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Abstract. L^2-solutions of the transmission problem, the Robin–transmission problem and the Dirichlet–transmission problem for the Brinkman system are studied by the integral equation method. The necessary and sufficient conditions for the solvability are given. The uniqueness of a solution is also studied.

Keywords. Brinkman system, transmission problem, single layer potential, double layer potential

MSC 2000: 35Q35, 35Q30

1 Introduction

The integral equation method is one of traditional methods in hydrodynamics ([3], [10],[11], [14], [15],[17]). This method is especially fruitful for transmission problems ([1], [5], [6], [7],[8], [9],[14]). In this paper we study the following transmission problem: Let $\Omega = \Omega_+ \subset R^m$, $m > 2$, be a bounded open set with Lipschitz boundary. Denote $\Omega_- = R^m \setminus \overline{\Omega}_+$, where $\overline{\Omega}_+$ is the closure of Ω_+. Let λ_+, λ_-, c_+ be non-negative constants and a_+, a_-, b_+, b_- positive constants. We study the transmission problem for the Brinkman system

$$-\Delta \mathbf{u}_\pm + \lambda_\pm \mathbf{u}_\pm + \nabla p_\pm = 0, \quad \nabla \cdot \mathbf{u}_\pm = 0 \quad \text{in} \quad \Omega_\pm,$$

$$a_+\mathbf{u}_+ - a_-\mathbf{u}_- = \mathbf{g}, \quad b_+T(\mathbf{u}_+, p_+)\mathbf{n}_+ - b_-T(\mathbf{u}_-, p_-)\mathbf{n}_+ + c_+\mathbf{u}_+ = \mathbf{f} \quad \text{on } \partial\Omega.$$

Here $\mathbf{g} \in W^{1,2}(\partial\Omega, R^m)$, $\mathbf{f} \in L^2(\partial\Omega, R^m)$. We look for an L^2-solution of the problem, i.e. the nontangential maximal functions of $\mathbf{u}_\pm$, $\nabla \mathbf{u}_\pm$ and $p_\pm$ are in $L^2(\partial\Omega)$ and the boundary conditions are fulfilled in the sense of the nontangential limit. This problem was studied in [14] for $c_+ = 0$, $\lambda_\pm = 0$, and in [6] for $a_\pm = b_\pm = 1$, $c_+ = 0$. We study the transmission problem for arbitrary $\lambda_\pm$, $a_\pm$, $b_\pm$ and c_+.

In all preceding papers the transmission problem is studied under additional condition concerning behaviour of $\mathbf{u}_-$ and p_- at infinity. To remove this additional condition we study behaviour of a solution of the Brinkman system at infinity and we prove the theorem of Liouville's type. From this we deduce that if the nontangential maximal function corresponding to $\mathbf{u}_-$ and p_- is in $L^2(\partial\Omega)$, then there exist $\mathbf{u}_\infty \in R^m$, $p_\infty \in R^1$ such that $\mathbf{u}_-(\mathbf{x}) \to \mathbf{u}_\infty$, $p_-(\mathbf{x}) \to p_\infty$ as $|\mathbf{x}| \to \infty$, and $|\mathbf{u}_-(\mathbf{x}) - \mathbf{u}_\infty(\mathbf{x})| = O(|\mathbf{x}|^{2-m})$, $|\nabla \mathbf{u}_-| + |p_-(\mathbf{x}) - p_\infty| = O(|\mathbf{x}|^{1-m})$.

At the end we study the Robin–transmission and the Dirichlet–transmission problems. Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $\Omega = \Omega_+$ be a bounded open set with Lipschitz boundary such that $\overline{\Omega} \subset G$. Denote $\Omega_- = G \setminus \overline{\Omega}$, and by $\mathbf{n}_\pm$ the outward unit normal of $\Omega_\pm$. Let $\lambda_\pm$, $c_\pm$ be non-negative constants and $a_\pm$, $b_\pm$ be positive constants. We study by the integral equation method the Robin–transmission problem for the Brinkman system

$$-\Delta \mathbf{u}_\pm + \lambda_\pm \mathbf{u}_\pm + \nabla p_\pm = 0, \quad \nabla \cdot \mathbf{u}_\pm = 0 \quad \text{in} \quad \Omega_\pm,$$

$$a_+\mathbf{u}_+ - a_-\mathbf{u}_- = \mathbf{g}, \quad b_+T(\mathbf{u}_+,p_+)\mathbf{n}_+ - b_-T(\mathbf{u}_-,p_-)\mathbf{n}_+ + c_+\mathbf{u}_+ = \mathbf{f} \quad \text{on } \partial\Omega,$$

$$T(\mathbf{u}_-,p_-)\mathbf{n}_- + c_-\mathbf{u}_- = \mathbf{h} \quad \text{on } \partial G.$$

Here $\mathbf{g} \in W^{1,2}(\partial\Omega, R^m)$, $\mathbf{f} \in L^2(\partial\Omega, R^m)$, $\mathbf{h} \in L^2(\partial G)$. We look for an L^2-solution of the problem, i.e. the nontangential maximal functions of $\mathbf{u}_\pm$, $\nabla\mathbf{u}_\pm$ and $p_\pm$ are in $L^2(\partial\Omega_-)$ and the boundary conditions are fulfilled in the sense of the nontangential limit. This problem was studied in ([5]) for $c_\pm = 0$, $a_\pm = b_\pm = 1$, $\lambda_+ = 0$.

Then the regular Dirichlet–transmission problem is studied by the integral equation method:

$$-\Delta\mathbf{u}_\pm + \lambda_\pm\mathbf{u}_\pm + \nabla p_\pm = 0, \quad \nabla\cdot\mathbf{u}_\pm = 0 \quad \text{in} \quad \Omega_\pm,$$

$$a_+\mathbf{u}_+ - a_-\mathbf{u}_- = \mathbf{g}, \quad b_+T(\mathbf{u}_+,p_+)\mathbf{n}_+ - b_-T(\mathbf{u}_-,p_-)\mathbf{n}_+ + c_+\mathbf{u}_+ = \mathbf{f} \quad \text{on } \partial\Omega,$$

$$\mathbf{u}_- = \mathbf{h} \quad \text{on } \partial G.$$

Here $\mathbf{g} \in W^{1,2}(\partial\Omega, R^m)$, $\mathbf{f} \in L^2(\partial\Omega, R^m)$, $\mathbf{h} \in W^{1,2}(\partial G)$. We look for an L^2-solution of the problem, i.e. the nontangential maximal functions of $\mathbf{u}_\pm$, $\nabla\mathbf{u}_\pm$ and $p_\pm$ are in $L^2(\partial\Omega_-)$ and the boundary conditions are fulfilled in the sense of the nontangential limit. This problem was studied in [8] for $a_\pm = b_\pm = 1$, $c_+ = 0$.

2 Formulation of the transmission problem

Let $\Omega = \Omega_+ \subset R^m$, $m > 2$, be a bounded open set with Lipschitz boundary. Denote $\Omega_- = R^m \setminus \overline{\Omega}_+$, where $\overline{\Omega}_+$ is the closure of Ω_+. Denote by $\mathbf{n} = \mathbf{n}_+ = \mathbf{n}^\Omega$ the outward unit normal of Ω_+. Let λ_+, λ_-, c_+ be non-negative constants and a_+, a_-, b_+, b_- positive constants. We shall study the transmission problem for the Brinkman system

$$-\Delta\mathbf{u}_\pm + \lambda_\pm\mathbf{u}_\pm + \nabla p_\pm = 0, \quad \nabla\cdot\mathbf{u}_\pm = 0 \quad \text{in} \quad \Omega_\pm, \tag{1}$$

$$a_+\mathbf{u}_+ - a_-\mathbf{u}_- = \mathbf{g}, \; b_+T(\mathbf{u}_+,p_+)\mathbf{n}_+ - b_-T(\mathbf{u}_-,p_-)\mathbf{n}_+ + c_+\mathbf{u}_+ = \mathbf{f} \quad \text{on } \partial\Omega. \tag{2}$$

If $\mathbf{u} = (u_1, \ldots, u_m)$ is a velocity field, p is a pressure, denote

$$T(\mathbf{u},p) = 2\hat{\nabla}\mathbf{u} - pI$$

the corresponding stress tensor. Here I denotes the identity matrix and

$$\hat{\nabla}\mathbf{u} = \frac{1}{2}[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]$$

is the strain tensor, with $(\nabla\mathbf{u})^T$ as the matrix transposed to $\nabla\mathbf{u} = (\partial_j u_k)$, $(k, j = 1, \ldots, m)$. Denote $\nabla\cdot\mathbf{u} = \partial_1 u_1 + \ldots + \partial_m u_m$ the divergence of $\mathbf{u}$.

Now we define an L^2-solution of the transmission problem. Let G be an open set with Lipschitz boundary. If $\mathbf{x} \in \partial G$, $a > 0$ denote the non-tangential approach region of opening a at the point $\mathbf{x}$ by

$$\Gamma_a^G(\mathbf{x}) := \{\mathbf{y} \in G; |\mathbf{x} - \mathbf{y}| < (1 + a) \operatorname{dist}(\mathbf{y}, \partial G)\}.$$

If now $\mathbf{v}$ is a vector function defined in G we denote the non-tangential maximal function of $\mathbf{v}$ on ∂G by

$$\mathbf{v}_G^*(\mathbf{x}) := \sup\{|\mathbf{v}(\mathbf{y})|; \mathbf{y} \in \Gamma_a^G(\mathbf{x})\}.$$

If $\mathbf{x} \in \partial G$, $\Gamma(\mathbf{x}) = \Gamma_a^G(\mathbf{x})$ then

$$\mathbf{v}(\mathbf{x}) = \lim_{\substack{\mathbf{y} \to \mathbf{x} \\ \mathbf{y} \in \Gamma(\mathbf{x})}} \mathbf{v}(\mathbf{y})$$

is the non-tangential limit of $\mathbf{v}$ with respect to G at $\mathbf{x}$.

