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Abstract Let $S^k : \mathbb{R}^n \to \mathbb{R}$ denote the kth elementary symmetric function of n variables ($1 \le k \le n$, $n = 1, \ldots$). The class of all infinitely differentiable functions $f : (0, \infty) \to \mathbb{R}$ is studied which satisfy the following condition:

$$\left.\begin{array}{l}\text{for all } n = 1, \ldots \text{ we have}\\ \qquad f(x_1)+\ldots+f(x_n) \le f(y_1)+\ldots+f(y_n)\\ \text{whenever } x_1, \ldots, x_n > 0 \text{ and } y_1, \ldots, y_n > 0 \text{ satisfy}\\ \qquad S^k(x_1, \ldots, x_n) \le S^k(y_1, \ldots, y_n) \quad \text{for} \quad k = 1, \ldots, n-1,\\ \qquad S^n(x_1, \ldots, x_n) = S^n(y_1, \ldots, y_n).\end{array}\right\} \quad \text{(C)}$$

Two sufficient conditions, themselves mutually equivalent, are given:

- the function $x \mapsto xf'(x)$, $x > 0$, has an analytic extension $\varphi : \mathbb{C} \sim (-\infty, 0] \to \mathbb{C}$ with nonnegative imaginary part $\operatorname{Im} \varphi(z)$ for all $z \in \mathbb{C}$ with $\operatorname{Im} z > 0$;
- there exists a nonnegative measure μ on $(0, \infty)$ with $\int_{(0,\infty)} y/(1+y^2)\, d\mu(y) < \infty$ and constants $a, c \in \mathbb{R}$ and $b \ge 0$, $d \ge 0$ such that

  $$f(x) = a + bx + c \log x + d/x + \int_{(0,\infty)} \left(\log\left[(x+y)/(1+y)\right] - \log x/(1+y^2)\right) d\mu(y)$$

  for all $x > 0$.

These two conditions are equivalent to (C) if the function $x \mapsto xf'(x)$ is bounded from below **or** from above (in particular if f is nondecreasing or nonincreasing). The proofs rely on classical results on Pick's functions. The question of characterizing the class (C) was raised in [11].

Key words Elementary symmetric functions · Bernstein's theorem · completely monotone functions · Stieltjes transform · matrix monotone functions · Pick class · Nevanlinna's representation

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Contents

1 Introduction

Motivated by questions related to the logarithmic strain measure and logarithmic energy in nonlinear elasticity extensively studied by P. Neff and collaborators [6–10], Pompe & Neff [11] raise the question of characterizing all functions $f : (0,\infty) \to \mathbb{R}$ which satisfy Condition (C) in Definition 1.1, below. For each positive integer n and each integer k with $0 \le k \le n$ let $S^k : \mathbb{R}^n \to \mathbb{R}$ be the kth elementary symmetric function of n variables,

$$S^0(x) = 1,$$

$$S^k(x) = \sum_{1 \le i_1 < \cdots < i_k \le n} x_{i_1} \cdots x_{i_n}, \quad x = (x_1, \ldots, x_n) \in \mathbb{R}^n, \quad \text{if} \quad k > 0.$$

To avoid repeated hypotheses, throughout the paper let $f : (0,\infty) \to \mathbb{R}$ be an infinitely differentiable function.

1.1 Definition. f is said to satisfy Condition (C) [respectively, (C^+)] if for all $n = 1, \ldots$ we have

$$f(x_1)+\ldots+f(x_n) \le f(y_1)+\ldots+f(y_n) \tag{1.1}$$

whenever x and $y \in (0,\infty)^n$ satisfy

in the case of (C): $\quad S^k(x) \le S^k(y) \;$ for $\; k = 1, \ldots, n-1, \; S^n(x) = S^n(y),$

in the case of (C^+): $\quad S^k(x) \le S^k(y) \;$ for $\; k = 1, \ldots, n.$

Clearly,

$$(C^+) \;\Rightarrow\; (C)$$

and the classes of functions satisfying Condition (C) [or (C^+)] form convex cones, i.e., if f_1, f_2 satisfy any of these two conditions then $t_1 f_1 + t_2 f_2$ satisfies the same condition for any $t_1 \ge 0$, $t_2 \ge 0$.

