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ON THE L^q-SOLUTION OF THE OSEEN-BRINKMAN TRANSMISSION PROBLEM AROUND A SOLID $(m-1)$-DIMENSIONAL OBSTACLE

MIRELA KOHR*, DAGMAR MEDKOVÁ†, AND WOLFGANG L. WENDLAND

ABSTRACT. The purpose of this paper is to develop a layer potential analysis in order to show the well-posedness result of a crack type transmission problem for the Oseen and Brinkman systems in open sets in $\mathbb{R}^m$ ($m \in \{2, 3\}$) with Lipschitz boundaries when the boundary data belong to some L^q-spaces.

1. INTRODUCTION

The layer potential methods have a well known role in the analysis of elliptic boundary value problems (see, e.g., [2, 5, 7, 14, 18, 19, 31]). Escauriaza and Mitrea [8] have developed a layer potential analysis for the transmission problems of the Laplace operator on Lipschitz domains in Euclidean setting. Fabes, Kenig and Verchota [9] have developed a layer potential method in order to show the solvability of the Dirichlet problem for the Stokes system on Lipschitz domains in $\mathbb{R}^n$, $n \geq 3$, with L^2-boundary data. Dahlberg, Kenig and Verchota [6] have studied the Dirichlet and Neumann problems for the Lamé system in Lipschitz domains in $\mathbb{R}^n$ $(n \geq 3)$. Mitrea and Wright [34] have exploited layer potential methods in order to analyze the main boundary value problems for the Stokes system in arbitrary Lipschitz domains in $\mathbb{R}^n$, $n \geq 2$. In [29] the author has obtained existence and uniqueness results for L^2-solutions of the transmission problem, the Robin-transmission problem and the Dirichlet-transmission problem for the Brinkman system in Lipschitz domains in $\mathbb{R}^n$ $(n \geq 3)$, by using the integral equation method. Mitrea, Mitrea and Qiang [33] have used layer potential theoretic methods to obtain well-posedness results for variable coefficient transmission problems in Lipschitz domains on nonsmooth manifolds. The authors in [16] have used a layer potential theoretic method in order to show the well-posedness of a mixed-transmission problem for two (linear) Brinkman systems on two adjacent Lipschitz domains in $\mathbb{R}^n$ $(n \geq 3)$ with linear transmission conditions on the interface between these domains and linear mixed Dirichlet-Robin conditions on another interface, which is the boundary of a bounded creased Lipschitz domain (see also [17]).

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Chkadua, Mikhailov and Natroshvili [3] have used localized direct segregated boundary-domain integral equations for variable coefficient transmission problems with interface crack corresponding to scalar second order elliptic partial differential equations in a bounded composite domain consisting of adjacent anisotropic subdomains having a common interface surface. The same authors in [4] have analyzed segregated direct boundary-domain integral equation (BDIE) systems associated with mixed, Dirichlet and Neumann boundary value problems (BVPs) for a scalar partial differential equation with variable coefficients of the Laplace type for domains with interior cuts (cracks). The authors have established the equivalence of BDIE's to such boundary value problems and the invertibility of the BDIE operators in the corresponding Sobolev spaces (see also [2]). Buchukuri, Chkadua, Duduchava and Natroshvili [1] have investigated three-dimensional interface crack problems for metallic-piezoelectric composite bodies with regard to thermal effects. Krutitskii [20, 21, 22, 23, 24] has studied boundary value problems for the Laplace equation on domains with cracks, by means of integral equation methods. A boundary problem with jump conditions of Robin type for the Stokes system has been studied in [30]. A boundary value problem with jump conditions of the Robin type for the Stokes system has been studied in [30]. A three-dimensional Stokes flow exterior to an open surface has been studied in [45] by means of a layer potential method. The authors in [44] have studied hypersingular integral equations on a curved open smooth arc in $\mathbb{R}^2$ that model either curved cracks in an elastic medium or the scattering of acoustic and elastic waves at a hard screen.

In this paper we develop a layer potential analysis in order to show the well-posedness result of a crack type transmission problem for the Oseen and Brinkman systems in open sets in $\mathbb{R}^m$ ($m \in \{2,3\}$) with Lipschitz boundaries when the boundary data belong to some L^q-spaces. The paper is organized as follows. The next section is devoted to some preliminaries results and to the formulation of the crack type transmission problem for the Oseen and Brinkman systems (see (2.4)-(2.9)), which will be investigated below, as well as to the definition of an L^q-solution of such a problem. Next, we present the main results of layer potential theory for the Stokes, Brinkman and Oseen systems on Lipschitz domains in Euclidean setting. The last section is devoted to the study of the solvability of the transmission problem with crack (2.4)-(2.9) in L^q spaces, with $q \in (1, q_0)$, for some $q_0 > 2$, by using a layer potential theoretic method. Our study is motivated by various applications, such as the transport in micro-cracked rocks or reservoirs, which require the analysis of the fluid flows in porous media containing cracks or fractures. The cracks influence the permeability of fractured rocks and porous materials (for further applications we refer the reader to [36] and the references therein).

2. Preliminaries

Let $\Omega \subseteq \mathbb{R}^m$ ($m \in \{2,3\}$) be an open set with compact Lipschitz boundary. Let $a > 0$. If $\mathbf{x} \in \partial\Omega$, then

$$\Gamma(\mathbf{x}) = \Gamma_a(\mathbf{x}) := \{\mathbf{y} \in \Omega : |\mathbf{x} - \mathbf{y}| < (1+a)\mathrm{dist}(\mathbf{y}, \partial\Omega)\}$$

denotes the nontangential approach region of opening a at the point $\mathbf{x}$. Also, for a given $\mathbf{v} : \Omega \to \mathbb{R}^m$, the nontangential maximal function of $\mathbf{v}$ at $\mathbf{x}$ is defined by

$$\mathbf{v}^*(\mathbf{x}) := \sup\{|\mathbf{v}(\mathbf{y})| : \mathbf{y} \in \Gamma(\mathbf{x})\}.$$

It is well known that if $q \in [1,\infty)$ and $\mathbf{v}^* \in L^q(\partial\Omega)$ for one choice of a, then such a property holds for arbitrary choice of a (see, e.g., [12, 31] and [39, p. 62]). Moreover, $|\mathbf{v}| \in L^q(\omega)$ for any bounded open subset ω of Ω (see [32, Lemma 4.1]). Next, define the nontangential limit of $\mathbf{v}$ at $\mathbf{x} \in \partial\Omega$ by

$$\mathrm{Tr}\ \mathbf{v}(\mathbf{x}) = \mathrm{Tr}_\Omega \mathbf{v}(\mathbf{x}) := \lim_{\Gamma(\mathbf{x})\ni\ \mathbf{y}\to\mathbf{x}} \mathbf{v}(\mathbf{y})$$

whenever such a limit exists. If $\mathbf{v}$ is defined in $\mathbb{R}^m \setminus \partial\Omega$, we denote by $\mathrm{Tr}_{\Omega^\pm}\mathbf{v}$ the nontangential limit of $\mathbf{v}$ with respect to $\Omega^+ = \Omega$ and $\Omega^- := \mathbb{R}^m \setminus \overline{\Omega}$, respectively.

2.1. **Crack type transmission problem for the Oseen and Brinkman systems.** Let $\Omega_B, \Omega_O \subseteq \mathbb{R}^m$ $(m \in \{2,3\})$ be open sets, not necessary connected, with compact Lipschitz boundaries, such that $\Omega_B \neq \emptyset$, $\Omega_B \cap \Omega_O = \emptyset$ and $\overline{\Omega}_B \cup \overline{\Omega}_O = \mathbb{R}^m$. Let $S_{if} := \partial\Omega_B = \partial\Omega_O$ be the interface of these sets. Further, we assume that the region Ω_B contains an interior crack. We define the crack as an $(m-1)$-dimensional, two-sided closed manifold S_{cB}. We assume that S_{cB} is a sub-manifold of a closed Lipschitz surface $\partial\omega_B$, which is the boundary of a bounded open set $\omega_B \subseteq \mathbb{R}^m$ such that $\overline{\omega}_B \subset \Omega_B$.

Let $q \in (1,\infty)$. For our purpose we need to consider the following spaces

$$L^q(S_{cB}) := \{f \in L^q(\partial\omega_B) : f = 0 \text{ on } \partial\omega_B \setminus S_{cB}\}, \tag{2.1}$$

$$L^q_1(S_{cB}) := \{f \in L^q_1(\partial\omega_B) : f = 0 \text{ on } \partial\omega_B \setminus S_{cB}\}. \tag{2.2}$$

Note that if $S_{cB} = \overline{\omega}$, where ω is an open subset with Lipschitz boundary of the manifold $\partial\omega_B$, then $L^q_1(S_{cB})$ is the closure of the set of all infinitely differentiable functions on $\partial\omega_B$ supported in ω with respect to some related norm.

Let c_O and α be positive constants, λ be a non-zero real constant. Let h_B, h_O, $h_\pm$ be non-negative Borel measurable matrices with bounded entries. Recall that a matrix valued function h of type $m \times m$ is non-negative if the following condition

$$\langle h(\mathbf{x})\theta, \theta\rangle \geq 0,\ \forall\ \mathbf{x}, \theta \in \mathbb{R}^m \tag{2.3}$$

is satisfied, where $\langle \cdot, \cdot \rangle$ means the inner product in $\mathbb{R}^n$.

In the sequel, we will study the existence of a solution for a *crack type transmission problem for the Oseen and Brinkman systems in L^q-spaces.* Such a problem requires to find some functions $((\mathbf{u}_B, \pi_B), (\mathbf{u}_O, \pi_O))$, which satisfy

- *The Brinkman and Oseen systems*:

$$\triangle \mathbf{u}_B - \alpha \mathbf{u}_B - \nabla \pi_B = \mathbf{0},\ \mathrm{div}\ \mathbf{u}_B = 0 \ \text{ in } \ \Omega_B \setminus S_{cB}, \tag{2.4}$$

$$\triangle \mathbf{u}_O - \lambda \partial_1 \mathbf{u}_O - \nabla \pi_O = 0,\ \mathrm{div}\ \mathbf{u}_O = 0 \ \text{ in } \ \Omega_O, \tag{2.5}$$

- *The transmission conditions on the interface S_{if}*:

$$\mathrm{Tr}_{\Omega_B} \mathbf{u}_B - \mathrm{Tr}_{\Omega_O} \mathbf{u}_O = \mathbf{g}_{if} \in L^q_1(S_{if}, \mathbb{R}^n), \tag{2.6}$$

$$\partial^0_{\nu_B;\Omega_B}(\mathbf{u}_B, \pi_B) - c_O \partial^\lambda_{\nu_B;\Omega_O}(\mathbf{u}_O, \pi_O) + h_B \mathrm{Tr}_{\Omega_B} \mathbf{u}_B + h_O \mathrm{Tr}_{\Omega_O} \mathbf{u}_O = \mathbf{f}_{if} \in L^q(S_{if}, \mathbb{R}^n), \tag{2.7}$$

- *The crack type boundary conditions on S_{cB}*:

$$\mathrm{Tr}_{\omega_B} \mathbf{u}_B - \mathrm{Tr}_{\Omega_B \setminus \overline{\omega}_B} \mathbf{u}_B = \mathbf{g}_{cB} \in L^q_1(S_{cB}, \mathbb{R}^n), \tag{2.8}$$

$$\begin{aligned} \partial^0_{\nu_B;\omega_B}(\mathbf{u}_B, \pi_B) - \partial^0_{\nu_B;\Omega_B\setminus\overline{\omega}_B}(\mathbf{u}_B, \pi_B) + h_+ \mathrm{Tr}_{\omega_B} \mathbf{u}_B \\ + h_- \mathrm{Tr}_{\Omega_B \setminus \overline{\omega}_B} \mathbf{u}_B = \mathbf{f}_{cB} \in L^q(S_{cB}, \mathbb{R}^n), \end{aligned} \tag{2.9}$$

where ∂_1 means the derivative in the direction of the x_1-axis, $\nu_{_B}$ stands for the unit normal vector to $\partial\Omega_B$ exterior to Ω_B on S_{if}, and for the the unit normal vector to $\partial\omega_B$ exterior to ω_B on $\partial\omega_B \supset S_{cB}$. Further,

$$\partial^{\beta}_{\nu;\mathfrak{D}}(\mathbf{u},\pi) := \left(-\pi\mathbb{I} + 2\widehat{\nabla}\mathbf{u}\right)\nu - \frac{\beta}{2}\nu_1\mathbf{u} \tag{2.10}$$

is the conormal derivative of the velocity and pressure fields $\mathbf{u} = (u_1, \ldots, u_m)$ and p, which are defined on an open set $\mathfrak{D} \subseteq \mathbb{R}^n$ with Lipschitz boundary. Here ν denotes the unit normal to $\partial\mathfrak{D}$, defined a.e. $\partial\mathfrak{D}$, and $\partial^{\beta}_{\nu;\mathfrak{D}}(\mathbf{u}, p)$ is defined a.e. on $\partial\mathfrak{D}$ in the sense of nontangential convergence. In addition, $\beta \in \mathbb{R}$, $\mathbb{I}$ is the identity matrix, and $\widehat{\nabla}\mathbf{u}$ is the symmetric part of $\nabla\mathbf{u}$.

In the case $\beta = \lambda$, $\partial^{\beta}_{\nu;\Omega_O}(\mathbf{v}, \pi)$ is the conormal derivative corresponding to the Oseen system (2.5), while, in the case $\beta = 0$, $\partial^{\beta}_{\nu;\Omega_B}(\mathbf{v}, \pi)$ is the conormal derivative for the Brinkman system (2.4).

Further, we assume that $h_B = 0$, $h_O = 0$, $\mathbf{g}_{if} = 0$, $\mathbf{f}_{if} = 0$ on $\mathbb{R}^m \setminus S_{if}$, and $h_{\pm} = 0$, $\mathbf{g}_{cB} = 0$, $\mathbf{f}_{cB} = 0$ on $\mathbb{R}^m \setminus S_{cB}$.

Definition 2.1. Let $q \in (1, \infty)$ be given. An *L^q-solution* of the boundary value problem (2.4)-(2.9) is an element $((\mathbf{u}_B, \pi_B), (\mathbf{u}_O, \pi_O))$ in the space

$$(C^{\infty}(\Omega_B \setminus S_{cB}, \mathbb{R}^n) \times C^{\infty}(\Omega_B \setminus S_{cB})) \times (C^{\infty}(\Omega_O, \mathbb{R}^n) \times C^{\infty}(\Omega_O)),$$

with the following properties:

- There exist the non-tangential limits of $\mathbf{u}_B$, $\nabla\mathbf{u}_B$, π_B, $\mathbf{u}_O$, $\nabla\mathbf{u}_O$, π_O almost every where on the interface S_{if}.
- The non-tangential maximal functions of $\mathbf{u}_B$, $\nabla\mathbf{u}_B$, π_B, $\mathbf{u}_O$, $\nabla\mathbf{u}_O$, π_O are q-integrable on S_{if}.
- There exist the non-tangential limits of $\mathbf{u}_B$, $\nabla\mathbf{u}_B$ and π_B, which correspond to ω_B and $\Omega_B \setminus \overline{\omega}_B$, respectively, a.e. on the interface crack S_{cB}.
- The non-tangential maximal functions of $\mathbf{u}_B$, $\nabla\mathbf{u}_B$, π_B are q-integrable on S_{cB}.
- The equations (2.4) and (2.5) are satisfied everywhere in $\Omega_B \setminus S_{cB}$ and Ω_O, respectively.
- The boundary conditions are fulfilled in the sense of non-tangential limit.

3. Layer potentials for the Stokes, Brinkman and Oseen systems

Let $\alpha > 0$. Then the Brinkman system

$$\begin{cases} (\Delta - \alpha\mathbb{I})\,\mathbf{u} - \nabla\pi = \mathbf{0} \ \text{ in } \ \Omega \\ \qquad \operatorname{div}\,\mathbf{u} = 0 \ \text{ in } \ \Omega \end{cases} \tag{3.1}$$

is a zero order perturbation of the Stokes system (when $\alpha = 0$) and describes the viscous incompressible fluids flows in porous media (see, e.g., [15]). Let $(\mathcal{G}^{\alpha}, \Pi^{\alpha}) \in \mathcal{D}'(\mathbb{R}^m, \mathbb{R}^m \times \mathbb{R}^m) \times \mathcal{D}'(\mathbb{R}^m, \mathbb{R}^m)$ denote the fundamental solution of the Brinkman system in $\mathbb{R}^m$ ($m \geq 2$), where $\mathcal{D}'(\mathbb{R}^m)$ is the space of distributions, i.e., the dual of $C_0^{\infty}(\mathbb{R}^n)$ equipped with the inductive limit topology. Thus,

$$(\Delta_{\mathbf{x}} - \alpha\mathbb{I})\mathcal{G}^{\alpha}(\mathbf{x}-\mathbf{y}) - \nabla_{\mathbf{x}}\Pi^{\alpha}(\mathbf{x}-\mathbf{y}) = -\delta_{\mathbf{y}}(\mathbf{x})\mathbb{I}, \quad \operatorname{div}_{\mathbf{x}}\mathcal{G}^{\alpha}(\mathbf{x}-\mathbf{y}) = 0, \tag{3.2}$$

where $\delta_{\mathbf{y}}$ is the Dirac distribution with mass at $\mathbf{y}$, and the subscript $\mathbf{x}$ added to a differential operator shows the action of the operator with respect to the variable $\mathbf{x}$. Let $\mathcal{G}^{\alpha}_{ij}$ be the components of the fundamental tensor $\mathcal{G}^{\alpha}$. If Π^{α}_j are the components of the fundamental pressure vector Π^α, then the components of the fundamental stress tensor $\mathbf{S}^\alpha(\cdot,\cdot)$ are defined by the relations

$$S^\alpha_{jk\ell}(\mathbf{x},\mathbf{y}) := -\Pi^\alpha_j(\mathbf{x}-\mathbf{y})\delta_{jk} + \frac{\partial \mathcal{G}^\alpha_{jk}(\mathbf{x}-\mathbf{y})}{\partial x_\ell} + \frac{\partial \mathcal{G}^\alpha_{\ell k}(\mathbf{x}-\mathbf{y})}{\partial x_j}. \tag{3.3}$$

In addition, if Λ_α is the fundamental pressure tensor with components Λ^α_{jk}, then

$$\Delta_{\mathbf{x}} S^\alpha_{jk\ell}(\mathbf{x},\mathbf{y}) - \alpha S^\alpha_{jk\ell}(\mathbf{x},\mathbf{y}) - \frac{\partial \Lambda^\alpha_{j\ell}(\mathbf{x},\mathbf{y})}{\partial x_k} = 0,\ \frac{\partial S^\alpha_{jk\ell}(\mathbf{x},\mathbf{y})}{\partial x_k} = 0,\ \mathbf{x}\neq\mathbf{y}. \tag{3.4}$$

The expressions of $(\mathcal{G}^\alpha,\Pi^\alpha)$ and $(\mathbf{S}^\alpha,\Lambda^\alpha)$ can be found in [42, Chapter 2] and [19, Chapter 2]. We omit them for the sake of brevity.