Let $\mathbf{g} \in W^{1,2}(\partial\Omega, R^m)$, $\mathbf{f} \in L^2(\partial\Omega, R^m)$. *We say that $\mathbf{u}_\pm$, $p_\pm$ defined on $\Omega_\pm$ is an L^2-solution of the transmission problem (1), (2) if $\mathbf{u}_\pm$, $p_\pm$ satisfy (1); $\mathbf{u}_\pm^*$, $p_\pm^*$, $(\nabla \mathbf{u})_\pm^*$ are from $L^2(\partial\Omega, R^1)$; for almost all $\mathbf{x} \in \partial\Omega$ there exist the non-tangential limits of $\mathbf{u}_\pm$, $\nabla \mathbf{u}_\pm$, $p_\pm$ at $\mathbf{x}$ and the condition (2) is fulfilled in the sense of the nontangential limit a.e. on $\partial\Omega$.*

3 The surface potentials

We shall look for a solution of the transmission problem by the integral equation method. The aim of this section is to assemble some basic facts on surface potentials for the Brinkman system.

For $\lambda \geq 0$ denote by $E^\lambda(\mathbf{x}) = \{E_{ij}^\lambda(\mathbf{x})\}_{i,j=1,\ldots,m}$, $Q^\lambda(\mathbf{x}) = \{Q_j^\lambda(\mathbf{x})\}_{j=1,\ldots,m}$ the fundamental matrix for the Brinkman system

$$-\Delta \mathbf{u} + \lambda \mathbf{u} + \nabla p = 0, \quad \nabla \cdot \mathbf{u} = 0 \tag{3}$$

such that $E^\lambda(\mathbf{x}) \to 0$, $Q^\lambda(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. If j is fixed, $\mathbf{u} = (E_{1j}, \ldots, E_{mj})$, $p = Q_j$ then $\mathbf{u}$, p is a solution of the Brinkman system (3) in $R^m \setminus \{0\}$. If $\lambda = 0$ then the fundamental matrix for the Stokes system is given by

$$E_{ij}^0(\mathbf{x}) = \frac{1}{2\omega_m} \left[\delta_{ij} \frac{|\mathbf{x}|^{2-m}}{m-2} + \frac{x_i x_j}{|\mathbf{x}|^m} \right], \qquad Q_j^0(\mathbf{x}) = \frac{x_j}{\omega_m |\mathbf{x}|^m},$$

where ω_m denotes the surface of the unit sphere in R^m. (See [17] or [14].) The fundamental matrix for $\lambda > 0$ are studies in Chapter 2 of [17]:

$$Q^\lambda(\mathbf{x}) = Q^0(\mathbf{x}),$$

$$E^\lambda_{ij}(\mathbf{x}) = \frac{1}{\omega_m}\left[\frac{\delta_{ij}}{|\mathbf{x}|^{m-2}} A_1(\sqrt{\lambda}|\mathbf{x}|) + \frac{x_i x_j}{|\mathbf{x}|^m} A_2(\sqrt{\lambda}|\mathbf{x}|)\right],$$

$$A_1(t) = \frac{t^{m/2-1}K_{m/2-1}(t)}{2^{m/2-1}\Gamma(m/2)} + \frac{t^{m/2-2}K_{m/2}(t)}{2^{m/2-1}\Gamma(m/2)} - \frac{1}{t^2},$$

$$A_2(t) = \frac{m}{t^2} - \frac{t^{m/2-1}K_{m/2+1}(t)}{2^{m/2-1}\Gamma(m/2)},$$

where K_ν is the modified Bessel function of order ν. If $\lambda > 0$ then

$$|E^\lambda(\mathbf{x})| = O(|\mathbf{x}|^{-m}), \quad |\nabla E^\lambda(\mathbf{x})| = O(|\mathbf{x}|^{1-m}) \qquad \text{as } |\mathbf{x}| \to \infty.$$

Since $E^\lambda \in \mathcal{C}^\infty(R^m \setminus \{0\}; R^{m\times m})$, $Q^\lambda \in \mathcal{C}^\infty(R^m \setminus \{0\}; R^m)$, we can define for $\mathbf{\Psi} \in L^2(\partial\Omega, R^m)$ the single layer potential with density $\mathbf{\Psi}$ by

$$(E^\lambda_\Omega \mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} E^\lambda(\mathbf{x}-\mathbf{y})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}) \tag{4}$$

and the corresponding pressure by

$$(Q^\lambda_\Omega \mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} Q^\lambda(\mathbf{x}-\mathbf{y})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}). \tag{5}$$

Then $E^\lambda_\Omega \mathbf{\Psi} \in \mathcal{C}^\infty(R^m \setminus \partial\Omega, R^m)$, $Q^\lambda_\Omega \mathbf{\Psi} \in \mathcal{C}^\infty(R^m \setminus \partial\Omega, R^1)$, $\nabla Q^\lambda_\Omega \mathbf{\Psi} - \Delta E^\lambda_\Omega \mathbf{\Psi} + \lambda E^\lambda_\Omega \mathbf{\Psi} = 0$, $\nabla \cdot E^\lambda_\Omega \mathbf{\Psi} = 0$ in $R^m \setminus \partial\Omega$.

$E^\lambda_\Omega \mathbf{\Psi}$ can be defined for almost all $\mathbf{x} \in \partial\Omega$ and $E^\lambda_\Omega \mathbf{\Psi}(\mathbf{x})$ is the non-tangential limit of $E^\lambda_\Omega \mathbf{\Psi}$. The nontangential maximal function of $E^\lambda_\Omega \mathbf{\Psi}$, $\nabla E^\lambda_\Omega \mathbf{\Psi}$, $Q^\lambda_\Omega \mathbf{\Psi}$ with respect to Ω_+ and Ω_- is in $L^2(\partial\Omega)$ (see [4], Lemma 2.1.4). Moreover, E^λ_Ω is a bounded linear operator from $L^2(\partial\Omega, R^m)$ to $W^{1,2}(\partial\Omega, R^m)$. (For $\lambda = 0$ see [14], for $\lambda > 0$ see for example [5].)

Denote

$$K^\lambda_\Omega(\mathbf{y}, \mathbf{x}) = -T_{\mathbf{x}}(E^\lambda(\mathbf{x}-\mathbf{y}), Q^\lambda(\mathbf{x}-\mathbf{y}))\mathbf{n}^\Omega(\mathbf{x}).$$

For $\mathbf{\Psi} \in L^2(\partial\Omega, R^m)$ define

$$K'_{\Omega,\lambda}\mathbf{\Psi}(\mathbf{x}) = \lim_{\epsilon \searrow 0} \int_{\partial\Omega \setminus B(\mathbf{x};\epsilon)} K^\lambda_\Omega(\mathbf{y}, \mathbf{x})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}),$$

where $B(\mathbf{x};\epsilon) = \{\mathbf{y}; |\mathbf{x}-\mathbf{y}| < \epsilon\}$. Then $K'_{\Omega,\lambda}$ is a bounded linear operator on $L^2(\partial\Omega, R^m)$. If $\mathbf{\Psi} \in L^2(\partial\Omega, R^m)$ then there exist the non-tangential limits $[\nabla E^\lambda_\Omega \mathbf{\Psi}(\mathbf{x})]_\pm$, $[Q^\lambda_\Omega \mathbf{\Psi})(\mathbf{x})]_\pm$ of $\nabla E^\lambda_\Omega \mathbf{\Psi}$, $Q^\lambda_\Omega \mathbf{\Psi}$ with respect to $\Omega_\pm$ at almost all $\mathbf{x} \in \partial\Omega$, and

$$[T(E^\lambda_\Omega \mathbf{\Psi}, Q^\lambda_\Omega \mathbf{\Psi})]_+ \mathbf{n}^\Omega = \frac{1}{2}\mathbf{\Psi} - K'_{\Omega,\lambda}\mathbf{\Psi}, \tag{6}$$

$$[T(E_\Omega^\lambda \mathbf{\Psi}, Q_\Omega^\lambda \mathbf{\Psi})]_- \mathbf{n}^\Omega = -\frac{1}{2}\mathbf{\Psi} - K'_{\Omega,\lambda}\mathbf{\Psi}. \tag{7}$$

(For $\lambda = 0$ see [14], for $\lambda > 0$ see for example [5]. See also [13].)

Now we define a double layer potential. For $\mathbf{\Psi} \in L^2(\partial\Omega, R^m)$ define in $R^m \setminus \partial\Omega$

$$(D_\Omega^\lambda \mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} K_\Omega^\lambda(\mathbf{x}, \mathbf{y})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}), \tag{8}$$

and the corresponding pressure by

$$(\Pi_\Omega^\lambda \mathbf{\Psi})(\mathbf{x}) = \int_{\partial\Omega} \Pi_\Omega^\lambda(\mathbf{x}, \mathbf{y})\mathbf{\Psi}(\mathbf{y})\ \mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}), \tag{9}$$

where

$$\Pi_\Omega^\lambda(\mathbf{x}, \mathbf{y}) = \frac{1}{\omega_m}\left\{-(\mathbf{y}-\mathbf{x})\frac{2m(\mathbf{y}-\mathbf{x})\cdot\mathbf{n}^\Omega(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^{m+2}} + \frac{2\mathbf{n}^\Omega(\mathbf{y})}{|\mathbf{y}-\mathbf{x}|^m} - \lambda\frac{|\mathbf{x}-\mathbf{y}|^{2-m}}{m-2}\mathbf{n}^\Omega(\mathbf{y})\right\}.$$

Then $D_\Omega^\lambda \mathbf{\Psi} \in \mathcal{C}^\infty(R^m \setminus \partial\Omega, R^m)$, $\Pi_\Omega^\lambda \mathbf{\Psi} \in \mathcal{C}^\infty(R^m \setminus \partial\Omega, R^1)$ and $\nabla \Pi_\Omega^\lambda \mathbf{\Psi} - \Delta D_\Omega^\lambda \mathbf{\Psi} + \lambda D_\Omega^\lambda \mathbf{\Psi} = 0$, $\nabla \cdot D_\Omega^\lambda \mathbf{\Psi} = 0$ in $R^m \setminus \partial\Omega$.

