oseen resolvent system:

For a given $\mathbf{f} \in [H^{1/2}(\partial\Omega_S)]^3$ and $\mathbf{g} \in [H^{-1/2}(\partial\Omega_S)]^3$ find a weak solution $\mathbf{v} \in [H^1(\Omega_S)]^3$ and $p \in L^2(\Omega_S)$ of the problem

$$\begin{aligned} -\eta\Delta\mathbf{v} + \lambda\mathbf{v} + \mathbf{k}\cdot\nabla\mathbf{v} + \nabla p = 0, \qquad \operatorname{div}\mathbf{v} = 0 \qquad & \text{in } \Omega_S,\\ \mathbf{v} = \mathbf{f} \qquad & \text{on } \partial\Omega_S \setminus \Gamma,\\ \mathbf{v}_\tau = \mathbf{f}_\tau \qquad & \text{on } \Gamma,\\ \Big[T_\eta(\mathbf{v},p)\mathbf{n} + b\,\mathbf{v} - \frac{1}{2}(\mathbf{k}\cdot\mathbf{n})\mathbf{v}\Big]_{\mathbf{n}} = \mathbf{g}_{\mathbf{n}} \qquad & \text{on } \Gamma, \end{aligned} \tag{10}$$

where $\mathbf{k} \in \mathbb{R}^3$ and η, $b > 0$ are constants.

Definition 3.1. *We say that* $(\mathbf{v}, p) \in [H^1(\Omega_S)]^3 \times L^2(\Omega_S)$ *is a weak solution of the problem* (10) *if the boundary conditions* $\mathbf{v} = \boldsymbol{f}$ *on* $\partial\Omega_S \setminus \Gamma$ *and* $\mathbf{v}_\tau = \mathbf{f}_\tau$ *on* Γ *are fulfilled in the sense of traces, and if it holds*

$$\begin{aligned}\int_{\Omega_S} \Big\{ 2\eta \boldsymbol{D}\mathbf{v} : \boldsymbol{D}\boldsymbol{\Phi} + \lambda \mathbf{v} \cdot \boldsymbol{\Phi} - p \operatorname{div} \boldsymbol{\Phi} + \frac{1}{2}\big[\boldsymbol{\Phi} \cdot (\mathbf{k} \cdot \nabla \mathbf{v}) - \mathbf{v} \cdot (\mathbf{k} \cdot \nabla \boldsymbol{\Phi})\big] \Big\} \, \mathrm{d}\mathbf{y} \\ + \int_{\partial\Omega_S} b \, \mathbf{v} \cdot \boldsymbol{\Phi} \, \mathrm{d}\sigma_{\mathbf{y}} = \langle \mathbf{g}, \boldsymbol{\Phi} \rangle_{H^{-1/2},\, H^{1/2}}\end{aligned} \tag{11}$$

for all $\boldsymbol{\Phi} \in \mathcal{V}_\Gamma$, *where*

$$\mathcal{V}_\Gamma = \big\{ \mathbf{v} \in [H^1(\Omega_S)]^3; \;\; \mathbf{v} = 0 \;\; \textit{on} \;\; \partial\Omega_S \setminus \Gamma, \;\; \mathbf{v}_\tau = 0 \; \textit{on} \;\; \Gamma \big\}.$$

The above integral relation follows from (10) with help of Green's formula, which also implies that a weak solution of the problem (10) is contained in the space

$$\begin{aligned}W_{\eta,\lambda,\mathbf{k}}(\Omega_S) = \big\{ (\mathbf{v}, p) \in H^1(\Omega_S)^3 \times L^2(\Omega_S); \;\; -\eta\Delta\mathbf{v} + \lambda\mathbf{v} + \mathbf{k} \cdot \nabla\mathbf{v} + \nabla p = 0, \\ \operatorname{div} \mathbf{v} = 0 \big\}.\end{aligned}$$

Moreover, if $\mathbf{v} \in H^2(\Omega_S)^3$ and $p \in H^1(\Omega_S)$, then all boundary conditions are satisfied in the sense of traces.

For the following we need some special Sobolev trace spaces on the boundary $\partial\Omega_S$, i.e. the space

$$V_\Gamma = \Big\{ \mathbf{v} \in [H^{1/2}(\partial\Omega_S)]^3; \;\; \mathbf{v} = 0 \;\text{ on } \partial\Omega_S \setminus \Gamma, \;\; \mathbf{v}_\tau = 0 \;\text{ on }\; \Gamma \Big\}$$

of traces of the space $\mathcal{V}_\Gamma$, and the space of restrictions

$$W_\Gamma = \Big\{ \big(\mathbf{v}\big|_{\partial\Omega_S \setminus \Gamma}, \; \mathbf{v}_\tau\big|_\Gamma \big); \;\; \mathbf{v} \in [H^{1/2}(\partial\Omega_S)]^3 \Big\},$$

the latter equipped with the norm

$$\|\mathbf{v}\|_{W_\Gamma} = \inf \big\{ \|\mathbf{u}\|_{H^{1/2}(\partial\Omega_S)}; \; \mathbf{u} \in [H^{1/2}(\partial\Omega_S)]^3, \mathbf{u} = \mathbf{v} \text{ on } \partial\Omega_S \setminus \Gamma, \; \mathbf{u}_\tau = \mathbf{v}_\tau \text{ on } \Gamma \big\}.$$

Since W_Γ coincides with the factor-space $[H^{1/2}(\partial\Omega_S)]^3/V_\Gamma$, it is a Banach space. By V_Γ' we denote the dual space of V_Γ. According to the theorem of Hahn-Banach this dual space can be represented by

$$V_\Gamma' = \Big\{ \mathbf{g}_{\mathbf{n}}\big|_\Gamma; \quad \mathbf{g} \in [H^{-1/2}(\partial\Omega_S)]^3 \Big\}.$$

According to [12, Theorem 6.9.2] there exists a bounded linear extension operator $E : H^{1/2}(\partial\Omega_S) \to H^1(\Omega_S)$ such that φ is the trace of $E\varphi$.