For $\alpha = 0$ we obtain the fundamental solution of the Stokes system. Next we use the notation $(\mathcal{G},\Pi)$ for the fundamental solution of the Stokes system. The components of $\mathcal{G}$ and Π are given by (see, e.g., [19, p. 38, 39]):

$$\begin{aligned} &\mathcal{G}_{jk}(\mathbf{x}) = \frac{1}{2\omega_m}\left\{\frac{\delta_{jk}}{(m-2)|\mathbf{x}|^{m-2}} + \frac{x_j x_k}{|\mathbf{x}|^m}\right\}, && m\geq 3\\ &\mathcal{G}_{jk}(\mathbf{x}) = \frac{1}{4\pi}\left\{\delta_{jk}\ln\frac{1}{|\mathbf{x}|} + \frac{x_j x_k}{|\mathbf{x}|^2}\right\}, && m=2,\\ &\Pi_j(\mathbf{x}) = \tfrac{1}{\sigma_n}\tfrac{x_j}{|\mathbf{x}|^m}, && m\geq 2,\end{aligned} \tag{3.5}$$

where σ_m is the area of the unit sphere in $\mathbb{R}^m$. Note that $\Pi^\alpha = \Pi$. The components of the stress and pressure tensors $\mathbf{S}$ and Λ are given by (see, e.g., [19, p. 38,39,132]):

$$S_{jk\ell}(\mathbf{x},\mathbf{y}) = \frac{m}{\omega_n}\frac{(x_j-y_j)(x_k-y_k)(x_\ell-y_\ell)}{|\mathbf{x}-\mathbf{y}|^{m+2}}, \tag{3.6}$$

$$\Lambda_{jk}(\mathbf{x},\mathbf{y}) = \frac{2}{\omega_m}\left(\frac{\delta_{jk}}{|\mathbf{x}-\mathbf{y}|^m} - m\frac{(x_j-y_j)(x_k-y_k)}{|\mathbf{x}-\mathbf{y}|^{n+2}}\right). \tag{3.7}$$

3.1. **Layer potentials of the Brinkman system.** Assume that $\Omega := \Omega_+ \subseteq \mathbb{R}^m$ $(m\geq 2)$ is a bounded open set with Lipschitz boundary $\partial\Omega$. Let $\Omega_- := \mathbb{R}^m\setminus\overline{\Omega}$. Let ν_ℓ, $\ell = 1,\dots,m$, be the components of the outward unit normal $\nu = \nu_\Omega$, which is defined a.e. on $\partial\Omega$. Let $\alpha\geq 0$, $q\in(1,\infty)$, $\mathbf{g}\in L^q(\partial\Omega,\mathbb{R}^m)$ and $\mathbf{h}\in L^q_1(\partial\Omega,\mathbb{R}^m)$. Then the single-layer potential for the Brinkman system $\mathbf{V}_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{g}$ and the corresponding pressure potential $Q^s_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{g}$ are given by

$$\begin{aligned} &\big(\mathbf{V}_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{g}\big)_j(\mathbf{x}) := \int_{\partial\Omega}\mathcal{G}^\alpha_{ij}(\mathbf{x}-\mathbf{y})g_j(\mathbf{y})\ \mathrm{d}\sigma(\mathbf{y}),\\ &\big(Q^s_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{g}\big)(\mathbf{x}) := \int_{\partial\Omega}\Pi^\alpha_j(\mathbf{x}-\mathbf{y})g_j(\mathbf{y})d\sigma(\mathbf{y}),\end{aligned}\qquad \mathbf{x}\in\mathbb{R}^m\setminus\partial\Omega. \tag{3.8}$$

Remark that $Q^s_{\partial\Omega}\mathbf{g} := Q^s_{\mathrm{Br};0;\partial\Omega}\mathbf{g} = Q^s_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{g}$.

The double-layer potential $\mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{h}$ and the corresponding pressure potential $Q^d_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{h}$ are given by

$$\begin{aligned} &\big(\mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{h}\big)_k(\mathbf{x}) := \int_{\partial\Omega}S^\alpha_{jk\ell}(\mathbf{x},\mathbf{y})\nu_\ell(\mathbf{y})h_j(\mathbf{y})d\sigma(\mathbf{y}),\\ &\big(Q^d_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{h}\big)(\mathbf{x}) := \int_{\partial\Omega}\Lambda^\alpha_{j\ell}(\mathbf{x},\mathbf{y})\nu_\ell(\mathbf{y})h_j(\mathbf{y})d\sigma(\mathbf{y}),\end{aligned}\qquad \mathbf{x}\in\mathbb{R}^m\setminus\partial\Omega, \tag{3.9}$$

and the principal value of $\mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega}$ is defined by

$$(\mathbf{K}_{\Omega;\mathrm{Br};\alpha}\mathbf{h})_k(\mathbf{x}) := \mathrm{p.v.}\int_{\partial\Omega}S^\alpha_{jk\ell}(\mathbf{x},\mathbf{y})\nu_\ell(\mathbf{y})h_j(\mathbf{y})d\sigma(\mathbf{y})\ \ \text{a.e. } \mathbf{x}\in\partial\Omega. \tag{3.10}$$

In view of (3.2) and (3.4), $(\mathbf{V}_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{g}, Q^s_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{g})$ and $(\mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{h}, Q^d_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{h})$ satisfy the Brinkman system in $\mathbb{R}^m \setminus \partial\Omega$. In addition, some of the well known properties of layer potentials for the Brinkman system are listed below (see, e.g., [9], [34, Propositions 4.2.5, 4.2.9, Corollary 4.3.2, Theorems 5.3.6, 5.4.1, 5.4.3, 10.5.3] in the case of the Stokes system, and [15, Lemma 3.1] in the case of the Brinkman system):

Lemma 3.1. *Let $\Omega := \Omega_+ \subseteq \mathbb{R}^m$ $(m \geq 2)$ be a bounded open set with Lipschitz boundary $\partial\Omega$. Let $\Omega_- := \mathbb{R}^m \setminus \overline{\Omega}$. Let $\alpha \geq 0$, $q \in (1,\infty)$, $\mathbf{h} \in L^q_1(\partial\Omega,\mathbb{R}^m)$ and $\mathbf{g} \in L^q(\partial\Omega,\mathbb{R}^m)$. Then the following formulas*

$$
\begin{aligned}
&\mathrm{Tr}^+_\Omega\big(\mathbf{V}_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{g}\big) = \mathrm{Tr}^-_\Omega\big(\mathbf{V}_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{g}\big) := \mathcal{V}_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{g},\\
&\mathrm{Tr}^\pm_\Omega(\mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{h}) = \left(\pm\frac{1}{2}\mathbb{I} + \mathbf{K}_{\mathrm{Br};\alpha;\partial\Omega}\right)\mathbf{h},
\end{aligned}
\tag{3.11}
$$

(3.12)

$$
\begin{aligned}
&\partial^0_{\nu;\Omega_\pm}\big(\mathbf{V}_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{g}, Q^s_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{g}\big) = \left(\pm\frac{1}{2}\mathbb{I} - \mathbf{K}'_{\mathrm{Br};\alpha;\partial\Omega}\right)\mathbf{g},\\
&\partial^0_{\nu;\Omega_+}\big(\mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{h}, Q^d_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{h}\big) = \partial^0_{\nu;\Omega_-}\big(\mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{h}, Q^d_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{h}\big) := \mathbf{D}_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{h}
\end{aligned}
$$

hold a.e. on $\partial\Omega$, where $\mathbf{K}'_{\mathrm{Br};\alpha;\partial\Omega}$ is the formal transpose of $\mathbf{K}_{\mathrm{Br};\alpha;\partial\Omega}$. In addition, the following operators are well-defined, linear and bounded

$$
\begin{aligned}
&\mathcal{V}_{\mathrm{Br};\alpha;\partial\Omega} : L^q(\partial\Omega,\mathbb{R}^m) \to L^q_1(\partial\Omega,\mathbb{R}^m)\\
&\mathbf{K}_{\mathrm{Br};\alpha;\partial\Omega} : L^q_1(\partial\Omega,\mathbb{R}^m) \to L^q_1(\partial\Omega,\mathbb{R}^m)\\
&\mathbf{K}'_{\mathrm{Br};\alpha;\partial\Omega} : L^q(\partial\Omega,\mathbb{R}^m) \to L^q(\partial\Omega,\mathbb{R}^m)\\
&\mathbf{D}_{\mathrm{Br};\alpha;\partial\Omega} : L^q_1(\partial\Omega,\mathbb{R}^m) \to L^q(\partial\Omega,\mathbb{R}^m).
\end{aligned}
\tag{3.13}
$$

In the case $\alpha = 0$, which corresponds to the Stokes system, we use the following notations for the layer potential operators $\mathbf{V}_{\partial\Omega} := \mathbf{V}_{\mathrm{Br};0;\partial\Omega}$, $\mathcal{V}_{\partial\Omega} := \mathcal{V}_{\mathrm{Br};0;\partial\Omega}$, $\mathbf{W}_{\partial\Omega} := \mathbf{W}_{\mathrm{Br};0;\partial\Omega}$, $Q^s_{\partial\Omega} := Q^s_{\mathrm{Br};0;\partial\Omega}$, $\mathbf{K}_{\partial\Omega} := \mathbf{K}_{\mathrm{Br};0;\partial\Omega}$ and $\mathbf{D}_{\partial\Omega} := \mathbf{D}_{\mathrm{Br};0;\partial\Omega}$.

The next lemma shows the compactness of the complementary layer potentials for the Stokes and Brinkman operator in L^q-spaces (see [15, Theorem 3.4]).

Lemma 3.2. *Let $\Omega \subseteq \mathbb{R}^m$ $(m \geq 2)$ be an open set with compact Lipschitz boundary $\partial\Omega$. Let $\alpha \geq 0$, $q \in (1,\infty)$. Then the following operators are compact:*

$$
\begin{aligned}
&\mathcal{V}_{\mathrm{Br};\alpha;\partial\Omega} - \mathcal{V}_{\partial\Omega} : L^q(\partial\Omega,\mathbb{R}^m) \to L^q_1(\partial\Omega,\mathbb{R}^m)\\
&\mathbf{K}_{\mathrm{Br};\alpha;\partial\Omega} - \mathbf{K}_{\partial\Omega} : L^q_1(\partial\Omega,\mathbb{R}^m) \to L^q_1(\partial\Omega,\mathbb{R}^m)\\
&\mathbf{K}'_{\mathrm{Br};\alpha;\partial\Omega} - \mathbf{K}'_{\partial\Omega} : L^q(\partial\Omega,\mathbb{R}^m) \to L^q(\partial\Omega,\mathbb{R}^m)\\
&\mathbf{D}_{\mathrm{Br};\alpha;\partial\Omega} - \mathbf{D}_{\partial\Omega} : L^q_1(\partial\Omega,\mathbb{R}^m) \to L^q(\partial\Omega,\mathbb{R}^m).
\end{aligned}
\tag{3.14}
$$

Finally, let us mention the behavior at infinity of the layer potentials for the Brinkman system (see, e.g., [19, Lemma 3.7.3], [42, Chapter 2]):

$$
\begin{matrix}
\left(\mathbf{V}_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{g}\right)(\mathbf{x}) & = O(|\mathbf{x}|^{-m})\\
\left(\mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{h}\right)(\mathbf{x}) & = O(|\mathbf{x}|^{1-m})
\end{matrix}
\quad \text{as } |\mathbf{x}| \to \infty.
\tag{3.15}
$$

In addition, if the density $\mathbf{h} \in L^q_1(\partial\Omega,\mathbb{R}^m)$ satisfies the condition

$$
\int_{\partial\Omega} \langle \mathbf{h}, \nu\rangle d\sigma = 0,
\tag{3.16}
$$

then

$$
\left(\mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{h}\right)(\mathbf{x}) = O(|\mathbf{x}|^{-m}) \;\text{ as } |\mathbf{x}| \to \infty.
\tag{3.17}
$$

Lemma 3.3. *Let $\Omega \subseteq R^m$ be an open set with compact Lipschitz boundary. Let $\alpha > 0$ and $\mathbf{f} \in L^1(\partial\Omega, R^m)$. Let β be a multiindex. Then*[1]

$$\left(\partial^\beta Q^s_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{f}\right)(\mathbf{x}) = O(|\mathbf{x}|^{1-m-|\beta|}) \quad as \quad |\mathbf{x}| \to \infty, \tag{3.18}$$

$$\begin{array}{ll} \left(Q^d_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{f}\right)(\mathbf{x}) = O(|\mathbf{x}|^{2-m}), & m \geq 3, \\ \left(Q^d_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{f}\right)(\mathbf{x}) = O(\ln(|\mathbf{x}|)), & m = 2, \end{array} \quad as \quad |\mathbf{x}| \to \infty. \tag{3.19}$$

If $|\beta| > 0$ then

$$\left(\partial^\beta Q^d_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{f}\right)(\mathbf{x}) = O(|\mathbf{x}|^{2-m-|\beta|}) \quad as \quad |\mathbf{x}| \to \infty. \tag{3.20}$$

Moreover, if the following condition

$$\int_{\partial\Omega} \langle \mathbf{f}, \nu \rangle \, d\sigma = 0 \tag{3.21}$$

holds and β is an arbitrary multiindex, then

$$\left(\partial^\beta Q^d_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{f}\right)(\mathbf{x}) = O(|\mathbf{x}|^{1-m-|\beta|}) \quad as \quad |\mathbf{x}| \to \infty. \tag{3.22}$$

Proof. By an straightforward computation one obtains the relation (3.18).

Further, let us consider the fundamental solution of the Laplace equation

$$h_\Delta(\mathbf{x}) = \begin{cases} -\frac{1}{2\pi} \ln |\mathbf{x}|, & m = 2, \\ \frac{1}{(m-2)\sigma_m} |\mathbf{x}|^{2-m}, & m > 2. \end{cases} \tag{3.23}$$

Taking into account the explicit representation of the fundamental tensor Λ^α_{jk} (see, e.g., [42, Chapter 2]) there exist some constants $c^\gamma_{jk} \in \mathbb{R}$ such that

$$\begin{aligned} Q^d_{\mathrm{Br};\alpha;\partial\Omega}\mathbf{f}(\mathbf{x}) = \int_{\partial\Omega} \Big[& \sum_{|\gamma|=2} \sum_{j,k=1}^m c^\gamma_{jk} \partial^\gamma h_\Delta(\mathbf{x}-\mathbf{y}) f_j(\mathbf{y}) \nu_k(\mathbf{y}) \\ & + \alpha h_\Delta(\mathbf{x}-\mathbf{y}) \langle \mathbf{f}(\mathbf{y}), \nu(\mathbf{y}) \rangle \Big] \, d\sigma(\mathbf{y}). \end{aligned} \tag{3.24}$$

By differentiating (3.24) and by using the relation

$$|\partial^\beta h_\Delta(\mathbf{x}-\mathbf{y})| \leq C_\beta |\mathbf{x}-\mathbf{y}|^{2-m-|\beta|}, \qquad |\beta| > 0$$

we obtain asymptotic formulas (3.19) and (3.20). In addition, if the condition (3.21) is satisfied, then a direct computation gives (3.22). We omit the details for the sake of brevity. □

3.2. **Layer potentials for the Oseen system.** Let $\lambda \in \mathbb{R}$. Let us now refer to the Oseen system

$$\Delta \mathbf{u} - \lambda \partial_1 \mathbf{u} - \nabla p = 0, \ \operatorname{div} \mathbf{u} = 0. \tag{3.25}$$

The fundamental solution $(O^\lambda, \Pi^\lambda_O)$ of such a system vanishing at infinity is well known (see [11], [19, Chapter 2], [28, Corollary 4.2]). Note that the fundamental pressure field of the Oseen system is the same as the fundamental pressure field of the Stokes system, i.e., $\Pi^\lambda_O = \Pi$. In addition, $O^0 = \mathcal{G}$, $O^\lambda_{jk} = O^\lambda_{kj} \in C^\infty(\mathbb{R}^m \setminus \{0\})$. If $\lambda \neq 0$ and β is a multi-index, then we have

$$\partial^\beta O^\lambda_{jk}(\mathbf{x}) = O(|\mathbf{x}|^{(1-m-|\beta|)/2}) \text{ as } |\mathbf{x}| \to \infty. \tag{3.26}$$

[1] ∂^β means the differential operator of order β, $\partial^\beta := \frac{\partial^{|\beta|}}{\partial^{\beta_1}\ldots\partial^{\beta_m}}$, where $\beta = (\beta_1, \ldots, \beta_m)$ and $|\beta| = \beta_1 + \ldots + \beta_m$.

If $|\mathbf{z}| \neq |z_1|$ then

$$\lim_{r\to\infty} |O^\lambda(r\mathbf{z})| r^{(m-1)/2} = 0. \tag{3.27}$$

If $r > 0$ and $q > 1 + \frac{1}{m}$ then

$$|\nabla O^\lambda_{jk}| \in L^q(\mathbb{R}^m \setminus B(0;r)). \tag{3.28}$$

Let us also mention the useful asymptotic formulas

$$|\partial^\beta \left(O^\lambda_{jk}(\mathbf{x}) - \mathcal{G}_{jk}(\mathbf{x})\right)| = O(|\mathbf{x}|^{-|\beta|}) \text{ as } |\mathbf{x}| \to 0,\ m = 3, \tag{3.29}$$

$$\begin{array}{l} |O^\lambda_{jk}(\mathbf{x}) - \mathcal{G}_{jk}(\mathbf{x})| = O(1) \text{ as } |\mathbf{x}| \to 0, \\ |\nabla\big(O^\lambda_{jk}(\mathbf{x}) - \mathcal{G}_{jk}(\mathbf{x})\big)| = O(\ln|\mathbf{x}|) \text{ as } |\mathbf{x}| \to 0, \\ |\partial^\beta\big(O^\lambda_{jk}(\mathbf{x}) - \mathcal{G}_{jk}(\mathbf{x})\big)| = O(|x|^{-|\beta|+1}) \text{ as } |\mathbf{x}| \to 0,\ |\beta| \geq 2, \end{array} \qquad m = 2. \tag{3.30}$$

A straightforward computation yields that

$$O^\lambda(-\mathbf{x}) = O^{-\lambda}(\mathbf{x}). \tag{3.31}$$

Now let $\Omega \subseteq \mathbb{R}^m$ $(m \in \{2,3\})$ be an open set with compact Lipschitz boundary and $\nu = \nu_\Omega$ be the outward unit normal of Ω. Let $q \in (1,\infty)$ and $\mathbf{\Psi} \in L^q(\partial\Omega, \mathbb{R}^m)$. Then the single-layer potential with density $\mathbf{\Psi}$ for the Oseen system is given by

$$\mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi}(\mathbf{x}) := \int_{\partial\Omega} O^\lambda(\mathbf{x}-\mathbf{y})\mathbf{\Psi}(\mathbf{y})\, d\sigma(\mathbf{y}).$$

The pair $(\mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi}, Q^s_{\partial\Omega}\mathbf{\Psi})$ is a solution of the Oseen system (3.25) in $\mathbb{R}^m \setminus \partial\Omega$.