Define

$$K_{\Omega,\lambda}\mathbf{\Psi}(\mathbf{x}) = \lim_{\epsilon \searrow 0} \int_{\partial\Omega\setminus B(\mathbf{x};\epsilon)} K_\Omega^\lambda(\mathbf{x}, \mathbf{y})\mathbf{\Psi}(\mathbf{y})\mathrm{d}\mathcal{H}_{m-1}(\mathbf{y}), \quad \mathbf{x} \in \partial\Omega.$$

Then $K_{\Omega,\lambda}$ is a bounded linear operator on $L^2(\partial\Omega; R^m)$ (adjoint to $K'_{\Omega,\lambda}$). There exists the nontangential limit $[D_\Omega^\lambda \mathbf{\Psi}]_+(\mathbf{x})$ of $D_\Omega^\lambda \mathbf{\Psi}$ with respect to Ω_+ and the nontangential limit $[D_\Omega^\lambda \mathbf{\Psi}]_-(\mathbf{x})$ of $D_\Omega^\lambda \mathbf{\Psi}$ with respect to Ω_- for almost all $\mathbf{x} \in \partial\Omega$ and

$$[D_\Omega^\lambda \mathbf{\Psi}]_+(\mathbf{x}) = \frac{1}{2}\mathbf{\Psi}(\mathbf{z}) + K_{\Omega,\lambda}\mathbf{\Psi}(\mathbf{z}), \quad [D_\Omega^\lambda \mathbf{\Psi}]_-(\mathbf{x}) = -\frac{1}{2}\mathbf{\Psi}(\mathbf{z}) + K_{\Omega,\lambda}\mathbf{\Psi}(\mathbf{z}). \tag{10}$$

If $\mathbf{\Psi} \in W^{1,2}(\partial\Omega, R^m)$ then $[|D_\Omega^\lambda \mathbf{\Psi}|]^*_{\Omega_\pm} + [|\nabla D_\Omega^\lambda \mathbf{\Psi}|]^*_{\Omega_\pm} \in L^2(\partial\Omega)$ and at almost all points of $\partial\Omega$ there exist the nontangential limits of $\nabla D_\Omega^\lambda \mathbf{\Psi}$ with respect to Ω_+ and with respect to Ω_-. Moreover, $[T(D_\Omega^\lambda \mathbf{\Psi}, \Pi_\Omega^\lambda \mathbf{\Psi})]_+ \mathbf{n}^\Omega = [T(D_\Omega^\lambda \mathbf{\Psi}, \Pi_\Omega^\lambda \mathbf{\Psi})]_- \mathbf{n}^\Omega$. (For $\lambda = 0$ see [14], for $\lambda > 0$ see for example [5].)

4 Behaviour at infinity

Proposition 4.1. *Let $\lambda \geq 0$, $u_1, \dots, u_k$ and p be tempered distributions in R^k, $k \geq 2$, $\mathbf{u} = (u_1, \dots, u_k)$. If $-\Delta\mathbf{u} + \lambda\mathbf{u} + \nabla p = 0$, $\nabla \cdot \mathbf{u} = 0$ in the sense of distributions in R^k, then $u_1, \dots, u_k$ and p are polynomials.*

Proof. Denote by $\mathcal{F}f$ the Fourier transformation of f. Since $-\Delta\mathbf{u} + \lambda\mathbf{u} + \nabla p = 0$, $\nabla \cdot \mathbf{u} = 0$, the Fourier transformation gives

$$|\mathbf{x}|^2 \mathcal{F}\mathbf{u}(\mathbf{x}) + \lambda\mathcal{F}\mathbf{u}(\mathbf{x}) + \mathbf{x}\mathcal{F}p(\mathbf{x}) = 0, \tag{11}$$

$$\mathbf{x}\cdot\mathcal{F}\mathbf{u}(\mathbf{x})=0. \tag{12}$$

Using (11), (12)

$$0=\mathbf{x}\cdot[(|\mathbf{x}|^2+\lambda)\mathcal{F}\mathbf{u}+\mathbf{x}\mathcal{F}p(\mathbf{x})]=|\mathbf{x}|^2\mathcal{F}p(\mathbf{x}).$$

Thus $\mathcal{F}p=0$ on $R^k\setminus\{0\}$. If $\mathbf{x}\in R^k\setminus\{0\}$ then

$$0=|\mathbf{x}|^2\mathcal{F}\mathbf{u}(\mathbf{x})+\lambda\mathcal{F}\mathbf{u}(\mathbf{x})+\mathbf{x}\mathcal{F}p(\mathbf{x})=(|\mathbf{x}|^2+\lambda)\mathcal{F}\mathbf{u}.$$

Therefore $\mathcal{F}u_j=0$ in $R^k\setminus\{0\}$. According to [16], Chapter II, §10, there exist $n\in N_0$ and constants a_α such that

$$\mathcal{F}u_j=\sum_{|\alpha|\le n}a_\alpha\partial^\alpha\delta_0.$$

Set

$$P_j(x)=\sum_{|\alpha|\le n}a_\alpha(-ix)^\alpha.$$

Then

$$\mathcal{F}P_j=\sum_{|\alpha|\le n}a_\alpha\mathcal{F}[(-ix)^\alpha 1]=\sum_{|\alpha|\le n}a_\alpha\partial^\alpha\delta_0=\mathcal{F}u_j.$$

Since the Fourier transform is an isomorphism on the space of tempered distributions we infer that $u_j=P_j$. Similarly for p.

Proposition 4.2. *Let $\mathbf{u}$, p be a bounded solution of the Brinkman system $-\Delta\mathbf{u}+\lambda\mathbf{u}+\nabla p=0$, $\nabla\cdot\mathbf{u}=0$ in $R^m\setminus F$, where F is a compact subset of R^m, $m>2$, $\lambda\ge 0$. Then there exist $p_\infty\in R^1$, $\mathbf{u}_\infty\in R^m$ such that $p(\mathbf{x})\to p_\infty$, $\mathbf{u}(\mathbf{x})\to\mathbf{u}_\infty$ as $|\mathbf{x}|\to\infty$. Moreover, $|p(\mathbf{x})-p_\infty|=O(|\mathbf{x}|^{1-m})$, $|\mathbf{u}(\mathbf{x})-\mathbf{u}_\infty|=O(|\mathbf{x}|^{2-m})$, $|\nabla\mathbf{u}(\mathbf{x})|=O(|\mathbf{x}|^{1-m})$ as $|\mathbf{x}|\to\infty$. If $\lambda>0$ then $\mathbf{u}_\infty=0$.*

Proof. Fix $\varphi\in\mathcal{C}^\infty(R^m)$ such that $\varphi=0$ on a neighbourhood of F and $\varphi=1$ on $R^m\setminus B(0;r)$ for some $r>0$. Define $\tilde{\mathbf{u}}=\varphi\mathbf{u}$, $\tilde p=\varphi p$ on $R^m\setminus F$; $\tilde{\mathbf{u}}=0$, $\tilde p=$ on F. Denote $(f_1,\dots,f_m)^T=-\Delta\tilde{\mathbf{u}}+\lambda\tilde{\mathbf{u}}+\nabla\tilde p$, $f_{m+1}=\nabla\cdot\tilde{\mathbf{u}}$, $\mathbf{f}=(f_1,\dots,f_{m+1})^T$. Define the $(m+1)\times(m+1)$ matrix function $\tilde E^\lambda$ by $\tilde E^\lambda_{ij}=E^\lambda_{ij}$, $\tilde E^\lambda_{m+1,j}=\tilde E^\lambda_{j,m+1}=Q^\lambda_j$ for $i,j\le m$, $\tilde E_{m+1,m+1}(\mathbf{x})=\delta(x)+\lambda|\mathbf{x}|^{2-m}/[(m-2)\omega_m]$. Denote $(v_1,\dots,v_m,q)^T=\tilde E^\lambda*\mathbf{f}$, $\mathbf{v}=(v_1,\dots,v_m)^T$, where $*$ means the convolution. Then $-\Delta\mathbf{v}+\lambda\mathbf{v}+\nabla q=(f_1,\dots,f_m)^T$, $\nabla\cdot\mathbf{v}=f_{m+1}$ by [17], §2.1. According to a behaviour of $\tilde E^\lambda$ at infinity we see that $|\mathbf{v}(\mathbf{x})|=O(|\mathbf{x}|^{2-m})$, $|\nabla\mathbf{v}(\mathbf{x})|+|q(\mathbf{x})|=O(|\mathbf{x}|^{1-m})$ as $|\mathbf{x}|\to\infty$. Since the functions u_j-v_j, $p-q$ are bounded, they are tempered distributions (see [2], Example 14.22). Since $-\Delta(\tilde{\mathbf{u}}-\mathbf{v})+\lambda(\tilde{\mathbf{u}}-\mathbf{v})+\nabla(\tilde p-q)=0$, $\nabla\cdot(\mathbf{u}-\mathbf{v})=0$ in R^m, Proposition 4.1 gives that $\tilde u_j-v_j$, $\tilde p-q$ are polynomials. Since $\tilde u_j-v_j$, $\tilde p-q$ are bounded there exist $p_\infty\in R^1$, $\mathbf{u}_\infty\in R^m$ such that $\tilde p-q=p_\infty$, $\tilde{\mathbf{u}}-\mathbf{v}=\mathbf{u}_\infty$. If $\lambda>0$ then $0=-\Delta(\tilde{\mathbf{u}}-\mathbf{v})+\lambda(\tilde{\mathbf{u}}-\mathbf{v})+\nabla(\tilde p-q)=\lambda\mathbf{u}_\infty$ and thus $\mathbf{u}_\infty=0$.