For $(\mathbf{v}, p) \in W_{\eta,\lambda,\mathbf{k}}(\Omega_S)$ we define $[T_\eta(\mathbf{v}, p)\mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n})\mathbf{v}]_{\mathbf{n}}|_\Gamma \in V_\Gamma'$ by

$$\begin{aligned}\Big\langle \Big[T_\eta(\mathbf{v}, p)\mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n})\mathbf{v} \Big]_{\mathbf{n}}\Big|_\Gamma, \boldsymbol{\Phi} \Big\rangle_{H^{-1/2},\, H^{1/2}} = \int_{\Omega_S} \Big\{ 2\eta \mathbf{D}\mathbf{v} : \mathbf{D}(E\boldsymbol{\Phi}) + (\lambda\mathbf{v}) \cdot (E\boldsymbol{\Phi}) \\ + \frac{1}{2}\big[(E\boldsymbol{\Phi}) \cdot (\mathbf{k} \cdot \nabla\mathbf{v}) - \mathbf{v} \cdot (\mathbf{k} \cdot \nabla E\boldsymbol{\Phi})\big] - p \operatorname{div}(E\boldsymbol{\Phi}) \Big\} \, \mathrm{d}\mathbf{y}\end{aligned}$$

for all $\boldsymbol{\Phi} \in V_\Gamma$.

In order for $\big[T_\eta(\mathbf{v}, p)\mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n})\mathbf{v}\big]_{\mathbf{n}}\big|_\Gamma$ to be well defined we shall show that its definition does not depend on the choice of the extension operator E.

Proposition 3.2. *If* $(\mathbf{v}, p) \in W_{\eta,\lambda,\mathbf{k}}(\Omega_S)$*, then* $\left[T_\eta(\mathbf{v}, p)\mathbf{n} - \frac{1}{2}(\mathbf{k}\cdot\mathbf{n})\mathbf{v}\right]_{\mathbf{n}}\big|_\Gamma$ *does not depend on the choice of the extension operator* E*.*

Proof. Consider $\mathbf{\Phi} \in V_\Gamma$ and $\tilde{\mathbf{\Phi}} \in [H^1(\Omega_S)]^3$ with $\tilde{\mathbf{\Phi}} = \mathbf{\Phi}$ on $\partial\Omega_S$. Then $E\mathbf{\Phi} - \tilde{\mathbf{\Phi}} \in [H^1_0(\Omega_S)]^3$. If $(\mathbf{v}, p) \in W_{\eta,\lambda,\mathbf{k}}(\Omega_S)$, then Green's formula implies

$$\int_{\Omega_S} \Big\{ 2\eta \mathbf{D}\mathbf{v} : \mathbf{D}(E\mathbf{\Phi} - \tilde{\mathbf{\Phi}}) + (\lambda \mathbf{v})\cdot(E\mathbf{\Phi} - \tilde{\mathbf{\Phi}}) \\ + \frac{1}{2}\big[(E\mathbf{\Phi} - \tilde{\mathbf{\Phi}})\cdot(\mathbf{k}\cdot\nabla\mathbf{v}) - \mathbf{v}\cdot(\mathbf{k}\cdot\nabla(E\mathbf{\Phi} - \tilde{\mathbf{\Phi}}))\big] - p\,\mathrm{div}(E\mathbf{\Phi} - \tilde{\mathbf{\Phi}})\Big\}\,\mathrm{d}\mathbf{y} = 0.$$

Hence $\left[T_\eta(\mathbf{v}, p)\mathbf{n} - \frac{1}{2}(\mathbf{k}\cdot\mathbf{n})\mathbf{v}\right]_{\mathbf{n}}\big|_\Gamma$ does not depend on the choice of the extension operator E. □

The following proposition ensures the unique solvability of the mixed Navier–Dirichlet problem (10) for the Oseen resolvent system.

Proposition 3.3. *For a constant* $b > 0$ *define*

$$U^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}^S, p^S) = \Big(\mathbf{v}^S\big|_{\partial\Omega_S\setminus\Gamma},\ \ \mathbf{v}^S_\tau\big|_\Gamma,\ \ \Big[T_\eta(\mathbf{v}, p)\mathbf{n} - \frac{1}{2}(\mathbf{k}\cdot\mathbf{n})\mathbf{v}\Big]_{\mathbf{n}}\Big|_\Gamma + b\mathbf{v}_{\mathbf{n}}\big|_\Gamma\Big).$$

Then $U^{b,\eta,\lambda,\mathbf{k}} : W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \to W_\Gamma \times V'_\Gamma$ *is an isomorphism.*

Hence for $\mathbf{f} \in [H^{1/2}(\partial\Omega_S)]^3$*,* $\mathbf{g} \in [H^{-1/2}(\partial\Omega_S)]^3$ *there exists a unique solution of the problem* (10)*, and* $(\mathbf{v}^S, p^S)$ *is a solution of the problem* (10) *if and only if* $(\mathbf{v}^S, p^S) \in W_{\eta,\lambda,\mathbf{k}}(\Omega_S)$ *and*

$$U^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}^S, p^S) \ = \ \big(\mathbf{f}\big|_{\partial\Omega_S\setminus\Gamma},\ \ \mathbf{f}_\tau\big|_\Gamma,\ \ \mathbf{g}_{\mathbf{n}}\big|_\Gamma\big).$$

Proof. Using the Green formula we obtain that $(\mathbf{v}^S, p^S)$ is a solution of problem (10) if and only if

$$(\mathbf{v}^S, p^S) \in W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \quad \text{and} \quad U^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}^S, p^S) = (\mathbf{f}|_{\partial\Omega_S\setminus\Gamma}\ \ \mathbf{f}_\tau|_\Gamma,\ \ \mathbf{g}_{\mathbf{n}}|_\Gamma).$$

Moreover, the definition of $U^{b,\eta,\lambda,\mathbf{k}}$ implies that

$$U^{b,\eta,\lambda,\mathbf{k}} : W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \to W_\Gamma \times V'_\Gamma$$

is a bounded linear operator.

