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Abstract

We study an hydrodynamical model describing the motion of thick astrophysical disks relying on compressible Navier-Sokes-Poisson system and we also suppose that the medium is electrically charged and we include energy exchanges through radiative transfer. Supposing that the system is rapidly rotating, we study the singular limit of the system when the Mach number, the Alfven number and Froude number go to zero and we prove convergence to a 3D incompressible MHD system with radiation with two stationary linear transport equations for transport of radiation intensity.

Key words: Navier-Stokes-Poisson system, magnetohydrodynamics, radiating transfer, rotation, accretion disk, weak solution.

1 Introduction

Our motivation in this work is the study of the equations describing objects called "accretion disk" which are quasi planar structures observed in various places in the universe. From a naive point of view, if a massive object attracts matter distributed around it through Newtonian gravitation in presence of a high angular momentum, the matter is not accreted isotropically around the central object but forms a disk around it. As the three main ingredients claimed by astrophysicists for explaining the existence of such objects are: gravitation, angular momentum and viscosity (see [22] [26] [27] for detailed presentations), a reasonable framework for their study seems to be a viscous selfgravitating rotating fluid.

In previous works we derived thin disks models [8] [9] corresponding to limit domains $\Omega_\varepsilon = \omega \times (0, \varepsilon)$ for $\varepsilon \to 0$. In the present one we consider a thick model where ϵ is no more small and replaced by 1 in the sequel.

The mathematical model we consider is basically the compressible heat conducting MHD system [5] describing the motion of a viscous charged fluid confined to the thick disk $\Omega = \omega \times (0, 1)$, where $\omega \in \mathbb{R}^2$ is a 2-D domain, moreover as we suppose a global rotation of the system, some new terms appear due to the change of frame and we also suppose that the fluid exchanges energy with radiation through radiative transfers (see [5] [7]).

More precisely, the non-dimensional system of equations giving the evolution of the mass density $\varrho = \varrho(t,x)$, the velocity field $\vec{u} = \vec{u}(t,x)$, the (divergence free) magnetic field $\vec{B} = \vec{B}(x,t)$, and the radiative intensity $I = I(x,t,\vec{\omega},\hat{\nu})$ as functions of the time $t \in (0,T)$, the spatial coordinate $x = (x_1,x_2,x_3) \in \Omega \subset \mathbb{R}^3$, and (for I) the angular and frequency variables $(\vec{\omega},\hat{\nu}) \in \mathcal{S}^2 \times \mathbb{R}_+$, reads as follows

$$\partial_t \varrho + \mathrm{div}_x(\varrho\vec{u}) = 0 \quad \text{ in } (0,T)\times\Omega, \tag{1.1}$$

$$\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x p + \varrho\vec{\chi}\times\vec{u}$$
$$= \mathrm{div}_x\mathbb{S} + \varrho\nabla\Psi - \varrho\nabla_x|\vec{\chi}\times\vec{x}|^2 + \vec{j}\times\vec{B} \qquad \text{in } (0,T)\times\Omega, \tag{1.2}$$

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e\vec{u}) + \mathrm{div}_x\vec{q} = \mathbb{S}:\nabla_x\vec{u} - p\,\mathrm{div}_x\vec{u} + \vec{j}\cdot\vec{E} - S_E \quad \text{ in } (0,T)\times\Omega, \tag{1.3}$$

$$\frac{1}{c}\partial_t I + \vec{\omega}\cdot\nabla_x I = S \quad \text{ in } (0,T)\times\Omega\times(0,\infty)\times\mathcal{S}^2. \tag{1.4}$$

$$\partial_t\vec{B} + \mathrm{curl}_x(\vec{B}\times\vec{u}) + \mathrm{curl}_x(\lambda\ \mathrm{curl}_x\vec{B}) = 0 \qquad \text{in } (0,T)\times\Omega. \tag{1.5}$$

$$-\Delta\Psi = 4\pi G(\eta\varrho + g) \quad \text{ in } (0,T)\times\Omega_\epsilon. \tag{1.6}$$

In the electromagnetic source terms, electric current $\vec{j}$ and electric field $\vec{E}$ are interrelated by *Ohm's law*

$$\vec{j} = \sigma(\vec{E} + \vec{u}\times\vec{B}),$$

and *Ampère's law*

$$\zeta\vec{j} = \mathrm{curl}_x\vec{B},$$

where $\zeta > 0$ is the (constant) magnetic permeability.