The goal of the present paper is to describe the classes of functions that satisfy Conditions (C) or (C^+). The description is almost complete, thanks to the powerful results on the following class of analytic functions.

1.2 Theorem (Pick functions). *Let* $\Pi_+ = \{z \in \mathbb{C} : \operatorname{Im} z > 0\}$ *be the upper complex halfplane. A function* $\varphi : \Pi_+ \to \mathbb{C}$ *is said to be a Pick function if it satisfies any of the following two equivalent conditions:*

(i) φ *is analytic and* $\operatorname{Im} \varphi(z) \ge 0$ *for all* $z \in \Pi_+$;

(ii) (R. Nevanlinna) *there exists a nonnegative measure* ν *on* $\mathbb{R}$ *with* $\int_{\mathbb{R}} (1 + \lambda^2)^{-1} d\nu(\lambda) < \infty$ *and constants* $b \ge 0$, $c \in \mathbb{R}$ *such that for each* $z \in \Pi_+$,

$$\varphi(z) = c + bz + \int_{\mathbb{R}} \big(1/(\lambda - z) - \lambda/(1+\lambda^2)\big)\, d\nu(\lambda). \tag{1.2}$$

A Pick function φ *admits an analytic continuation across an open interval* $I \subset \mathbb{R}$ *into the lower halfplane if and only if the measure* ν *in* (1.2) *satisfies* $\nu(I) = 0$.

We refer to [4; Chapter II], [12; Chapter 2], [2; Chapter V] and [1; Sections 59 & 60] for proofs, for other equivalent characterizations of Pick's functions, and for their relationship to Löwner's theory of monotone matrix functions.

The main message of this note is that Condition (C) *is closely related to the following condition.*

1.3 Definition. f is said to satisfy Condition (G) if the function $x \mapsto xf'(x)$ has an analytic extension $\varphi : \mathbb{C} \sim (-\infty, 0] \to \mathbb{C}$ whose restriction to Π_+ is a Pick function.

Our results are as follows: if

$$\left.\begin{array}{c}\text{either } xf'(x) \geq c \text{ for all } x > 0 \text{ and some } c \in \mathbb{R} \\ \text{or } xf'(x) \leq c \text{ for all } x > 0 \text{ and some } c \in \mathbb{R},\end{array}\right\} \qquad (1.3)$$

then $$\text{(G)} \quad \Leftrightarrow \quad \text{(C)} \qquad (1.4)$$

while generally $$\text{(G)} \quad \Rightarrow \quad \text{(C)} \qquad (1.5)$$

and $$(\text{C}^+) \quad \Leftrightarrow \quad \text{(G)} \quad \& \quad f \quad \text{is nondecreasing.}$$

Section 4 gives a synoptic summary of main results, but we first describe the underlying notions and their relationships in more detail.

In Section 2 we integrate Nevanlinna's formula to obtain a representation formula for functions satisfying Condition (G). Then we give an elementary proof that this representation implies Condition (C). The mentioned representation is easily established for the basic example of [11], viz., $f(x) = \log^2 x$, $x > 0$, thus providing a proof that $\log^2 x$ satisfies Condition (C) alternative to that of Borisov [3]. In Section 3 we establish the relationship of Condition (C) to completely monotone functions and Bernstein's representations of such functions. Using these ideas, a new proof is given of the necessary condition for (C) established in [11; Theorem 2.4]. The main goal of that section, however, is the proof that under (1.3), (C) conversely implies (G). This is achieved by using a theorem by Widder [17; Theorem 18b, p. 366], which fits our needs almost miraculously. Some examples are in Table 4.3.

A future paper will detailize the multifaceted relationships of Condition (C) to the theory of Pick's functions outlined above, and to the theory of majorization [5], whose relevance to continuum mechanics has been first pointed out in [15; Introduction].