As next we have to show that $U^{b,\eta,\lambda,\mathbf{k}} : W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \to W_\Gamma \times V'_\Gamma$ is an isomorphism.

We suppose first that $\lambda \equiv 0$ and $\mathbf{k} = 0$. In this case it is shown in [18] that the problem (10) is uniquely solvable and

$$U^{b,\eta,0,0} : W_{\eta,0,0}(\Omega_S) \to W_\Gamma \times V'_\Gamma$$

is an isomorphism.

We prove the proposition for arbitrary λ and $\mathbf{k}$ using a compact perturbation argument. One of the problem in the analysis of the system (10) is that the operator $U^{b,\eta,\lambda,\mathbf{k}}$ is defined on the space $W_{\eta,\lambda,\mathbf{k}}(\Omega_S)$ that depends on η, λ and $\mathbf{k}$. To overcome this problem we extend these operators on some common space. Denote

$$H(\Omega_S, \mathrm{div}) = \big\{\mathbf{v} \in [H^1(\Omega_S)]^3;\quad \mathrm{div}\,\mathbf{v} = 0\big\}\,.$$

Define the operator $T_1^{b,\eta,\lambda,\mathbf{k}} : H(\Omega_S, \mathrm{div}) \times L^2(\Omega_S) \to \mathcal{V}'_\Gamma$ by

$$\langle T_1^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v},p), \mathbf{\Phi}\rangle = \int_{\Omega_S} \Big[2\eta \mathbf{D}\mathbf{v} : \mathbf{D}\mathbf{\Phi} + \lambda \mathbf{v}\cdot\mathbf{\Phi} - p \,\mathrm{div}\,\mathbf{\Phi} \\ + \frac{1}{2}\big[\mathbf{\Phi}\cdot(\mathbf{k}\cdot\nabla\mathbf{v}) - \mathbf{v}\cdot(\mathbf{k}\cdot\nabla\mathbf{\Phi})\big]\Big]\,\mathrm{d}\mathbf{y} + \int_{\partial\Omega_S} b\,\mathbf{v}\cdot\mathbf{\Phi}\,\mathrm{d}\sigma_{\mathbf{y}}$$

for $\mathbf{\Phi} \in \mathcal{V}_\Gamma$, and the operator $T_2^{b,\eta,\lambda,\mathbf{k}} : H(\Omega_S, \mathrm{div}) \times L^2(\Omega_S) \to W_\Gamma \times \mathcal{V}'_\Gamma$ by

$$T_2^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v},p) = \Big(\mathbf{v}\big|_{\partial\Omega_S\setminus\Gamma},\ \mathbf{v}_\tau\big|_\Gamma,\ T_1^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v},p)\Big).$$

Notice that

$$T_2^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v},p) = (\mathbf{f},\mathbf{g})$$

for $\mathbf{f} \in W_\Gamma$ and $\mathbf{g} \in \mathcal{V}'_\Gamma$ means that

$$\begin{aligned} -\eta\Delta\mathbf{v} + \lambda\mathbf{v} + \mathbf{k}\cdot\nabla\mathbf{v} + \nabla p = \mathbf{g}, \qquad \mathrm{div}\,\mathbf{v} = 0 \qquad & \text{in } \Omega_S,\\ \mathbf{v} = \mathbf{f} \qquad & \text{on } \partial\Omega_S\setminus\Gamma,\\ \mathbf{v}_\tau = \mathbf{f}_\tau, \quad \big[T_\eta(\mathbf{v},p)\mathbf{n} + b\mathbf{v} - \frac{1}{2}(\mathbf{k}\cdot\mathbf{n})\mathbf{v}\big]_{\mathbf{n}} = \mathbf{g}_{\mathbf{n}} \qquad & \text{on } \Gamma. \end{aligned}$$

In the following we show that

$$T_2^{b,\eta,\lambda,k} : H(\Omega_S, \mathrm{div}) \times L^2(\Omega_S) \to W_\Gamma \times \mathcal{V}'_\Gamma$$

is an isomorphism.

To do so, first suppose that $\lambda \equiv 0$, $\mathbf{k} = 0$. If $T_2^{b,\eta,0,0}(\mathbf{v},p) = 0$ then $(\mathbf{v},p)$ is a solution of the problem (10) with $\lambda \equiv 0$, $\mathbf{k} = 0$ and with homogeneous boundary conditions. For this case we have proved in [18] that $(\mathbf{v},p) = \mathbf{0}$.