For $\mathbf{y} \in \partial\Omega$ and $\mathbf{x} \in \mathbb{R}^m \setminus \{\mathbf{y}\}$ we consider the matrix type kernel

$$\mathbb{K}^{\mathrm{Os};\lambda;\partial\Omega} := \partial^\lambda_{\nu;\partial\Omega}\left(O^\lambda(\cdot - \mathbf{y}), Q(\cdot - \mathbf{y})\right),$$

with the components

$$\begin{aligned} K^{\mathrm{Os};\lambda;\partial\Omega}_{j,k}(\mathbf{x},\mathbf{y}) = \big\langle \nu(\mathbf{y}), \nabla_{\mathbf{y}} O^\lambda_{jk}(\mathbf{x}-\mathbf{y})\big\rangle + \sum_{i=1}^m \nu_i(\mathbf{y}) \frac{\partial}{\partial y_k} O^\lambda_{ji}(\mathbf{x}-\mathbf{y}) \\ + \nu_k(\mathbf{y})\Pi_j(\mathbf{x}-\mathbf{y}) + \frac{\lambda\nu_1(\mathbf{y})}{2} O^\lambda_{jk}(\mathbf{x}-\mathbf{y}). \end{aligned} \tag{3.32}$$

In addition, we consider the expression

$$\begin{aligned} \Pi^{\mathrm{Os};\lambda;\partial\Omega}_k(\mathbf{x},\mathbf{y}) : \quad = \quad & \langle \nu(\mathbf{y}), \nabla_{\mathbf{y}}\Pi_k(\mathbf{x}-\mathbf{y})\rangle + \sum_{i=1}^m \nu_i(\mathbf{y}) \frac{\partial}{\partial y_k}\Pi_i(\mathbf{x}-\mathbf{y}) \\ & -\lambda\nu_k(\mathbf{y})\Pi_1(\mathbf{x}-\mathbf{y}) + \frac{\lambda\nu_1(\mathbf{y})}{2}\Pi_k(\mathbf{x}-\mathbf{y}). \end{aligned} \tag{3.33}$$

Let $\mathbf{\Psi} \in L^q(\partial\Omega, \mathbb{R}^m)$. Then the Oseen double-layer potential with density $\mathbf{\Psi}$ is defined by

$$(\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi})(\mathbf{x}) := \int_{\partial\Omega} \langle \mathbb{K}^{\mathrm{Os};\lambda;\partial\Omega}(\mathbf{x},\mathbf{y}), \mathbf{\Psi}(\mathbf{y})\rangle d\sigma(\mathbf{y}),\ \mathbf{x} \in \mathbb{R}^m \setminus \partial\Omega,$$

and the corresponding pressure potential is given by

$$(Q^d_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi})(\mathbf{x}) := \int_{\partial\Omega} \langle \Pi^{\mathrm{Os};\lambda;\partial\Omega}(\mathbf{x}-\mathbf{y}), \mathbf{\Psi}(\mathbf{y})\rangle d\sigma(\mathbf{y}),\ \mathbf{x} \in \mathbb{R}^m \setminus \partial\Omega.$$

By an straightforward algebra we obtain that

$$\mathbf{W}_{\mathrm{Os};0;\partial\Omega}\mathbf{\Psi} = \mathbf{W}_{\partial\Omega}\mathbf{\Psi},\ Q^d_{\mathrm{Os};0;\partial\Omega}\mathbf{\Psi} = Q^d_{\partial\Omega}\mathbf{\Psi} = Q^d_{\mathrm{Br};0;\partial\Omega}\mathbf{\Psi}.$$

In addition, the pair $(\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi}, Q^d_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi})$ is a solution of the Oseen system (3.25) in $\mathbb{R}^m \setminus \partial\Omega$ (see, e.g., [28, Proposition 5.7] in the case $\lambda = 1$, while for $\lambda > 0$ the formula follows with similar arguments. In addition, the formula (3.31) shows the result for arbitrary λ).

Now for $\mathbf{x} \in \partial\Omega$ we consider the boundary layer potentials

$$(\mathbf{K}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi})(\mathbf{x}) := \lim_{\epsilon\to 0}\int_{\partial\Omega\setminus B(\mathbf{x},\epsilon)} K^{\Omega,Os,\lambda}(\mathbf{x},\mathbf{y})\mathbf{\Psi}(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{y}),$$

$$(\mathbf{K}'_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi})(\mathbf{x}) := \lim_{\epsilon\to 0}\int_{\partial\Omega\setminus B(\mathbf{x},\epsilon)} K^{\Omega,Os,\lambda}(\mathbf{y},\mathbf{x})\mathbf{\Psi}(\mathbf{y})\, \mathrm{d}\sigma(\mathbf{y}),$$

and note that $\mathbf{K}_{\mathrm{Os};0;\partial\Omega} = \mathbf{K}_{\partial\Omega} = \mathbf{K}_{\mathrm{Br};0;\partial\Omega}$, $\mathbf{K}'_{\mathrm{Os};0;\partial\Omega} = \mathbf{K}'_{\partial\Omega} = \mathbf{K}'_{\mathrm{Br};0;\partial\Omega}$.

Also let $\mathcal{V}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi}$ be the boundary version of the single-layer potential $\mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi}$, i.e., the restriction of $\mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi}$ onto $\partial\Omega$. Note that if $\mathbf{\Psi} \in L^q(\partial\Omega, \mathbb{R}^m)$ then the non-tangential limit of $\mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi}$ coincides with $\mathcal{V}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi}$ a.e. on $\partial\Omega$. In addition, there exists a constant $C \equiv C(\Omega, q) > 0$ such that the non-tangential maximal function $(\mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi})^*$ satisfies the inequality

$$\| \left(\mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi}\right)^* \|_{L^q(\partial\Omega)} \le C\|\mathbf{\Psi}\|_{L^q(\partial\Omega,\mathbb{R}^m)}.$$

Moreover, we have the inequality

$$\|(\nabla \mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi})^*\|_{L^q(\partial\Omega)} \le C\|\mathbf{\Psi}\|_{L^q(\partial\Omega,\mathbb{R}^n)},$$

and $\nabla \mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi}$ has the non-tangential limit at almost all points of $\partial\Omega$, and

$$\partial^\lambda_{\nu;\Omega_\pm}(\mathbf{V}_{Os;\alpha;\partial\Omega}\mathbf{\Psi}, Q^s_{\partial\Omega}\mathbf{\Psi}) = \pm\frac{1}{2}\mathbf{\Psi} - \mathbf{K}'_{\Omega;\mathrm{Os};\lambda}\mathbf{\Psi}. \tag{3.34}$$

In addition, the layer potential operators $\mathcal{V}_{\mathrm{Os};\lambda;\partial\Omega} : L^q(\partial\Omega, \mathbb{R}^m) \to L^q(\partial\Omega, \mathbb{R}^m)$ and $\mathbf{K}_{\mathrm{Os};\lambda;\partial\Omega} : L^q(\partial\Omega, \mathbb{R}^m) \to L^q(\partial\Omega, \mathbb{R}^m)$ are linear and bounded operators (see, e.g., [34] in the case of the Stokes system, and [28, Lemma 5.1, Propositions 5.2, 5.3] in the case of the Oseen system).

In addition, if $\mathbf{\Psi} \in L^q(\partial\Omega, \mathbb{R}^n)$ then $\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi}$ has a non-tangential limit at almost all points of $\partial\Omega$, and there exists a constant $C_0 \equiv C_0(\Omega, q) > 0$, such that

$$\| \left(\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi}\right)^* \|_{L^q(\partial\Omega)} \le C_0\|\mathbf{\Psi}\|_{L^q(\partial\Omega,\mathbb{R}^m)},$$

$$\mathrm{Tr}_{\Omega_\pm}\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi} = \pm\frac{1}{2}\mathbf{\Psi} + \mathbf{K}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi} \tag{3.35}$$

(see, e.g., [28, Proposition 5.8, Lemma 5.3, Lemma 5.4]).

The following result shows the compactness of the complementary double-layer potential operator for the Oseen and Stokes systems.

Proposition 3.4. *Let $\Omega \subseteq \mathbb{R}^m$ $(m \in \{2, 3\})$ be an open set with compact Lipschitz boundary. Let $\lambda \in \mathbb{R}$ and $q \in (1, \infty)$. Then the complementary double-layer potential operator $\mathbf{K}_{\mathrm{Os};\lambda;\partial\Omega} - \mathbf{K}_{\partial\Omega} : L^q(\partial\Omega, \mathbb{R}^m) \to L^q(\partial\Omega, \mathbb{R}^m)$ is compact.*

Proof. The kernels of such an operator is weakly singular, as the formulas (3.29), (3.30) show, and the desired compactness is a direct consequence of this property (see, e.g., [10, §4.5.2, Satz 2]). □

For the problems we are going to investigate below we need the following useful result (see [43, Theorem 1.12]).

Lemma 3.5. *If $\Omega \subseteq \mathbb{R}^m$ $(m \geq 2)$ is an open set with compact Lipschitz boundary, then there exists a sequence of open sets $\{\Omega_j\}_{j\geq 1}$ in $\mathbb{R}^m$ with compact boundaries of class C^∞, such that*

- $\overline{\Omega}_j \subset \Omega$.
- *There exist a constant $a > 0$ and some homeomorphisms $\Lambda_j : \partial\Omega \to \partial\Omega_j$, such that $\Lambda_j(\mathbf{y}) \in \Gamma_a(\mathbf{y})$ for any $\mathbf{y} \in \partial\Omega$ and $j \geq 1$, and, in addition, $\sup\{|\mathbf{y} - \Lambda_j(\mathbf{y})| : \mathbf{y} \in \partial\Omega\} \to 0$ as $j \to \infty$.*
- *There are positive functions ω_j on $\partial\Omega$ bounded away from zero and infinity uniformly in j such that for any measurable set $E \subset \partial\Omega$,*
$$\int_E \omega_j \, \mathrm{d}\sigma = \int_{\Lambda_j(E)} 1 \, \mathrm{d}\sigma,$$
and so that $\omega_j \to 1$ pointwise a.e., and in every space $L^q(\partial\Omega)$, $1 \leq q < \infty$.
- *For any $q \in [1,\infty)$, the normal vector to Ω_j, $\nu(\Lambda_j(\mathbf{y}))$, converges pointwise a.e., and also in every space $L^q(\partial\Omega, \mathbb{R}^m)$, to $\nu(\mathbf{y}) := \nu_{\partial\Omega}(\mathbf{y})$, as $j \to \infty$.*

Proposition 3.6. *Let $\lambda \in \mathbb{R}$ and $m \in \{2,3\}$. Let $\Omega \subseteq \mathbb{R}^m$ be an open set with compact Lipschitz boundary. Then there exists a constant $q_0 \in (2,\infty)$ such that for any $q \in (1, q_0)$ the following properties hold:*

(*a*) *The complementary double-layer potential operator*
$$\mathbf{K}_{\mathrm{Os};\lambda;\partial\Omega} - \mathbf{K}_{\partial\Omega} : L^q_1(\partial\Omega, \mathbb{R}^m) \to L^q_1(\partial\Omega, \mathbb{R}^m)$$
is linear and compact.

(*b*) *For $\mathbf{\Psi} \in L^q_1(\partial\Omega, \mathbb{R}^m)$, there exists the non-tangential limit of $\nabla \mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi}$ at almost all points of $\partial\Omega$, and there exists a constant $C \equiv C(\Omega, q) > 0$ such that*
$$\|(\nabla \mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{\Psi})^*\|_{L^q(\partial\Omega)} \leq C\|\mathbf{\Psi}\|_{L^q(\partial\Omega,\mathbb{R}^m)}. \tag{3.36}$$

(*c*) *The non-tangential limit of $\partial_l\big(\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega} - \mathbf{W}_{\partial\Omega}\big)$ is a linear and compact operator from $L^q_1(\partial\Omega, \mathbb{R}^m)$ to $L^q(\partial\Omega, \mathbb{R}^m)$.*

If $\partial\Omega$ is of class C^1 then all of the above properties $(a) - (c)$ hold for $q_0 = \infty$.

Proof. We can suppose that $\partial\Omega \subset B(0;1)$. Let $\Gamma(1), \ldots, \Gamma(k)$ be all components of $\partial\Omega$. Denote by $\mathcal{S}_{\Gamma(j)}f$ the harmonic single layer potential corresponding to $\Gamma(j)$ and by S_j its restriction to $\Gamma(j)$. It is well known that there exists $q_j > 2$ such that $S_j : L^q(\Gamma(j)) \to L^q_1(\Gamma(j))$ is an isomorphism for $1 < q < q_j$. If $\Gamma(j)$ is of class C^1 then $q(j) = \infty$ (see [13, Theorem 2.2.22]). Let $q_0 = \min\{q_1, \ldots, q_k\}$. Let $\psi_j \in C^\infty(\mathbb{R}^m)$ be compactly supported functions such that $\psi_j = 1$ on a neighborhood of $\Gamma(j)$ and $\psi_j = 0$ on a neighborhood of $\partial\Omega \setminus \Gamma(j)$. Fix $q \in (1, q_0)$ and define
$$A_\Omega f := \sum_{j=1}^k \psi_j \mathcal{S}_{\Gamma(j)} S_j^{-1} f, \quad f \in L^q_1(\partial\Omega).$$
According to properties of a harmonic single layer potential ([13, Theorem 2.2.13]), A_Ω is a linear operator from $L^q_1(\partial\Omega)$ to $C^\infty(\Omega)$, f is the non-tangential limit of $A_\Omega f$ at almost all points of $\partial\Omega$, there exists the non-tangential limit of $\nabla A_\Omega f$ at almost all points of $\partial\Omega$, and there exists a constant $C_1 \equiv C_1(\Omega, q) > 0$ such that
$$\|(A_\Omega f)^*\|_{L^q(\partial\Omega)} + \|(\nabla A_\Omega f)^*\|_{L^q(\partial\Omega)} \leq C_1\|f\|_{L^q_1(\partial\Omega)}.$$
Let Ω_j be a sequence of open sets from Lemma 3.5. For $\mathbf{x} \in \Omega$ choose $\varphi_{\mathbf{x}} \in \mathcal{C}^\infty(\mathbb{R}^m)$ such that $\varphi_{\mathbf{x}} = 0$ at vicinity of $\mathbf{x}$ and $\varphi_{\mathbf{x}} = 1$ on a neighborhood of $\partial\Omega$. In view of the

Lebesgue Lemma and of the Green-Gauss Theorem we obtain for $\mathbf{f} \in L_1^q(\partial\Omega, \mathbb{R}^m)$, with $1 < q < q_0$, that

$$
\begin{aligned}
&\partial_l \mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{f}(\mathbf{x}) = \lim_{j\to\infty} \int_{\partial\Omega_j} \Big(\partial^{\lambda}_{\nu;\partial\Omega_j}(\partial_l O^{\lambda}(\mathbf{x}-\mathbf{y}), \partial_l \Pi(\mathbf{x}-\mathbf{y}))\Big) \varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y})\ d\sigma(\mathbf{y}) \\
&= \lim_{j\to\infty} \Bigg\{ \int_{\Omega_j} \Big(-(\Delta \partial_l O^{\lambda}(\mathbf{x}-\mathbf{y}))(\varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y})) - 2\widehat{\nabla}\partial_l O^{\lambda}(\mathbf{x}-\mathbf{y}) \cdot \widehat{\nabla}(\varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y})) \\
&\quad + (\nabla \partial_l \Pi(\mathbf{x}-\mathbf{y}))(\varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y})) + \partial_l \Pi(\mathbf{x}-\mathbf{y}) \nabla \cdot (\varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y}))\Big) d\mathbf{y} \\
&\quad - \int_{\partial\Omega_j} \frac{\lambda}{2} \nu_1^{\Omega} \partial_l O^{\lambda}(\mathbf{x}-\mathbf{y})] \varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y} d\sigma(\mathbf{y}) \Bigg\} \\
&= \lim_{j\to\infty} \Bigg\{ \int_{\Omega_j} [-2\widehat{\nabla}\partial_l O^{\lambda}(\mathbf{x}-\mathbf{y}) \cdot \widehat{\nabla}(\varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y})) + \partial_l \Pi(\mathbf{x}-\mathbf{y}) \nabla \cdot (\varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y})) \\
&\quad + \lambda(\partial_1 \partial_l O^{\lambda}(\mathbf{x}-\mathbf{y}))(\varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y}))] d\mathbf{y} - \int_{\partial\Omega_j} \frac{\lambda}{2} \nu_1^{\Omega} \partial_l O^{\lambda}(\mathbf{x}-\mathbf{y})] \varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y}) d\sigma(\mathbf{y}) \Bigg\} \\
&= \lim_{j\to\infty} \Bigg\{ \int_{\Omega_j} [2\widehat{\nabla} O^{\lambda}(\mathbf{x}-\mathbf{y}) \cdot \widehat{\nabla}\partial_l(\varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y})) - \Pi(\mathbf{x}-\mathbf{y}) \nabla \cdot \partial_l(\varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y})) \\
&\quad - \lambda(\partial_1 O^{\lambda}(\mathbf{x}-\mathbf{y}))\partial_l(\varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y}))]\ d\mathbf{y} + \int_{\partial\Omega_j} \Big[-\nu_l^{\Omega} 2\widehat{\nabla} O^{\lambda}(\mathbf{x}-\mathbf{y}) \cdot \widehat{\nabla}(\varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y})) \\
&\quad + \nu_l^{\Omega} \Pi(\mathbf{x}-\mathbf{y}) \nabla \cdot (\varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y})) + \lambda(\nu_l^{\Omega} \partial_1 O^{\lambda}(\mathbf{x}-\mathbf{y}))(\varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y})) \\
&\quad - \frac{\lambda}{2} \nu_1^{\Omega} \partial_l O^{\lambda}(\mathbf{x}-\mathbf{y}) \varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y}) \Big]\ d\sigma(\mathbf{y}) \Bigg\} \\
&= \lim_{j\to\infty} \int_{\partial\Omega_j} \Big(-\partial^0_{\nu}(O^{\lambda}(\mathbf{x}-\mathbf{y}), \Pi(\mathbf{x}-\mathbf{y}))\partial_l(\varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y})) \\
&\quad - 2\nu_l^{\Omega} \widehat{\nabla} O^{\lambda}(\mathbf{x}-\mathbf{y}) \cdot \widehat{\nabla}(\varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y})) + \nu_l^{\Omega} \Pi(\mathbf{x}-\mathbf{y}) \nabla \cdot (\varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y})) \\
&\quad + \lambda(\nu_l^{\Omega} \partial_1 O^{\lambda}(\mathbf{x}-\mathbf{y}))(\varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y})) - \frac{\lambda}{2} \nu_1^{\Omega} \partial_l O^{\lambda}(\mathbf{x}-\mathbf{y}) \varphi_{\mathbf{x}}(\mathbf{y}) A_\Omega \mathbf{f}(\mathbf{y}) \Big)\ d\sigma(\mathbf{y}) \\
&= -\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}(\partial_l A_\Omega \mathbf{f})(\mathbf{x}) - \frac{\lambda}{2} \mathbf{V}_{Os;\alpha;\partial\Omega}(\nu_1^{\Omega} \partial_l A_\Omega \mathbf{f})(\mathbf{x}) \\
&\quad + \int_{\partial\Omega} \Big(-2\nu_l^{\Omega} \widehat{\nabla} O^{\lambda}(\mathbf{x}-\mathbf{y}) \cdot \widehat{\nabla}(A_\Omega \mathbf{f}(\mathbf{y})) + \nu_l^{\Omega} \Pi(\mathbf{x}-\mathbf{y}) \nabla \cdot (A_\Omega \mathbf{f}(\mathbf{y})) \Big)\ d\sigma(\mathbf{y}) \\
&\quad + \lambda \partial_1 \mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}(\nu_l f)(\mathbf{x}) - \frac{\lambda}{2} \partial_l \mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}(\nu_1 f)(\mathbf{x}).
\end{aligned}
$$

The last equality and the properties of the Oseen layer potential $\mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}$ and $\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}$ imply (3.36) and the existence of the non-tangential limit of $\partial_l \mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{f}$ at almost all points of $\partial\Omega$.