2 Integral representation of f, proof of (G) ⇒ (C), and $\log^2 x$

From (1.2) we obtain the following central representation of functions satisfying Condition (G):

2.1 Theorem. *The following three conditions are equivalent:*

(i) *f satisfies Condition* (G);

(ii) *there exists a nonnegative measure μ on $(0,\infty)$ with $\int_{(0,\infty)} y/(1+y^2)\, d\mu(y) < \infty$ and constants a, $c \in \mathbb{R}$, $b \geq 0$ and $d \geq 0$ such that*

$$\begin{aligned} f(x) &= a + bx + c\log x + d/x \\ &\quad + \int_{(0,\infty)} \big(\log\left[(x+y)/(1+y)\right] - \log x/(1+y^2)\big)\, d\mu(y) \end{aligned} \qquad (2.1)$$

for all $x > 0$;

(iii) *there exists a nonnegative measure ρ on $(0,\infty)$ with $\int_{(0,\infty)} (1+y^2)^{-1}\, d\rho(y) < \infty$ and constants $b \geq 0$, $c \in \mathbb{R}$, $d \geq 0$ such that*

$$xf'(x) = c + bx - d/x - \int_{(0,\infty)} \big(1/(x+y) - y/(1+y^2)\big)\, d\rho(y), \qquad (2.2)$$

$x > 0$.

Proof (i) ⇔ (iii): Let φ be the Pick extension of $x \mapsto xf'(x)$. By Theorem 1.2, φ has the representation (1.2) with ν a nonnegative measure supported by $(-\infty, 0]$ and the constants b, c as above. Defining the measure ν' on $\mathbb{R}$ by $\nu' = \nu - \nu(\{0\})\delta_0$ where δ_0 is Dirac's measure at 0, and a measure ρ by $\rho(B) = \nu'(-B)$ for any Borel subset B of $\mathbb{R}$, we see that (1.2) reduces to (2.2) with $d = \nu(\{0\})$ and vice versa.

(ii) ⇔ (iii): Equation (2.2) provides

$$f'(x) = c/x + b - d/x^2 + \int_{(0,\infty)} [1/(x+y) - 1/[x(1+y^2)]]\, d\mu(y)$$

where $d\mu(y) = d\rho(y)/y$. Integrating this equality with respect to x from 1 to $x > 0$ provides

$$f(x) - f(1) = c\log x + b(x-1) + d/x$$
$$+ \int_{(0,\infty)} [\log[(x+y)/(1+y)] - \log x/(1+y^2)]\, d\mu(y),$$

which is (2.1) with $a = f(1) - b$. Conversely, a differentiation of (2.1) yields (2.2). □

2.2 Proof of (G) ⇒ (C). Let f satisfy (G) so that it is of the form (2.1) by Theorem 2.1. To prove that f satisfies (C), we note that in view of $b \ge 0$, $d \ge 0$ and $\mu \ge 0$, it suffices to verify that Condition (C) is satisfied by each of the functions

$$f(x) = \pm 1, \quad f(x) = x, \quad f(x) = \pm\log x, \quad f(x) = 1/x, \tag{2.3}$$

$$f_y(x) = \log[(x+y)/(1+y)] - (\log x)/(1+y^2) \tag{2.4}$$

where $y > 0$ is a parameter. This is immediate for the functions $(2.3)_{1-3}$; in the case $(2.3)_4$ we have

$$f(x_1) + \ldots + f(x_n) = S^{n-1}(x)/S^n(x),$$

which is an increasing function of S^{n-1} at constant S^n. Similarly,

$$f_y(x_1) + \ldots + f_y(x_n) = \log((x_1 + y) \cdots (x_n + y)/(1+y)^n)$$
$$-(\log(x_1 \cdots x_n))/(1+y^2)$$
$$= \log\Big(\sum_{i=0}^{n} y^{n-i} S^i(x)/(1+y)^n\Big) - (\log S^n(x))/(1+y^2)$$

which is an increasing function of $S^1, \ldots, S^{n-1}$ at constant S^n. □

The representation (2.1) identifies the functions in (2.3) and (2.4) as basic functions satisfying Condition (C). A general function f is obtained by a synthesis of these.