Let now $(\mathbf{f},\mathbf{g}) \in W_\Gamma \times \mathcal{V}'_\Gamma$ be given. Then $\mathbf{g} \in [H^1(\Omega_S)^3]'$ and according to [11, Chapter IV, Theorem 1.1] there exists $(\tilde{\mathbf{v}},\tilde{p}) \in [H^1(\Omega_S)]^3 \times L^2(\Omega_S)$ such that

$$\begin{aligned} -\eta\Delta\tilde{\mathbf{v}} + \nabla\tilde{p} = \mathbf{g}, \quad \mathrm{div}\,\tilde{\mathbf{v}} = 0 \qquad & \text{in } \Omega_S,\\ \tilde{\mathbf{v}} = 0 \qquad & \text{on } \partial\Omega_S. \end{aligned}$$

This means that $T_1^{b,\eta,0,0}(\tilde{\mathbf{v}},\tilde{p}) - \mathbf{g}$ is supported on $\partial\Omega_S$, and hence $\mathbf{g} - T_1^{b,\eta,0,0}(\tilde{\mathbf{v}},\tilde{p}) \in V'_\Gamma$. Moreover, in [18] we have proved that there exists $(\mathbf{u},\pi) \in W_{\eta,0,0}(\Omega_S)$ such that

$$\Big(\mathbf{u}\big|_{\partial\Omega_S\setminus\Gamma}, \mathbf{u}_\tau\big|_\Gamma\Big) = \Big(\mathbf{f}\big|_{\partial\Omega_S\setminus\Gamma}, \mathbf{f}_\tau\big|_\Gamma\Big) \quad \text{and} \quad T_1^{b,\eta,0,0}(\mathbf{u},\pi) = \mathbf{g} - T_1^{b,\eta,0,0}(\tilde{\mathbf{v}},\tilde{p}).$$

Consider $\mathbf{v} = \tilde{\mathbf{v}} + \mathbf{u}$ and $p = \tilde{p} + \pi$. Then from the definition of $\tilde{\mathbf{v}}$ and $\mathbf{u}$ we obtain $\mathrm{div}\,\mathbf{v} = 0$ and

$$T_2^{b,\eta,0,0}(\mathbf{v},p) = \Big(\mathbf{f}\big|_{\partial\Omega_S\setminus\Gamma}, \mathbf{f}_\tau\big|_\Gamma, \mathbf{g}\Big).$$

This implies that

$$T_2^{b,\eta,0,0} : H(\Omega_S, \mathrm{div}) \times L^2(\Omega_S) \to W_\Gamma \times \mathcal{V}'_\Gamma$$

is an isomorphism.

Next we consider a general matrix function λ with bounded entries and $\mathbf{k} \in \mathbb{R}^3$, such that λ satisfies the assumption (8). Then, Sobolev's imbedding theorem ensures that

$$T_2^{b,\eta,\lambda,\mathbf{k}} - T_2^{b,\eta,0,0} : H(\Omega_S, \mathrm{div}) \times L^2(\Omega_S) \to W_\Gamma \times \mathcal{V}'_\Gamma$$

is a compact operator. Since $T_2^{b,\eta,0,0} : H(\Omega_S, \mathrm{div}) \times L^2(\Omega_S) \to W_\Gamma \times \mathcal{V}'_\Gamma$ is an isomorphism, the operator $T_2^{b,\eta,\lambda,\mathbf{k}} : H(\Omega_S, \mathrm{div}) \times L^2(\Omega_S) \to W_\Gamma \times \mathcal{V}'_\Gamma$ is Fredholm with index 0.

Consider now $(\mathbf{v}, p) \in H(\Omega_S, \mathrm{div}) \times L^2(\Omega_S)$ such that $T_2^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}, p) = 0$. Then $(\mathbf{v}, p) \in W_{\eta,\lambda,\mathbf{k}}(\Omega_S)$, $\mathbf{v} = 0$ on $\partial\Omega_S \setminus \Gamma$, and $\mathbf{v}_\tau = 0$ on Γ. Since $\mathrm{div}\, \mathbf{v} = 0$, we have

$$0 = \langle T_1{}^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}, p), \mathbf{v}\rangle = \int_{\Omega_S} \left[2\eta|\mathbf{D}\mathbf{v}|^2 + (\lambda\mathbf{v}) \cdot \mathbf{v}\right] \, \mathrm{d}\boldsymbol{y} + \int_{\partial\Omega_S} b|\mathbf{v}|^2 \, \mathrm{d}\sigma_{\mathbf{y}}.$$

Thus $\mathbf{D}\mathbf{v} \equiv 0$ and $\mathbf{v} = 0$ on $\partial\Omega_S$. Since $\mathbf{D}\mathbf{v} \equiv 0$, we have that $\mathbf{v}$ is linear by [17, Lemma 3.1]. Since $\Delta\mathbf{v} = 0$ in Ω and $\mathbf{v} = 0$ on $\partial\Omega$, we infer that $\mathbf{v} \equiv 0$. So, $(\mathbf{v}, p)$ is a solution of the problem (10) with $\lambda \equiv 0$, $\mathbf{k} = 0$, $\mathbf{f} \equiv 0$, $\mathbf{g} \equiv 0$. Therefore $p \equiv 0$. Hence, the operator $T_2^{b,\eta,\lambda,\mathbf{k}} : H(\Omega_S, \mathrm{div}) \times L^2(\Omega_S) \to W_\Gamma \times \mathcal{V}'_\Gamma$ is a Fredholm operator with index 0 and with trivial kernel, so it is an isomorphism.

