On well-posedness of problems arising in dynamics of fluids

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Weak solutions to abstract conservation laws

Abstract conservation law

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{F}(\mathbf{U}) = 0 \text{ or "artificial" viscosity } (\varepsilon \Delta_x \mathbf{U})$$

Linear field equation

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{V} = 0$$

Nonlinear constitutive relation

$$\mathbb{V} = \mathbb{F}(\mathbf{U})$$

Entropy inequalities

$$\partial_t E(\mathbf{U}) + \mathrm{div}_x \mathbb{F}_E(\mathbf{U}) \leq 0, \ E \text{ convex}$$

Compactness vs. oscillations

Family of bounded solutions (approximate, numerical)

$$\{\mathbf{U}_\varepsilon\}_{\varepsilon>0},\ \mathbf{U}_\varepsilon \rightharpoonup \mathbf{U} \text{ weakly-star in } L^\infty$$

$$\Leftrightarrow$$

$$\int_B \mathbf{U}_\varepsilon \to \int_B \mathbf{U} \text{ for any } B$$

Weak convergence of non-linear composition

$$G(\mathbf{U}_\varepsilon) \rightharpoonup \overline{G(\mathbf{U})},\ \overline{G(\mathbf{U})} \neq G(\mathbf{U}) \text{ in general}$$

$$G \text{ convex } \Rightarrow G(\mathbf{U}) \leq \overline{G(\mathbf{U})}$$

$$G \text{ strictly convex, } \overline{G(\mathbf{U})} = G(\mathbf{U}) \Leftrightarrow \mathbf{U}_\varepsilon \to \mathbf{U} \text{ a.a.}$$

Compensated compactness vs. convex integration

Compensated compactness

The *linear* constraints imposed on the derivatives by the field equations combined with the *nonlinear* constitutive relations prevent oscillations. Successful in 1-D geometries

Oscillatory solutions

The oscillations are in fact *present* in the families of solutions to non-linear conservation laws in higher space dimensions

Lower semicontinuity of the energy

$$\liminf_{\varepsilon \to 0} |\mathbf{U}_\varepsilon|^2 \boxed{>} |\mathbf{U}|^2$$

Basic ideas

Constructing oscillatory solutions

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{V} = 0, \ \ \mathbb{V} = \mathbb{F}(\mathbf{U})$$

"Implicit" constitutive relation

$$|\mathbf{U}|^2 \leq G(\mathbf{U}, \mathbb{V}) \leq e, \ \ \mathbb{V} = \mathbb{F}(\mathbf{U}) \Leftrightarrow |\mathbf{U}|^2 = G(\mathbf{U}, \mathbb{V}) = e$$

Subsolutions

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{V} = 0, \ \ G(\mathbf{U}, \mathbb{V}) \boxed{<} e$$

Oscillatory lemma

Subsolution

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{V} = 0,\ |\mathbf{U}|^2 \leq G(\mathbf{U}, \mathbb{V}) < e$$

Oscillatory perturbation

$$\partial_t \mathbf{u}_\varepsilon + \mathrm{div}_x \mathbb{V}_\varepsilon = 0,\ \mathbf{u}_\varepsilon, \mathbb{V}_\varepsilon \text{ compactly supported}$$

$$G\left(\mathbf{U} + \mathbf{u}_\varepsilon, \mathbb{V} + \mathbb{V}_\varepsilon\right) < e,\ \mathbf{u}_\varepsilon \rightharpoonup 0$$

$$\liminf_{\varepsilon \to 0} \int_B |\mathbf{u}_\varepsilon|^2 \geq \int_B \Lambda\left(e - G(\mathbf{U}, \mathbb{V})\right),\ \Lambda(Z) > 0 \text{ for } Z > 0$$

$$\Rightarrow$$

$$\liminf_{\varepsilon \to 0} \int_B |\mathbf{U} + \mathbf{u}_\varepsilon|^2 \geq \int_B |\mathbf{U}|^2 + \int_B \Lambda\left(e - G(\mathbf{U}, \mathbb{V})\right)$$

Incompressible Euler [DeLellis, Székelyhidi]

Incompressible Euler system

$$\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U} \otimes \mathbf{U}) + \nabla_x \Pi = 0,\ \mathrm{div}_x \mathbf{U} = 0, N = 2, 3$$

Equivalent formulation

$$\partial_t \mathbf{U} + \mathrm{div}_x \mathbb{V} = 0,\ \mathrm{div}_x \mathbf{U} = 0,\ \mathbf{U} \otimes \mathbf{U} - \frac{1}{3}|\mathbf{U}|^2 \mathbb{I} = \mathbb{V}$$

Convex integration

$$\frac{1}{2}|\mathbf{U}|^2 \leq \frac{N}{2}\lambda_{\max}\left[\mathbf{U} \otimes \mathbf{U} - \mathbb{V}\right] \equiv G(\mathbf{U}, \mathbb{V}) < e,\ \mathbb{V} \in R^{3\times 3}_{0,\mathrm{sym}}$$