5 Solution of the transmission problem

Put $\tilde{b}_\pm = b_\pm/a_\pm$, $\tilde{c}_+ = c_+/a_+$. If $\tilde{\mathbf{u}}_\pm = a_\pm \mathbf{u}_\pm$, $\tilde{p}_\pm = a_\pm p_\pm$ then $\mathbf{u}_\pm$, $p_\pm$ is an L^2-solution of the transmission problem (1), (2) if and only if $\tilde{\mathbf{u}}_\pm$, $\tilde{p}_\pm$ is an L^2-solution of the transmission problem

$$-\Delta\tilde{\mathbf{u}}_\pm + \lambda_\pm \tilde{\mathbf{u}}_\pm + \nabla \tilde{p}_\pm = 0, \quad \nabla\cdot\tilde{\mathbf{u}}_\pm = 0 \quad \text{in} \quad \Omega_\pm, \tag{13}$$

$$\tilde{\mathbf{u}}_+ - \tilde{\mathbf{u}}_- = \mathbf{g}, \quad \tilde{b}_+ T(\tilde{\mathbf{u}}_+, \tilde{p}_+)\mathbf{n} - \tilde{b}_- T(\tilde{\mathbf{u}}_-, \tilde{p}_-)\mathbf{n} + \tilde{c}_+ \tilde{\mathbf{u}}_+ = \mathbf{f} \quad \text{on } \partial\Omega. \tag{14}$$

Let $\mathbf{\Phi} \in W^{1,2}(\partial\Omega, R^m)$, $\mathbf{\Psi} \in L^2(\partial\Omega, R^m)$. Put

$$\tilde{\mathbf{u}}_\pm = D_\Omega^{\lambda_\pm}\mathbf{\Phi} + E_\Omega^{\lambda_\pm}\mathbf{\Psi}, \quad \tilde{p}_\pm = \Pi_\Omega^{\lambda_\pm}\mathbf{\Phi} + Q_\Omega^{\lambda_\pm}\mathbf{\Psi} \quad \text{in } \Omega_\pm, \tag{15}$$

$$\tau_1^{\lambda_+,\lambda_-}(\mathbf{\Phi},\mathbf{\Psi}) = \mathbf{\Phi} + K_{\Omega,\lambda_+}\mathbf{\Phi} - K_{\Omega,\lambda_-}\mathbf{\Phi} + E_\Omega^{\lambda_+}\mathbf{\Psi} - E_\Omega^{\lambda_-}\mathbf{\Psi},$$

$$\tau_2^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(\mathbf{\Phi},\mathbf{\Psi}) = \tilde{b}_+[\mathbf{\Psi} - K'_{\Omega,\lambda_+}] - \tilde{b}_-[-\mathbf{\Psi} - K'_{\Omega,\lambda_-}] + \tilde{c}_+ E_\Omega^{\lambda_+}\mathbf{\Psi}$$
$$+\tilde{b}_+[T(D_\Omega^{\lambda_+}\mathbf{\Phi}, \Pi_\Omega^{\lambda_+}\mathbf{\Phi})]_+\mathbf{n}^\Omega - \tilde{b}_-[T(D_\Omega^{\lambda_-}\mathbf{\Phi}, \Pi_\Omega^{\lambda_-}\mathbf{\Phi})]_-\mathbf{n}^\Omega.$$

The operator $\tau^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+} = [\tau_1^{\lambda_+,\lambda_-}, \tau_2^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}]$ is a bounded linear operator on $W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m)$. The functions $\tilde{\mathbf{u}}_\pm$, $\tilde{p}_\pm$ given by (15) are an L^2-solution of the transmission problem (13), (14) such $\tilde{\mathbf{u}}_-(\mathbf{x}) \to 0$, $\tilde{p}_-(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$ if and only if $\tau^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,c_+}(\mathbf{\Phi},\mathbf{\Psi}) = [\mathbf{g},\mathbf{f}]$.

Lemma 5.1. *Denote* $\mathcal{R}_m = \{\mathbf{v}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}; \mathbf{b} \in R^m, A = (a_{ij})$ *an antisymmetric matrix, i.e.* $a_{ij} = -a_{ji}\}$ *the space of rigid motions. Let* $\mathbf{u} \in \mathcal{R}_m$, $M = \{\mathbf{x}; \mathbf{u}(\mathbf{x}) = 0\}$. *If* $\mathcal{H}_{m-1}(M) > 0$ *then* $\mathbf{u} \equiv 0$.

Proof. There exist a matrix $A = (a_{ij})$ with $a_{ij} = -a_{ji}$ and $\mathbf{b} \in R^m$ such that $\mathbf{u}(\mathbf{x}) = A\mathbf{x} + \mathbf{b}$. Suppose first $a_{ij} \neq 0$ for some indices i, j. Denote $L_i = \{\mathbf{x}; a_{i1}x_1 + \ldots + a_{im}x_m + b_i = 0\}$, $L_j = \{\mathbf{x}; a_{j1}x_1 + \ldots + a_{jm}x_m + b_i = 0\}$. Since $a_{ii} = a_{jj} = 0$, $a_{ji} = -a_{ij} \neq 0$ we have $\mathcal{H}_{m-1}(L_i \cap L_j) = 0$. This contradicts to $M \subset L_i \cap L_j$. Hence $A = 0$ and $\mathbf{u}$ is constant. $M \neq \emptyset$ forces $\mathbf{u} \equiv 0$.

Proposition 5.2. *Let* $\mathbf{u}_\pm$, $p_\pm$ *be an* L^2*-solution for the transmission problem (1), (2). If* $\mathbf{f} = 0$, $\mathbf{g} = 0$ *and* $\mathbf{u}_-(\mathbf{x}) \to 0$, $p_-(\mathbf{x}) \to 0$ *as* $|\mathbf{x}| \to \infty$ *then* $\mathbf{u}_\pm \equiv 0$, $p_\pm \equiv 0$.

Proof. $|p(\mathbf{x})| = O(|\mathbf{x}|^{1-m})$, $|\mathbf{u}(\mathbf{x})| = O(|\mathbf{x}|^{2-m})$, $|\nabla\mathbf{u}(\mathbf{x})| = O(|\mathbf{x}|^{1-m})$ as $|\mathbf{x}| \to \infty$ (see Proposition 4.2). Using Green's formula

$$0 = \int_{\partial\Omega} \mathbf{u}_+ \cdot [b_+ T(\mathbf{u}_+, p_+)\mathbf{n} - b_- T(\mathbf{u}_-, p_-)\mathbf{n} + c_+\mathbf{u}_+] \, \mathrm{d}\mathcal{H}_{m-1}$$

$$= b_+ \int\limits_{\partial\Omega_-} \mathbf{u}_+ \cdot T(\mathbf{u}_+, p_+)\mathbf{n}^{\Omega_+} \, d\mathcal{H}_{m-1} + \int\limits_{\partial\Omega_-} c_+ |\mathbf{u}_+|^2 \, d\mathcal{H}_{m-1}$$

$$+ \lim_{r\to\infty} b_- \frac{a_-}{a_+} \int\limits_{\partial(\Omega_- \cap B(0;r))} \mathbf{u}_- \cdot T(\mathbf{u}_-, p_-)\mathbf{n}^{\Omega_-} = b_+ \int\limits_{\Omega_+} [2|\hat\nabla \mathbf{u}_+|^2 + \lambda_+ |\mathbf{u}_+|^2]$$

$$+ \int\limits_{\partial\Omega_-} c_+ |\mathbf{u}_+|^2 \, d\mathcal{H}_{m-1} + \frac{b_- a_-}{a_+} \int\limits_{\Omega_+} [2|\hat\nabla \mathbf{u}_+|^2 + \lambda_- |\mathbf{u}_+|^2] \, d\mathcal{H}_m.$$

Denote $u = u_\pm$ on $\Omega_\pm$. Then $\hat\nabla \mathbf{u} = 0$ in $R^m \setminus \partial\Omega$. Denote by $\omega_0, \omega_1, \dots, \omega_k$ all components of $R^m \setminus \partial\Omega$, where ω_0 is the unbounded component. According to [12], Lemma 3.1 there exist antisymmetric matrices A^j and vectors $\mathbf{B}^j$ such that $\mathbf{u}(\mathbf{x}) = A^j\mathbf{x} + \mathbf{B}^j$ in ω_j. Since $\mathbf{u}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$, we deduce that $\mathbf{u} = 0$ in ω_0. If $\partial\omega_0 \cap \partial\omega_j \neq \emptyset$ then the condition $a_+\mathbf{u}_+ = a_-\mathbf{u}_-$ gives that $A^j\mathbf{x} + \mathbf{B}^j = 0$ on $\partial\omega_0 \cap \partial\omega_j$. Lemma 5.1 gives that $A^j\mathbf{x} + \mathbf{B}^j \equiv 0$. We can continue by this way and prove that $\mathbf{u} = 0$.

Proposition 5.3. *The operator $\tau^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+}$ is an isomorphism on the space $W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m)$.*

Proof. The operator $\tau^{0,0,\tilde b_+,\tilde b_-,0}$ is a Fredholm operator with index 0 on $W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m)$ by [14]. If $\lambda \geq 0$ then $K_{\Omega,\lambda} - K_{\Omega,0}$ is compact on $W^{1,2}(\partial\Omega, R^m)$, $K'_{\Omega,\lambda} - K'_{\Omega,0}$ is compact on $L^2(\partial\Omega, R^m)$, $E^\lambda_\Omega - E^0_\Omega$ is a compact operator from $L^2(\partial\Omega, R^m)$ to $W^{1,2}(\partial\Omega, R^m)$ (see [5], Theorem 3.4). Since E^0_Ω is a bounded operator from $L^2(\partial\Omega, R^m)$ to $W^{1,2}(\partial\Omega, R^m)$, it is a compact linear operator on $L^2(\partial\Omega, R^m)$. Thus $\tau^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+} - \tau^{0,0,\tilde b_+,\tilde b_-,0}$ is a compact operator on $W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m)$. Hence $\tau^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,c_+}$ is a Fredholm operator with index 0. Therefore it is enough to prove that $\tau^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+}$ is injective.

Let $(\mathbf{\Phi}, \mathbf{\Psi}) \in W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m)$, $\tau^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+}(\mathbf{\Phi}, \mathbf{\Psi}) = 0$. Let $\tilde{\mathbf{u}}_\pm$, $\tilde p_\pm$ be given by (15). Then $\tilde{\mathbf{u}}_\pm$, $\tilde p_\pm$ is an L^2-solution of the problem (13), (14) with $\mathbf{g} = 0$, $\mathbf{f} = 0$ such $\tilde{\mathbf{u}}_-(\mathbf{x}) \to 0$, $\tilde p_-(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Proposition 5.2 gives that $\tilde{\mathbf{u}}_\pm = 0$, $\tilde p_\pm = 0$. Thus $\tilde{\mathbf{u}}_\pm$, $\tilde p_\pm$ is an L^2-solution of the problem (13),

$$\tilde{\mathbf{u}}_+ - \tilde{\mathbf{u}}_- = 0, \quad T(\tilde{\mathbf{u}}_+, \tilde p_+)\mathbf{n} - T(\tilde{\mathbf{u}}_-, \tilde p_-)\mathbf{n} = 0 \quad \text{on } \partial\Omega.$$

Denote $\tilde\lambda_+ = \lambda_-$, $\tilde\lambda_- = \lambda_+$,

$$\mathbf{v}_+ = D^{\lambda_-}_\Omega \mathbf{\Phi} + E^{\lambda_-}_\Omega \mathbf{\Psi}, \quad q_+ = \Pi^{\lambda_-}_\Omega \mathbf{\Phi} + Q^{\lambda_-}_\Omega \mathbf{\Psi}, \quad \text{in } \Omega_+,$$

$$\mathbf{v}_- = -D^{\lambda_+}_\Omega \mathbf{\Phi} - E^{\lambda_+}_\Omega \mathbf{\Psi}, \quad q_+ = -\Pi^{\lambda_+}_\Omega \mathbf{\Phi} - Q^{\lambda_+}_\Omega \mathbf{\Psi}, \quad \text{in } \Omega_-.$$