Let us now denote by $L_l \mathbf{f}$ the non-tangential limit of $\partial_l(\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{f} - \mathbf{W}_{\partial\Omega}\mathbf{f})$, and by $\mathbf{H}_{l,\lambda}\mathbf{f}$ the non-tangential limit of $\partial_l \mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{f}$. Then $\mathbf{H}_{l,\lambda}$ is a linear and bounded operator on $L^q(\partial\Omega, \mathbb{R}^m)$. We have proved the following equality

$$
L_{\ell}(\mathbf{x}) = -\left(\mathbf{K}_{\mathrm{Os};\lambda;\partial\Omega} - \mathbf{K}_{\partial\Omega}\right)(\partial_{\ell} A_\Omega \mathbf{f})(\mathbf{x}) - \frac{\lambda}{2}\left(\mathcal{V}_{Os;\lambda;\partial\Omega} - \mathcal{V}_{\partial\Omega}\right)(\nu_{\ell} \partial_l A_\Omega \mathbf{f})(\mathbf{x})
$$

$$+\int_{\partial\Omega}\left\langle -2\nu_\ell\left(\widehat{\nabla}O^\lambda(\mathbf{x}-\mathbf{y})-\widehat{\nabla}\mathcal{G}(\mathbf{x}-\mathbf{y})\right),\widehat{\nabla}(A_\Omega\mathbf{f}(\mathbf{y}))\right\rangle\, d\sigma(\mathbf{y})$$

$$+\lambda\mathbf{H}_{1,\lambda}(\nu_\ell\mathbf{f})(\mathbf{x})-\frac{\lambda}{2}\mathbf{H}_{\ell,\lambda}(\nu_1\mathbf{f})(\mathbf{x}), \tag{3.37}$$

where $\mathbf{H}_{1,\lambda}$ is a bounded operator on $L^q(\partial\Omega,\mathbb{R}^m)$. The operator $\mathbf{f}\mapsto\nu_1\mathbf{f}$ is compact from $L^q_1(\partial\Omega)$ to $L^q(\partial\Omega)$. Thus, $\lambda\mathbf{H}_{1,\lambda}(\nu_\ell\mathbf{f})$ is compact from $L^q_1(\partial\Omega)$ to $L^q(\partial\Omega)$. Similar arguments imply that the operator $\lambda\mathbf{H}_{l,\lambda}(\nu_1\mathbf{f})$ is also compact. All remaining operators in (3.37) are integral operators with weakly singular kernels as those of $\nabla A_\Omega\mathbf{f}$ (see (3.29), (3.30)). Hence, $L_l : L^q_1(\partial\Omega,\mathbb{R}^m)\to L^q(\partial\Omega,\mathbb{R}^m)$ is compact.

We now turn to prove the compactness of the operator $\mathbf{K}_{\mathrm{Os};\lambda;\partial\Omega}-\mathbf{K}_{\partial\Omega}$. Indeed, such an operator is linear and compact on $L^q(\partial\Omega,\mathbb{R}^m)$ as Proposition 3.4 shows. In addition, note that for any $\mathbf{f}\in L^q_1(\partial\Omega,\mathbb{R}^m)$, $(\mathbf{K}_{\mathrm{Os};\lambda;\partial\Omega}-\mathbf{K}_{;\partial\Omega})\mathbf{f}$ is the non-tangential limit of $\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{f}-\mathbf{W}_{;\partial\Omega}\mathbf{f}$. By considering the tangential derivative operator $\partial_{\tau_{ij}}:=\nu_i\partial_j-\nu_j\partial_i$, we deduce that $\partial_{\tau_{ij}}\left(\mathbf{K}_{\mathrm{Os};\lambda;\partial\Omega}-\mathbf{K}_{\partial\Omega}\right)=\nu_iL_j-\nu_jL_i$ is a compact operator from $L^q_1(\partial\Omega,\mathbb{R}^m)$ to $L^q(\partial\Omega,\mathbb{R}^m)$. Hence, $\partial_\ell\left(\mathbf{K}_{\mathrm{Os};\lambda;\partial\Omega}-\mathbf{K}_{\partial\Omega}\right)$ is a compact operator on $L^q_1(\partial\Omega,\mathbb{R}^m)$, as asserted. □

Remark 3.7. By using a similar argument as above, we can show Proposition 3.6 for any dimension $m\geq 2$. For any $m\geq 2$ and $q_0=\infty$ we can also show this result by using the theory of pseudodifferential operators (see, e.g., [15, Theorem 3.4]).

Next, we mention the main properties of the pressure potential associated to an Oseen double-layer potential.

Proposition 3.8. *Let $\Omega\subseteq\mathbb{R}^m$ $(m\in\{2,3\})$ be an open set with compact Lipschitz boundary. Let $\lambda\in\mathbb{R}$ and $q\in(1,\infty)$. If $\mathbf{f}\in L^q_1(\partial\Omega,\mathbb{R}^m)$ then the layer potential $Q^d_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{f}$ has a non-tangential limit at almost all points of $\partial\Omega$, and there exists a constant $C\equiv C(\Omega,q)>0$ such that*

$$\|(Q^d_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{f})^*\|_{L^q(\partial\Omega)}\leq C\|\mathbf{f}\|_{L^q_1(\partial\Omega,\mathbb{R}^m)}.$$

The non-tangential limit of $Q^d_{\mathrm{Os};\lambda;\partial\Omega}-Q^d_{\mathrm{Os};0;\partial\Omega}$ is a linear and compact operator from $L^q_1(\partial\Omega,\mathbb{R}^m)$ to $L^q(\partial\Omega)$.

Proof. In the case $\lambda=0$ we refer the reader to [34]. In the case $\lambda\neq 0$ we use the following equality

$$Q^d_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{f}-Q^d_{\mathrm{Os};0;\partial\Omega}\mathbf{f}=-\lambda Q^s_{\partial\Omega}\left(\langle\nu,\mathbf{f}\rangle,0,0\right)+\frac{\lambda}{2}Q^s_{\partial\Omega}\nu_1\mathbf{f},$$

which, together with the boundedness of $Q^s_{\partial\Omega}$ and the compactness of the embedding $L^q_1(\partial\Omega,\mathbb{R}^m)\hookrightarrow L^q(\partial\Omega,\mathbb{R}^m)$, implies the compactness of the complementary layer potential operator $\mathrm{Tr}\big(Q^d_{\mathrm{Os};\lambda;\partial\Omega}-Q^d_{\mathrm{Os};0;\partial\Omega}\big):L^q_1(\partial\Omega,\mathbb{R}^m)\to L^q(\partial\Omega)$. □

The next result gives the behaviour at infinity of a bounded solution of the Oseen system (see [28, Theorem 6.8]).

Lemma 3.9. *Let $m\in\{2,3\}$. Assume that $\Omega\subseteq\mathbb{R}^m$ is an open set such that $\mathbb{R}^m\setminus\Omega$ is compact. Let $(\mathbf{u},\pi)$ be a bounded solution of the Oseen system (3.25) in Ω. Then there are two constants $\pi_\infty\in\mathbb{R}$ and $\mathbf{u}_\infty\in\mathbb{R}^m$ such that $\mathbf{u}(\mathbf{x})\to\mathbf{u}_\infty$, $\pi(\mathbf{x})\to\pi_\infty$ as $|\mathbf{x}|\to\infty$. Moreover, if β is a multiindex, then $|\partial^\beta(\mathbf{u}(\mathbf{x})-\mathbf{u}_\infty)|=O(|\mathbf{x}|^{(1-m-|\beta|)/2})$, $|\partial^\beta(\pi(\mathbf{x})-\pi_\infty)|=O(|\mathbf{x}|^{1-m-|\beta|})$ as $|\mathbf{x}|\to\infty$. Moreover, $r^{(m-1)/2}\mathbf{u}(r\mathbf{x})\to 0$ as $r\to\infty$, whenever $|\mathbf{x}|\neq|x_1|$.*

The following result gives the direct layer potential representation of a solution of the Oseen system in an open set with Lipschitz boundary (in the case of the Stokes system we refer the reader to [34, Proposition 4.4.1]).

Proposition 3.10. *Let $\Omega \subseteq \mathbb{R}^m$ be an open set with compact Lipschitz boundary. Let $q \in (1,\infty)$. Assume that the pair $(\mathbf{u},\pi)$ is a solution of the Oseen system (2.5) in Ω, such that the the nontangential maximal functions of $\mathbf{u}$, $\nabla\mathbf{u}$ and π with respect to Ω belong to the space $L^q(\partial\Omega)$ and there exist nontangential limits of $\mathbf{u}$, $\nabla\mathbf{u}$ and π at almost all points of $\partial\Omega$. If Ω is an unbounded set, we assume in addition that $\mathbf{u}(\mathbf{x}) \to 0$ and $\pi(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Then the following Green representation formulas hold*

$$\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\left(\mathrm{Tr}_\Omega\, \mathbf{u}\right) + \mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}\left(\partial^\lambda_{\nu;\Omega}(\mathbf{u},\pi)\right) = \begin{cases} \mathbf{u} & in\ \ \Omega, \\ 0 & in\ \ \mathbb{R}^m\setminus\overline{\Omega}, \end{cases} \tag{3.38}$$

$$Q^d_{\mathrm{Os};\lambda;\partial\Omega}\left(\mathrm{Tr}_\Omega\, \mathbf{u}\right) + Q^s_{\partial\Omega}(\partial^\lambda_{\nu;\Omega}(\mathbf{u},\pi)) = \begin{cases} \pi & in\ \ \Omega, \\ 0 & in\ \ \mathbb{R}^m\setminus\overline{\Omega}. \end{cases} \tag{3.39}$$

Proof. First, we assume that Ω is bounded, and consider some open sets Ω_j as in Lemma 3.5. Then by the Green representation formula for each of the sets Ω_j one obtains the relations (3.38) and (3.39) for such sets (see [11, §VII.6]). Finally, by means of the Lebesgue lemma we obtain the formula (3.38) for the set Ω.

Now we assume that Ω is an unbounded set, and consider the bounded set $B(0;r)\cap\Omega$, for some $r>0$ sufficiently large. Then by using the formulas (3.38) and (3.39) for such a set, by letting $r\to\infty$, and by using the Lebesque lemma, Lemma 3.9 and the properties of the fundamental solution of the Oseen system, we obtain the formulas (3.38) and (3.39) in the case of the set Ω. □

Corollary 3.11. *Let $\Omega := \Omega_+ \subseteq \mathbb{R}^m$ be a bounded open set with Lipschitz boundary. Let $\Omega_- := \mathbb{R}^m\setminus\overline{\Omega}$. For $q\in(1,\infty)$ fixed, assume that the functions $(\mathbf{u}_\pm,\pi_\pm)$ satisfy the Oseen system (3.25) in $\Omega_\pm$, such that $\mathbf{u}_-(\mathbf{x})\to 0$ and $\pi_-(\mathbf{x})\to 0$ as $|\mathbf{x}|\to\infty$, and $\mathbf{u}^*_\pm$, $(\nabla\mathbf{u}_\pm)^*$, $(\pi_\pm)^* \in L^q(\partial\Omega)$. Suppose that there exist the nontangential limits of $\mathbf{u}_\pm$, $\nabla\mathbf{u}_\pm$ and $\pi_\pm$ at almost all points of $\partial\Omega$. Then the following formulas*

$$\mathbf{u}_\pm = \mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\left(\mathrm{Tr}^+_\Omega\mathbf{u}_+ - \mathrm{Tr}^-_\Omega\mathbf{u}_-\right) + \mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}\left(\partial^\lambda_{\nu;\Omega_+}(\mathbf{u}_+,\pi_+) - \partial^\lambda_{\nu;\Omega_-}(\mathbf{u}_-,\pi_-)\right)$$
$$\pi_\pm = Q^d_{\mathrm{Os};\lambda;\partial\Omega}\left(\mathrm{Tr}^+_\Omega\mathbf{u}_+ - \mathrm{Tr}^-_\Omega\mathbf{u}_-\right) + Q^s_{\partial\Omega}\left(\partial^\lambda_{\nu;\Omega_+}(\mathbf{u}_+,\pi_+) - \partial^\lambda_{\nu;\Omega_-}(\mathbf{u}_-,\pi_-)\right)$$

hold in $\Omega_\pm$.

Proof. By using the formulas (3.38) and (3.39) in each of the domains Ω_+ and Ω_-, respectively, and adding them, we obtain the desired relations. □

The next result shows that the conormal derivative of an Oseen double-layer potential is continuous a.e. on $\partial\Omega$ (in the case of the Stokes system we refer the reader to [34, Corollary 4.3.2], and for the Brinkman system to [15, Lemma 3.1]).

Corollary 3.12. *Let $\Omega \subseteq \mathbb{R}^m$ $(m\geq 2)$ be an open set with compact Lipschitz boundary $\partial\Omega$. Let $\lambda\in\mathbb{R}$ and $q_0>2$ be the constant provided by Proposition 3.6. Let $q\in(1,q_0)$ and $\mathbf{h}\in L^q_1(\partial\Omega,\mathbb{R}^n)$. Then the following equality*[2]

$$\partial^\lambda_{\nu;\Omega_+}\left(\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{h}, Q^d_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{h}\right) = \partial^\lambda_{\nu;\Omega_-}\left(\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{h}, Q^d_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{h}\right) := \mathbf{D}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{h}$$

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[2] The conormal derivatives below exist a.e. on $\partial\Omega$ and are understood in the sense of nontangential limits.

holds a.e. on $\partial\Omega$. In addition, the operator

$$\mathbf{D}_{\mathrm{Os};\lambda;\partial\Omega} : L^q_1(\partial\Omega, \mathbb{R}^m) \to L^q(\partial\Omega, \mathbb{R}^m), \tag{3.40}$$

is linear and bounded.

Proof. Let us consider the Oseen double-layer potential $\mathbf{u}_\pm := \mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{h}$ and its associated pressure potential $\pi_\pm := Q^d_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{h}$ in $\Omega_\pm$. Then by the jump formulas (3.35) we obtain that $\mathrm{Tr}_{\Omega_+}\mathbf{u}_+ - \mathrm{Tr}_{\Omega_-}\mathbf{u} = \mathbf{h}$. In addition, by Propositions 3.6 and 3.8, $(\nabla \mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{h})^*$, $(Q^d_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{h})^* \in L^q(\partial\Omega)$. Let us now consider the difference $\mathbf{f} := \partial^\lambda_{\nu;\Omega_+}\left(\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{h}, Q^d_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{h}\right) - \partial^\lambda_{\nu;\Omega_-}\left(\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{h}, Q^d_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{h}\right)$. By Corollary 3.11

$$\begin{aligned} \mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{h} &= \mathbf{u}_\pm = \mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{h} + \mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{f}, \qquad \text{in } \mathbb{R}^m \setminus \partial\Omega. \\ Q^d_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{h} &= \pi_\pm = Q^d_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{h} + Q^s_{\partial\Omega}\mathbf{f} \text{ in } \mathbb{R}^m \setminus \partial\Omega. \end{aligned}$$

Consequently, $\mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{f} = 0$, $Q^s_{\partial\Omega}\mathbf{f} = 0$ in $\mathbb{R}^m \setminus \partial\Omega$. Finally, taking into account these relations and by using the formulas (3.34), we obtain that

$$\mathbf{f} = \partial^\lambda_{\nu;\Omega_+}\left(\mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{f}, Q^s_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{f}\right) - \partial^\lambda_{\nu;\Omega_-}\left(\mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{f}, Q^s_{\mathrm{Os};\lambda;\partial\Omega}\mathbf{f}\right) = 0,$$

and the proof is complete. □

Corollary 3.13. *Let $m \in \{2, 3\}$. Let $\Omega \subseteq \mathbb{R}^m$ be an open set with compact Lipschitz boundary. Then there exists a constant $q_0 \in (2, \infty)$ such that for any $q \in (1, q_0)$ the operators*

$$\Psi \longmapsto \partial^0_{\nu;\Omega}(\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega}\Psi, Q^d_{\mathrm{Os};\lambda;\partial\Omega}\Psi) - \mathbf{D}_{\partial\Omega}\Psi \tag{3.41}$$

$$\mathbf{D}_{\mathrm{Os};\lambda;\partial\Omega} - \mathbf{D}_{\partial\Omega} \tag{3.42}$$

are linear and compact from $L^q_1(\partial\Omega, \mathbb{R}^m)$ to $L^q(\partial\Omega, \mathbb{R}^m)$. If $\partial\Omega$ is of class C^1, then the compactness property of the operators (3.41) and (3.42) holds for $q_0 = \infty$.

Proof. The compactness of the operator (3.41) is a direct consequence of Proposition 3.6 and Proposition 3.8, while the compactness of the operator (3.42) follows from the fact that the single-layer potential operator $\mathcal{V}_{\mathrm{Os};\lambda;\partial\Omega}$ is a compact operator on the space $L^q(\partial\Omega, \mathbb{R}^m)$. □

Lemma 3.14. *Let $m \in \{2, 3\}$. Let $\Omega \subseteq \mathbb{R}^m$ be an open set with compact Lipschitz boundary. Let $\lambda \in \mathbb{R}$, $\lambda \neq 0$, and $q \in (1, \infty)$. Then the complementary single-layer potential operator for the Oseen and Stokes systems*

$$\mathcal{V}_{\mathrm{Os};\lambda;\partial\Omega} - \mathcal{V}_{\partial\Omega} : L^q(\partial\Omega, \mathbb{R}^m) \to L^q_1(\partial\Omega, \mathbb{R}^m)$$

is linear and compact.

Proof. Let us consider the derivatives of the complementary kernel for the Oseen and Stokes single-layer potentials, $\mathbb{R}_k := \partial_k\left(\mathcal{O}^\lambda - \mathcal{G}\right)$, as well as the corresponding layer potential

$$\mathbf{R}_k\mathbf{f}(\mathbf{x}) = \int_{\partial\Omega} \mathbb{R}_k(\mathbf{x}, \mathbf{y})\mathbf{f}(\mathbf{y})d\sigma(\mathbf{y}),\ \mathbf{x} \in \mathbb{R}^m \setminus \partial\Omega.$$

Let $\mathcal{R}_k\mathbf{f}$ be the restriction of $\mathbf{R}_k\mathbf{f}$ onto $\partial\Omega$. It is known that if $\mathbf{f} \in L^q(\partial\Omega, \mathbb{R}^m)$, then $\mathcal{R}_k\mathbf{f}$ is a nontangential limit of $\mathbf{R}^k\mathbf{f}$ a.e. on $\partial\Omega$ (see [28, Proposition 5.2]). In addition, in view of the fact that $\mathbb{R}_k(\mathbf{x}, \mathbf{y}) = O(|\mathbf{x} - \mathbf{y}|^{2-m})$, $O^\lambda(\mathbf{x}, \mathbf{y}) - \mathcal{G}(\mathbf{x}, \mathbf{y}) = O(|\mathbf{x} - \mathbf{y}|^{2-m})$ as $|\mathbf{x} - \mathbf{y}| \to 0$, the operators $\mathcal{R}_k$, $\mathcal{V}_{\mathrm{Os};\lambda;\partial\Omega} - \mathcal{V}_{\partial\Omega}$ are compact on

$L^q(\partial\Omega,\mathbb{R}^m)$ (see, e.g., [10, §4.5.2, Satz 2]). Further, by considering the tangential derivatives $\partial\tau_{ij}=\nu_i\partial_j-\nu_j\partial_i$, we deduce that the equality

$$\partial_{\tau_{ij}}\left(\mathcal{V}_{\mathrm{Os};\lambda;\partial\Omega}-\mathcal{V}_{\partial\Omega}\right)=\nu_i\mathcal{R}_j-\nu_j\mathcal{R}_i,$$

which shows that the operator $\partial_{\tau_{ij}}\left(\mathcal{V}_{\mathrm{Os};\lambda;\partial\Omega}-\mathcal{V}_{\partial\Omega}\right)$ is compact on $L^q(\partial\Omega,\mathbb{R}^m)$. Thus, $\mathcal{V}_{\mathrm{Os};\lambda;\partial\Omega}-\mathcal{V}_{\partial\Omega}:L^q(\partial\Omega,\mathbb{R}^m)\to L^q_1(\partial\Omega,\mathbb{R}^m)$ is compact, as asserted. □

4. Existence and uniqueness for the transmission problem with crack for the Oseen and Brinkman systems

The purpose of this section is to study the solvability of the crack type transmission problem (2.4)-(2.9).