In (1.6) Ψ is the gravitational potential and the corresponding source term in (1.2) is the Newton force $\varrho\nabla\Psi$. G is the Newton constant and g is a given function, modelling an external gravitational effect. Supposing that ϱ is extended by 0 outside Ω we have

$$\Psi(t,x) = G\int_\Omega K(x-y)(\eta\varrho(t,y) + g(y))\ dy,$$

where $K(x) = \frac{1}{|x|}$, and the parameter η may take the values 0 or 1: for $\eta = 1$ selfgravitation is present and for $\eta = 0$ gravitation only acts as an external field (some astrophysicists consider selfgravitation of accretion disks as small compared to the external attraction by a given massive central object modeled by g [27]).

We also assume that the system is globally rotating at uniform velocity χ around the vertical direction $\vec{e}_3$ and we note $\vec{\chi} = \chi\vec{e}_3$. Then Coriolis acceleration term $\varrho\vec{\chi}\times\vec{u}$ appears in the system, together with the centrifugal force term $\varrho\nabla_x|\vec{\chi}\times\vec{x}|^2$ (see [3]).

In (1.5) $\lambda = \lambda(\vartheta) > 0$ is the magnetic diffusivity of the fluid.

Observe that we consider here the simplified model studied in [11] where radiation does not appear in the momentum equation. Only appears the source S_E in the energy equation

$$S_E(t,x) = \int_{\mathcal{S}^2}\int_0^\infty S(t,x,\vec{\omega},\nu)\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu.$$

The symbol $p = p(\varrho,\vartheta)$ denotes the thermodynamic pressure and $e = e(\varrho,\vartheta)$ is the specific internal energy, interrelated through Maxwell's relation

$$\frac{\partial e}{\partial\varrho} = \frac{1}{\varrho^2}\left(p(\varrho,\vartheta) - \vartheta\frac{\partial p}{\partial\vartheta}\right). \tag{1.7}$$

Furthermore, $\mathbb{S}$ is the viscous stress tensor determined by

$$\mathbb{S} = \mu\left(\nabla_x \vec{u} + \nabla_x^t \vec{u} - \frac{2}{3}\mathrm{div}_x \vec{u}\right) + \eta\ \mathrm{div}_x \vec{u}\ \mathbb{I}, \tag{1.8}$$

where the shear viscosity coefficient $\mu = \mu(\vartheta) > 0$ and the bulk viscosity coefficient $\eta = \eta(\vartheta) \geq 0$ are effective functions of the temperature. Similarly, $\vec{q}$ is the heat flux given by Fourier's law

$$\vec{q} = -\kappa \nabla_x \vartheta, \tag{1.9}$$

with the heat conductivity coefficient $\kappa = \kappa(\vartheta) > 0$. Finally,

$$S = S_{a,e} + S_s, \tag{1.10}$$

where

$$S_{a,e} = \sigma_a\Big(B(\nu,\vartheta) - I\Big),\ S_s = \sigma_s\left(\tilde{I} - I\right). \tag{1.11}$$

In this formula $\tilde{I} := \frac{1}{4\pi}\int_{\mathcal{S}^2} I(\cdot,\vec{\omega})\ \mathrm{d}\vec{\omega}$ and $B(\nu,\vartheta) = 2h\nu^3 c^{-2}\left(e^{\frac{h\nu}{k\vartheta}} - 1\right)^{-1}$ is the radiative equilibrium function where h and k are the Planck and Boltzmann constants, $\sigma_a = \sigma_a(\nu,\vartheta) \geq 0$ is the absorption coefficient and $\sigma_s = \sigma_s(\nu,\vartheta) \geq 0$ is the scattering coefficient. More restrictions on these structural properties of constitutive quantities will be imposed in Section 2 below.