Using boundary behaviour of potentials we obtain on $\partial\Omega$

$$\mathbf{v}_+ = \mathbf{\Phi} + \tilde{\mathbf{u}}_- = \mathbf{\Phi},$$

$$\mathbf{v}_- = -[-\mathbf{\Phi} + \tilde{\mathbf{u}}_+] = \mathbf{\Phi},$$

$$[T(\mathbf{v}_+, q_+)\mathbf{n}^\Omega]_+ = \mathbf{\Psi} + [T(\tilde{\mathbf{u}}_-, \tilde{p}_-)\mathbf{n}^\Omega]_- = \mathbf{\Psi},$$

$$[T(\mathbf{v}_-, q_-)\mathbf{n}^\Omega]_- = -[-\mathbf{\Psi} + [T(\tilde{\mathbf{u}}_+, \tilde{p}_+)\mathbf{n}^\Omega]_+ = \mathbf{\Psi}.$$

Therefore $\mathbf{v}_\pm$, $q_\pm$ is a solution of the transmission problem

$$-\Delta\mathbf{v}_\pm + \tilde{\lambda}_\pm \mathbf{v}_\pm + \nabla q_\pm = 0, \quad \nabla \cdot \mathbf{v}_\pm = 0 \quad \text{in} \quad \Omega_\pm,$$

$$\mathbf{v}_+ - \mathbf{v}_- = 0, \quad T(\mathbf{v}_+, q_+)\mathbf{n} - T(\mathbf{v}_-, q_-)\mathbf{n} = 0 \quad \text{on } \partial\Omega,$$

$$\mathbf{v}_-(\mathbf{x}) \to 0, \quad q_-(\mathbf{x}) \to 0 \quad \text{as } |\mathbf{x}| \to \infty.$$

Proposition 5.2 gives that $\mathbf{v}_\pm \equiv 0$, $q_\pm \equiv 0$. We have on $\partial\Omega$

$$\mathbf{\Phi} = \mathbf{v}_+ = 0,$$

$$\mathbf{\Psi} = [T(\mathbf{v}_+, q_+)\mathbf{n}^\Omega]_+ = 0.$$

Theorem 5.4. *Let $\mathbf{g} \in W^{1,2}(\partial\Omega, R^m)$, $\mathbf{f} \in L^2(\partial\Omega, R^m)$. Then there exists an L^2-solution of the transmission problem (1), (2). If $\mathbf{u}_\pm$, $p_\pm$ is an L^2-solution of the problem then there exist $p_\infty \in R^1$, $\mathbf{u}_\infty \in R^m$ such that $\mathbf{u}_-(\mathbf{x}) \to \mathbf{u}_\infty$, $p_-(\mathbf{x}) \to p_\infty$ as $|\mathbf{x}| \to \infty$. If $\lambda_- > 0$ then $\mathbf{u}_\infty = 0$. Fix $p_\infty \in R^1$, $\mathbf{u}_\infty \in R^m$. If $\lambda_- > 0$ suppose that $\mathbf{u}_\infty = 0$. Then there exists a unique L^2-solution $\mathbf{u}_\pm$, $p_\pm$ of the transmission problem (1), (2) such that $\mathbf{u}_-(\mathbf{x}) \to \mathbf{u}_\infty$, $p_-(\mathbf{x}) \to p_\infty$ as $|\mathbf{x}| \to \infty$.*

Proof. If $\mathbf{u}_\pm$, $p_\pm$ is an L^2-solution of the problem then there exist $p_\infty \in R^1$, $\mathbf{u}_\infty \in R^m$ such that $\mathbf{u}_-(\mathbf{x}) \to \mathbf{u}_\infty$, $p_-(\mathbf{x}) \to p_\infty$ as $|\mathbf{x}| \to \infty$. If $\lambda_- > 0$ then $\mathbf{u}_\infty = 0$. (See Proposition 4.2.)

Fix $p_\infty \in R^1$, $\mathbf{u}_\infty \in R^m$. If $\lambda_- > 0$ suppose that $\mathbf{u}_\infty = 0$. Put $\mathbf{u}_- = \mathbf{v}_- + \mathbf{u}_\infty$, $\mathbf{u}_+ = \mathbf{v}+$, $p_- = q_- + p_\infty$, $p_+ = q_+$. Then $\mathbf{u}_\pm$, $p_\pm$ is a solution of the problem (1), (2), $\mathbf{u}_-(\mathbf{x}) \to \mathbf{u}_\infty$, $p_-(\mathbf{x}) \to p_\infty$ if and only if $\mathbf{v}_\pm$, $q_\pm$ is a solution of the transmission problem (1),

$$a_+\mathbf{v}_+ - a_-\mathbf{v}_- = \mathbf{g} + a_-\mathbf{u}_\infty, \; b_+T(\mathbf{v}_+, q_+)\mathbf{n} - b_-T(\mathbf{v}_-, q_-)\mathbf{n} + c_+\mathbf{v}_+ = \mathbf{f} - b_-p_\infty\mathbf{n},$$

$\mathbf{v}_-(\mathbf{x}) \to 0$, $q_-(\mathbf{x}) \to 0$. According to Proposition 5.3 there exist $\mathbf{\Phi} \in W^{1,2}(\partial\Omega, R^m)$, $\mathbf{\Psi} \in L^2(\partial\Omega, R^m)$ such that

$$\mathbf{v}_\pm = a_\pm^{-1}[D_\Omega^{\lambda_\pm}\mathbf{\Phi} + E_\Omega^{\lambda_\pm}\mathbf{\Psi}], \quad q_\pm = a_\pm^{-1}[\Pi_\Omega^{\lambda_\pm}\mathbf{\Phi} + Q_\Omega^{\lambda_\pm}\mathbf{\Psi}] \quad \text{in } \Omega_\pm$$

is a solution of the problem. The uniqueness of a solution follows from Proposition 5.2.

6 Robin–transmission problem

Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $\Omega = \Omega_+$ be a bounded open set with Lipschitz boundary such that $\overline{\Omega} \subset G$. Denote $\Omega_- = G \setminus \overline{\Omega}$, and by $\mathbf{n}_\pm$ the outward unit normal of $\Omega_\pm$. Let $\lambda_\pm$, $c_\pm$ be non-negative constants and $a_\pm$, $b_\pm$ be positive constants. We shall study the Robin–transmission problem for the Brinkman system (1), (2) accompanied with the condition

$$T(\mathbf{u}_-, p_-)\mathbf{n}_- + c_- \mathbf{u}_- = \mathbf{h} \quad \text{on } \partial G. \tag{16}$$

Let $\mathbf{g} \in W^{1,2}(\partial\Omega, R^m)$, $\mathbf{f} \in L^2(\partial\Omega, R^m)$, $\mathbf{h} \in L^2(\partial G, R^m)$. *We say that* $\mathbf{u}_\pm$, $p_\pm$ *defined on* $\Omega_\pm$ *is an* L^2*-solution of the Robin–transmission problem (1), (2), (16) if* $\mathbf{u}_\pm$, $p_\pm$ *satisfy (1);* $\mathbf{u}^*_\pm$, $p^*_\pm$, $(\nabla \mathbf{u})^*_\pm$ *are from* $L^2(\partial\Omega_\pm, R^1)$*; for almost all* $\mathbf{x} \in \partial\Omega_\pm$ *there exist the non-tangential limits of* $\mathbf{u}_\pm$, $\nabla\mathbf{u}_\pm$, $p_\pm$ *at* $\mathbf{x}$ *and the conditions (2), (16) are fulfilled in the sense of the nontangential limit a.e. on* $\partial\Omega_-$*.*

Put $\tilde{b}_\pm = b_\pm / a_\pm$, $\tilde{c}_+ = c_+/a_\pm$. If $\tilde{\mathbf{u}}_\pm = a_\pm \mathbf{u}_\pm$, $\tilde{p}_\pm = a_\pm p_\pm$ then $\mathbf{u}_\pm$, $p_\pm$ is an L^2-solution of the Robin–transmission problem (1), (2), (16) if and only if $\tilde{\mathbf{u}}_\pm$, $\tilde{p}_\pm$ is an L^2-solution of the Robin–transmission problem (13), (14),

$$T(\tilde{\mathbf{u}}_-, \tilde{p}_-)\mathbf{n}_- + c_- \tilde{\mathbf{u}}_- = a_- \mathbf{h} \quad \text{on } \partial G. \tag{17}$$

Let $\mathbf{\Phi} \in W^{1,2}(\partial\Omega, R^m)$, $\mathbf{\Psi} \in L^2(\partial\Omega, R^m)$, $\mathbf{\Theta} \in L^2(\partial G, R^m)$. Let $\tilde{\mathbf{u}}_+$, $\tilde{p}_+$ be given by (15),

$$\tilde{\mathbf{u}}_- = D^{\lambda_-}_\Omega \mathbf{\Phi} + E^{\lambda_-}_\Omega \mathbf{\Psi} + E^{\lambda_-}_G \mathbf{\Theta}, \quad \tilde{p}_- = \Pi^{\lambda_-}_\Omega \mathbf{\Phi} + Q^{\lambda_-}_\Omega \mathbf{\Psi} + Q^{\lambda_-}_G \mathbf{\Theta} \quad \text{in } \Omega_-. \tag{18}$$

Then $\tilde{\mathbf{u}}_\pm$, $\tilde{p}_\pm$ is an L^2-solution of the Robin–transmission problem (13), (14), (17) if and only if

$$R^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+,c_-}(\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}) = [\mathbf{g}, \mathbf{f}, a_- \mathbf{h}],$$

where

$$R^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+,c_-}(\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}) = [\tau_1^{\lambda_+,\lambda_-}(\mathbf{\Phi}, \mathbf{\Psi}) - E^{\lambda_-}_G \mathbf{\Theta},$$
$$\tau_2^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(\mathbf{\Phi}, \mathbf{\Psi}) - \tilde{b}_- T(E^{\lambda_-}_G \mathbf{\Theta}, Q^{\lambda_-}_G \mathbf{\Theta})\mathbf{n}_+,$$
$$\frac{1}{2}\mathbf{\Theta} - K'_{G,\lambda_-}\mathbf{\Theta} + T(E^{\lambda_-}_\Omega \mathbf{\Psi} + D^{\lambda_-}_\Omega \mathbf{\Phi}, Q^{\lambda_-}_G \mathbf{\Psi})\mathbf{n}_- + c_-(E^{\lambda_-}_G \mathbf{\Theta} + E^{\lambda_-}_\Omega \mathbf{\Psi} + D^{\lambda_-}_\Omega \mathbf{\Phi}].$$