2.3 Example. The paper [11] is mainly motivated by the (then) conjectural statement that the function

$$f(x) = \log^2 x, \quad x > 0,$$

satisfies Condition (C). The authors show in [11] that Inequalities (1.1) hold if $1 \le n \le 4$ and a recent proof of Borisov [3] shows that the conjecture is true. An alternative proof, which does not require any knowledge, is to realize that $\log^2 x$

admits the representation (2.1) and to invoke the above verification that (2.1) implies (C). Specifically, one has (2.1) with $a = b = c = d = 0$ and $d\mu(y) = 2dy/y$ if $y > 0$ and $d\mu(y) = 0$ otherwise, i.e.,

$$\log^2 x = 2 \int_{(0,\infty)} \left(\log\left[(x+y)/y\right] - (\log x)/(1+y^2)\right) dy/y, \quad x > 0.$$

(This is elementary to verify: the equality plainly holds for $x = 1$ and the differentiation followed by a subsequent multiplicstion by $x/2$ leads to

$$\log x = -\int_{(0,\infty)} \left((x+y)^{-1} - y/(1+y^2)\right) dy$$

which is verified by an easy integration.) □

3 Bernstein, Stieltjes, and the proof of (C) ⇒ (G)

To summarize, we have proved Implication (1.5). As already mentioned, the converse implication, and hence the equivalence (1.4), is currently available only under (1.3). The main theme of the present section is just the proof of (C) ⇒ (G). This will bring us to another relevant class of functions, see Definition 3.3, below. We first derive a differential consequence of (C) for completeness; cf. [11] for similar considerations.

3.1 Proposition. *If f satisfies Condition* (C) *then for any positive integer n and any $x = (x_1, \ldots, x_n) \in (0,\infty)^n$ with mutually distinct components we have*

$$\sum_{i=1}^{n} (-x_i)^k \omega_i(x) f'(x_i) \ge 0, \quad k = 1, \ldots, n-1, \tag{3.1}$$

where

$$\omega_i(x) = 1 \Big/ \prod_{\substack{m=1 \\ m \ne i}}^{n} (x_m - x_i), \quad i = 1, \ldots, n.$$

Proof Let $\Phi : (0,\infty)^n \to \mathbb{R}$ and $S : \mathbb{R}^n \to \mathbb{R}^n$ be defined by

$$\Phi(x_1, \ldots, x_n) = f(x_1) + \ldots + f(x_n), \quad x_1, \ldots, x_n > 0,$$
$$S(x) = (S^1(x), \ldots, S^n(x)), \quad x \in \mathbb{R}^n.$$

Let $\nabla S(x)$ be the derivative of S at x, i.e., a linear transformation from $\mathbb{R}^n$ to $\mathbb{R}^n$ with the matrix $\nabla S(x) = [J_{ij}]$, $J_{ij} = S^i_{,j}(x)$. For each $x \in \mathbb{R}^n$, let

$$Z(x) = [Z_{ij}(x)] \quad \text{with} \quad Z_{ij}(x) = x_i^{n-j}$$

be the Vandermonde matrix and let

$$\Omega(x) = \operatorname{diag}(\omega_1(x), \ldots, \omega_n(x)).$$

By [14; Lemma 2.4], $\nabla S(x)$ is nonsingular if and only if the components of x are distinct and then

$$\nabla S(x)^{-1} = \Omega(x) Z(-x). \tag{3.2}$$

If $\sigma = \sigma(t) : [0,\varepsilon) \to \mathbb{R}^n$ is a smooth curve with $\sigma(0) = S(x)$ then by the inverse function theorem if ε small enough, there exists a smooth curve $\xi = \xi(t) : [0,\varepsilon) \to (0,\infty)^n$ with $\xi(0) = x$ and $\sigma(t) = S(\xi(t))$ for $t \in [0,\varepsilon)$. Equation (3.2) gives

$$\frac{d}{dt}\Phi(\xi(t))\Big|_{t=0} = \sum_{j=1}^{n} p_j \sum_{i=1}^{n}(-x_i)^{n-j}\omega_i(x)\Phi_{,\,i}(x)$$

where $p = (p_1, \dots, p_n) = \dot\sigma(0)$. Assuming that $\sigma_i(t) > \sigma(0)$, $i = 1, \dots, n-1$, $\sigma_n(t) = \sigma(0)$, for $t \in [0, \varepsilon)$, we obtain from Condition (C) the inequality $\Phi(\xi(t)) \ge \Phi(x)$. Differentiating, we obtain $d\Phi(\xi(t))/dt|_{t=0} \ge 0$ and consequently the arbitrariness of $p_1 \ge 0, \dots, p_{n-1} \ge 0$ gives