Pressure control

$$\Pi = -\frac{1}{3}|\mathbf{U}|^2,\ \frac{1}{2}|\mathbf{U}|^2 = e$$

Typical results

Good news

The problem possesses a *global-in-time* solution for *any* initial data

Bad news

The problem possesses *infinitely many* solutions for any initial data

What's wrong? ... more bad news

"Many" solutions violate the energy conservation **but** there is a "large" set of initial data for which the problem admits infinitely many energy conserving (dissipating) solutions

Savage-Hutter model for avalanches

Unknowns

flow height . . . . $h = h(t,x)$

depth-averaged velocity . . . . $\mathbf{u} = \mathbf{u}(t,x)$

$$\partial_t h + \mathrm{div}_x(h\mathbf{u}) = 0$$

$$\partial_t(h\mathbf{u}) + \mathrm{div}_x(h\mathbf{u} \otimes \mathbf{u}) + \nabla_x(ah^2) = h\left(-\gamma\frac{\mathbf{u}}{|\mathbf{u}|} + \mathbf{f}\right)$$

Periodic boundary conditions

$$\Omega = \left([0,1]|_{\{0,1\}}\right)^2$$

Transformation - Step I

Helmholtz decomposition

$$h\mathbf{u} = \mathbf{v} + \mathbf{V} + \nabla_x \Psi$$

where

$$\mathrm{div}_x \mathbf{v} = 0, \ \int_\Omega \Psi \ \mathrm{d}x = 0, \ \int_\Omega \mathbf{v} \ \mathrm{d}x = 0, \ \mathbf{V} \in R^2$$

Fixing h and the potential Ψ

$$\partial_t h + \Delta \Psi = 0$$

$$h(0,\cdot) = h_0, \ -\partial_t h(0,\cdot) = \Delta \Psi_0$$

Problem I

Equation

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{V} + \nabla_x \Psi) \otimes (\mathbf{v} + \mathbf{V} + \nabla_x \Psi)}{h} + \left(a h^2 + \partial_t \Psi \right) \mathbb{I} \right)$$

$$+ \partial_t \mathbf{V}$$

$$= h \left(-\gamma \frac{\mathbf{v} + \mathbf{V} + \nabla_x \Psi}{|\mathbf{v} + \mathbf{V} + \nabla_x \Psi|} + \mathbf{f} \right),$$

Constraints and initial conditions

$$\mathrm{div}_x \mathbf{v} = 0, \ \int_\Omega \mathbf{v}(t, \cdot) \ \mathrm{d}x = 0$$

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{V}(0) = \mathbf{V}_0$$

Transformation - Step II

Prescribing the kinetic energy

$$\frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}+\nabla_x\Psi|^2}{h} = E \equiv \Lambda(t) - ah^2 - \partial_t\Psi$$

Problem II

$$\partial_t\mathbf{v} + \partial_t\mathbf{V}$$

$$+\mathrm{div}_x\left(\frac{(\mathbf{v}+\mathbf{V}+\nabla_x\Psi)\otimes(\mathbf{v}+\mathbf{V}+\nabla_x\Psi)}{h} - \frac{1}{2}\frac{|\mathbf{v}+\mathbf{V}+\nabla_x\Psi|^2}{h}\mathbb{I}\right)$$

$$= -\gamma\left(\frac{h}{2E}\right)^{1/2}(\mathbf{v}+\mathbf{V}+\nabla_x\Psi) + h\mathbf{f}$$

Transformation - Step III

Determining function V

$$\partial_t \mathbf{V} - \left[\frac{1}{|\Omega|} \int_\Omega \gamma \left(\frac{h}{2E} \right)^{1/2} \mathrm{d}x \right] \mathbf{V}$$

$$= \frac{1}{|\Omega|} \int_\Omega \left[\gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \nabla_x \Psi) + h\mathbf{f} \right] \mathrm{d}x, \;\; \mathbf{V}(0) = \mathbf{V}_0$$

Problem III

Equation

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi) \odot (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi)}{h} \right)$$

$$= -\gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi)$$

$$+ \frac{1}{|\Omega|} \int_\Omega \gamma \left(\frac{h}{2E} \right)^{1/2} (\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi) \ \mathrm{d}x + h\mathbf{f} - \frac{1}{|\Omega|} \int_\Omega h\mathbf{f} \ \mathrm{d}x$$

$$\mathbf{v} \odot \mathbf{w} = \mathbf{v} \otimes \mathbf{w} - \frac{1}{2} \mathbf{v} \cdot \mathbf{w} \mathbb{I}$$

Transformation - Step IV

Solving elliptic problem

$$\mathrm{div}_x \mathbb{M} \equiv \mathrm{div}_x \left(\nabla_x \mathbf{m} + \nabla_x^t \mathbf{m} - \mathrm{div}_x \mathbf{m} \mathbb{I} \right)$$

$$= -\gamma \left(\frac{h}{2E} \right)^{1/2} \left(\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi \right)$$

$$+ \frac{1}{|\Omega|} \int_\Omega \gamma \left(\frac{h}{2E} \right)^{1/2} \left(\mathbf{v} + \mathbf{V}[\mathbf{v}] + \nabla_x \Psi \right) \ \mathrm{d}x + h\mathbf{f} - \frac{1}{|\Omega|} \int_\Omega h\mathbf{f} \ \mathrm{d}x,$$

$$\int_\Omega \mathbb{M}(t, \cdot) \ \mathrm{d}x = 0 \text{ for any } t \in [0, T].$$