Lemma 6.1. *The operator* $R^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+,c_-}$ *is a Fredholm operator with index 0 on* $W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m) \times L^2(\partial G, R^m)$*.*

Proof. $R : (\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}) \mapsto [\tau_1^{\lambda_+,\lambda_-}(\mathbf{\Phi}, \mathbf{\Psi}), \tau_2^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(\mathbf{\Phi}, \mathbf{\Psi}), \frac{1}{2}\mathbf{\Theta} - K'_{G,0}\mathbf{\Theta}]$ is a Fredholm operator with index 0 on $W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m) \times L^2(\partial G, R^m)$ by [14] and Proposition 5.3. If $\lambda \geq 0$ then $K'_{G,\lambda} - K'_{G,0}$ is compact on $L^2(\partial G, R^m)$,

$E^\lambda_G - E^0_G$ is a compact operator from $L^2(\partial G, R^m)$ to $W^{1,2}(\partial G, R^m)$ (see [5], Theorem 3.4). Since E^0_G is a bounded operator from $L^2(\partial G, R^m)$ to $W^{1,2}(\partial G, R^m)$, it is a compact linear operator on $L^2(\partial G, R^m)$. Thus $R^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+,c_-} - R$ is a compact operator. Hence $R^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+,c_-}$ is a Fredholm operator with index 0.

Lemma 6.2. *Let $\tilde{\mathbf{u}}_+$, $\tilde p_+$ be given by (15), and $\tilde{\mathbf{u}}_-$, $\tilde p_-$ by (18). If $\tilde{\mathbf{u}}_\pm = 0$, $\tilde p_\pm = 0$ in $\Omega_\pm$ then $\mathbf{\Phi} = 0$, $\mathbf{\Psi} = 0$, $\mathbf{\Theta} = 0$.*

Proof. Define

$$\mathbf{v} = D^{\lambda_-}_\Omega \mathbf{\Phi} + E^{\lambda_-}_\Omega \mathbf{\Psi} + E^{\lambda_-}_G \mathbf{\Theta}, \quad q = \Pi^{\lambda_-}_\Omega \mathbf{\Phi} + Q^{\lambda_-}_\Omega \mathbf{\Psi} + Q^\lambda_G \mathbf{\Theta} \quad \text{in } \omega = R^m \setminus \overline{G}.$$

Continuity of a single layer potential gives that $\mathbf{v} = \mathbf{u}_- = 0$ on ∂G. Since $\mathbf{v}(\mathbf{x}) = O(|\mathbf{x}|^{2-m})$, $|\nabla \mathbf{v}(\mathbf{x})| + |q(\mathbf{x})| = O(|\mathbf{x}|^{1-m})$ as $|\mathbf{x}| \to \infty$ then Green's formula gives

$$0 = \int_{\partial\omega} \mathbf{v} \cdot T(\mathbf{v}, q)\mathbf{n}^\omega \, \mathrm{d}\mathcal{H}_{m-1} = \int_\omega [|2\hat\nabla \mathbf{v}|^2 + \lambda_- |\mathbf{v}|^2] \, \mathrm{d}\mathcal{H}_m.$$

Since $\hat\nabla \mathbf{v} = 0$ we have $\mathbf{v} \in \mathrm{R}_m$ by [12], Lemma 3.1. Behaviour of potentials at infinity gives that $\mathbf{v}(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. This forces that $\mathbf{v} \equiv 0$. Since $\nabla q = \Delta \mathbf{v} - \lambda_- \mathbf{v} = 0$ we deduce that q is constant. Behaviour of potentials at infinity gives that $q \equiv 0$.

By virtue of (6) and (7)

$$\mathbf{\Theta} = T(\tilde{\mathbf{u}}_-, \tilde p_-)\mathbf{n}_- - T(\mathbf{v}, q)\mathbf{n}_- = 0.$$

Denote $\omega_+ = \Omega_+$, $\omega_- = R^m \setminus \overline{\omega}_+$. If $\tilde{\mathbf{u}}_\pm$, $\tilde p_\pm$ is given by (15) in $\omega_\pm$ then $\tilde{\mathbf{u}}_\pm$, $\tilde p_\pm$ is an L^2-solution of the transmission problem

$$-\Delta \tilde{\mathbf{u}}_\pm + \lambda_\pm \tilde{\mathbf{u}}_\pm + \nabla \tilde p_\pm = 0, \quad \nabla \cdot \tilde{\mathbf{u}}_\pm = 0 \quad \text{in} \quad \omega_\pm,$$

$$\tilde{\mathbf{u}}_+ - \tilde{\mathbf{u}}_- = 0, \quad \tilde b_+ T(\tilde{\mathbf{u}}_+, \tilde p_+)\mathbf{n}_+ - \tilde b_- T(\tilde{\mathbf{u}}_-, \tilde p_-)\mathbf{n}_+ + \tilde c_+ \tilde{\mathbf{u}}_+ = 0 \quad \text{on } \partial\omega_+.$$

In particular, $\tau_2^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+}(\mathbf{\Phi}, \mathbf{\Psi}) = 0$. Proposition 5.3 gives that $\mathbf{\Phi} = 0$, $\mathbf{\Psi} = 0$.

Proposition 6.3. *Let $\mathbf{u}_\pm$, $p_\pm$ be an L^2-solution of the Robin–transmission problem (1), (2), (16) with $\mathbf{g} = 0$, $\mathbf{f} = 0$, $\mathbf{h} = 0$.*

- *If $\lambda_+ + \lambda_- + c_+ + c_- > 0$ then $\mathbf{u}_\pm \equiv 0$, $p_\pm \equiv 0$.*
- *If $\lambda_+ + \lambda_- + c_+ + c_- = 0$ then $p_\pm \equiv 0$ and there exists a rigid motion $\mathbf{v} \in \mathcal{R}_m$ such that $\mathbf{u}_\pm = \mathbf{v}/a_\pm$.*

Proof. Using Green's formula

$$0 = b_-^{-1} \int_{\partial\Omega} \mathbf{u}_- \cdot [b_+ T(\mathbf{u}_+, p_+)\mathbf{n}_+ - b_- T(\mathbf{u}_-, p_-)\mathbf{n}_+ + c_+ \mathbf{u}_+] \, \mathrm{d}\mathcal{H}_{m-1}$$

$$+ \int_{\partial G} \mathbf{u}_- \cdot [T(\mathbf{u}_-, p_-)\mathbf{n}_- + c_- \mathbf{u}_-] \, \mathrm{d}\mathcal{H}_{m-1} = \int_{\Omega_-} [2|\hat\nabla \mathbf{u}_-|^2 + \lambda_- |\mathbf{u}_-|^2] \, \mathrm{d}\mathcal{H}_m$$

$$+ \frac{a_+ b_+}{a_- b_-} \int_{\Omega_+} [2|\hat\nabla \mathbf{u}_+|^2 + \lambda_+ |\mathbf{u}_+|^2] \, \mathrm{d}\mathcal{H}_m + \int_{\partial G} c_- |\mathbf{u}_-|^2 \, \mathrm{d}\mathcal{H}_{m-1} + \int_{\partial\Omega} \frac{c_+ a_+ |\mathbf{u}_+|^2}{a_-} \, \mathrm{d}\mathcal{H}_{m-1}.$$

Thus $\hat\nabla \mathbf{u}_\pm = 0$, $\lambda_\pm \mathbf{u}_\pm = 0$ in $\Omega_\pm$, $c_+ \mathbf{u}_+ = 0$ on $\partial\Omega$, $c_- \mathbf{u}_- = 0$ on ∂G. Define $\mathbf{v} = a_\pm \mathbf{u}_\pm$ on $\Omega_\pm$. Denote by $\omega_1, \dots, \omega_k$ all components of $G \setminus \partial\Omega$. According to [12], Lemma 3.1 there exist antisymmetric matrices A^j and vectors $\mathbf{B}^j$ such that $\mathbf{v}(\mathbf{x}) = A^j \mathbf{x} + \mathbf{B}^j$ in ω_j. If $\partial\omega_j \cap \partial\omega_i \neq \emptyset$, $\omega_j \subset \Omega_+$, $\omega_i \subset \Omega_-$ then $a_+ \mathbf{u}_+ - a_- \mathbf{u}_- = 0$ gives $(A^j \mathbf{x} + \mathbf{B}^j) - (A^i \mathbf{x} + \mathbf{B}^i) = 0$ on $\partial\omega_j \cap \partial\omega_i$. Lemma 5.1 gives that $(A^j \mathbf{x} + \mathbf{B}^j) - (A^i \mathbf{x} + \mathbf{B}^i) = 0$ in R^m. Thus $\mathbf{v} \in \mathcal{R}_m$. If $\lambda_+ + \lambda_- + c_+ + c_- > 0$ then Lemma 5.1 gives that $\mathbf{v} \equiv 0$.

Since $\nabla p_\pm = \Delta \mathbf{u}_\pm - \lambda_\pm \mathbf{u}_\pm = 0$ there exist constant $d_1, \dots, d_k$ such that $p = d_j$ on ω_j, where $p = p_\pm$ on $\Omega_\pm$. If $\partial\omega_j \cap \partial\omega_i \neq \emptyset$, $\omega_j \subset \Omega_+$, $\omega_i \subset \Omega_-$ then $0 = b_+ T(\mathbf{u}_+, p_+)\mathbf{n}_+ - b_- T(\mathbf{u}_-, p_-)\mathbf{n}_+ + c_+ \mathbf{u}_+ = (b_i d_i - b_+ d_j)\mathbf{n}_+$. Therefore there is a constant d such that $p_\pm = d/b_\pm$. On ∂G we have $0 = T(\mathbf{u}_-, p_-)\mathbf{n}_- = -d\mathbf{n}_-/b_-$. This gives $d = 0$.