System (1.1 - 1.6) is supplemented with the boundary conditions:

$$\vec{u}|_{\partial\Omega} = 0,\ \vec{q}\cdot\vec{n}|_{\partial\Omega} = 0,\ \vec{B}\cdot\vec{n}|_{\partial\Omega} = 0,\ \vec{E}\times\vec{n}|_{\partial\Omega} = 0, \tag{1.12}$$

$$I(t,x,\nu,\vec{\omega}) = 0 \text{ for } x \in \partial\Omega,\ \vec{\omega}\cdot\vec{n} \leq 0, \tag{1.13}$$

where $\vec{n}$ denotes the outer normal vector to $\partial\Omega$.

Let us mention that there are already existing works in this field but not in the case of rotating fluid with radiation. We can mention some of existing works. First one was done by Kukučka [18] when Mach and Alpfen number go to zero in the case of bounded domain. In [25] Novotný and his investigated the problem in the case of strong stratification. See also work of Trivisa et al. [19] or work of Wang et al.[14],or works of Jiang et al.[16, 17, 15].

The paper is organized as follows.

In Section 2, we list the principal hypotheses imposed on constitutive relations, introduce the concept of weak solution to problem (1.1 - 1.13), and state the existence result for our model. In Section 3 we compute the formal asymptotics of the problem. Uniform bounds imposed on weak solutions by the data are derived in Section 4. The convergence Theorem is proved in Section 5. Existence of a solution for the target system is briefly given in the Appendix.

2 Hypotheses and stability result

We consider the pressure in the form

$$p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4,\ a > 0, \tag{2.1}$$

where $P : [0, \infty) \to [0, \infty)$ is a given function with the following properties:

$$P \in C^1[0, \infty), \; P(0) = 0, \; P'(Z) > 0 \text{ for all } Z \geq 0, \tag{2.2}$$

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z \geq 0, \tag{2.3}$$

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{2.4}$$

After Maxwell's equation (1.7), the specific internal energy e is

$$e(\varrho, \vartheta) = \frac{3}{2}\vartheta \left(\frac{\vartheta^{3/2}}{\varrho}\right) P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + a\frac{\vartheta^4}{\varrho}, \tag{2.5}$$

and the associated specific entropy reads

$$s(\varrho, \vartheta) = M\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{4a}{3}\frac{\vartheta^3}{\varrho}, \tag{2.6}$$

with

$$M'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0.$$

A new feature of the present paper (see below) will be the explicit introduction of the entropy for the photon gas.

The transport coefficients μ, η, κ and λ are continuously differentiable functions of the absolute temperature such that

$$0 < c_1(1+\vartheta) \leq \mu(\vartheta), \; \mu'(\vartheta) < c_2, \; 0 \leq \eta(\vartheta) \leq c(1+\vartheta), \tag{2.7}$$

$$0 < c_1(1+\vartheta^3) \leq \kappa(\vartheta), \; \lambda(\vartheta) \leq c_2(1+\vartheta^3) \tag{2.8}$$

for any $\vartheta \geq 0$. Moreover we assume that σ_a, σ_s, B are continuous functions of ν, ϑ such that

$$0 \leq \sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta), |\partial_\vartheta \sigma_a(\nu,\vartheta)|, |\partial_\vartheta \sigma_s(\nu,\vartheta)| \leq c_1, \tag{2.9}$$

$$0 \leq \sigma_a(\nu,\vartheta)B(\nu,\vartheta), |\partial_\vartheta\{\sigma_a(\nu,\vartheta)B(\nu,\vartheta)\}| \leq c_2, \tag{2.10}$$

$$\sigma_a(\nu,\vartheta), \sigma_s(\nu,\vartheta), \sigma_a(\nu,\vartheta)B(\nu,\vartheta) \leq h(\nu), \; h \in L^1(0,\infty). \tag{2.11}$$

for all $\nu \geq 0$, $\vartheta \geq 0$, where $c_{1,2,3}$ are positive constants.

Let us recall some definitions introduced in [10].