Let us mention the following useful remark. Assume that $(\mathbf{u}_B,\pi_B,\mathbf{u}_O,\pi_O)$ is an L^q-solution of the transmission problem (2.4)–(2.9). Since the relations (2.8) and (2.9) hold on $\partial\omega_B$, we deduce that $\mathbf{f}_{if}\in L^q(S_{if},\mathbb{R}^m)$, $\mathbf{g}_{if}\in L^q_1(S_{if},\mathbb{R}^m)$, $\mathbf{f}_{cB}\in L^q(S_{cB},\mathbb{R}^m)$ and $\mathbf{g}_{cB}\in L^q_1(S_{cB},\mathbb{R}^m)$.

4.1. Uniqueness of the solution of the crack type transmission problem.

As an intermediary step in the analysis of the transmission problem (2.4)-(2.9), we are going to show that a solution of such a problem is not always unique. In order to obtain uniqueness we have to add some additional conditions at infinity. On the other hand, if $m=2$ and Ω_B is unbounded, then the transmission problem is not solvable for all boundary data, and a necessary condition for solvability is required.

We first investigate the behavior of an L^q-solution of the transmission problem (2.4)-(2.9) at infinity (see [29, Proposition 4.1]).

Lemma 4.1. *Let $\alpha\geq 0$. Assume that $\mathbf{u}=(u_1,\ldots,u_m)$ and π are tempered distributions in $\mathbb{R}^m$ $(m\geq 2)$, which satisfy the Brinkman system*

$$\triangle\mathbf{u}-\alpha\mathbf{u}-\nabla\pi=0,\ \mathrm{div}\ \mathbf{u}=0$$

in the sense of distributions in $\mathbb{R}^m$. Then $u_1,\ldots,u_m$ and π are polynomials.

An intermediary result in our analysis is the Green representation formula below of a solution of the Brinkman system (3.1) in $\Omega_B\setminus S_{cB}$.

Proposition 4.2. *Let $q\in(1,\infty)$ and $\alpha>0$. Let $(\mathbf{u},\pi)$ be a solution of the Brinkman system (3.1) in $\Omega_B\setminus S_{cB}$. Assume that*

- (a) *There exist non-tangential limits of $\mathbf{u}$, $\nabla\mathbf{u}$, π with respect to ω_B and $\mathfrak{D}_B:=\Omega_B\setminus\overline{\omega}_B$ respectively, at almost all points of $\partial\omega_B$.*
- (b) *There exist non-tangential limits of $\mathbf{u}$, $\nabla\mathbf{u}$, π at almost all points of $\partial\Omega_B$.*
- (c) *The non-tangential maximal functions of $\mathbf{u}$, $\nabla\mathbf{u}$, π with respect to ω_B belong to $L^q(\partial\omega_B)$, and the non-tangential maximal functions of $\mathbf{u}$, $\nabla\mathbf{u}$ and π with respect to $\mathfrak{D}_B$ belong to $L^q(\partial\mathfrak{D}_B)$.*

If Ω_B is unbounded, we require that $\mathbf{u}(\mathbf{x})\to 0$ and $\pi(\mathbf{x})\to 0$ as $|\mathbf{x}|\to\infty$.

Then the following representation formulas hold

(4.1)
$$\mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega_B}\left(\mathrm{Tr}_{\Omega_B}\mathbf{u}\right)+\mathbf{V}_{\mathrm{Br};\alpha;\partial\Omega_B}\left(\partial^0_{\nu_B;\Omega}(\mathbf{u},\pi)\right)+\mathbf{W}_{\mathrm{Br};\alpha\partial\omega_B}\left(\mathrm{Tr}_{\omega_B}\mathbf{u}-\mathrm{Tr}_{\Omega\setminus\overline{\omega}_B}\mathbf{u}\right)$$
$$+\mathbf{V}_{\mathrm{Br};\alpha;\partial\omega_B}\left(\partial^0_{\nu_B;\omega_B}(\mathbf{u},\pi)-\partial^0_{\nu_B;\Omega\setminus\overline{\omega}_B}(\mathbf{u},\pi)\right)=\begin{cases}\mathbf{u} & in\ \ \Omega_B\setminus S_{cB},\\ 0 & in\ \ \mathbb{R}^m\setminus\overline{\Omega}_B.\end{cases}$$

(4.2)
$$Q^d_{\mathrm{Br};\alpha;\partial\Omega_B}\left(\mathrm{Tr}_{\Omega_B}\mathbf{u}\right)+Q^s_{\mathrm{Br};\alpha;\partial\Omega_B}\left(\partial^0_{\nu_B;\Omega}(\mathbf{u},\pi)\right)+Q^d_{\mathrm{Br};\alpha\partial\omega_B}\left(\mathrm{Tr}_{\omega_B}\mathbf{u}-\mathrm{Tr}_{\Omega\setminus\overline{\omega}_B}\mathbf{u}\right)$$
$$+Q^s_{\mathrm{Br};\alpha;\partial\omega_B}\left(\partial^0_{\nu_B;\omega_B}(\mathbf{u},\pi_B)-\partial^0_{\nu_B;\Omega\setminus\overline{\omega}_B}(\mathbf{u},\pi)\right)=\begin{cases}\pi & in\ \ \Omega_B\setminus \overline{S}_{cB},\\ 0 & in\ \ \mathbb{R}^m\setminus\overline{\Omega}_B.\end{cases}$$

If $m=2$ and Ω_B is unbounded, then

$$\int_{\partial\Omega_B}\langle \mathrm{Tr}_{\Omega_B}\mathbf{u},\nu_B\rangle\, d\sigma+\int_{S_{cB}}\left\langle\left(\mathrm{Tr}_{\omega_B}\mathbf{u}-\mathrm{Tr}_{\Omega_B\setminus\overline{\omega}_B}\mathbf{u}\right),\nu_B\right\rangle d\sigma=0. \tag{4.3}$$

Proof. First we assume that Ω_B is bounded and $S_{cB}=\emptyset$. Let $\{\Omega_j\}_{j\geq 1}$ be a sequence of open sets as in Lemma3.5. For such domains the representation formulas (4.1) and (4.2) hold (see [42, p. 60]). Then by means of the Lebesgue lemma we obtain the formulas (4.1) and (4.2) for Ω_B, as well.

Next, we assume that Ω_B is unbounded and $S_{cB}=\emptyset$. Let $r>0$, sufficiently large, such that $\partial\Omega_B\subset B(0;r)$. Then, by applying the formulas (4.1) and (4.2) in the bounded set $\Omega_B\cap B(0;r)$, we obtain

$$\mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega_B}\left(\mathrm{Tr}_{\Omega_B}\mathbf{u}\right)+\mathbf{V}_{\mathrm{Br};\alpha;\partial\Omega_B}\left(\partial^0_{\nu;\Omega_B}(\mathbf{u},\pi_B)\right)-\mathbf{u}_B$$
$$=-\mathbf{W}_{\mathrm{Br};\alpha;\partial B(0;r)}\left(\mathrm{Tr}_{B(0;r)}\mathbf{u}\right)-\mathbf{V}_{\mathrm{Br};\alpha;B(0;r)}\left(\partial^0_{\nu;B(0;r)}(\mathbf{u},\pi)\right),$$

$$Q^d_{\mathrm{Br};\alpha;\partial\Omega_B}\left(\mathrm{Tr}_{\Omega_B}\mathbf{u}\right)+Q^s_{\mathrm{Br};\alpha;\partial\Omega_B}\left(\partial^0_{\nu;\Omega_B}(\mathbf{u},\pi)\right)-\pi_B$$
$$=-Q^d_{\mathrm{Br};\alpha;\partial B(0;r)}\left(\mathrm{Tr}_{B(0;r)}\mathbf{u}\right)-Q^s_{\mathrm{Br};\alpha;B(0;r)}\left(\partial^0_{\nu;B(0;r)}(\mathbf{u},\pi)\right)$$

in $\Omega_B\cap B(0;r)$. Consequently, if we define the functions

$$\mathbf{v}=\begin{cases}\mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega_B}\left(\mathrm{Tr}_{\Omega_B}\mathbf{u}\right)+\mathbf{V}_{\mathrm{Br};\alpha;\partial\Omega_B}\left(\partial^0_{\nu;\Omega_B}(\mathbf{u},\pi)\right)-\mathbf{u} & \text{in }\ \Omega_B\\ -\mathbf{W}_{\mathrm{Br};\alpha;\partial B(0;r)}\left(\mathrm{Tr}_{B(0;r)}\mathbf{u}\right)-\mathbf{V}_{\mathrm{Br};\alpha;B(0;r)}\left(\partial^0_{\nu;B(0;r)}(\mathbf{u},\pi)\right) & \text{in }\ \mathbb{R}^m\setminus\Omega_B\end{cases}$$

$$p=\begin{cases}Q^d_{\mathrm{Br};\alpha;\partial\Omega_B}\left(\mathrm{Tr}_{\Omega_B}\mathbf{u}\right)+Q^s_{\mathrm{Br};\alpha;\partial\Omega_B}\left(\partial^0_{\nu;\Omega_B}(\mathbf{u},\pi)\right)-\pi & \text{in }\ \Omega_B\\ -Q^d_{\mathrm{Br};\alpha;\partial B(0;r)}\left(\mathrm{Tr}_{B(0;r)}\mathbf{u}\right)-Q^s_{\mathrm{Br};\alpha;B(0;r)}\left(\partial^0_{\nu;B(0;r)}(\mathbf{u},\pi)\right) & \text{in }\ \mathbb{R}^m\setminus\Omega_B,\end{cases}$$

then $(\mathbf{v},p)$ is a solution of the Brinkman system (3.1) in $\mathbb{R}^m$. Since $\mathbf{v}(\mathbf{x})\to 0$, $p(\mathbf{x})=O(\ln|\mathbf{x}|)$ as $|\mathbf{x}|\to\infty$, the entries $v_1,\ldots,v_m$, p are tempered distributions. Then by Lemma 4.1 $v_1,\ldots,v_m$ and p are polynomials. Since $\mathbf{v}(\mathbf{x})\to 0$ we deduce that $\mathbf{v}=0$ in $\mathbb{R}^m$, and hence $\nabla p=0$. Therefore, there exists a constant $c\in\mathbb{R}$ such that $p=c$ in $\mathbb{R}^m$.

Now, we use the explicit representation of the layer potential $Q^d_{\mathrm{Br};\alpha;\partial\Omega_B}\left(\mathrm{Tr}_{\Omega_B}\mathbf{u}_B\right)$ (see [42, Chapter 2]). In the case $m=3$, such a representation implies that $p(\mathbf{x})\to 0$ as $|\mathbf{x}|\to\infty$. Hence, $c=0$, i.e., $p=0$ in $\mathbb{R}^3$. Let us now assume that $m=2$. Then the above mentioned representation implies that

$$p(x)-\frac{\lambda}{2\pi}\left(\ln\frac{1}{|\mathbf{x}|}\right)\int_{\partial\Omega_B}\langle\mathrm{Tr}_{\Omega_B}\mathbf{u}_B,\nu_B\rangle d\sigma\to 0\ \text{ as }\ |\mathbf{x}|\to\infty.$$

Since p is bounded at infinity, we deduce that the relation (4.3) holds. Thus, $p(x)\to 0$ as $|\mathbf{x}|\to\infty$. Finally, the property that p is a constant implies that $p=0$ in $\mathbb{R}^2$.

Let us now assume that $S_{cB}\neq\emptyset$. Recall that $\nu_B=\nu_{\Omega_B}$ on $\partial\Omega_B$, and $\nu_B=\nu_{\omega_B}=-\nu_{\Omega_B\setminus\overline{\omega_B}}$ on $\partial\omega_B$. We use the formulas (4.1) and (4.2) for the sets ω_B and

$\Omega\setminus\overline{\omega}_B$. Adding them we obtain the formulas (4.1) and (4.2). In addition, if $m=2$ and Ω_B is unbounded, we have proved that

$$\int_{\partial(\Omega_B\setminus\overline{\omega}_B)}\langle \mathrm{Tr}_{\Omega_B\setminus\overline{\omega}_B}\mathbf{u},\nu_{\Omega_B\setminus\overline{\omega_B}}\rangle d\sigma=0, \tag{4.4}$$

while the divergence theorem implies that

$$\int_{\partial\omega_B}\langle \mathrm{Tr}_{\omega_B}\mathbf{u},\nu_B\rangle\ \mathrm{d}\sigma=0. \tag{4.5}$$

Finally, by adding the relations (4.4) and (4.5) and taking in mind that $\mathbf{u}$ is continuous on $\partial\omega_B\setminus S_{cB}$, we obtain the property (4.3), and the proof is complete. $\square$

A useful result in our analysis is given below[3].

Lemma 4.3. *Let $\alpha>0$ be given. Then the following formulas hold in $\mathbb{R}^m$ $(m\geq 2)$*

$$\mathcal{G}^\alpha(\mathbf{x})=O(|\mathbf{x}|^{-m})\quad as\quad |\mathbf{x}|\to\infty, \tag{4.6}$$

$$\nabla\mathcal{G}^\alpha(\mathbf{x})=\begin{cases} O(|\mathbf{x}|^{1-m}), & m>2,\\ O(|\mathbf{x}|^{-m}), & m=2,\end{cases}\quad as\quad |\mathbf{x}|\to\infty. \tag{4.7}$$

Proof. The asymptotic formula (4.6) is a direct consequence of the expression of the fundamental tensor $\mathcal{G}^\alpha$ (see, e.g., [42, (2.14), Lemma 2.11].

Next we turn to show the formula (4.7). In the case $m>2$, such a formula follows from [29, Proposition 4.2].

Let us now consider the case $m=2$. Denote by $\widehat{g}$ the Fourier transformation of g. Then by applying the Fourier transformation to the system (3.2) we obtain that

$$(|\mathbf{y}|^2+\alpha)\widehat{\mathcal{G}}^\alpha_{jk}(\mathbf{y})+iy_j\widehat{\Pi}_k(\mathbf{y})=\delta_{jk},\ j,k=1,\dots,m, \tag{4.8}$$

$$y_1\widehat{\mathcal{G}}^\alpha_{1k}(\mathbf{y})+\cdots+y_m\widehat{\mathcal{G}}^\alpha_{mk}(\mathbf{y})=0. \tag{4.9}$$

By (4.8) we deduce that

$$\|(|\mathbf{y}|^2+\alpha)(\widehat{\mathcal{G}}^\alpha_{1k}(\mathbf{y}),\dots,\widehat{\mathcal{G}}^\alpha_{mk}(\mathbf{y}))+i\mathbf{y}\widehat{\Pi}_k(\mathbf{y})\|=1,\ k=1,\dots,m, \tag{4.10}$$

where $\|\cdot\|$ denotes the usual norm in $\mathbb{C}^m$. In addition, the divergence type equation (4.9) shows that the inner product in $\mathbb{C}^m$ of the vectors $(|\mathbf{y}|^2+\alpha)(\widehat{\mathcal{G}}^\alpha_{1k}(\mathbf{y}),\dots,\widehat{\mathcal{G}}^\alpha_{mk}(\mathbf{y}))$ and $i\mathbf{y}\widehat{\Pi}_k(\mathbf{y})$ is equal to zero. Consequently, the relation (4.10) implies that

$$\|(|\mathbf{y}|^2+\alpha)(\widehat{\mathcal{G}}^\alpha_{1k}(\mathbf{y}),\dots,\widehat{\mathcal{G}}^\alpha_{mk}(\mathbf{y}))\|\leq 1,$$

and hence

$$|\widehat{\mathcal{G}}^\alpha_{jk}(\mathbf{y})|\leq\frac{1}{|\mathbf{y}|^2+\alpha}. \tag{4.11}$$

In the case $\alpha=0$ we obtain the relations

$$|\mathbf{y}|^2\widehat{\mathcal{G}}_{jk}(\mathbf{y})+iy_j\widehat{\Pi}_k(\mathbf{y})=\delta_{jk},\ j,k=1,\dots,m, \tag{4.12}$$

$$|\widehat{\mathcal{G}}_{jk}(\mathbf{y})|\leq\frac{1}{|\mathbf{y}|^2}. \tag{4.13}$$

In addition, by a straightforward computation based on the relations (4.8) and (4.12) we obtain that

$$\widehat{\mathcal{G}}^\alpha_{jk}(\mathbf{y})-\widehat{\mathcal{G}}_{jk}(\mathbf{y})=-\frac{\alpha}{|\mathbf{y}|^2+\alpha}\widehat{\mathcal{G}}_{jk}(\mathbf{y}).$$

[3]Recall that $\mathcal{G}^\alpha$ is the fundamental tensor of the Brinkman system in $\mathbb{R}^m$ $(m\geq 2)$.

In addition, the relations (4.11), (4.13) imply the inequality

$$|\widehat{\mathcal{G}}^{\alpha}_{jk}(\mathbf{y})-\widehat{\mathcal{G}}_{jk}(\mathbf{y})|\leq \min\left\{\frac{2}{|\mathbf{y}|^2},\frac{\alpha}{|\mathbf{y}|^2(|\mathbf{y}|^2+\alpha)}\right\}.$$

Therefore, in the case $m=2$ and for any $p\in(1,2)$, we obtain that

$$y_\ell\left(\widehat{\mathcal{G}}^{\alpha}_{jk}-\widehat{\mathcal{G}}_{jk}\right)\in L^p(\mathbb{R}^2),\ y_s y_\ell\left(\widehat{\mathcal{G}}^{\alpha}_{jk}-\widehat{\mathcal{G}}_{jk}\right)\in L^p(\mathbb{R}^2),\ \ell,s=1,2. \tag{4.14}$$

Let $p'>2$ such that $\frac{1}{p}+\frac{1}{p'}=1$. Then the relations (4.14) imply that

$$\partial_\ell(\mathcal{G}^{\alpha}_{jk}-\mathcal{G}_{jk})\in L^{p'}(\mathbb{R}^2),\ \partial_s\partial_\ell(\mathcal{G}^{\alpha}_{jk}-\mathcal{G}_{jk})\in L^{p'}(\mathbb{R}^2) \tag{4.15}$$

(see, e.g., [41, Theorem 1.18.8]).

Now we consider a function $\varphi\in C^\infty(\mathbb{R}^2)$ such that $\varphi=0$ in a small neighborhood of the origin, and $\varphi(\mathbf{x})=1$ for $|\mathbf{x}|>1$. Since $p'>2$ and $\partial_\ell\mathcal{G}(\mathbf{x})=O(|\mathbf{x}|^{-1})$, $\partial_s\partial_\ell\mathcal{G}(\mathbf{x})=O(|\mathbf{x}|^{-2})$ as $|\mathbf{x}|\to\infty$, we deduce that $\varphi\partial_\ell\mathcal{G}^{\alpha}_{jk}\in L_1^{p'}(\mathbb{R}^2)$. Since $p'>2$, [40, Theorem 6.7] shows that $\partial_\ell\mathcal{G}^{\alpha}_{jk}(\mathbf{x})\to 0$ as $|\mathbf{x}|\to\infty$. On the other hand, by considering for each $k=1,2$ the functions $\mathbf{u}:=(\partial_\ell\mathcal{G}^{\alpha}_{1k},\partial_\ell\mathcal{G}^{\alpha}_{2k})$ and $\pi:=\Pi_k$, we deduce that $(\mathbf{u},\pi)$ is a solution of the Brinkman system (3.1) in the exterior of the unit ball $B_1^c:=\{\mathbf{x}\in\mathbb{R}^2:|\mathbf{x}|>1\}$, which vanishes at infinity. Then by applying Proposition 4.2 to such a solution, we obtain the Green representation formula

$$\mathbf{u}=\mathbf{W}_{\mathrm{Br};\alpha;\partial B_1^c}(\mathrm{Tr}_{B_1^c}\mathbf{u})+\mathbf{V}_{\mathrm{Br};\alpha;\partial B_1^c}\left(\partial^0_{\nu;B_1^c}(\mathbf{u},\pi)\right)\ \text{ in }\ B_1^c, \tag{4.16}$$

and

$$\int_{|\mathbf{x}|=1}\langle\mathrm{Tr}_{B_1^c}\mathbf{u},\nu\rangle d\sigma=0.$$

Finally, by (4.16) and the behavior of the single- and double-layer potentials of the Brinkman system at infinity we obtain the asymptotic formula $\mathbf{u}(\mathbf{x})=O(|\mathbf{x}|^{-2})$ as $|\mathbf{x}|\to\infty$, and the proof is complete. □

The next result shows that any bounded solution of the Brinkman system in the exterior of a compact set in $\mathbb{R}^m$ vanishes at infinity (up to a constant pressure).