Abstract formulation

Variable coefficients "Euler system"

$$\partial_t \mathbf{v} + \mathrm{div}_x \left(\frac{(\mathbf{v} + \mathbf{H}[\mathbf{v}]) \odot (\mathbf{v} + \mathbf{H}[\mathbf{v}])}{h[\mathbf{v}]} + \mathbb{M}[\mathbf{v}] \right) = 0$$

$$\mathrm{div}_x \mathbf{v} = 0,$$

Kinetic energy

$$\frac{1}{2} \frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]} = E[\mathbf{v}]$$

Data

$$\mathbf{v}(0, \cdot) = \mathbf{v}_0, \ \mathbf{v}(T, \cdot) = \mathbf{v}_T$$

Abstract operators

Boundedness

b maps bounded sets in $L^\infty((0,T)\times\Omega;R^N)$ on bounded sets in $C_b(Q,R^M)$

Continuity

$$b[\mathbf{v}_n] \to b[\mathbf{v}] \text{ in } C_b(Q;R^M) \text{ (uniformly for } (t,x)\in Q \text{)}$$

whenever

$$\mathbf{v}_n \to \mathbf{v} \text{ in } C_{\rm weak}([0,T];L^2(\Omega;R^N))$$

Causality

$\mathbf{v}(t,\cdot)=\mathbf{w}(t,\cdot)$ for $0\le t\le\tau\le T$ implies $b[\mathbf{v}]=b[\mathbf{w}]$ in $[(0,\tau]\times\Omega]$

Results

Result (A)

The set of subsolutions is non-empty $\Rightarrow$ there exists infinitely many weak solutions of the problem with the same initial data

Initial energy jump

$$\frac{1}{2}\frac{|\mathbf{v}_0 + \mathbf{H}[\mathbf{v}_0]|^2}{h[\mathbf{v}_0]} \boxed{<} \liminf_{t\to 0} \frac{1}{2}\frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]}$$

Result (B)

The set of subsolutions is non-empty $\Rightarrow$ there exists a dense set of times such that the values $\mathbf{v}(t)$ give rise to non-empty subsolution set with

$$\frac{1}{2}\frac{|\mathbf{v}_0 + \mathbf{H}[\mathbf{v}_0]|^2}{h[\mathbf{v}_0]} \boxed{=} \liminf_{t\to 0} \frac{1}{2}\frac{|\mathbf{v} + \mathbf{H}[\mathbf{v}]|^2}{h[\mathbf{v}]}$$

Application to Savage-Hutter model

Theorem

(i) Let the initial data

$$h_0 \in C^2(\Omega),\ \mathbf{u}_0 \in C^2(\Omega; R^2), h_0 > 0 \text{ in } \Omega$$

be given, and let $\mathbf{f}$ and a be smooth.
Then the Savage-Hutter system admits infinitely many weak solutions in $(0, T) \times \Omega$.

(ii) Let $T > 0$ and

$$h_0 \in C^2(\Omega),\ h_0 > 0$$

be given.
Then there exists

$$\mathbf{u}_0 \in L^\infty(\Omega; R^2)$$

such that the Savage-Hutter system admits infinitely many weak solutions in $(0, T) \times \Omega$ satisfying the energy inequality.

Example II, Euler-Fourier system

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x(\varrho \vartheta) = 0$$

Internal energy balance

$$\frac{3}{2}\Big[\partial_t(\varrho\vartheta) + \mathrm{div}_x(\varrho\vartheta\mathbf{u})\Big] - \Delta\vartheta = -\varrho\vartheta\mathrm{div}_x\mathbf{u}$$

Example III, Euler-Korteweg-Poisson system

Mass conservation - equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equations - Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho)$$

$$= \boxed{\varrho \nabla_x \left(K(\varrho)\Delta_x \varrho + \frac{1}{2} K'(\varrho)|\nabla_x \varrho|^2 \right)} - \varrho \mathbf{u} + \varrho \nabla_x V$$

Poisson equation

$$\Delta_x V = \varrho - \overline{\varrho}$$

Example IV, Euler-Cahn-Hilliard system

Model by Lowengrub and Truskinovsky

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p_0(\varrho, c) = \mathrm{div}_x \left(\varrho \nabla_x c \otimes \nabla_x c - \frac{\varrho}{2} |\nabla_x c|^2 \mathbb{I} \right)$$

Cahn-Hilliard equation

$$\partial_t(\varrho c) + \mathrm{div}_x(\varrho c \mathbf{u}) = \Delta \left(\mu_0(\varrho, c) - \frac{1}{\varrho} \mathrm{div}_x \left(\varrho \nabla_x c \right) \right)$$