- In the weak formulation of the Navier-Stokes-Fourier system the *equation of continuity* (1.1) is replaced by its (weak) renormalized version [4] represented by the family of integral identities

$$\int_0^T \int_\Omega \left(\Big(\varrho + b(\varrho)\Big)\partial_t\varphi + \Big(\varrho + b(\varrho)\Big)\vec{u}\cdot\nabla_x\varphi + \Big(b(\varrho) - b'(\varrho)\varrho\Big)\mathrm{div}_x\vec{u}\;\varphi \right) \,\mathrm{d}x\,\mathrm{d}t = -\int_\Omega \Big(\varrho_0 + b(\varrho_0)\Big)\varphi(0,\cdot)\,\mathrm{d}x \tag{2.12}$$

satisfied for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$, and any $b \in C^\infty[0,\infty)$, $b' \in C_c^\infty[0,\infty)$, where (2.12) implicitly includes the initial condition $\varrho(0,\cdot)=\varrho_0$.

- Similarly, the *momentum equation* (1.2) is replaced by

$$\int_0^T \int_\Omega \Big((\varrho\vec{u})\cdot\partial_t\varphi + (\varrho\vec{u}\otimes\vec{u}) : \nabla_x\varphi + p\,\mathrm{div}_x\varphi + (\varrho\vec{\chi}\times\vec{u})\cdot\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t \tag{2.13}$$

$$= \int_0^T \int_\Omega \Big(\mathbb{S} : \nabla_x\varphi - \varrho\nabla_x\Psi\cdot\varphi - (\vec{j}\times\vec{B})\cdot\varphi - \varrho\nabla_x|\vec{\chi}\times\vec{x}|^2\cdot\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t - \int_\Omega (\varrho\vec{u})_0\cdot\varphi(0,\cdot)\,\mathrm{d}x$$

for any $\varphi \in C_c^\infty([0,T)\times\Omega;\mathbb{R}^3)$. As usual, for (2.13) to make sense, the field $\vec{u}$ must belong to a certain Sobolev space with respect to the spatial variable we require that

$$\vec{u} \in L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3)), \tag{2.14}$$

where (2.14) already includes the no-slip boundary condition (1.12).

- The *magnetic equation* (1.5) is replaced by

$$\int_0^T \int_\Omega \Big(\vec{B}\cdot\partial_t\varphi - (\vec{B}\times\vec{u} + \lambda\mathrm{curl}_x\vec{B})\cdot\mathrm{curl}_x\varphi \Big)\,\mathrm{d}x\,\mathrm{d}t + \int_\Omega \vec{B}_0\cdot\varphi(0,\cdot)\,\mathrm{d}x = 0, \tag{2.15}$$

to be satisfied for any vector field $\varphi \in \mathcal{D}([0,T)\times R^3)$.

Here, according the boundary conditions, one has to take

$$\vec{B}_0 \in L^2(\Omega),\ \mathrm{div}_x\vec{B}_0 = 0 \text{ in } \mathcal{D}'(\Omega),\ \vec{B}_0\cdot\vec{n}|_{\partial\Omega} = 0. \tag{2.16}$$

Following Theorem 1.4 in [30], $\vec{B}_0$ belongs to the closure of all solenoidal functions from $\mathcal{D}(\Omega)$ with respect to the L^2−norm.

Anticipating (see (2.28) below) we see that

$$\vec{B} \in L^\infty(0,T;L^2(\Omega;R^3)),\ \mathrm{curl}_x\vec{B} \in L^2(0,T;L^2(\Omega;R^3))$$

and we deduce from (2.15) that

$$\mathrm{div}_x\vec{B}(t) = 0 \text{ in } \mathcal{D}'(\Omega),\ \vec{B}(t)\cdot\vec{n}|_{\partial\Omega} = 0 \text{ for a.a. } t\in(0,T).$$

In particular, using Theorem 6.1 in [12], we conclude

$$\vec{B} \in L^2(0,T;W^{1,2}(\Omega;R^3)),\ \mathrm{div}_x\vec{B}(t) = 0,\ \vec{B}\cdot\vec{n}|_{\partial\Omega} = 0 \text{ for a.a. } t\in(0,T). \tag{2.17}$$

- From (1.2) and (1.3) we have the *energy conservation law*

$$\partial_t\Big(\frac{1}{2}\varrho|\vec{u}|^2+\varrho e+\frac{1}{2\mu}|\vec{B}|^2\Big)+\mathrm{div}_x\Big((\frac{1}{2}\varrho|\vec{u}|^2+\varrho e+p)\vec{u}+\vec{E}\times\vec{B}-\mathbb{S}\vec{u}+\vec{q}\Big) = \varrho\nabla_x\Psi\cdot\vec{u}+\varrho\nabla_x|\vec{\chi}\times\vec{x}|^2\cdot\vec{u}-S_E. \tag{2.18}$$

Let us rearrange the right hand side.