From the definition of $T_1^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}, p)$ we have that $T_1^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}, p)$ is supported on $\partial\Omega_S$ if and only if $(\mathbf{v}, p) \in W_{\eta,\lambda,\mathbf{k}}(\Omega_S)$. Hence, $T_1^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}, p) \in V'_\Gamma$ if and only if $(\mathbf{v}, p) \in W_{\eta,\lambda,\mathbf{k}}(\Omega_S)$. Then, since $T_2^{b,\eta,\lambda,\mathbf{k}} : H(\Omega_S, \mathrm{div}) \times L^2(\Omega_S) \to W_\Gamma \times \mathcal{V}'_\Gamma$ is an isomorphism, we have that $U^{b,\eta,\lambda,\mathbf{k}} = T_2^{b,\eta,\lambda,\mathbf{k}} : W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \to W_\Gamma \times V'_\Gamma$ is also an isomorphism. □

4. Robin type problem

The second auxiliary problem is the following Robin problem, related to the generalized Darcy equations,

$$\mathrm{div}(K\nabla p) = 0 \quad \text{in } \Omega_D, \tag{12}$$

$$\mathbf{n}^D \cdot K\nabla p + p = f \quad \text{on } \partial\Omega_D. \tag{13}$$

Here $\mathbf{n}^D$ is the unit exterior normal of Ω_D and $f \in H^{-1/2}(\partial\Omega_D)$. For $p \in H^1(\Omega_D)$ we define

$$\langle \partial_K^D p, \varphi\rangle \equiv \int_{\Omega_D} (K\,\nabla p) \cdot \nabla\varphi \, \mathrm{d}\mathbf{y}, \qquad \varphi \in H^1(\Omega_D). \tag{14}$$

We remark that $\partial_K^D p$ is supported on $\partial\Omega_D$, i.e. $\partial_K^D p \in H^{-1/2}(\partial\Omega_D)$, if and only if p is a solution of (12) in the sense of distributions (a weak solution). If this is true then $\partial_K^D p$ has a meaning of the conormal derivative $\mathbf{n}^D \cdot K(x)\nabla p$ from (13), i.e. $\partial_K^D p = \mathbf{n}^D \cdot K\nabla p$.

Definition 4.1. *We say that $p \in H^1(\Omega_D)$ is a solution of the Robin problem* (12)–(13) *if*

$$\langle U_K p, \varphi\rangle := \langle \partial_K^D p, \varphi\rangle + \int_{\partial\Omega_D} p\,\varphi \, \mathrm{d}\sigma_{\mathbf{y}} = \langle f, \varphi\rangle \qquad \forall\varphi \in H^1(\Omega_D). \tag{15}$$

Proposition 4.2. *The operator*

$$U_K : H^1(\Omega_D; K) \to H^{-1/2}(\partial\Omega_D)$$

is an isomorphism, where

$$H^1(\Omega_D; K) = \left\{p \in H^1(\Omega_D); \quad \mathrm{div}(K\nabla p) = 0\right\}.$$

Hence if $f \in H^{-1/2}(\partial\Omega_D)$ then there exists a unique solution of the Robin problem (12)–(13)*, and $u \in H^1(\Omega_D)$ is a solution of this problem if and only if $u \in H^1(\Omega_D; K)$ and $U_K u = f$.*

Proof. From (14) and (15), and the assumption on K, we have that $\langle U_K p, \varphi\rangle$ defines an inner product on $H^1(\Omega_D)$ and the corresponding norm is equivalent to the original norm in $H^1(\Omega_D)$. According to the Riesz representation theorem $U_K : H^1(\Omega_D) \to [H^1(\Omega_D)]'$ is an isomorphism. The space $H^{-1/2}(\partial\Omega_D)$ is a closed subspace of the dual space $[H^1(\Omega_D)]'$ formed by distributions from $[H^1(\Omega_D)]'$ supported on $\partial\Omega_D$. Since $H^1(\Omega_D; K) = U_K^{-1}([H^1(\Omega_D)]')$, we obtain the assertion stated in the proposition. □

5. Generalized Oseen–Darcy problem

In this section we shall study the original coupled problem (1)–(6). Since $\mathbf{v}^D\cdot\mathbf{n} = \mathbf{v}^S\cdot\mathbf{n} + h$ on Γ, we can rewrite (6) as

$$\Big[T_\eta(\mathbf{v}^S, p^S)\mathbf{n} - \frac{1}{2}(\mathbf{k}\cdot\mathbf{n})\mathbf{v}^S\Big]\cdot\mathbf{n} + \beta p^D + (\alpha+\gamma)\mathbf{v}^S\cdot\mathbf{n} = \tilde{\mathbf{g}}\cdot\mathbf{n}$$

or in the vector form

$$\Big[T_\eta(\mathbf{v}^S, p^S)\mathbf{n} - \frac{1}{2}(\mathbf{k}\cdot\mathbf{n})\mathbf{v}^S\Big]_{\mathbf{n}} + \beta p^D\mathbf{n} + (\alpha+\gamma)\mathbf{v}^S_{\mathbf{n}} = \tilde{\mathbf{g}}_{\mathbf{n}},$$

where $\tilde{\mathbf{g}}_{\mathbf{n}} = \mathbf{g}_{\mathbf{n}} - h\,\mathbf{n}$. Instead of this transmission condition we can consider a bit more general condition

$$\Big[T_\eta(\mathbf{v}^S, p^S)\mathbf{n} - \frac{1}{2}(\mathbf{k}\cdot\mathbf{n})\mathbf{v}^S\Big]_{\mathbf{n}} + \beta p^D\mathbf{n} + (A\mathbf{v}^S)_{\mathbf{n}} = \tilde{\mathbf{g}}_{\mathbf{n}} \quad \text{on } \Gamma, \tag{16}$$

where A is a matrix function of the type 3×3 with $A_{ij}\in L^\infty(\partial\Omega_S)$, which is nonnegative on Γ, i.e.

$$\big(A(\mathbf{x})\boldsymbol{\xi}\big)\cdot\boldsymbol{\xi} \ge 0 \qquad \forall\,\mathbf{x}\in\Gamma,\ \ \forall\,\boldsymbol{\xi}\in\mathbb{R}^3. \tag{17}$$

We shall study the transmission problem (1)–(5), (16) under assumption $\tilde{\mathbf{g}} \in [H^{-1/2}(\partial\Omega_S)]^3$, $h\in H^{-1/2}(\partial\Omega_D)$, and $\mathbf{f}\in[H^{1/2}(\Omega_S)]^3$.