Proposition 4.4. *Let $\mathfrak{D}\subseteq\mathbb{R}^m$ $(m\geq 2)$ be a compact set and $\alpha>0$. Let $(\mathbf{u},\pi)$ be a bounded solution of the Brinkman system (3.1) in $\mathbb{R}^m\setminus\mathfrak{D}$. Let β be a multiindex. Then there exists a constant π_∞ such that*

$$\mathbf{u}(\mathbf{x})\to 0,\ \pi(\mathbf{x})\to\pi_\infty \text{ as } |\mathbf{x}|\to\infty,\quad m\geq 2,$$

$$\left.\begin{array}{ll}|\partial^\beta(\pi(\mathbf{x})-\pi_\infty)|=O(|\mathbf{x}|^{2-m-|\beta|}),\ |\partial^\beta\mathbf{u}(\mathbf{x})|=O(|\mathbf{x}|^{1-m}), & m>2,\\ |\partial^\beta(\pi(\mathbf{x})-\pi_\infty)|=O(|\mathbf{x}|^{1-m-|\beta|}),\ |\partial^\beta\mathbf{u}(\mathbf{x})|=O(|\mathbf{x}|^{-m}), & m=2,\end{array}\right.\ \text{as}\ \ |\mathbf{x}|\to\infty.$$

Proof. Let $\varphi\in C^\infty(\mathbb{R}^m)$ such that $\varphi=0$ in a neighborhood of $\mathfrak{D}$ and $\varphi=1$ outside a compact set which contains $\mathfrak{D}$. Also let

$$\mathbf{f}:=-[\triangle(\varphi\mathbf{u})-\alpha\varphi\mathbf{u}-\nabla(\varphi\pi)],\quad g:=\operatorname{div}\mathbf{u}. \tag{4.17}$$

Let $\tilde{\mathbb{E}}$ be an $(m+1)\times(m+1)$ matrix type function with the entries

$$\begin{aligned}&\tilde{E}_{jk}:=\mathcal{G}^{\alpha}_{jk},\quad j,k=1,\dots m,\\ &\tilde{E}_{j,m+1}=\tilde{E}_{m+1,j}=\Pi_j,\quad j=1,\dots,m,\\ &\tilde{E}_{m+1,m+1}(\mathbf{x})=\delta_0(\mathbf{x})+\frac{\alpha}{\sigma_m}\left\{\begin{array}{ll}\ln\frac{1}{|\mathbf{x}|}, & m=2,\\ \frac{|\mathbf{x}|^{2-m}}{m-2}, & m>2.\end{array}\right.\end{aligned}$$

Now, we define the fields

$$(\mathbf{v}(\mathbf{x}), p(\mathbf{x}))^\top := \int_{\mathbb{R}^m} \tilde{\mathbb{E}}(\mathbf{x}-\mathbf{y})(\mathbf{f}, g)^\top d\mathbf{y}. \tag{4.18}$$

Since $\tilde{\mathbb{E}}$ is a fundamental solution of the Brinkman system $-[\triangle \mathbf{w} - \alpha \mathbf{w} - \nabla r] = \mathbf{F}$, div $\mathbf{w} = \mathbf{G}$ by [42, §2.1], and $\mathbf{v}$ is a Newtonian potential for the Brinkman system in $\mathbb{R}^m$, we deduce that

$$\mathbf{f} = -[\triangle \mathbf{v} - \alpha \mathbf{v} - \nabla p], \quad g = \operatorname{div} \mathbf{v}. \tag{4.19}$$

By (4.17) and (4.19) the pair $(\mathbf{w}, \rho) := (\mathbf{v} - \varphi \mathbf{u}, p - \varphi \pi)$ is a solution of the Brinkman system (3.1) in $\mathbb{R}^m$. Then by Lemma 4.1 all entries $w_1, \ldots, w_m$ and ρ are polynomials. In addition, since $(\mathbf{w}(\mathbf{x}), \rho(\mathbf{x})) = o(|\mathbf{x}|)$ as $|\mathbf{x}| \to \infty$ by Lemma 4.3, there exist constants $\mathbf{w}_\infty \in \mathbb{R}^m$ and $\pi_\infty \in \mathbb{R}$ such that $\mathbf{w} = \mathbf{w}_\infty$, $\rho = \pi_\infty$ in $\mathbb{R}^m$. But $0 = \triangle \mathbf{w} - \alpha \mathbf{w} - \nabla \rho = -\alpha \mathbf{w}_\infty$, and hence $\mathbf{w}_\infty = 0$. On the other hand, the boundedness of ρ at infinity yields that

$$\int_{\mathbb{R}^m} g \; d\mathbf{y} = 0.$$

Then, by using again Lemma 4.3, we obtain that

$$\lim_{|\mathbf{x}| \to \infty} \mathbf{v}(\mathbf{x}) = 0, \quad \lim_{|\mathbf{x}| \to \infty} \nabla \mathbf{v}(\mathbf{x}) = 0, \quad \lim_{|\mathbf{x}| \to \infty} p(\mathbf{y}) = 0. \tag{4.20}$$

Now, by means of the relations (4.20), $\mathbf{w} = 0$ and $\rho = \pi_\infty$ in $\mathbb{R}^n$, and $(\mathbf{w}, \rho) = (\mathbf{v} - \varphi \mathbf{u}, p - \varphi \pi)$, as well as the fact that $\varphi = 1$ outside a bounded set, we deduce that

$$\lim_{|\mathbf{x}| \to \infty} \mathbf{u}(\mathbf{x}) = 0, \quad \lim_{|\mathbf{x}| \to \infty} \nabla \mathbf{u}(\mathbf{x}) = 0, \quad \lim_{|\mathbf{x}| \to \infty} \pi(\mathbf{x}) = \pi_\infty.$$

Moreover, Lemma 4.3 implies that

$$\begin{array}{r} \mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^{1-m}), \ \nabla \mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^{-m}) \ \text{ for } m = 2, \\ \nabla \mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^{1-m}) \ \text{ for } m > 2 \end{array} \quad \text{as } \ |\mathbf{x}| \to \infty.$$

Let $r > 0$ sufficiently large. Let $\Omega_r := \{\mathbf{y} \in \mathbb{R}^m : |\mathbf{y}| > r\}$. By Proposition 4.2, $\mathbf{u}$ and π can be written as

$$\mathbf{u} = \mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega_r}\left(\mathrm{Tr}_{\Omega_r} \mathbf{u}\right) + \mathbf{V}_{\mathrm{Br};\alpha;\partial\Omega_r}\left(\partial^0_{\nu;\Omega_r}(\mathbf{u}, \pi)\right) \quad \text{in } \ \Omega_r, \tag{4.21}$$

$$\pi = Q^d_{\mathrm{Br};\alpha;\partial\Omega_r}\left(\mathrm{Tr}_{\Omega_r} \mathbf{u}\right) + Q^s_{\mathrm{Br};\alpha;\partial\Omega_r}\left(\partial^0_{\nu;\Omega_r}(\mathbf{u}, \pi)\right) \quad \text{in } \ \Omega_r. \tag{4.22}$$

In addition, if $m = 2$ then

$$\int_{\partial\Omega_r} \langle \mathrm{Tr}_{\Omega_r} \mathbf{u}, \nu \rangle \; \mathrm{d}\sigma = 0. \tag{4.23}$$

Finally, by Lemma 3.3 and the formula (4.22), we obtain that

$$\partial^\beta \pi(\mathbf{x}) = O(|\mathbf{x}|^{2-m-|\beta|}) \ \text{ as } \ |\mathbf{x}| \to \infty, \ m > 2.$$

However, if $m = 2$ then Lemma 3.3 and the formulas (4.21) and (4.22) imply that

$$\partial^\beta \pi(\mathbf{x}) = O(|\mathbf{x}|^{1-m-|\beta|}), \ \mathbf{u}(\mathbf{x}) = O(|\mathbf{x}|^{-m}) \ \text{ as } \ |\mathbf{x}| \to \infty.$$

The remaining part of Proposition 4.4 can be obtained by an induction argument, which we omit for the sake of brevity. □

Next we provide the Green formula for an L^2–solution of the Brinkman system and of the Oseen system, respectively.

Lemma 4.5. *Let $\Omega \subseteq \mathbb{R}^m$ $(m \geq 2)$ be an open set with compact Lipschitz boundary and the exterior unit normal ν. Let $(\mathbf{u}, \pi)$ be a solution of the Brinkman system (3.1) in Ω, such that $\mathbf{u}^*, (\nabla \mathbf{u})^*, \pi^* \in L^2(\partial\Omega)$ and there exist the nontangential limits of $\mathbf{u}$, $\nabla \mathbf{u}$ and π at almost all points of $\partial\Omega$. In addition, if Ω is unbounded, we assume that $\mathbf{u}(\mathbf{x}) \to 0$, $\pi(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Then $(\mathbf{u}, \pi)$ satisfies the following Green formula*

$$\int_{\partial\Omega} \left\langle \mathrm{Tr}_\Omega \mathbf{u}, \partial^0_{\nu;\Omega}(\mathbf{u}, \pi) \right\rangle d\sigma = \int_\Omega \left\{ 2|\hat{\nabla}\mathbf{u}|^2 + \alpha|\mathbf{u}|^2 \right\} d\mathbf{x}. \tag{4.24}$$

Proof. First we assume that Ω is bounded. Let $\{\Omega_j\}_{j\geq 1}$ be a sequence of open sets as in Lemma 3.5. For each Ω_j the Green formula (4.24) holds. Then by means of the Lebesgue Lemma we obtain the formula (4.24) for Ω, as well.

Now we assume that Ω is unbounded. Let $r > 0$ be sufficiently large. Then the formula (4.24) holds for the bonded set $\Omega_r := \Omega \cap B(0; r)$. Finally, letting $r \to \infty$ in such a formula and by means of Proposition 4.4 we also get the Green formula (4.24) for the unbounded set Ω, and the proof is complete. □

Lemma 4.6. *Let $\Omega \subseteq \mathbb{R}^m$ $(m \geq 2)$ be an open set with compact Lipschitz boundary. Let ν be the exterior unit normal to $\partial\Omega$. Let $\lambda \in \mathbb{R}$. Let $(\mathbf{u}, \pi)$ be a solution of the Oseen system (3.25) in Ω, such that $\mathbf{u}^*, (\nabla \mathbf{u})^*, \pi^* \in L^2(\partial\Omega)$ and there exist the nontangential limits of $\mathbf{u}$, $\nabla \mathbf{u}$ and π at almost all points of $\partial\Omega$. In addition, if Ω is unbounded, we assume that $\mathbf{u}(\mathbf{x}) \to 0$, $\pi(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Then $(\mathbf{u}, \pi)$ satisfies the following Green formula*

$$\int_{\partial\Omega} \left\langle \mathrm{Tr}_\Omega \mathbf{u}, \partial^\lambda_{\nu;\Omega}(\mathbf{u}, \pi) \right\rangle d\sigma = 2\int_\Omega |\hat{\nabla}\mathbf{u}|^2 d\mathbf{x}. \tag{4.25}$$

Proof. The formula (4.25) follows by means of Lemma 3.9 and [28, Lemma 6.2]. □

Next we show the uniqueness of an L^q-solution of the crack type transmission problem (2.5)-(2.9).

Proposition 4.7. *Let $q \in (1, \infty)$. Let $\big((\mathbf{u}_B, \pi_B), (\mathbf{u}_O \pi_O)\big)$ be an L^q-solution of the homogeneous crack type transmission problem (2.4) − (2.9). If Ω_B is unbounded, we assume that $\mathbf{u}_B(\mathbf{x}) \to 0$, $\pi_B(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. If Ω_O is unbounded, we assume that $\mathbf{u}_O(\mathbf{x}) \to 0$, $\pi_O(\mathbf{x}) \to 0$ as $|\mathbf{x}| \to \infty$. Then $\mathbf{u}_B \equiv \mathbf{0}$ and $\pi_B \equiv 0$ in $\Omega_B \setminus S_{cB}$, $\mathbf{u}_O \equiv \mathbf{0}$ and $\pi_O \equiv 0$ in Ω_O.*

Proof. First, we consider the case $q \geq 2$. In view of the fact that $((\mathbf{u}_B, \pi_B), (\mathbf{u}_O \pi_O))$ is a solution of the homogeneous transmission problem (2.5) − (2.9), we obtain the boundary conditions

$$\mathrm{Tr}_{\Omega_B} \mathbf{u}_B = \mathrm{Tr}_{\Omega_O} \mathbf{u}_O \text{ on } S_{if}, \;\; \mathrm{Tr}_{\omega_B} \mathbf{u}_B = \mathrm{Tr}_{\Omega_B \setminus \overline{\omega}_B} \mathbf{u}_B = \mathbf{u}_B \text{ on } \partial\omega_B.$$

Remember that ν_B is the exterior unit normal to Ω_B on S_{if}, and the exterior unit normal to ω_B on $\partial\omega_B$. Then by applying Lemma 4.6 for $(\mathbf{u}_O, \pi_O)$ in Ω_O, and Lemma 4.5 for $(\mathbf{u}_B, \pi_B)$ in ω_B and $\Omega_B \setminus \overline{\omega}_B$, respectively, and adding the resulting formulas, we obtain

$$0 = \int_{\partial\Omega_O} \langle \mathrm{Tr}_{\Omega_O} \mathbf{u}_O, \partial^0_{\nu_B;\Omega_B}(\mathbf{u}_B, \pi_B) - c_O \partial^\lambda_{\nu_B;\Omega_O}(\mathbf{u}_O, \pi_O) + h_B \mathrm{Tr}_{\Omega_B} \mathbf{u}_B + h_O \mathrm{Tr}_{\Omega_O} \mathbf{u}_O \rangle \, d\sigma$$
$$+ \int_{\partial\omega_B} \left\langle \mathrm{Tr}_{\omega_B} \mathbf{u}_B, \partial^0_{\nu_B;\omega_B}(\mathbf{u}_B, \pi_B) - \partial^0_{\nu_B;\Omega_B \setminus \overline{\omega}_B}(\mathbf{u}_B, \pi_B) + (h_+ + h_-) \mathrm{Tr}_{\omega_B} \mathbf{u}_B \right\rangle d\sigma$$

$$=c_O\int_{\Omega_O} 2|\hat\nabla \mathbf{u}_O|^2\,\mathrm{d}\mathbf{x}+\int_{S_{if}}\langle (h_B+h_O)\mathrm{Tr}_{\Omega_O}\mathbf{u}_O,\mathrm{Tr}_{\Omega_O}\mathbf{u}_O\rangle\,d\sigma$$
$$+\int_{\Omega_B\setminus S_{cB}}\left(2|\hat\nabla \mathbf{u}_B|^2+\alpha|\mathbf{u}_B|^2\right)d\mathbf{x}+\int_{S_{cB}}\langle (h_++h_-)\mathrm{Tr}_{\omega_B}\mathbf{u}_B,\mathrm{Tr}_{\omega_B}\mathbf{u}_B\rangle\,d\sigma.$$

Therefore $\mathbf{u}_B=0$ in Ω_B and $\hat\nabla\mathbf{u}_O=0$ in Ω_O. The second relation implies that $\mathbf{u}_O$ is linear on each component of Ω_O (see [27, Lemma 3.1]), and hence $\mathbf{u}_O$ is a harmonic function. Then by the transmission condition $\mathrm{Tr}_{\Omega_O}\mathbf{u}_O=\mathrm{Tr}_{\Omega_B}\mathbf{u}_B=0$ on $\partial\Omega_O$. If Ω_O is unbounded then $\mathbf{u}_O(\mathbf{x})\to 0$ as $|\mathbf{x}|\to\infty$. By the maximum principle we obtain that $\mathbf{u}_O=0$ in Ω_O. In addition, the Brinkman and Oseen equations imply that the functions π_B and π_O are locally constant in Ω_B and Ω_O, respectively. Since $\mathbf{u}_B=0$, the crack type condition (2.9) implies that π_B has no jump on the crack S_{cB}, and the transmission condition (2.7) yields that $\pi_B=c_O\pi_O$ on the interface S_{if}. If Ω_O (Ω_B) is unbounded then $\pi_O(\mathbf{x})\to 0$ ($\pi_B(\mathbf{x})\to 0$) as $|\mathbf{x}|\to\infty$, respectively. Consequently, we deduce that $\pi_B=0$ in Ω_B and $\pi_O=0$ in Ω_O.