As the gravitational potential Ψ is determined by equation (1.6) considered on the whole space $\mathbb{R}^3$, the density ϱ being extended to be zero outside Ω we observe that $\int_\Omega \varrho\nabla_x\Psi\cdot\vec{u}\,\mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega \varrho\Psi\,\mathrm{d}x$.

In the same stroke $\int_\Omega \varrho \nabla_x |\vec{\chi}\times\vec{x}|^2 \cdot \vec{u}\ \mathrm{d}x = -\frac{\mathrm{d}}{\mathrm{d}t}\frac{1}{2}\int_\Omega \varrho |\vec{\chi}\times\vec{x}|^2\ \mathrm{d}x$.

Denoting now by E^R the radiative energy given by

$$E^R(t,x) = \frac{1}{c}\int_{\mathcal{S}^2}\int_0^\infty I(t,x,\vec{\omega},\nu)\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu, \tag{2.19}$$

and integrating the radiative transfer equation (1.5), we get

$$\partial_t \int_\Omega E^R\ \mathrm{d}x + \int_0^\tau \int\int_{\partial\Omega\times\mathcal{S}^2,\ \vec{\omega}\cdot\vec{n}\geq 0}\int_0^\infty \vec{\omega}\cdot\vec{n} I(t,x,\vec{\omega},\nu)\ \mathrm{d}\nu\ \mathrm{d}\vec{\omega}\ \mathrm{d}S_x\ \mathrm{d}t = \int_\Omega S_E\ \mathrm{d}x.$$

Using boundary conditions, we deduce the identity

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \Big(\frac{1}{2}\varrho|\vec{u}|^2 + \varrho e + \frac{1}{2\mu}|\vec{B}|^2 - \frac{1}{2}\varrho\Psi + \frac{1}{2}\varrho|\vec{\chi}\times\vec{x}|^2 + E^R\Big)\ \mathrm{d}x + \int\int_{\partial\Omega\times\mathcal{S}^2,\ \vec{\omega}\cdot\vec{n}\geq 0}\int_0^\infty \vec{\omega}\cdot\vec{n} I(t,x,\vec{\omega},\nu)\ \mathrm{d}\nu\ \mathrm{d}\vec{\omega}\ \mathrm{d}S_x = 0. \tag{2.20}$$

- Finally, dividing (1.3) by ϑ and using Maxwell's relation (1.7), we obtain the *entropy equation*

$$\partial_t(\varrho s) + \mathrm{div}_x(\varrho s \vec{u}) + \mathrm{div}_x\left(\frac{\vec{q}}{\vartheta}\right) = \varsigma, \tag{2.21}$$

where

$$\varsigma = \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta} + \frac{\lambda}{\zeta}|\mathrm{curl}_x\vec{B}|^2\right) - \frac{S_E}{\vartheta}, \tag{2.22}$$

where the first term $\varsigma_m := \frac{1}{\vartheta}\left(\mathbb{S} : \nabla_x\vec{u} - \frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta} + \frac{\lambda}{\zeta}|\mathrm{curl}_x\vec{B}|^2\right)$ is the (positive) electromagnetic matter entropy production.