Definition 5.1. *We say that* $(\mathbf{v}^S, p^S)\in[H^1(\Omega_S)]^3\times L^2(\Omega_S)$ *and* $(\mathbf{v}^D, p^D)\in[L^2(\Omega_D)]^3\times H^1(\Omega_D)$ *is a solution of the problem* (1)–(5), (16) *if*

- (i) $(\mathbf{v}^S, p^S)$ *is a solution of the system* (1) *in the sense of distributions, i.e.* $(\mathbf{v}^S, p^S)\in W_{\eta,\lambda,\mathbf{k}}(\Omega_S)$,
- (ii) $(\mathbf{v}^D, p^D)$ *is a solution of the system* (2) *in the sense of distributions, i.e.* $p^D\in H^1(\Omega_D;K)$ *and* $\mathbf{v}^D = -K\nabla p^D$ *in* Ω_D,
- (iii) *the boundary conditions* $\mathbf{v}^S = \mathbf{f}$ *on* $\partial\Omega_S\setminus\Gamma$ *and* $\mathbf{v}^S_\tau = \mathbf{f}_\tau$ *on* Γ *are satisfied in the sense of traces,*
- (iv) $\big[T_\eta(\mathbf{v}^S, p^S)\mathbf{n} - \frac{1}{2}(\mathbf{k}\cdot\mathbf{n})\mathbf{v}^S\big]_{\mathbf{n}}\big|_\Gamma + \beta p^D\mathbf{n}\chi_\Gamma\big|_\Gamma + [A\mathbf{v}^S]_{\mathbf{n}}\big|_\Gamma = \tilde{\mathbf{g}}_{\mathbf{n}}\big|_\Gamma$ *and* $\partial^D_K p^D = h + \mathbf{v}^S\cdot\mathbf{n}\chi_\Gamma$.

Here χ_Γ is the characteristic function of Γ.

First we show the uniqueness of a solution $(\mathbf{v}^S, p^S)\in[H^1(\Omega_S)]^3\times L^2(\Omega_S)/\mathbb{R}$ and $(\mathbf{v}^D, p^D)\in[L^2(\Omega_D)]^3\times H^1(\Omega_D)/\mathbb{R}$ of the problem (1)–(5), (16).

Proposition 5.2. *Let* $(\mathbf{v}^S, p^S)\in[H^1(\Omega_S)]^3\times L^2(\Omega_S)$ *and* $(\mathbf{v}^D, p^D)\in[L^2(\Omega_D)]^3\times H^1(\Omega_D)$ *be a solution of the problem* (1)–(5), (16) *with* $\mathbf{f}\equiv 0$, $h\equiv 0$, *and* $\tilde{\mathbf{g}}\equiv 0$. *Then there exists a constant* c *such that* $p^S = \beta c$, $\mathbf{v}^S\equiv 0$, $\mathbf{v}^D\equiv 0$, *and* $p^D = c$.

On the other hand, if $p^S = \beta c$, $\mathbf{v}^S \equiv 0$, $\mathbf{v}^D \equiv 0$, $p^D = c$ *for some constant* c, *then* $(\mathbf{v}^S, p^S, \mathbf{v}^D, p^D)$ *is a solution of the problem* (1)–(5), (16) *with* $\mathbf{f} \equiv 0$, $h \equiv 0$, *and* $\tilde{\mathbf{g}} \equiv 0$.

Proof. Since $\mathbf{v}^S = 0$ on $\partial\Omega_S \setminus \Gamma$ and $\mathbf{v}^S_\tau = 0$ on Γ, we have $\mathbf{v}^S \in \mathcal{V}_\Gamma$. Since $\mathbf{v}^S_\tau = 0$ on Γ, we have $\mathbf{v}^S \cdot (A\mathbf{v}^S)_\mathbf{n} = \mathbf{v}^S \cdot (A\mathbf{v}^S)$ on Γ. Using the fact that

$$\partial^D_K p^D = \mathbf{v}^S \cdot \mathbf{n}\chi_\Gamma$$

and

$$\Big[T_\eta(\mathbf{v}^S, p^S)\mathbf{n} - \frac{1}{2}(\mathbf{k}\cdot\mathbf{n})\mathbf{v}^S\Big]_\mathbf{n}\Big|_\Gamma + \beta p^D \mathbf{n}\chi_\Gamma\big|_\Gamma + \big(A\mathbf{v}^S\big)_\mathbf{n}\big|_\Gamma = 0,$$