$$\sum_{i=1}^{n}(-x_i)^k\omega_i(x)\Phi_{,\,i}(x) \ge 0, \quad k = 1, \dots, n-1,$$

which is to (3.1). □

Let $h : (0,\infty) \to \mathbb{R}$. If $x > 0$, $r > 0$ and m is a nonnegative integer, we define the mth iterated difference of h at x corresponding to the increment r by

$$\Delta_r^m h(x) = \sum_{l=0}^{m}(-1)^l\binom{m}{l}h(x + lr).$$

3.2 Theorem (Bernstein's theorem; see, e.g., [13; Theorem 4.8]). *If $h : (0,\infty) \to \mathbb{R}$ is a function then the following conditions are equivalent:*

(i) *h is infinitely differentiable and*

$$(-1)^m h^{(m)}(x) \ge 0, \quad m = 0, 1, \dots, \quad x > 0;$$

(ii) *we have*

$$(-1)^m\Delta_r^m h(x) \ge 0$$

for any nonnegative integer m, any $x > 0$ and $r > 0$;

(iii) *there exists a nonnegative measure v on $[0,\infty)$ such that*

$$h(x) = \int_{[0,\infty)} e^{-tx}\,dv(t)$$

for every $x > 0$ where the measure v is such that the integral on the right hand side converges for every $x > 0$.

3.3 Definition. A function h is said to be completely monotone if it satisfies Conditions (i)–(iii) in Theorem 3.2.

3.4 Proposition. *Let k be a nonnegative integer and let $h : (0,\infty) \to \mathbb{R}$ be class k function such that*

$$(-1)^m\Delta_r^m h(x) \ge 0$$

for any integer $m \ge k$, any $x > 0$ and $r > 0$. Then $(-1)^k h^{(k)}$ is completely monotone.

Proof By induction on k. The case $k = 0$ is clear. Let $k \ge 1$. The inspection of [17; Proof of Theorem 6, p. 150] shows that if $q : (-\infty, 0) \to \mathbb{R}$ satisfies

$$\Delta_r^m q(x) \ge 0$$

for some positive integer m and all $x < 0$, $r > 0$ such that $x + mr < 0$ then

$$\Delta_r^{m-1}q(x) \le \Delta_r^{m-1}q(y)$$

whenever $x < y < 0$ and $y + mr < 0$. The differentiation then gives

$$\Delta_r^{m-1} q'(x) \ge 0 \tag{3.3}$$

for every $x < 0$ and $r > 0$ such that $x + (m-1)r < 0$. Let h satisfy the hypothesis of the theorem. By (3.3) with $q(x) = h(-x)$ then h' satisfies

$$-\Delta_r^m h'(x) \ge 0$$

for any integer $m \ge k - 1$, any $x > 0$ and $r > 0$. The induction hypothesis gives that $-(-1)^{k-1}(h')^{(k-1)} \equiv (-1)^k h^{(k)}$ is completely monotone. □

We are now ready to use completely monotone functions to derive the following necessary condition for (C) by Pompe & Neff [11; Theorem 2.4], whose original proof is different.

3.5 Proposition. *If f satisfies* (C) *then for each $k = 1, \dots,$ the function $(x^k f'(x))^{(k)}$ is completely monotone, i.e.,*

$$(-1)^m (x^k f'(x))^{(k+m)} \ge 0, \quad x > 0, \quad m = 0, \dots \tag{3.4}$$

Proof Let $m \ge k$ be a given integer and let $x > 0$ and $r > 0$. We apply Inequality (3.1) to $n = m + 1$ and $x = (x_1, \dots, x_n) \in (0, \infty)^n$ given by

$$x_i = x + (i-1)r, \quad i = 1, \dots, n.$$

Observing that

$$\omega_i(x) = (-1)^{i+1} r^{-n} \binom{n-1}{i-1} / (n-1)!, \quad i = 1, \dots, n,$$

and writing $g_k(x) := x^k f'(x)$, we see that (3.1) reduces to

$$\begin{aligned} \sum_{i=1}^{n} (-1)^{i+k} \binom{n-1}{i-1} g_k(x + (i-1)r) &\equiv (-1)^{k-1} \sum_{l=0}^{m} (-1)^l \binom{m}{l} g_k(x + lr) \\ &\equiv (-1)^{k-1} \Delta_r^m g_k(x) \ge 0, \end{aligned}$$

which holds for all $m \ge k$. The conclusion follows from Proposition 3.4. □

The proof that if f satisies the bounddness (1.3), then Condition (3.4) implies (G) [and hence also (C)] is based on the following remarkable theorem of Widder, proved by his inversion formula for the Stieltjes transform in [16], [17; Theorem 10a, p. 348].