Let now $1<q\le 2$. It is sufficient to prove that $\big((\mathbf{u}_B,\pi_B),(\mathbf{u}_O\pi_O)\big)$ is an L^2-solution of the homogeneous crack type transmission problem (2.4) – (2.9). Denote

$$\mathbf{g}:=\mathbf{u}_O,\qquad \mathbf{f}=\partial^\lambda_{-\nu_B}(\mathbf{u}_O,\pi_O)\qquad \text{on }\partial\Omega_O,$$
$$\mathbf{\Phi}:=[\mathbf{u}_B]_+,\qquad \mathbf{\Psi}:=[\partial^O_{\nu_B}(\mathbf{u}_B,\pi_B)]_+-[\partial^O_{\nu_B}(\mathbf{u}_B,\pi_B)]_-\qquad \text{on }\partial\omega_B.$$

Then $\mathbf{u}_B=\mathbf{g}$ on S_{if}. Since $h_+=h_-=0$ on $\partial\Omega\setminus S_{cB}$ we have

$$\mathbf{\Psi}+(h_++h_-)\mathbf{\Phi}=0. \tag{4.26}$$

(2.7) gives $\partial^0_{\nu_B}(\mathbf{u}_B,\pi_B)=-c_0\mathbf{f}-(h_O+h_B)\mathbf{g}$ on S_{if}. According to Proposition 4.2

$$\mathbf{u}_B=\mathbf{W}_{Br;\alpha;\partial\Omega_B}\mathbf{g}-c_O\mathbf{V}_{Br;\alpha;\partial\Omega_B}\mathbf{f}-\mathbf{V}_{Br;\alpha;\partial\Omega_B}(h_O+h_B)\mathbf{g}+\mathbf{V}_{Br;\alpha;\partial\omega_B}\mathbf{\Psi}, \tag{4.27}$$

$$\pi_B=Q^d_{Br;\alpha;\partial\Omega_B}\mathbf{g}-c_OQ^s_{Br;\alpha;\partial\Omega_B}\mathbf{f}-Q^s_{Br;\alpha;\partial\Omega_B}(h_O+h_B)\mathbf{g}+Q^s_{Br;\alpha;\partial\omega_B}\mathbf{\Psi}. \tag{4.28}$$

If we go to $\partial\omega_B$ in (4.27), we obtain

$$\mathbf{\Phi}-\mathbf{W}_{Br;\alpha;\partial\Omega_B}\mathbf{g}+c_O\mathbf{V}_{Br;\alpha;\partial\Omega_B}\mathbf{f}+\mathbf{V}_{Br;\alpha;\partial\Omega_B}(h_O+h_B)\mathbf{g}-\mathcal{V}_{Br;\alpha;\partial\omega_B}\mathbf{\Psi}=0. \tag{4.29}$$

According to Proposition 3.10

$$\mathbf{u}_O=\mathbf{W}_{Os;\lambda;\partial\Omega_O}\mathbf{g}+\mathbf{V}_{Os;\alpha;\partial\Omega_O}\mathbf{f}, \tag{4.30}$$

$$\pi_O=Q^d_{Os;\lambda;\partial\Omega_O}\mathbf{g}+Q^s_{\partial\Omega_O}\mathbf{f}. \tag{4.31}$$

If we go to S_{if} in (4.27) and (4.30) we obtain

$$\mathbf{g}=\frac12\mathbf{g}+\mathbf{K}_{Br;\alpha;\partial\Omega_B}\mathbf{g}-c_O\mathcal{V}_{Br;\alpha;\partial\Omega_B}\mathbf{f}-\mathcal{V}_{Br;\alpha;\partial\Omega_B}(h_O+h_B)\mathbf{g}+\mathbf{V}_{Br;\alpha;\partial\omega_B}\mathbf{\Psi},$$
$$\mathbf{g}=\frac12\mathbf{g}+\mathbf{K}_{Os;\lambda;\partial\Omega_O}\mathbf{g}+\mathcal{V}_{Os;\lambda;\partial\Omega_O}\mathbf{f}.$$

Multiply the second equation by c_o and add to the first equation. (Remind that $\mathcal{V}_{Br;\alpha;\partial\Omega_B}=\mathcal{V}_{Br;\alpha;\partial\Omega_O}$, $\mathbf{K}_{Br;\alpha;\partial\Omega_B}=-\mathbf{K}_{Br;\alpha;\partial\Omega_O}$.)

$$\begin{aligned}&-\tfrac{1+c_0}{2}\mathbf{g}-\mathbf{K}_{Br;\alpha;\partial\Omega_O}\mathbf{g}+c_O\mathbf{K}_{Os;\lambda;\partial\Omega_O}\mathbf{g}+c_O(\mathcal{V}_{Os;\lambda;\partial\Omega_O}\mathbf{f}-\mathcal{V}_{Br;\alpha;\partial\Omega_O}\mathbf{f})\\&-\mathcal{V}_{Br;\alpha;\partial\Omega_B}(h_O+h_B)\mathbf{g}+\mathbf{V}_{Br;\alpha;\partial\omega_B}\mathbf{\Psi}=0.\end{aligned} \tag{4.32}$$

Calculating the conormal derivative from (4.30), (4.31) we obtain

$$\mathbf{f}=\mathbf{D}_{Os;\lambda;\partial\Omega_O}\mathbf{g}+\frac12\mathbf{f}-K'_{Os;\lambda;\partial\Omega_O}\mathbf{f}. \tag{4.33}$$

Define

$$\begin{aligned} T_1(\mathbf{\Psi},\mathbf{\Phi},\mathbf{g},\mathbf{f}) = \ & \mathbf{\Psi} + (h_+ + h_-)\mathbf{\Phi}, \\ T_2(\mathbf{\Psi},\mathbf{\Phi},\mathbf{g},\mathbf{f}) = \ & \mathbf{\Phi} - \mathbf{W}_{Br;\alpha;\partial\Omega_B}\mathbf{g} + c_O\mathbf{V}_{Br;\alpha;\partial\Omega_B}\mathbf{f} + \mathbf{V}_{Br;\alpha;\partial\Omega_B}(h_O + h_B)\mathbf{g} \\ & -\mathcal{V}_{Br;\alpha;\partial\omega_B}\mathbf{\Psi}, \\ T_3(\mathbf{\Psi},\mathbf{\Phi},\mathbf{g},\mathbf{f}) = \ & -\tfrac{1+c_0}{2}\mathbf{g} - \mathbf{K}_{Br;\alpha;\partial\Omega_O}\mathbf{g} c_O \mathbf{K}_{Os;\lambda;\partial\Omega_O}\mathbf{g} - \mathcal{V}_{Br;\alpha;\partial\Omega_B}(h_O + h_B)\mathbf{g} \\ & + c_O(\mathcal{V}_{Os;\lambda;\partial\Omega_O}\mathbf{f} - \mathcal{V}_{Br;\alpha;\partial\Omega_O}\mathbf{f}) + \mathbf{V}_{Br;\alpha;\partial\omega_B}\mathbf{\Psi}, \\ T_4(\mathbf{\Psi},\mathbf{\Phi},\mathbf{g},\mathbf{f}) = \ & \mathbf{D}_{Os;\lambda;\partial\Omega_O}\mathbf{g} - \tfrac{1}{2}\mathbf{f} - K'_{Os;\lambda;\partial\Omega_O}\mathbf{f} \end{aligned}$$

Then $T = (T_1, T_2, T_3, T_4)$ is a bounded operator on $L^q(\partial\omega_B,\mathbb{R}^m) \times L^q_1(\partial\omega_B,\mathbb{R}^m) \times L^q_1(\partial\Omega_O,\mathbb{R}^m) \times L^q(\partial\Omega_O,\mathbb{R}^m)$. Sobolev imbedding theorem and compactness of complementary layer potentials force that T is a compact perturation of the operator $\tilde{T} = (\tilde{T}_1, \tilde{T}_2, \tilde{T}_3, \tilde{T}_4)$, where

$$\tilde{T}_1(\mathbf{\Psi},\mathbf{\Phi},\mathbf{g},\mathbf{f}) = \mathbf{\Psi},$$

$$\tilde{T}_2(\mathbf{\Psi},\mathbf{\Phi},\mathbf{g},\mathbf{f}) = \mathbf{\Phi} - \mathcal{V}_{Br;\alpha;\partial\omega_B}\mathbf{\Psi},$$

$$\tilde{T}_3(\mathbf{\Psi},\mathbf{\Phi},\mathbf{g},\mathbf{f}) = -\frac{1+c_0}{2}\mathbf{g} + (c_O - 1)\mathbf{K}_{\partial\Omega_O}\mathbf{g}$$

$$\tilde{T}_4(\mathbf{\Psi},\mathbf{\Phi},\mathbf{g},\mathbf{f}) = \mathbf{D}_{\partial\Omega_O}\mathbf{g} - \frac{1}{2}\mathbf{f} - K'_{\partial\Omega_O}\mathbf{f}$$

Since the equation $\mathbf{\Phi} - \mathcal{V}_{Br;\alpha;\partial\omega_B}\mathbf{\Psi} = \varphi$ can be rewritern as $\mathbf{\Phi} = \mathcal{V}_{Br;\alpha;\partial\omega_B}\mathbf{\Psi} + \varphi$, the operator $(\tilde{T}_1, \tilde{T}_2)$ is an invertible operator on $L^q(\partial\omega_B,\mathbb{R}^m) \times L^q_1(\partial\omega_B,\mathbb{R}^m)$. The operator $\tilde{T}_3$ is a Fredholm operator with index 0 on $L^q_1(\partial\Omega_O,\mathbb{R}^m)$ by [34, Theorem 9.1.3]. The operator $-\frac{1}{2}I - K'_{\partial\Omega_O}$ is a Fredholm operator with index 0 on $L^q(\partial\Omega_O,\mathbb{R}^m)$ by [34, Theorem 9.1.11]. Since the equation $\mathbf{D}_{\partial\Omega_O}\mathbf{g} - \frac{1}{2}\mathbf{f} - K'_{\partial\Omega_O}\mathbf{f} = \varphi$ can be rewrtitern as $-\frac{1}{2}\mathbf{f} - K'_{\partial\Omega_O}\mathbf{f} = \varphi - \mathbf{D}_{\partial\Omega_O}\mathbf{g}$ the operator $(\tilde{T}_3, \tilde{T}_4)$ is a Fredholm operator with index 0 on $L^q_1(\partial\Omega_O,\mathbb{R}^m) \times L^q(\partial\Omega_O,\mathbb{R}^m)$. Since $\tilde{T}$ is a Fredholm operator with index 0 and $T - \tilde{T}$ is compact, the operator T is a Fredholm operator with index 0 on $L^q(\partial\omega_B,\mathbb{R}^m) \times L^q_1(\partial\omega_B,\mathbb{R}^m) \times L^q_1(\partial\Omega_O,\mathbb{R}^m) \times L^q(\partial\Omega_O,\mathbb{R}^m)$ for arbitrary $1 < q \le 2$. Since $T(\mathbf{\Psi},\mathbf{\Phi},\mathbf{g},\mathbf{f}) = 0$ by (4.26), (4.29), (4.32) and (4.33), we have $(\mathbf{\Psi},\mathbf{\Phi},\mathbf{g},\mathbf{f}) \in L^2(\partial\omega_B,\mathbb{R}^m) \times L^2_1(\partial\omega_B,\mathbb{R}^m) \times L^2_1(\partial\Omega_O,\mathbb{R}^m) \times L^2(\partial\Omega_O,\mathbb{R}^m)$ (see [26, Lemma 5] or [34]). The representation (4.27), (4.28), (4.30), (4.31) gives that $\big((\mathbf{u}_B, \pi_B), (\mathbf{u}_O, \pi_O)\big)$ is an L^2-solution of the transmission problem.

□

4.2. **Existence of a solution of the crack type transmission problem.** Next we are concerning with the existence of a solution of the crack type transmission problem (2.4)-(2.9). We determine a solution of such a problem, which vanishes at infinity, in the form

$$\begin{aligned} \mathbf{u}_B &= \mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Phi} + \mathbf{V}_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Psi} + \mathbf{W}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Upsilon} + \mathbf{V}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Theta}, \\ \pi_B &= Q^d_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Phi} + Q^s_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Psi} + Q^d_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Upsilon} + Q^s_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Theta}, \\ \mathbf{u}_O &= \mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Phi} + \mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Psi}, \\ \pi_O &= Q^d_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Phi} + Q^s_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Psi}, \end{aligned} \tag{4.34}$$

with unknown densities $(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta}) \in \mathcal{X}_q$, where

$$\mathcal{X}_q := L^q_1(S_{if},\mathbb{R}^m) \times L^q(S_{if},\mathbb{R}^m) \times L^q_1(S_{cB},\mathbb{R}^m) \times L^q(S_{cB},\mathbb{R}^m). \tag{4.35}$$

If $m = 2$ and Ω_B is unbounded we require that $(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta}) \in \mathcal{Y}_q$, where

$$\mathcal{Y}_q := \left\{ (\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta}) \in \mathcal{X}_q : \int_{S_{if}} \langle \mathbf{\Phi}, \nu_B \rangle d\sigma + \int_{S_{cB}} \langle \mathbf{\Phi}, \nu_B \rangle d\sigma = 0 \right\}. \tag{4.36}$$

Note that the potentials $\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Phi}$, $\mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Psi}$, $Q^d_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Phi}$, $Q^s_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Psi}$ are analyzed in Ω_O, i.e., outside their reference set Ω_B. The potentials $\mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Phi}$, $\mathbf{V}_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Psi}$, $Q^d_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Phi}$, $Q^s_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Psi}$ are well-defined and infinitely differentiable in a neighborhood of S_{cB}, i.e., strictly inside their reference set Ω_B. The potentials $\mathbf{W}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Upsilon}$, $\mathbf{V}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Theta}$, $Q^d_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Upsilon}$, $Q^s_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Theta}$ are well-defined and infinitely differentiable in a neighborhood of S_{if}, i.e., strictly outside their reference set ω_B. By imposing the transmission and crack type conditions (2.6)-(2.9)) and the required boundary behavior to the layer potential representations (4.34), we deduce that $(\mathbf{u}_B,\pi_B,\mathbf{u}_O,\pi_O)$ is an L^q-solution of the crack type transmission problem (2.4)-(2.9) if and only if

$$\mathcal{T}^{\alpha;\lambda}(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})=(\mathbf{g}_{if},\mathbf{f}_{if},\mathbf{g}_{cB},\mathbf{f}_{cB}),\tag{4.37}$$

where $\mathcal{T}^{\alpha;\lambda}=(\mathcal{T}_1^{\alpha;\lambda},\mathcal{T}_2^{\alpha;\lambda},\mathcal{T}_3^{\alpha;\lambda},\mathcal{T}_4^{\alpha;\lambda})$ is a bounded linear operator on $\mathcal{X}_q$, having the following entries

(4.38)
$$\begin{aligned}
\mathcal{T}_1^{\alpha;\lambda}(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})=&\left(\tfrac12\mathbb{I}+\mathbf{K}_{\mathrm{Br};\alpha;\partial\Omega_B}\right)\mathbf{\Phi}+\mathcal{V}_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Psi}+\mathbf{W}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Upsilon}+\mathbf{V}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Theta}\\
&-\left[\left(-\tfrac12\mathbb{I}+\mathbf{K}_{\mathrm{Os};\lambda;\partial\Omega_B}\right)\mathbf{\Phi}+\mathcal{V}_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Psi}\right],\\
\mathcal{T}_2^{\alpha;\lambda}(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})=&\,\mathbf{D}_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Phi}+\left(\tfrac12\mathbb{I}+\mathbf{K}'_{\mathrm{Br};\alpha;\partial\Omega_B}\right)\mathbf{\Psi}\\
&+\partial^0_{\nu_B;\partial\Omega_B}\left(\mathbf{W}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Upsilon}+\mathbf{V}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Theta},Q^d_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Upsilon}+Q^s_{\partial\omega_B}\mathbf{\Theta}\right)\\
&-c_O\left[\left(-\tfrac12\mathbb{I}+\mathbf{K}'_{\mathrm{Os};\lambda;\partial\Omega_B}\right)\mathbf{\Psi}+\mathbf{D}_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Phi}\right]\\
&+h_B\Big(\left(\tfrac12\mathbb{I}+\mathbf{K}_{\mathrm{Br};\alpha;\partial\Omega_B}\right)\mathbf{\Phi}+\mathcal{V}_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Psi}+\mathbf{W}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Upsilon}\\
&+\mathbf{V}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Theta}\Big)+h_O\left[\left(-\tfrac12\mathbb{I}+\mathbf{K}_{\mathrm{Os};\lambda;\partial\Omega_B}\right)\mathbf{\Phi}+\mathcal{V}_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Theta}\right],\\
\mathcal{T}_3^{\alpha;\lambda}(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})=&\,\mathbf{\Upsilon},\\
\mathcal{T}_4^{\alpha;\lambda}(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})=&\,\mathbf{\Theta}+(h_++h_-)\Big(\mathcal{V}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Theta}+\mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Phi}+\mathbf{V}_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Psi}\Big)\\
&+h_+\left(\tfrac12\mathbb{I}+\mathbf{K}_{\mathrm{Br};\alpha;\partial\omega_B}\right)\mathbf{\Upsilon}+h_-\left(-\tfrac12\mathbb{I}+\mathbf{K}_{\mathrm{Br};\alpha;\partial\omega_B}\right)\mathbf{\Upsilon}.
\end{aligned}$$

Note that $\mathcal{T}^{\alpha;\lambda}$ is a linear operator on $\mathcal{X}_q$, as we have $h_+=h_-=0$ on $\partial\omega\setminus S_{cB}$. In addition, we mention the following useful result.

Lemma 4.8. *Let X, Y be Banach spaces, $T:X\to Y$ be a bounded linear operator. If T is a Fredholm operator with index zero then there exists a finite-dimensional linear operator $P:X\to Y$ such that $T+P$ is an isomorphism.*

Proof. Let $X_0=\{x\in X: Tx=0\}$. According to [38, Lemma 5.1] there exists a closed subspace X_1 of X such that $X=X_0\oplus X_1$. Denote by P_1 the projection of X onto X_0 along X_1. Since the index of T is 0 there exists a subspace Z of Y of the same dimension like X_0 such that $Y=Z\oplus T(X)$. Then there exists an isomorphism P_2 of X_0 onto Z. Let $P:=P_2\circ P_1$. Since P is a compact operator, the operator $T+P:X\to Y$ is Fredholm with index zero. Assume now that $(T+P)x=0$. Since $Px\in Z$ and $Y=Z\oplus T(X)$, we deduce that $Tx=0$, $Px=0$. Since $Tx=0$, we have $x\in X_0$. Therefore, $0=Px=P_2x$. Since $P_2:X_0\to Z$ is an isomorphism we deduce that $x=0$. Therefore, $T+P:X\to Y$ is a Fredholm operator with index zero and trivial kernel, and hence such an operator is an isomorphism. □

Proposition 4.9. *Let $\mathcal{X}_q$ and $\mathcal{Y}_q$ be the spaces defined in (4.35) and (4.36). Then there exists $q_0>2$ such that for any $q\in(1,q_0)$ the operators $\mathcal{T}^{\alpha;\lambda}:\mathcal{X}_q\to\mathcal{X}_q$ given* *by* (4.38) *is an isomorphism. If* $m=2$ *and* Ω_B *is unbounded then* $\mathcal{T}^{\alpha;\lambda}:\mathcal{Y}_q\to\mathcal{Y}_q$ *is an isomorphism. If the interface* S_{if} *is of class* C^1 *then* $q_0=\infty$*.*

Proof. First we show that there exists $q_0>2$ such that the operator $\mathcal{T}^{\alpha;\lambda}:\mathcal{X}_q\to\mathcal{X}_q$ is Fredholm with index zero for all $q\in(1,q_0)$. In fact, by [34, Theorem 9.1.11, Theorem 10.5.3] and [25], there is $q_0>2$ such that $\frac{1+c_0}{2}\mathbb{I}+(1-c_0)\mathbf{K}'_{\Omega_B}:\mathcal{X}_q\to\mathcal{X}_q$ is a Fredholm operator with index zero for any $q\in(1,q_0)$. Moreover, if S_{if} is of C^1 class, then $q_0=\infty$. Let $q\in(1,q_0)$. By Lemma 4.8 there exists a finite-dimensional linear operator $P:\mathcal{X}_q\to\mathcal{X}_q$ such that $\frac{1+c_0}{2}\mathbb{I}+(1-c_0)\mathbf{K}'_{\Omega_B}+P:\mathcal{X}_q\to\mathcal{X}_q$ is an isomorphism. Then define the operator $\mathcal{T}=(\mathcal{T}_1,\mathcal{T}_2,\mathcal{T}_3,\mathcal{T}_4)$ by

$$
\begin{aligned}
&\mathcal{T}_1(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})=\mathbf{\Phi},\\
&\mathcal{T}_2(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})=\left[\tfrac{1+c_0}{2}\mathbb{I}+(1-c_0)\mathbf{K}'_{\partial\Omega_B}+P\right]\mathbf{\Psi}+\mathbf{D}_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Phi}-c_O\mathbf{D}_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Phi},\\
&\mathcal{T}_3(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})=\mathbf{\Upsilon},\\
&\mathcal{T}_4(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})=\mathbf{\Theta}+h_+\Big(\tfrac12\mathbb{I}+\mathbf{K}_{\mathrm{Br};\alpha;\partial\omega_B}\Big)\mathbf{\Upsilon}+h_-\Big(-\tfrac12\mathbb{I}+\mathbf{K}_{\mathrm{Br};\alpha;\partial\omega_B}\Big)\mathbf{\Upsilon}.
\end{aligned}
$$

Since $h_+=h_-=0$ on $\partial\Omega\setminus S_{cB}$, the operator $\mathcal{T}$ is linear and bounded on $\mathcal{X}_q$.