Theorem 6.4. *Let $\lambda_+ + \lambda_- + c_+ + c_- > 0$. Then $R^{\lambda_+, \lambda_-, \tilde b_+, \tilde b_-, \tilde c_+, c_-}$ is an isomorphism on $W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m) \times L^2(\partial G, R^m)$. Let $\mathbf{g} \in W^{1,2}(\partial\Omega, R^m)$, $\mathbf{f} \in L^2(\partial\Omega, R^m)$, $\mathbf{h} \in L^2(\partial G, R^m)$. Then there exists a unique L^2-solution $\mathbf{u}_\pm$, $p_\pm$ of the Robin–transmission problem (1), (2), (16). Moreover, $\mathbf{u}_\pm \in H^{3/2}(\Omega_\pm, R^m)$, $p_\pm \in H^{1/2}(\Omega_\pm)$ and*

$$\|\mathbf{u}_+\|_{H^{3/2}(\Omega_+)} + \|\mathbf{u}_-\|_{H^{3/2}(\Omega_-)} + \|p_+\|_{H^{1/2}(\Omega_+)} + \|p_-\|_{H^{1/2}(\Omega_1)}$$

$$\leq C[\|\mathbf{g}\|_{W^{1,2}(\partial\Omega, R^m)} + \|\mathbf{f}\|_{L^2(\partial\Omega, R^m)} + \|\mathbf{h}\|_{L^2(\partial G, R^m)},]$$

where C does not depend on $\mathbf{g}$, $\mathbf{f}$ and $\mathbf{h}$.

Proof. $R^{\lambda_+, \lambda_-, \tilde b_+, \tilde b_-, \tilde c_+, c_-}$ is a Fredholm operator with index 0 by Lemma 6.1. Let $R^{\lambda_+, \lambda_-, \tilde b_+, \tilde b_-, \tilde c_+, c_-}(\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}) = 0$. Let $\tilde{\mathbf{u}}_+$, $\tilde p_+$ be given by (15), and $\tilde{\mathbf{u}}_-$, $\tilde p_-$ by (18). Then $\tilde{\mathbf{u}}_\pm = 0$, $\tilde{\mathbf{p}}_\pm = 0$ by Proposition 6.3. Lemma 6.2 gives $\mathbf{\Phi} = 0$, $\mathbf{\Psi} = 0$, $\mathbf{\Theta} = 0$. Since $R^{\lambda_+, \lambda_-, \tilde b_+, \tilde b_-, \tilde c_+, c_-}$ is a Fredholm operator with index 0, we infer that $R^{\lambda_+, \lambda_-, \tilde b_+, \tilde b_-, \tilde c_+, c_-}$ is an isomorphism.

Let $\mathbf{g} \in W^{1,2}(\partial\Omega, R^m)$, $\mathbf{f} \in L^2(\partial\Omega, R^m)$, $\mathbf{h} \in L^2(\partial G, R^m)$ be fixed. Put

$$(\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}) = (R^{\lambda_+, \lambda_-, \tilde b_+, \tilde b_-, \tilde c_+, c_-})^{-1}[\mathbf{g}, \mathbf{f}, a_- \mathbf{h}].$$

Define $\tilde{\mathbf{u}}_+$, $\tilde{p}_+$ by (15), and $\tilde{\mathbf{u}}_-$, $\tilde{p}_-$ by (18). Then $\tilde{\mathbf{u}}_\pm$, $\tilde{p}_\pm$ is an L^2-solution of the Robin–transmission problem (13), (14), (17). Denoting $\mathbf{u}_\pm = \tilde{\mathbf{u}}_\pm / a_\pm$, $p_\pm = \tilde{p}_\pm / a_\pm$ we obtain an L^2 solution of the problem (1), (2), (16). The uniqueness follows from Proposition 6.3. The rest is a consequence of the fact that $E^{\lambda_\pm}_{\Omega_\pm} : L^2(\partial\Omega_\pm, R^m) \to H^{3/2}(\Omega_\pm, R^m)$, $D^{\lambda_\pm}_{\Omega_\pm} : W^{1,2}(\partial\Omega_\pm, R^m) \to H^{3/2}(\Omega_\pm, R^m)$, $Q^{\lambda_\pm}_{\Omega_\pm} : L^2(\partial\Omega_\pm, R^m) \to H^{1/2}(\Omega_\pm, R^m)$, $\Pi^{\lambda_\pm}_{\Omega_\pm} : W^{1,2}(\partial\Omega_\pm, R^m) \to H^{1/2}(\Omega_\pm, R^m)$ are bounded linear operators (see [5] and [14]).

Theorem 6.5. *Let $\lambda_+ = \lambda_- = c_+ = c_- = 0$, $\mathbf{g} \in W^{1,2}(\partial\Omega, R^m)$, $\mathbf{f} \in L^2(\partial\Omega, R^m)$, $\mathbf{h} \in L^2(\partial G, R^m)$. Then there exists an L^2-solution $\mathbf{u}_\pm$, $p_\pm$ of the Robin–transmission problem (1), (2), (16) if and only if*

$$\int_{\partial\Omega} \mathbf{v} \cdot \mathbf{f} \, \mathrm{d}\mathcal{H}_{m-1} + \int_{\partial G} b_- \mathbf{v} \cdot \mathbf{h} \, \mathrm{d}\mathcal{H}_{m-1} = 0 \quad \forall \mathbf{v} \in \mathcal{R}_m. \tag{19}$$

The general from of an L^2-solution of the problem (1), (2), (16) is

$$\mathbf{u}_\pm + \mathbf{v}/a_\pm, \ p\pm, \qquad \mathbf{v} \in \mathcal{R}_m. \tag{20}$$

Proof. Let $\mathbf{u}_\pm$, $p_\pm$ be an L^2-solution of the Robin–transmission problem (1), (2), (16), $\mathbf{v} \in \mathcal{R}_m$. Then

$$\int_{\partial\Omega_\pm} \mathbf{v} \cdot T(\mathbf{u}_\pm, p_\pm) \mathbf{n}^{\Omega_\pm} \, \mathrm{d}\mathcal{H}_{m-1} = 0$$

(see [14]). Thus

$$0 = b_+ \int_{\partial\Omega_+} \mathbf{v} \cdot T(\mathbf{u}_+, p_+) \mathbf{n}_+ + b_- \int_{\partial\Omega_-} \mathbf{v} \cdot T(\mathbf{u}_-, p_-) \mathbf{n}_- = \int_{\partial\Omega} \mathbf{v} \cdot f + \int_{\partial G} b_- \mathbf{v} \cdot h.$$

Denote by X^{b_-} the space of $[\mathbf{g}, \mathbf{f}, \mathbf{h}] \in X = W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m) \times L^2(\partial G, R^m)$ satisfying (19). We have proved that $R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0}(X) \subset X^{\tilde{b}_-}$. Therefore $\operatorname{codim} R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0}(X) \geq \operatorname{codim} X^{\tilde{b}_-} = \dim \mathcal{R}_m$.

Let $[\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}] \in \operatorname{Ker} R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0}$. Let $\tilde{\mathbf{u}}_+$, $\tilde{p}_+$ be given by (15), and $\tilde{\mathbf{u}}_-$, $\tilde{p}_-$ by (18). According to Proposition 6.3 there exists $\mathbf{v} \in \mathcal{R}_m$ such that $\tilde{\mathbf{u}}_\pm = \mathbf{v}$, $\tilde{p}_\pm = 0$. If $\mathbf{v} = 0$ then $\mathbf{\Phi} = 0$, $\mathbf{\Psi} = 0$, $\mathbf{\Theta} = 0$ by Lemma 6.2. Thus $\dim \operatorname{Ker} R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0} \leq \dim \mathcal{R}_m$. Since $R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0}$ is a Fredholm operator with index 0 by Lemma 6.1, we deuce that $\dim \operatorname{Ker} R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0} = \operatorname{codim} R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0}(X) = \dim \mathcal{R}_m$. Therefore $R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0}(X) = X^{\tilde{b}_-}$.

Let now $[\mathbf{g}, \mathbf{f}, \mathbf{h}] \in X$. We have proved that there exist $[\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}]$ such that $R^{0,0,\tilde{b}_+,\tilde{b}_-,0,0}[\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}] = [\mathbf{g}, \mathbf{f}, a_- \mathbf{h}]$. Let $\tilde{\mathbf{u}}_+$, $\tilde{p}_+$ be given by (15), and $\tilde{\mathbf{u}}_-$, $\tilde{p}_-$

by (18), $\mathbf{u}_\pm = \tilde{\mathbf{u}}_\pm / a_\pm$, $p_\pm = \tilde{p}_\pm / a_\pm$. Then $\mathbf{u}_\pm$, $p_\pm$ is an L^2-solution of the Robin–transmission problem (1), (2), (16). Easy calculation yields that (20) gives another solution of the problem. Proposition 6.3 gives that each solution of the problem is of the form (20).

7 Regular Dirichlet–transmission problem

Let $G \subset R^m$ be a bounded domain with connected Lipschitz boundary, $\Omega = \Omega_+$ be a nonempty bounded open set with Lipschitz boundary such that $\overline{\Omega} \subset G$. Denote $\Omega_- = G \setminus \overline{\Omega}$, and by $\mathbf{n}_\pm$ the outward unit normal of $\Omega_\pm$. Let $\lambda_\pm$, c_+ be non-negative constants and $a_\pm$, $b_\pm$ be positive constants. We shall study the regular Dirichlet–transmission problem for the Brinkman system (1), (2) accompanied with the condition

$$\mathbf{u}_- = \mathbf{h} \quad \text{on } \partial G. \tag{21}$$

Let $\mathbf{g} \in W^{1,2}(\partial\Omega, R^m)$, $\mathbf{f} \in L^2(\partial\Omega, R^m)$, $\mathbf{h} \in W^{1,2}(\partial G, R^m)$. *We say that* $\mathbf{u}_\pm$, $p_\pm$ *defined on* $\Omega_\pm$ *is an* L^2*-solution of the regular Dirichlet–transmission problem (1), (2), (21) if* $\mathbf{u}_\pm$, $p_\pm$ *satisfy (1);* $\mathbf{u}^*_\pm$, $p^*_\pm$, $(\nabla \mathbf{u})^*_\pm$ *are from* $L^2(\partial\Omega_\pm, R^1)$*; for almost all* $\mathbf{x} \in \partial\Omega_\pm$ *there exist the non-tangential limits of* $\mathbf{u}_\pm$, $\nabla\mathbf{u}_\pm$, $p_\pm$ *at* $\mathbf{x}$ *and the conditions (2), (21) are fulfilled in the sense of the nontangential limit a.e. on* $\partial\Omega_-$.