we obtain

$$\begin{aligned}
0 &= \Big\langle \big[T_\eta(\mathbf{v}^S, p^S)\mathbf{n} - \frac{1}{2}(\mathbf{k}\cdot\mathbf{n})\mathbf{v}^S\big]_\mathbf{n}\big|_\Gamma + \beta p^D \mathbf{n}\chi_\Gamma\big|_\Gamma + (A\mathbf{v}^S)_\mathbf{n}\big|_\Gamma, \mathbf{v}^S \Big\rangle \\
&= \int_{\Omega_S} \big[2\eta|\mathbf{D}\,\mathbf{v}^S|^2 + (\lambda\mathbf{v}^S)\cdot\mathbf{v}^S\big]\mathrm{d}\mathbf{y} + \beta\langle\partial^D_K p^D, p^D\rangle + \int_\Gamma \mathbf{v}^S \cdot (A\mathbf{v}^S)_\mathbf{n}\mathrm{d}\sigma_\mathbf{y} \\
&= \int_{\Omega_S} \big[2\eta|\mathbf{D}\,\mathbf{v}^S|^2 + (\lambda\mathbf{v}^S)\cdot\mathbf{v}^S\big]\mathrm{d}\mathbf{y} + \beta\int_{\Omega_D} (K\nabla p^D)\cdot\nabla p^D\mathrm{d}\mathbf{y} + \int_\Gamma \mathbf{v}^S \cdot (A\mathbf{v}^S)\mathrm{d}\sigma_\mathbf{y} \\
&\geq \int_{\Omega_S} 2\eta|\mathbf{D}\,\mathbf{v}^S|^2\mathrm{d}\mathbf{y} + a\beta\int_{\Omega_D} |\nabla p^D|^2\mathrm{d}\mathbf{y}.
\end{aligned}$$

From the last inequality we deduce that $\nabla p^D = 0$ in Ω_D and $\mathbf{D}\,\mathbf{v}^S = 0$ in Ω_S. Hence p^D is constant on each component of Ω and $\mathbf{v}^D = -K\nabla p^D = 0$.

According to (5) we have $\mathbf{v}^S \cdot \mathbf{n} = \mathbf{v}^D \cdot \mathbf{n} = 0$ on Γ. Since $\mathbf{v}^S = 0$ on $\partial\Omega_S \setminus \Gamma$ and $\mathbf{v}^S_\tau = 0$ on Γ, we infer that $\mathbf{v}^S = 0$ on $\partial\Omega_S$. Using the fact that $\mathbf{D}\,\mathbf{v}^S \equiv 0$, we obtain that the functions $\mathbf{v}^S_j$, for $j = 1, 2, 3$, are affine (see [16, Lemma 6]), and therefore harmonic. The maximum principle for harmonic functions yields that $\mathbf{v}^S_j \equiv 0$, for $j = 1, 2, 3$.
Using the equation (1) and the fact that $\mathbf{v}^S \equiv 0$, we obtain $\nabla p^S = \eta\Delta\mathbf{v}^S - \lambda\mathbf{v}^S - \mathbf{k}\cdot\nabla\mathbf{v} = 0$. Thus there exists a constant $\tilde{c}$ such that $p^S = \tilde{c}$.

We have proved that p^D is constant on each component of Ω, $\mathbf{v}^D \equiv 0$, $\mathbf{v}^S \equiv 0$ and $p^S \equiv \tilde{c}$. Then, using the boundary conditions $0 = [T_\eta(\mathbf{v}^S, p^S)\mathbf{n}^S - \frac{1}{2}(\mathbf{k}\cdot\mathbf{n})\mathbf{v}^S]\cdot\mathbf{n} + \beta p^D + (A\mathbf{v}^D)\cdot\mathbf{n} = -\tilde{c} + \beta p^D$ on Γ, we conclude that $p^D = \tilde{c}/\beta$. Considering $c = \beta\tilde{c}$ we obtain the first statement of the proposition.

Substituting $p^S = \beta c$, $\mathbf{v}^S \equiv 0$, $\mathbf{v}^D \equiv 0$, $p^D = c$, with some constant c, into (1)–(5), (16), we obtain that the equations are satisfied for $\mathbf{f} \equiv 0$, $h \equiv 0$, and $\tilde{\mathbf{g}} \equiv 0$. □

Lemma 5.3. *Let* $\tilde{\mathbf{g}} \in [H^{-1/2}(\partial\Omega_S)]^3$, $\mathbf{f} \in [H^{1/2}(\partial\Omega_S)]^3$, $h \in H^{-1/2}(\partial\Omega_D)$. *If there exists a solution of the problem* (1)–(5), (16), *then*

$$\langle h, 1\rangle = \int_{\partial\Omega_S\setminus\Gamma} \mathbf{f}\cdot\mathbf{n}^S\ \mathrm{d}\sigma_\mathbf{y}. \tag{18}$$

Proof. Let $(\mathbf{v}^S, p^S) \in [H^1(\Omega_S)]^3 \times L^2(\Omega_S)$ and $(\mathbf{v}^D, p^D) \in [L^2(\Omega_D)]^3 \times H^1(\Omega_D)$ be a solution of the problem (1)–(5), (16). For $\varphi \equiv 1$ we have

$$\langle\partial^D_K p^D, 1\rangle = \int_{\Omega_D} (K\nabla p^D)\cdot\nabla\varphi\ \mathrm{d}\mathbf{y} = 0.$$

Since $\operatorname{div}\mathbf{v}^S = 0$, Green's theorem yields

$$\int_{\partial\Omega_S} \mathbf{v}^S \cdot \mathbf{n}^S \, \mathrm{d}\sigma_{\mathbf{y}} = 0, \tag{19}$$

see e.g. [11, Chapter 4]. Using the fact that $\partial_K^D p^D = \mathbf{v}^D \cdot \mathbf{n}$ and the equality (19), we obtain

$$\begin{aligned} 0 = \langle \partial_K^D p^D, 1\rangle &= \langle h, 1\rangle + \int_\Gamma \mathbf{v}^S \cdot \mathbf{n}^S \, \mathrm{d}\sigma_{\mathbf{y}} - \int_{\partial\Omega_S} \mathbf{v}^S \cdot \mathbf{n}^S \, \mathrm{d}\sigma_{\mathbf{y}} \\ &= \langle h, 1\rangle - \int_{\partial\Omega_S\setminus\Gamma} \mathbf{f} \cdot \mathbf{n}^S \, \mathrm{d}\sigma_{\mathbf{y}}. \end{aligned} \tag{20}$$