3.6 Theorem ([17; Theorem 18b, p. 366]). *If $h : (0, \infty) \to \mathbb{R}$ is an infinitely differentiable function then h has a representation*

$$h(x) = b + \bar{c}/x + \int_{(0,\infty)} (x + y)^{-1} \, d\mu(y), \quad x > 0, \tag{3.5}$$

where $b \ge 0$, $\bar{c} \ge 0$, and μ is a nonnegative measure on $(0, \infty)$ if and only if h satisfies

$$h \ge 0, \tag{3.6}$$

$$(-1)^{k-1} (x^k h(x))^{2k-1} \ge 0, \quad x > 0, \quad k = 1, \dots \tag{3.7}$$

We have slightly rephrased Widder's original formulation by separating a possible atom at 0 of Widder's measure and moving it into the term $\bar{c}/x$. The passage

$$\mu \mapsto s(x) = \int_{(0,\infty)} (x + y)^{-1} \, d\mu(y)$$

is called the Stieltjes transformation.

3.7 Proof of the implication (C) ⇒ (G) under (1.3). Let f satisfy (C) and $(1.3)_1$. One finds that the function

$$h(x) = f'(x) - c/x \quad x > 0,$$

satisfies (3.6) and (3.7) since for $k \ge 1$ we have

$$(-1)^{k-1}(x^k h(x))^{(2k-1)} \;=\; (-1)^{k-1}(x^k f'(x))^{(2k-1)} \ge 0,$$

by (3.4) with $m = k-1$. Thus Theorem 3.6 provides the representation

$$f'(x) = b + \hat{c}/x + \int_{(0,\infty)} (x+y)^{-1}\, d\mu(y), \quad x > 0$$

where $\hat{c} = c + \bar{c}$. Integrating from 1 to $x > 0$, we obtain

$$f(x) = a + bx + \hat{c}\log x + \int_{(0,\infty)} \log\left[(x+y)/(1+y)\right] d\mu(y)$$

with $a = f(1) - b$. The convergence of (3.5) for $x = 1$ gives

$$\int_{(0,\infty)} (1+y)^{-1}\, d\mu(y) < \infty$$

which implies

$$\int_{(0,\infty)} y/(1+y^2)\, d\mu(y) < \infty, \qquad \int_{(0,\infty)} (1+y^2)^{-1}\, d\mu(y) < \infty.$$

Thus we have (2.1) with

$$c = \hat{c} + \int_{(0,\infty)} (1+y^2)^{-1}\, d\mu(y).$$

Next let f satisfy (C) and $(1.3)_2$. We reduce this case to the previous one. Let $\bar{f} : (0,\infty) \to \mathbb{R}$ be defined by $\bar{f}(x) = f(1/x)$, $x > 0$. Let us see that $\bar{f}$ satisfies Condition (C). Indeed, let $\xi = (x_1, \ldots, x_n)$ and $\eta = (y_1, \ldots, y_n) \in (0,\infty)^n$ satisfy

$$S^k(\xi) \le S^k(\eta) \ \text{ for } \ k = 1, \ldots, n-1, \quad S^n(\xi) = S^n(\eta)$$

and prove that

$$\bar{f}(x_1) + \ldots + \bar{f}(x_n) \le \bar{f}(y_1) + \ldots + \bar{f}(y_n). \tag{3.8}$$

The easily verifiable formulas

$$S^k(\bar{\xi}) = S^{n-k}(\xi)/S^n(\xi),$$

$0 \le k \le n$, where $\bar{\xi} := (1/x_1, \ldots, 1/x_n)$, imply that $\bar{\xi}$ and $\bar{\eta} := (1/y_1, \ldots, 1/y_n)$ satisfy

$$S^k(\bar{\xi}) \le S^k(\bar{\eta}) \ \text{ for } \ k = 1, \ldots, n-1, \quad S^n(\bar{\xi}) = S^n(\bar{\eta}).$$