Further, observe that the equation $\mathcal{T}(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})=(\mathbf{\Lambda}_1,\mathbf{\Lambda}_2,\mathbf{\Lambda}_3,\mathbf{\Lambda}_4)$ is equivalent to $\mathbf{\Phi}=\mathbf{\Lambda}_1$, $[\frac{1+c_0}{2}\mathbb{I}+(1-c_0)\mathbf{K}'_{\Omega_B}+P]\mathbf{\Psi}=\mathbf{\Lambda}_2-\mathbf{D}_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Lambda}_1-c_O\mathbf{D}_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Lambda}_1$, $\mathbf{\Upsilon}=\mathbf{\Lambda}_3$, $\mathbf{\Theta}=\mathbf{\Lambda}_4-h_+(\frac12\mathbb{I}+\mathbf{K}_{\mathrm{Br};\alpha;\partial\omega_B})\mathbf{\Lambda}_3-h_-(-\frac12\mathbb{I}+\mathbf{K}_{\mathrm{Br};\alpha;\partial\omega_B})\mathbf{\Lambda}_3$. Therefore, $\mathcal{T}:\mathcal{X}_q\to\mathcal{X}_q$ is an isomorphism. Next we show that the operator $\mathcal{T}^{\alpha;\lambda}:\mathcal{X}_q\to\mathcal{X}_q$ is a Fredholm operator with index 0. To this purpose, note the useful relations

$$
\begin{aligned}
(\mathcal{T}_1^{\alpha;\lambda}-\mathcal{T}_1)(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})=&\left(\mathbf{K}_{\mathrm{Br};\alpha;\partial\Omega_B}-\mathbf{K}_{\mathrm{Os};\alpha;\partial\Omega_B}\right)\mathbf{\Phi}+\mathbf{W}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Upsilon}\\
&+\mathbf{V}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Theta}+(\mathcal{V}_{\mathrm{Br};\alpha;\partial\Omega_B}-\mathcal{V}_{\mathrm{Os};\lambda;\partial\Omega_B})\mathbf{\Psi},\\
(\mathcal{T}_2^{\alpha;\lambda}-\mathcal{T}_2)(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})=&\ [(\mathbf{K}'_{\mathrm{Br};\alpha;\partial\Omega_B}-\mathbf{K}_{\partial\Omega_B})-c_0(\mathbf{K}'_{\mathrm{Os};\lambda;\partial\Omega_B}-\mathbf{K}_{\partial\Omega_B})]\mathbf{\Psi}\\
&+\partial^0_{\nu_B;\partial\Omega_B}(\mathbf{W}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Upsilon}+\mathbf{V}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Theta},Q^d_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Upsilon}\\
&+Q^s_{\partial\omega_B}\mathbf{\Theta})+h_B\Big(\left(\tfrac12\mathbb{I}+\mathbf{K}_{\mathrm{Br};\alpha;\partial\Omega_B}\right)\mathbf{\Phi}+\mathcal{V}_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Psi}\\
&+\mathbf{W}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Upsilon}+\mathbf{V}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Theta}\Big)\\
&+h_O\left[\left(-\tfrac12\mathbb{I}+\mathbf{K}_{\mathrm{Os};\lambda;\partial\Omega_B}\right)\mathbf{\Phi}+\mathcal{V}_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Theta}\right]-P\mathbf{\Psi},\\
(\mathcal{T}_3^{\alpha;\lambda}-\mathcal{T}_3)(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})=&\ \mathbf{0},\\
(\mathcal{T}_4^{\alpha;\lambda}-\mathcal{T}_4)(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})=&\ (h_++h_-)\Big(\mathcal{V}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Theta}+\mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Phi}+\mathbf{V}_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Psi}\Big).
\end{aligned}
$$

The operator $\mathcal{T}-\mathcal{T}^{\alpha;\lambda}:\mathcal{X}_q\to\mathcal{X}_q$ is linear and compact, as the Sobolev imbedding theorem and compactness of complementary potentials show (see Lemma 3.2, Proposition 3.4, Proposition 3.6 and Lemma 3.14). Therefore, the operator $\mathcal{T}^{\alpha;\lambda}:\mathcal{X}_q\to\mathcal{X}_q$ is Fredholm with index zero, as well.

Let $G\subseteq\mathbb{R}^m$ be a bounded open set with Lipschitz boundary. Let $\varphi\in L^q_1(\partial G,\mathbb{R}^n)$. By the divergence theorem we obtain the equality

$$\int_{\partial G}\left\langle\left(\frac12\mathbb{I}+\mathbf{K}_{\mathrm{Br};\alpha;\partial G}\right)\varphi,\nu_\Omega\right\rangle d\sigma=\int_{\partial G}\langle\mathrm{Tr}_G\left(\mathbf{W}_{\mathrm{Os};\lambda;\partial G}\varphi\right),\nu_G\rangle d\sigma=0.$$

In addition, we have the equality

$$
\begin{aligned}
\int_{\partial G}\left\langle\left(-\frac12\mathbb{I}+\mathbf{K}_{\mathrm{Br};\lambda;\partial G}\right)\varphi,\nu_G\right\rangle d\sigma=&\int_{\partial G}\left\langle\left(\frac12\mathbb{I}+\mathbf{K}_{\mathrm{Br};\alpha;\partial G}\right)\varphi,\nu_G\right\rangle d\sigma\\
&-\int_{\partial G}\langle\varphi,\nu_G\rangle\,d\sigma=-\int_{\partial G}\langle\varphi,\nu_G\rangle\,d\sigma.
\end{aligned}
\tag{4.39}
$$

Therefore, $\left(-\frac{1}{2}\mathbb{I}+\mathbf{K}_{\mathrm{Br};\lambda;\partial G}\right)\varphi$ is orthogonal to ν_G if and only if φ is orthogonal to ν_G. Similar results hold for $\left(\pm\frac{1}{2}\mathbb{I}+\mathbf{K}_{G;Os,\lambda}\right)\varphi$.

Now we turn to show that the operator $\mathcal{T}^{\alpha;\lambda}:\mathcal{X}_q\to\mathcal{X}_q$ is one-to-one. For this purpose, assume that $(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})\in\mathcal{X}_q$ satisfies the equation $\mathcal{T}^{\alpha;\lambda}(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})=0$. Then $\mathbf{\Upsilon}=0$. Let $((\mathbf{u}_B,\pi_B),(\mathbf{u}_O,\pi_O))$ be the layer potentials defined as in (4.34). If Ω_B is bounded or $m=3$ then $((\mathbf{u}_B,\pi_B),(\mathbf{u}_O,\pi_O))$ is an L^q-solution of the homogeneous crack type transmission problem (2.4)-(2.9), which vanishes at infinity. By Proposition 4.7 we obtain that $\mathbf{u}_B=0$, $\pi_B=0$ in Ω_B and $\mathbf{u}_O=0$, $\pi_O=0$ in Ω_O. Suppose now that $m=2$ and Ω_B is unbounded. Then

$$\left(\frac{1}{2}\mathbb{I}+\mathbf{K}_{\mathrm{Br};\alpha;\partial\Omega_B}\right)\mathbf{\Phi}=\left(-\frac{1}{2}\mathbb{I}+\mathbf{K}_{\mathrm{Os};\lambda;\partial\Omega_B}\right)\mathbf{\Phi}\\ -\mathcal{V}_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Psi}-\mathbf{V}_{\mathrm{Br};\alpha;\partial\omega_B}\mathbf{\Theta}+\mathcal{V}_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Psi}$$

on S_{if}. The divergence theorem implies that the right-hand side in the above equality is orthogonal to ν_B on S_{if}. Then by the orthogonality result above (4.39), $\mathbf{\Phi}$ is orthogonal to ν_B on S_{if}. Therefore $((\mathbf{u}_B,\pi_B),(\mathbf{u}_O,\pi_O))$ is an L^q-solution of the homogeneous crack type transmission problem (2.4)-(2.9), which vanishes at infinity. By Proposition 4.7 we obtain that $\mathbf{u}_B=0$, $\pi_B=0$ in Ω_B and $\mathbf{u}_O=0$, $\pi_O=0$ in Ω_O.

Since $\mathbf{u}_B=0$, $\pi_B=0$, the crack type condition (2.9) implies that $\mathbf{\Theta}=\mathbf{0}$. Let us define the layer potential

$$\begin{array}{l}\mathbf{v}_B:=\mathbf{W}_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Phi}+\mathbf{V}_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Psi}\\ p_B:=Q^d_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Phi}+Q^s_{\mathrm{Br};\alpha;\partial\Omega_B}\mathbf{\Psi}\end{array}\quad\text{in }\mathbb{R}^m\setminus S_{if},\tag{4.40}$$

$$\begin{array}{l}\mathbf{v}_O=\mathbf{W}_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Phi}+\mathbf{V}_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Psi}\\ p_O:=Q^d_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Phi}+Q^s_{\mathrm{Os};\lambda;\partial\Omega_B}\mathbf{\Psi}\end{array}\quad\text{in }\mathbb{R}^m\setminus S_{if}.\tag{4.41}$$

We turn to show that $\mathbf{v}_B=0$, $p_B=0$, $\mathbf{v}_O=0$, $p_O=0$ on $\mathbb{R}^m\setminus S_{if}$. To this purpose note that $\mathbf{\Upsilon}=0$ and $\mathbf{\Theta}=0$ imply $\mathbf{v}_B=\mathbf{u}_B=0$, $p_B=\pi_B=0$ in Ω_B, and $\mathbf{v}_O=\mathbf{u}_O=0$, $p_O=\pi_O=0$ in Ω_O. In addition, the jump formulas of the boundary layer potentials and their conormal derivatives imply that

$$\mathrm{Tr}_{\Omega_O}\mathbf{v}_B=\mathrm{Tr}_{\Omega_O}\mathbf{v}_B-\mathrm{Tr}_{\Omega_B}\mathbf{v}_B=-\mathbf{\Phi},\tag{4.42}$$

$$\partial^0_{\nu_B;\Omega_O}(\mathbf{v}_B,p_B)=\partial^0_{\nu_B;\Omega_O}(\mathbf{v}_B,\pi_B)-\partial^0_{\nu_B;\Omega_B}(\mathbf{v}_B,p_B)=-\mathbf{\Psi},\tag{4.43}$$

$$\mathrm{Tr}_{\Omega_B}\mathbf{v}_O=\mathrm{Tr}_{\Omega_B}\mathbf{v}_O-\mathrm{Tr}_{\Omega_O}\mathbf{v}_O=\mathbf{\Phi},\tag{4.44}$$

$$\partial^\lambda_{\nu_B;\Omega_B}(\mathbf{v}_O,p_O)=\partial^\lambda_{\nu_B;\Omega_B}(\mathbf{v}_O,\pi_O)-\partial^\lambda_{\nu_B;\Omega_O}(\mathbf{v}_O,p_O)=\mathbf{\Psi}.\tag{4.45}$$

In addition, if Ω_B is bounded then the Divergence Theorem gives

$$\int_{\partial\Omega_B}\langle\mathbf{\Phi},\nu_B\rangle\,d\sigma=\int_{\partial\Omega_B}\langle\mathrm{Tr}_{\Omega B}\mathbf{v}_B,\nu_B\rangle\,d\sigma=0.$$

Now the relations (4.42)-(4.45) show that $(\mathbf{v}_B,p_B,\mathbf{v}_O,p_O)$ is an L^q-solution of the transmission problem

$$\begin{cases}(\Delta-\alpha\mathbb{I})\mathbf{v}_B-\nabla p_B=\mathbf{0},\ \mathrm{div}\ \mathbf{v}_B=0\ \text{ in }\ \Omega_O,\\ (\Delta-\lambda\partial_1)(-\mathbf{v}_O)-\nabla(-p_O)=0,\ \mathrm{div}\ \mathbf{v}_O=0\ \text{ in }\ \Omega_B,\\ \mathrm{Tr}_{\Omega_O}\mathbf{v}_B-\mathrm{Tr}_{\Omega_B}(-\mathbf{v}_O)=\mathbf{0}\ \text{ on }\ S_{if},\\ \partial^0_{\nu_B;\Omega_O}(\mathbf{v}_B,p_B)-\partial^\lambda_{\nu_B;\Omega_B}(-\mathbf{v}_O,-p_O)=\mathbf{0}\ \text{ on }\ S_{if},\end{cases}\tag{4.46}$$

which vanishes at infinity. Then by Proposition 4.7 we obtain that $\mathbf{v}_B=\mathbf{0}$, $p_B=0$ in Ω_O, and $-\mathbf{v}_O=\mathbf{0}$, $-p_O=0$ in Ω_B. In addition, the relations (4.44), (4.45) imply that $\mathbf{\Phi}=0$ and $\mathbf{\Psi}=0$. Consequently, the Fredholm operator of index zero $\mathcal{T}^{\alpha;\lambda}:\mathcal{X}_q\to\mathcal{X}_q$ is one-to-one, and hence an isomorphism.

Finally, we analyze the case $m=2$ and Ω_B unbounded. Let us suppose that $(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})\in\mathcal{Y}_q$. Put $(\mathbf{g}_{if},\mathbf{f}_{if},\mathbf{g}_{cB},\mathbf{f}_{cB})=\mathcal{T}^{\alpha;\lambda}(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})$. If $(\mathbf{u}_B,\pi_B,\mathbf{u}_O,\pi_O)$ is given by the formula (4.34) then $(\mathbf{u}_B,\pi_B,\mathbf{u}_O,\pi_O)$ is a solution of the problem (2.4)-(2.9) vanishing at infinity. Proposition 4.2 implies that

$$\int_{\partial\Omega_B}\langle \mathrm{Tr}_{\Omega_B}\mathbf{u}_B,\nu_B\rangle\,d\sigma+\int_{S_{cB}}\langle \mathbf{g}_{cB},\nu_B\rangle\,d\sigma=0.$$

In addition, by the Divergence Theorem we obtain the equality

$$\int_{\partial\Omega_B}\langle \mathrm{Tr}_{\Omega_O}\mathbf{u}_O,\nu_B\rangle\,d\sigma=0.$$

Subtracting the above equalities we obtain that $(\mathbf{g}_{if},\mathbf{f}_{if},\mathbf{g}_{cB},\mathbf{f}_{cB})\in\mathcal{Y}_q$. Since $\mathcal{T}^{\alpha;\lambda}(\mathcal{Y}_q)\subset\mathcal{Y}_q$ and $\mathcal{T}^{\alpha;\lambda}:\mathcal{X}_q\to\mathcal{X}_q$ is an isomorphism, we conclude that the operator $\mathcal{T}^{\alpha;\lambda}:\mathcal{Y}_q\to\mathcal{Y}_q$ is an isomorphism, as asserted. □

The main result related to the transmission problem (2.4)-(2.9) is given below.

Theorem 4.10. *Let $\Omega_B,\Omega_O\subseteq\mathbb{R}^m$ $(m\in\{2,3\})$ be open sets with compact Lipschitz boundaries, such that $\Omega_B\neq\emptyset$, $\Omega_B\cap\Omega_O=\emptyset$ and $\overline{\Omega}_B\cup\overline{\Omega}_O=\mathbb{R}^m$. Let $S_{if}:=\partial\Omega_B=\partial\Omega_O$. Let $\omega_B\subseteq\mathbb{R}^m$ be a bounded open set with a closed Lipschitz surface $\partial\omega_B$. Let $S_{cB}\subset\partial\omega_B$ be closed. Then there exists a number $q_0>2$ such that for any $q\in(1,q_0)$ the following properties holds.*

1. *Assume that Ω_B is bounded.*
   - (a) *Then there exists an L^q-solution of the crack type transmission problem (2.4)-(2.9) if and only if $(\mathbf{g}_{if},\mathbf{f}_{if},\mathbf{g}_{cB},\mathbf{f}_{cB})\in\mathcal{X}_q$.*
   - (b) *Let $(\mathbf{g}_{if},\mathbf{f}_{if},\mathbf{g}_{cB},\mathbf{f}_{cB})\in\mathcal{X}_q$ be given. If $((\mathbf{u}_B,\pi_B),(\mathbf{u}_O,\pi_O))$ is an L^q-solution of the problem (2.4)-(2.9), then there exist two constants $\mathbf{u}_\infty\in\mathbb{R}^m$ and $\pi_\infty\in\mathbb{R}$ such that $\mathbf{u}_O(\mathbf{x})\to\mathbf{u}_\infty$, $\pi_O(\mathbf{x})\to\pi_\infty$ as $|\mathbf{x}|\to\infty$.*
   - (c) *Let $\mathbf{u}_\infty\in\mathbb{R}^m$ and $\pi_\infty\in\mathbb{R}$ be given. Let $(\mathbf{\Phi},\mathbf{\Psi},\mathbf{\Upsilon},\mathbf{\Theta})$ be equal to*

     $$(\mathcal{T}^{\alpha;\lambda})^{-1}(\mathbf{g}_{if}+\mathbf{u}_\infty,\mathbf{f}_{if}-c_0(\pi_\infty\nu+\lambda\nu_1\mathbf{u}_\infty)-h_O\mathbf{u}_\infty,\mathbf{g}_{cB},\mathbf{f}_{cB}),$$

     where $\mathcal{T}^{\alpha;\lambda}$ is the isomorphism (4.38). Let $\mathbf{u}_B$, π_B, $\mathbf{u}_O$ and π_O be the layer potentials given by (4.34). Let $\mathbf{v}_O=\mathbf{u}_O+\mathbf{u}_\infty$, $p_O=\pi_O+\pi_\infty$. Then $((\mathbf{u}_B,\pi_B),(\mathbf{v}_O,p_O))$ is the unique L^q-solution of the crack type transmission problem (2.4)-(2.9), such that

     $$\mathbf{v}_O(\mathbf{x})\to\mathbf{u}_\infty,\ \ p_O(\mathbf{x})\to\pi_\infty\quad as\quad |\mathbf{x}|\to\infty.$$
2. *Assume that Ω_B is unbounded.*
   - (a) *If $m=3$ then there exists an L^q-solution of the crack type transmission problem (2.4)-(2.9) if and only if $(\mathbf{g}_{if},\mathbf{f}_{if},\mathbf{g}_{cB},\mathbf{f}_{cB})\in\mathcal{X}_q$.*
   - (b) *If $m=2$ then there exists an L^q-solution of the crack type transmission problem (2.4)-(2.9) if and only if $(\mathbf{g}_{if},\mathbf{f}_{if},\mathbf{g}_{cB},\mathbf{f}_{cB})\in\mathcal{Y}_q$.*
   - (c) *Let $(\mathbf{g}_{if},\mathbf{f}_{if},\mathbf{g}_{cB},\mathbf{f}_{cB})\in\mathcal{X}_q$ be given. In the case $m=2$, we assume that $(\mathbf{g}_{if},\mathbf{f}_{if},\mathbf{g}_{cB},\mathbf{f}_{cB})\in\mathcal{Y}_q$. If $\big((\mathbf{u}_B,\pi_B),(\mathbf{u}_O,\pi_O)\big)$ is an L^q-solution of the crack type transmission problem (2.4)-(2.9), then there exists a constant $\pi_\infty\in\mathbb{R}$ such that $\pi_B(\mathbf{x})\to\pi_\infty$ as $|\mathbf{x}|\to\infty$.*

(d) *Let $\pi_\infty \in \mathbb{R}$ be given. Let*

$$(\mathbf{\Phi}, \mathbf{\Psi}, \mathbf{\Upsilon}, \mathbf{\Theta}) = (\mathcal{T}^{\alpha;\lambda})^{-1}(\mathbf{g}_{if}, \mathbf{f}_{if} + \pi_\infty \nu, \mathbf{g}_{cB}, \mathbf{f}_{cB}).$$

Let $\mathbf{u}_B$, π_B, $\mathbf{u}_O$ and π_O be the layer potentials given by (4.34). Also let $p_B = \pi_O + \pi_\infty$. Then $((\mathbf{u}_B, p_B), (\mathbf{u}_O, \pi_O))$ is the unique L^q-solution of the crack type transmission problem (2.4)-(2.9), such that

$$p_B(\mathbf{x}) \to \pi_\infty \ \text{as} \ |\mathbf{x}| \to \infty.$$

(3) *If S_{if} is of class C^1 then $q_0 = \infty$.*

Proof. The results of Theorem 4.10 follow by means of Proposition 4.9, Proposition 4.7, Lemma 3.9 and Proposition 4.4. □

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