□

Theorem 5.4. *Consider* $\tilde{\mathbf{g}} \in [H^{-1/2}(\partial\Omega_S)]^3$, $\mathbf{f} \in [H^{1/2}(\partial\Omega_S)]^3$, $h \in H^{-1/2}(\partial\Omega_D)$. *Then there exists a solution of the problem* (1)–(5), (16) *if and only if the compatibility condition* (18) *is satisfied.*

Proof. We notice that $(\mathbf{v}^S, p^S)$ and $(\mathbf{v}^D, p^D)$ is a solution of the problem (1)–(5), (16) if and only if $(\mathbf{v}^S, p^S) \in W_{\eta,\lambda,\mathbf{k}}(\Omega_S)$, $p^D \in H^1(\Omega_D; K)$ and $\mathbf{v}^D = -K\nabla p^D$ together with

$$U(\mathbf{v}^S, p^S, p^D) = \left(\mathbf{f}\big|_{\partial\Omega_S\setminus\Gamma}, \mathbf{f}_\tau\big|_\Gamma, \tilde{\mathbf{g}}_{\mathbf{n}}\big|_\Gamma, h\right),$$

where the operator $U : W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \times H^1(\Omega_D; K) \to W_\Gamma \times V_\Gamma' \times H^{-1/2}(\partial\Omega_D)$ is given by

$$\begin{aligned} U(\mathbf{v}^S, p^S, p^D) = \Big(\mathbf{v}^S\big|_{\partial\Omega_S\setminus\Gamma},\ \mathbf{v}_\tau^S\big|_\Gamma,\ &\big[T_\eta(\mathbf{v}^S, p^S)\mathbf{n} - \frac{1}{2}(\mathbf{k}\cdot\mathbf{n})\mathbf{v}^S\big]_{\mathbf{n}}\big|_\Gamma \\ &+\beta p^D \mathbf{n}\big|_\Gamma + \big[A\mathbf{v}^S\big]_{\mathbf{n}}\big|_\Gamma,\ \partial_K^D p^D - \mathbf{v}^S \cdot \mathbf{n}\chi_\Gamma\Big). \end{aligned}$$

We also define the operator $\tilde U : W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \times H^1(\Omega_D; K) \to W_\Gamma \times V_\Gamma' \times H^{-1/2}(\partial\Omega_D)$ as

$$\begin{aligned} &\tilde U(\mathbf{v}^S, p^S, p^D) \\ &= \Big(\mathbf{v}^S\big|_{\partial\Omega_S\setminus\Gamma}, \mathbf{v}_\tau^S\big|_\Gamma, \big[T_\eta(\mathbf{v}^S, p^S)\mathbf{n} - \frac{1}{2}(\mathbf{k}\cdot\mathbf{n})\mathbf{v}^S\big]_{\mathbf{n}}\big|_\Gamma + \big[\mathbf{v}_{\mathbf{n}}^S + \beta p^D\mathbf{n}\big]\big|_\Gamma, \partial_K^D p^D + p^D\Big). \end{aligned}$$

Notice that

$$(\tilde U - U)(\mathbf{v}^S, p^S, p^D) = \Big(0, 0, \mathbf{v}_{\mathbf{n}}^S\big|_\Gamma - \big[A\mathbf{v}^S\big]_{\mathbf{n}}\big|_\Gamma, \mathbf{v}^S\cdot\mathbf{n}\chi_\Gamma + p^D\Big).$$

Consider fixed $[\mathbf{F}, \mathbf{G}] \in W_\Gamma \times V_\Gamma'$ and $h \in H^{-1/2}(\partial\Omega_D)$. According to Proposition 4.2 there exists a unique $p^D \in H^1(\Omega_D; K)$ such that $\partial_K^D p^D + p^D = h$. Proposition 3.3 ensures the existence of a unique $(\mathbf{v}^S, p^S) \in W_{\eta,\lambda,\mathbf{k}}(\Omega_S)$ such that

$$\Big(\mathbf{v}^S\big|_{\partial\Omega_S\setminus\Gamma}, \mathbf{v}_\tau^S\big|_\Gamma, \big[T_\eta(\mathbf{v}^S, p^S)\mathbf{n} - \frac{1}{2}(\mathbf{k}\cdot\mathbf{n})\mathbf{v}^S\big]_{\mathbf{n}}\big|_\Gamma + \mathbf{v}_{\mathbf{n}}^S\big|_\Gamma\Big) = \Big(\mathbf{F}, \big[\mathbf{G}_{\mathbf{n}} - \beta p^D\mathbf{n}\big]\big|_\Gamma\Big).$$

Hence $\tilde U : W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \times H^1(\Omega_D; K) \to W_\Gamma \times V_\Gamma' \times H^{-1/2}(\partial\Omega_D)$ is an isomorphism.

The operator $U - \tilde U : W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \times H^1(\Omega_D; K) \to W_\Gamma \times V_\Gamma' \times H^{-1/2}(\partial\Omega_D)$ is compact by the Sobolev imbedding theorem. Thus the operator U is a Fredholm operator with index 0. By Proposition 5.2 we have that the dimension of the kernel of U is equal to 1. Therefore the range of U is a subspace of $W_\Gamma \times V_\Gamma' \times H^{-1/2}(\partial\Omega_D)$ of the codimension 1. Hence, using Lemma 5.3 we deduce that there exists a solution of the problem (1)–(5), (16) if and only if (18) holds true. □

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