By hypothesis, f satisfies (C) and hence

$$f(1/x_1) + \ldots + f(1/x_n) \le f(1/y_1) + \ldots + f(1/y_n),$$

i.e., (3.8) holds, which proves that $\bar{f}$ satisfies Condition (C). One easily notes that

$$xf'(x) = -(1/x)\bar{f}'(1/x); \tag{3.9}$$

therefore, since f satisfies $(1.3)_2$, $\bar{f}$ satisfies $(1.3)_1$. Hence $\bar{f}$ satisfies (G) by the preceding part of the proof, i.e., $x \mapsto x\bar{f}'(x)$, $x > 0$, has an analytic extension

$\bar{\varphi} : \mathbb{C} \sim (-\infty, 0] \to \mathbb{C}$. Equation (3.9) shows that then $x \mapsto xf'(x)$, $x > 0$, has an analytic extension φ given by

$$\varphi(z) = -\bar{\varphi}(1/z), \quad z \in \mathbb{C} \sim (-\infty, 0]$$

and $\operatorname{Im} \varphi(z) = -\operatorname{Im} \bar{\varphi}(1/z)$. Since $\bar{\varphi}$ is a Pick function, it satisfies $\operatorname{Im} \bar{\varphi}(1/z) \leq 0$ whenever $\operatorname{Im}(1/z) < 0$ by the reflection principle. Thus φ is a Pick function as well; i.e., $x \mapsto xf'(x)$ satisfies (G). □

We close this section with the following remark on producing new functions satisfying (C) or (C$^+$) from old ones.

3.8 Remark. *If f satisfies*

$$(-1)^m (x^l f'(x))^{(l+m)} \geq 0, \quad x > 0 \tag{3.10}$$

for each

$$l,\ m = 0, \ldots, \quad l + m \geq 1,$$

then for any $a \geq 0$, the function $f_a : (0, \infty) \to \mathbb{R}$ given by $f_a(x) = f(x + a)$, $x > 0$, satisfies Condition (G) *and hence also* (C).

Proof It suffices to prove that $x \mapsto xf_a'(x)$ is bounded from below and that $(x^k f_a'(x))^{(k)}$ is completely monotone for each $k = 1, \ldots,$ i.e.,

$$(-1)^m (x^k f'(x + a))^{(k+m)} \geq 0, \quad x > 0, \quad m = 0, \ldots,\ k = 1, \ldots \tag{3.11}$$

We first apply (3.10) with $l = 0$, $m = 1$ and then with $l = 1$, $m = 0$ to find that

$$f''(x) \leq 0, \quad f'(x) + xf''(x) \geq 0$$

which gives $f'(x) \geq 0$. Thus $x \mapsto xf'(x)$ is bounded from below by 0.

To prove (3.11), we note

$$\begin{aligned}(-1)^m (x^k f'(x + a))^{(k+m)} &= (-1)^m \big[((x + a) - a)^k f'(x + a) \big]^{(k+m)} \\ &= (-1)^m \Big[\sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} a^{k-l} (x + a)^l f'(x + a) \Big]^{(k+m)} \\ &= \sum_{l=0}^{k} \binom{k}{l} a^{k-l} \big[(-1)^{k+m-l} (x + a)^l f'(x + a) \big]^{(l+k+m-l)}.\end{aligned}$$

Each term in the last square bracket is nonnegative since it is (3.10) with $m \geq 0$ replaced by $k + m - l \geq 0$. □

4 Summary of main results; examples

We summarize Sections 1–3 as follows.

4.1 Theorem. *Consider the following three assertions*:

(i) *f satisfies condition* (G);

(ii) *f satisfies Condition* (C);

(iii) *for each $k = 1, \ldots$, the function $x \mapsto (x^k f'(x))^{(k)}$ is completely monotone.*

Then

$$\text{(i)} \Rightarrow \text{(ii)} \Rightarrow \text{(iii)}.$$

If additionally $x \mapsto xf'(x)$ is bounded from below or from above then

$$\text{(i)} \Leftrightarrow \text{(ii)} \Leftrightarrow \text{(iii)}.$$

The following theorem deals with Condition (C^+).