In order to identify the second term in (2.22), let us recall [1] the formula for the entropy of a photon gas

$$s^R = -\frac{2k}{c^3}\int_0^\infty\int_{\mathcal{S}^2} \nu^2\left[n\log n - (n+1)\log(n+1)\right] d\vec{\omega} d\nu, \tag{2.23}$$

where $n = n(I) = \frac{c^2 I}{2h\nu^3}$ is the occupation number. Defining the radiative entropy flux

$$\vec{q}^R = -\frac{2k}{c^2}\int_0^\infty\int_{\mathcal{S}^2} \nu^2\left[n\log n - (n+1)\log(n+1)\right]\vec{\omega}\ d\vec{\omega} d\nu, \tag{2.24}$$

and using the radiative transfer equation, we get the equation

$$\partial_t s^R + \mathrm{div}_x\vec{q}^R = -\frac{k}{h}\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\log\frac{n}{n+1}\ S\ d\vec{\omega} d\nu =: \varsigma^R. \tag{2.25}$$

Checking the identity $\log\frac{n(B)}{n(B)+1} = \frac{h\nu}{k\vartheta}$ with $B = B(\vartheta,\nu)$ the Planck's function, and using the definition of S, the right-hand side of (2.25) rewrites

$$\varsigma^R = \frac{S_E}{\vartheta} - \frac{k}{h}\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(B)}{n(B)+1}\right]\sigma_a(B-I)\ d\vec{\omega} d\nu$$
$$-\frac{k}{h}\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1} - \log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(\tilde{I}-I)\ d\vec{\omega} d\nu,$$

where we used the hypothesis that the transport coefficients $\sigma_{a,s}$ do not depend on $\vec{\omega}$. So we obtain finally

$$\partial_t \left(\varrho s + s^R\right) + \mathrm{div}_x \left(\varrho s \vec{u} + \vec{q}^R\right) + \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta}\right) = \varsigma + \varsigma^R. \tag{2.26}$$

and equation (2.21) is replaced in the weak formulation by the inequality

$$\int_0^T \int_\Omega \left((\varrho s + s^R)\partial_t \varphi + \varrho s \vec{u} \cdot \nabla_x \varphi + (\frac{\vec{q}}{\vartheta} + \vec{q}^R) \cdot \nabla_x \varphi \right) \, \mathrm{d}x \, \mathrm{d}t \tag{2.27}$$

$$\leq - \int_\Omega (\varrho s + s^R)_0 \varphi(0,\cdot) \, \mathrm{d}x - \int_0^T \int_\Omega \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} + \frac{\lambda}{\zeta} |\mathrm{curl}_x \vec{B}|^2 \right) \varphi \, \mathrm{d}x \, \mathrm{d}t$$

$$- \frac{k}{h} \int_0^T \int_\Omega \left[\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(B)}{n(B)+1} \right] \sigma_a (B - I) \, d\vec{\omega} d\nu \right.$$

$$\left. + \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu} \left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1} \right] \sigma_s (\tilde{I} - I) \, d\vec{\omega} d\nu \right] \varphi \, \mathrm{d}x \mathrm{d}x \, \mathrm{d}t$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, $\varphi \geq 0$, where the sign of all the terms in the right hand side may be controlled.

- Since replacing equation (1.3) by inequality (2.27) would result in a formally under-determined problem, system (2.12), (2.13), (2.27) must be supplemented with the *total energy balance*

$$\int_\Omega \left(\frac{1}{2} \varrho |\vec{u}|^2 + \varrho e(\varrho, \vartheta) + \frac{1}{2\mu} |\vec{B}|^2 - \frac{1}{2} \varrho \Psi + \frac{1}{2} \varrho |\vec{\chi} \times \vec{x}|^2 + E^R \right) (\tau, \cdot) \, \mathrm{d}x \tag{2.28}$$

$$+ \int_0^\tau \int \int_{\partial\Omega \times \mathcal{S}^2, \ \vec{\omega}\cdot\vec{n} \geq 0} \int_0^\infty \vec{\omega} \cdot \vec{n} I(t, x, \vec{\omega}, \nu) \, \mathrm{d}\nu \, \mathrm{d}\vec{\omega} \, \mathrm{d}S_x \, \mathrm{d}t$$

$$= \int_\Omega \left(\frac{1}{2\varrho_0} |(\varrho \vec{u})_0|^2 + (\varrho e)_0 \frac{1}{2\mu} |\vec{B}_0|^2 - \frac{1}{2} \varrho_0 \Psi_0 + \frac{1}{2} \varrho_0 |\vec{\chi} \times \vec{x}|^2 + E_0^R \right) \, \mathrm{d}x,$$

where E_0^R is given by

$$E_0^R(x) = \frac{1}{c} \int_{\mathcal{S}^2} \int_0^\infty I(0, x, \vec{\omega}, \nu) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu.$$