Put $\tilde{b}_\pm = b_\pm / a_\pm$, $\tilde{c}_+ = c_+ / a_+$. If $\tilde{\mathbf{u}}_\pm = a_\pm \mathbf{u}_\pm$, $\tilde{p}_\pm = a_\pm p_\pm$ then $\mathbf{u}_\pm$, $p_\pm$ is an L^2-solution of the regular Dirichlet–transmission problem (1), (2), (21) if and only if $\tilde{\mathbf{u}}_\pm$, $\tilde{p}_\pm$ is an L^2-solution of the regular Dirichlet–transmission problem (13), (14),

$$\tilde{\mathbf{u}}_- = a_- \mathbf{h} \quad \text{on } \partial G. \tag{22}$$

Let $\mathbf{\Phi} \in W^{1,2}(\partial\Omega, R^m)$, $\mathbf{\Psi} \in L^2(\partial\Omega, R^m)$, $\mathbf{\Theta} \in L^2(\partial G, R^m)$. Let $\tilde{\mathbf{u}}_+$, $\tilde{p}_+$ be given by (15), and $\tilde{\mathbf{u}}_-$, $\tilde{p}_-$ be given by (18). Then $\tilde{\mathbf{u}}_\pm$, $\tilde{p}_\pm$ is an L^2-solution of the regular Dirichlet–transmission problem (13), (14), (22) if and only if

$$R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}) = [\mathbf{g}, \mathbf{f}, a_- \mathbf{h}],$$

where

$$R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}) = [\tau_1^{\lambda_+,\lambda_-}(\mathbf{\Phi}, \mathbf{\Psi}) - E_G^{\lambda_-}\mathbf{\Theta}, \tau_2^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(\mathbf{\Phi}, \mathbf{\Psi})$$
$$-\tilde{b}_- T(E_G^{\lambda_-}\mathbf{\Theta}, Q_G^{\lambda_-}\mathbf{\Theta})\mathbf{n}_+, D_\Omega^{\lambda_-}\mathbf{\Phi} + E_\Omega^{\lambda_-}\mathbf{\Psi} + E_G^{\lambda_-}\mathbf{\Theta}].$$

Proposition 7.1. *Let* $\mathbf{u}_\pm$, $p_\pm$ *be an* L^2*-solution of the regular Dirichlet–transmission problem (1), (2), (21) with* $\mathbf{g} = 0$, $\mathbf{f} = 0$, $\mathbf{h} = 0$. *Then there exists a constant* c *such that* $\mathbf{u}_\pm = 0$, $p_\pm = c/b_\pm$.

Proof. Using Green's formula

$$0 = b_-^{-1} \int_{\partial\Omega} \mathbf{u}_- \cdot [b_+ T(\mathbf{u}_+, p_+)\mathbf{n}_+ - b_- T(\mathbf{u}_-, p_-)\mathbf{n}_+ + c_+ \mathbf{u}_+] \, d\mathcal{H}_{m-1}$$

$$+\int_{\partial G} \mathbf{u}_- \cdot T(\mathbf{u}_-, p_-)\mathbf{n}_- \, \mathrm{d}\mathcal{H}_{m-1} = \int_{\Omega_-} [2|\hat\nabla \mathbf{u}_-|^2 + \lambda_-|\mathbf{u}_-|^2] \, \mathrm{d}\mathcal{H}_m$$

$$+\frac{a_+ b_+}{a_- b_-} \int_{\Omega_+} [2|\hat\nabla \mathbf{u}_+|^2 + \lambda_+|\mathbf{u}_+|^2] \, \mathrm{d}\mathcal{H}_m + \int_{\partial\Omega} \frac{c_+ a_+ |\mathbf{u}_+|^2}{a_-} \, \mathrm{d}\mathcal{H}_{m-1}.$$

Thus $\hat\nabla \mathbf{u}_\pm = 0$. According to [12], Lemma 3.1 there exist an antisymmetric matrix A and a vector $\mathbf{B}$ such that $\mathbf{u}_-(\mathbf{x}) = A\mathbf{x} + \mathbf{B}$. Since $\mathbf{u}_- = 0$ on ∂G, Lemma 5.1 gives that $\mathbf{u}_- = 0$. Since $\nabla p_- = \Delta \mathbf{u}_- - \lambda_- \mathbf{u}_- = 0$ there exists a constant c such that $p_- = c/b_-$. Let ω be a component of Ω_+. According to [12], Lemma 3.1 there exist an antisymmetric matrix A and a vector $\mathbf{B}$ such that $\mathbf{u}_+(\mathbf{x}) = A\mathbf{x} + \mathbf{B}$ in ω. Since $\mathbf{u}_+ = a_-\mathbf{u}_-/a_+ = 0$ on $\partial\omega$, Lemma 5.1 gives that $\mathbf{u}_+ = 0$ in ω. Since $\nabla p_+ = \Delta \mathbf{u}_+ - \lambda_+ \mathbf{u}_+ = 0$ there exists a constant C such that $p_+ = C$ in ω. We have $0 = b_+ T(\mathbf{u}_+, p_+)\mathbf{n}_+ - b_- T(\mathbf{u}_-, p_-)\mathbf{n}_+ + c_+ \mathbf{u}_+ = -b_+ C\mathbf{n}_+ + b_-(c/b_-)\mathbf{n}_+$ on $\partial\omega$. Hence $p_+ = C = c/b_+$.

Theorem 7.2. *Let $\mathbf{g} \in W^{1,2}(\partial\Omega, R^m)$, $\mathbf{f} \in L^2(\partial\Omega, R^m)$, $\mathbf{h} \in W^{1,2}(\partial G, R^m)$. There there exists an L^2-solution $\mathbf{u}_\pm$, $p_\pm$ of the regular Dirichlet–transmission problem (1), (2), (21) if and only if*

$$\int_{\partial\Omega} \mathbf{n}_+ \cdot \mathbf{g} \, \mathrm{d}\mathcal{H}_{m-1} + a_- \int_{\partial G} \mathbf{n}_- \cdot \mathbf{h} \, \mathrm{d}\mathcal{H}_{m-1} = 0 \tag{23}$$

The general form of a solution of the problem is $\mathbf{u}_\pm$, $p_\pm + c/b_\pm$, where c is a constant.

Proof. Suppose that $\mathbf{u}_\pm$, $p_\pm$ be an L^2-solution $\mathbf{u}_\pm$, $p_\pm$ of the regular Dirichlet–transmission problem (1), (2), (21). Then

$$0 = a_+ \int_{\partial\Omega} \mathbf{n}_+ \cdot \mathbf{u}_+ + a_- \int_{\partial G} \mathbf{n}_- \cdot \mathbf{u}_- = \int_{\partial\Omega} \mathbf{n}_+ \cdot \mathbf{g} \, \mathrm{d}\mathcal{H}_{m-1} + a_- \int_{\partial G} \mathbf{n}_- \cdot \mathbf{h} \, \mathrm{d}\mathcal{H}_{m-1}.$$

$R : (\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}) \mapsto [\tau_1^{\lambda_+,\lambda_-}(\mathbf{\Phi}, \mathbf{\Psi}), \tau_2^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+}(\mathbf{\Phi}, \mathbf{\Psi}), E_G^0 \mathbf{\Theta}]$ is a Fredholm operator with index 0 from $X = W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m) \times L^2(\partial G, R^m)$ to the space $Y = W^{1,2}(\partial\Omega, R^m) \times L^2(\partial\Omega, R^m) \times W^{1,2}(\partial G, R^m)$ by [14] and Proposition 5.3. If $\lambda \geq 0$ then $E_G^\lambda - E_G^0$ is a compact operator from $L^2(\partial G, R^m)$ to $W^{1,2}(\partial G, R^m)$ (see [5], Theorem 3.4). Thus $R_D^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+} - R$ is a compact operator. Hence $R_D^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+}$ is a Fredholm operator from X to Y with index 0. Denote by $Z(a_-)$ the set of all $[\mathbf{g}, \mathbf{f}, \mathbf{h}] \in Y$ satisfying (23). We have proved that $R_D^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+}(X) \subset Z(1)$. Thus $\operatorname{codim} R_D^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+}(X) \geq 1$.

Let now $R_D^{\lambda_+,\lambda_-,\tilde b_+,\tilde b_-,\tilde c_+}(\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}) = 0$. Let $\tilde{\mathbf{u}}_+$, $\tilde p_+$ be given by (15), and $\tilde{\mathbf{u}}_-$, $\tilde p_-$ be given by (18). Then $\tilde{\mathbf{u}}_\pm$, $\tilde p_\pm$ is an L^2-solution of the regular Dirichlet–transmission problem (13), (14), (22) with $\mathbf{g} = 0$, $\mathbf{f} = 0$, $\mathbf{h} = 0$. Proposition 7.1

gives that there exists a constant c such that $\mathbf{u}_\pm = 0$, $p_\pm = c/\tilde{b}_\pm$. If $c = 0$ then $\mathbf{\Phi} = 0$, $\mathbf{\Psi} = 0$, $\mathbf{\Theta} = 0$ by Lemma 6.2. Therefore $\dim \operatorname{Ker} R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+} \leq 1$. Hence $1 \leq \operatorname{codim} R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(X) = \dim \operatorname{Ker} R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+} \leq 1$. This forces $R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(X) = Z(1)$.

Suppose now that (23) is fulfilled. We have proved that there exists $[\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}] \in X$ such that $R_D^{\lambda_+,\lambda_-,\tilde{b}_+,\tilde{b}_-,\tilde{c}_+}(\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Theta}) = [\mathbf{g}, \mathbf{f}, a_-\mathbf{h}]$. Let $\tilde{\mathbf{u}}_+$, $\tilde{p}_+$ be given by (15), and $\tilde{\mathbf{u}}_-$, $\tilde{p}_-$ be given by (18). Then $\tilde{\mathbf{u}}_\pm$, $\tilde{p}_\pm$ is an L^2-solution of the regular Dirichlet–transmission problem (13), (14), (22). So $\mathbf{u}_\pm = \tilde{\mathbf{u}}_\pm/a_\pm$, $p_\pm = \tilde{p}_\pm/a_\pm$ is an L^2-solution of (1), (2), (21). If c is a constant, then easy calculation gives that $\mathbf{u}_\pm$, $p_\pm + c/b_\pm$ is a solution of the problem, too. Proposition 7.1 gives that each solution of the problem has this form.

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