4.2 Theorem. *The following three assertions are equivalent*:
(i) *f satisfies Condition* (C^+);
(ii) *$f' \geq 0$ and f satisfies* (G);
(iii) *there exists a nonnegative measure μ on $(0,\infty)$ with $\int_{(0,\infty)}(1+y^2)^{-1}\,d\mu(y) < \infty$ and constants $a \in \mathbb{R}$, $b \geq 0$ and $\bar{c} \geq 0$ such that*

$$f(x) = a + bx + \bar{c}\log x + \int_{(0,\infty)} \log\left[(x+y)/(1+y)\right] d\mu(y), \quad x > 0;$$

(iv) *$f' \geq 0$ and for each $k = 1,\ldots,$ the function $x \mapsto (x^k f'(x))^{(k)}$ is completely monotone.*

We have equivalences here. We also see that f satisfies Condition (C^+) if it satisfies Condition (C) and is nondecreasing.

Proof (i) $\Rightarrow$ (iv): If f satisfies (C^+) then Inequality (1.1) for $n = 1$ says that f is nondecreasig: $f'(x) \geq 0$, $x > 0$. Also, if f satisfies (C^+), it also satisfies (C) and hence for each $k = 1,\ldots,$ the function $x \mapsto (x^k f'(x))^{(k)}$ is completely monotone by Theorem 4.1.

(iv) $\Rightarrow$ (iii): Noting that if (iii) holds, then the function $h := f'$ satisfies hypothesis of Theorem 3.6. Hence we have the representation

$$f'(x) = b + \bar{c}/x + \int_{(0,\infty)} (x+y)^{-1}\,d\mu(y), \quad x > 0,$$

with $b \geq 0$, $\bar{c} \geq 0$ and μ a nonnegative measure on $(0,\infty)$. Integrating in the same way as in Subsection 3.7 we obtain (iii).

(iii) $\Leftrightarrow$ (ii): This follows from the equivalence (ii) $\Leftrightarrow$ (i) in Theorem 2.1.

(iii) $\Rightarrow$ (i): One just modifies the proof of (G) $\Rightarrow$ (C) in Subsection 2.2. As $b \geq 0$, $\bar{c} \geq 0$ and $\mu \geq 0$, it suffices to verify that each of the functions

$$f(x) = \pm 1, \quad f(x) = x, \quad f(x) = \log x,$$

$$f_y(x) = \log\left[(x+y)/(1+y\right]$$

where $y > 0$ is a parameter, satisfies (C^+). This is almost the same as in the proof in Subsection 2.2 and the details are left to the reader. □

4.3 Examples. The classes $\mathscr{C}$ and $\mathscr{C}^+$ of functions satisfying Conditions (C) and (C^+) are large: by Theorem 4.1(i), $\mathscr{C}$ is parametrized by three constants and a nonnegative measure. Alternatively, $\mathscr{C}$ is parametrized by Pick's functions, via the associated function $xf'(x)$. The Pick class contains all complete Bernstein functions [13], some of which are tabulated in [13] together with associated representations (2.2). In Table 4.3, below, we discuss some examples with a particular attention to those conjectured in [11]. Let P be the set of all functions g on $(0,\infty)$ which admit an analytic extension to a Pick function and let $g_k(x) := x^k f'(x)$, $k = 1,\dots$, and $g := g_1$.

#	$f(x)$		in $\mathscr{C}$?	in $\mathscr{C}^+$?	$g(x)$	Why?
1	$\log^2 x$		yes	no	$\log x$	$g \in P$, (standard, e.g., [4; p. 19], f not monotone)
2	$\log x$		yes	yes	1	$g \in P$, $f' \geq 0$ (obvious)
3	x^p	$-\infty < p < -1$,	no	no	px^p	$g_k^{(k)}$ is not completely monotone for $k \geq -p$
4	x^p	$-1 \leq p < 0$	yes	no	px^p	$g \in P$, $f' < 0$ (obvious)
5	x^p	$0 \leq p \leq 1$,	yes	yes	px^p	$g \in P$, $f' \geq 0$ (obvious)
6	x^p	$1 < p < \infty$,	no	no	px^p	$g_k^{(k)}$ is not completely monotone for $k \geq p$

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