Concerning the transport equation (1.4), it can be extended to the whole physical space $\mathbb{R}^3$ provided we set $\sigma_a(x, \nu, \vartheta) = 1_\Omega \sigma_a(\nu, \vartheta)$ and $\sigma_s(x, \nu, \vartheta) = 1_\Omega \sigma_s(\nu, \vartheta)$ and take the initial distribution $I_0(x, \vec{\omega}, \nu)$ to be zero for $x \in \mathbb{R}^3 \setminus \Omega$. Accordingly, for any fixed $\vec{\omega} \in \mathcal{S}^2$, equation (1.4) can be viewed as a linear transport equation defined in $(0,T) \times \mathbb{R}^3$, with a right-hand side S. With the above mentioned convention, extending $\vec{u}$ to be zero outside Ω, we may therefore assume that both ϱ and I are defined on the whole physical space $\mathbb{R}^3$.

Definition 2.1 *We say that $\varrho, \vec{u}, \vartheta, \vec{B}, I$ is a weak solution of problem (1.1 - 1.6) if*

$$\varrho \geq 0, \ \vartheta > 0 \textit{ for a.a. } (t,x) \times \Omega, \ I \geq 0 \textit{ a.a. in } (0,T) \times \Omega \times \mathcal{S}^2 \times (0, \infty),$$

$$\varrho \in L^\infty(0,T; L^{5/3}(\Omega)), \ \vartheta \in L^\infty(0,T; L^4(\Omega)),$$

$$\vec{u} \in L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)),$$
$$\vartheta \in L^2(0,T; W^{1,2}(\Omega)), \quad \vartheta \in L^\infty(0,T; L^4(\Omega))$$
$$\vec{B} \in L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)),$$
$$I \in L^\infty((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)),\ I \in L^\infty(0,T; L^1(\Omega\times\mathcal{S}^2\times(0,\infty)),$$

and if ϱ, $\vec{u}$, ϑ, $\vec{B}$, I satisfy the integral identities (2.12), (2.13), (2.27), (2.15), (2.28), together with the transport equation (1.4).

The stability result of [7] reads now

Theorem 2.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1 - 2.6), and that the transport coefficients μ, λ, κ, σ_a, and σ_s comply with (2.7 - 2.11).*

Let $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, \vec{B}_\varepsilon, I_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to problem (1.1 - 1.13) in the sense of Definition 2.1 such that

$$\varrho_\varepsilon(0,\cdot) \equiv \varrho_{\varepsilon,0} \to \varrho_0 \ \textit{in}\ L^{5/3}(\Omega), \tag{2.29}$$

$$\int_\Omega \left(\frac{1}{2}\varrho_\varepsilon|\vec{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon)\frac{1}{2\mu}|\vec{B}_\varepsilon|^2 - \frac{1}{2}\varrho_\varepsilon\Psi_\varepsilon + \frac{1}{2}\varrho_\varepsilon|\vec{\chi}\times\vec{x}|^2 + E_{R,\varepsilon}\right)(0,\cdot)\ \mathrm{d}x \tag{2.30}$$
$$\equiv \int_\Omega \left(\frac{1}{2\varrho_{0,\varepsilon}}|(\varrho\vec{u})_{0,\varepsilon}|^2 + (\varrho e)_{0,\varepsilon} + E_{R,0,\varepsilon}\right)\ \mathrm{d}x \le E_0,$$
$$\int_\Omega [\varrho_\varepsilon s(\varrho_\varepsilon,\vartheta_\varepsilon) + s^R(I_\varepsilon)](0,\cdot)\ \mathrm{d}x \equiv \int_\Omega (\varrho s + s^R)_{0,\varepsilon}\ \mathrm{d}x \ge S_0,$$

and

$$0 \le I_\varepsilon(0,\cdot) \equiv I_{0,\varepsilon}(\cdot) \le I_0,\ |I_{0,\varepsilon}(\cdot,\nu)| \le h(\nu)\ \textit{for a certain}\ h \in L^1(0,\infty).$$

Then

$$\varrho_\varepsilon \to \varrho\ \textit{in}\ C_{\mathrm{weak}}([0,T]; L^{5/3}(\Omega)),$$
$$\vec{u}_\varepsilon \to \vec{u}\ \textit{weakly in}\ L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)),$$
$$\vartheta_\varepsilon \to \vartheta\ \textit{weakly in}\ L^2(0,T; W^{1,2}(\Omega)),$$
$$\vec{B}_\varepsilon \to \vec{B}\ \textit{weakly in}\ L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)),$$

and

$$I_\varepsilon \to I\ \textit{weakly-(*) in}\ L^\infty((0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty)),$$

at least for suitable subsequences, where $\{\varrho, \vec{u}, \vartheta, \vec{B}, I\}$ is a weak solution of problem (1.1 - 1.6).

3 Formal scaling analysis

In order to identify the appropriate limit regime we perform a general scaling, denoting by $L_{ref},\ T_{ref},\ U_{ref},\ \rho_{ref},\ \vartheta_{ref},\ p_{ref},\ e_{ref},\ \mu_{ref},\ \lambda_{ref},\ \kappa_{ref}$, the reference hydrodynamical quantities (length, time, velocity, density, temperature, pressure, energy, viscosity, conductivity), by $I_{ref},\ \nu_{ref},\ \sigma_{a,ref},\ \sigma_{s,ref}$, the reference radiative quantities (radiative intensity, frequency, absorption and scattering coefficients), by χ_{ref} the reference rotation velocity, and by $\zeta_{ref},\ B_{ref}$ the reference electrodynamic quantities (permeability and magnetic induction).

We also assume the compatibility conditions $p_{ref} \equiv \rho_{ref} e_{ref},\ \ \nu_{ref} = \frac{k_B \vartheta_{ref}}{h},\ \ I_{ref} = \frac{2h\nu_{ref}^3}{c^2}$, $\lambda = \frac{\lambda_{ref}}{L_{ref} U_{ref}}$ and we denote by $Sr := \frac{L_{ref}}{T_{ref} U_{ref}},\ Ma := \frac{U_{ref}}{\sqrt{p_{ref}/\rho_{ref}}},\ Re := \frac{U_{ref}\rho_{ref} L_{ref}}{\mu_{ref}},\ Pe := \frac{U_{ref} p_{ref} L_{ref}}{\vartheta_{ref}\kappa_{ref}},\ Fr := \frac{U_{ref}}{\sqrt{G\rho_{ref} L_{ref}^2}},\ \mathcal{C} := \frac{c}{U_{ref}}$, the Strouhal, Mach, Reynolds, Péclet, Froude and "infrarelativistic" dimensionless numbers corresponding to hydrodynamics, by $Ro := \frac{U_{ref}}{\chi_{ref} L_{ref}}$ the Rossby number, by $Al := \frac{U_{ref}\rho_{ref}^{1/2}\zeta_{ref}^{1/2}}{B_{ref}}$ the Alfven number and by $\mathcal{L} := L_{ref}\sigma_{a,ref},\ \mathcal{L}_s := \frac{\sigma_{s,ref}}{\sigma_{a,ref}},\ \mathcal{P} := \frac{2k_B^4\vartheta_{ref}^4}{h^3c^3\ \rho_{ref} e_{ref}}$, various dimensionless numbers corresponding to radiation.

Using these scalings and using carets to symbolize renormalized variables we get

$$S = \frac{I_{ref}}{L_{ref}}\ \hat{S},$$

where

$$\hat{S} = \mathcal{L}\hat{\sigma}_a \left(B(\hat{\nu},\hat{\vartheta}) - \hat{I}\right) + \mathcal{L}\mathcal{L}_s\hat{\sigma}_s \left(\frac{1}{4\pi}\int_{\mathcal{S}^2} \hat{I}(\cdot,\vec{\omega})\ \mathrm{d}\vec{\omega} - \hat{I}\right).$$

Omitting the carets in the following, we get first the scaled equation for I, in the region $(0,T)\times\Omega\times(0,\infty)\times\mathcal{S}^2$

$$\frac{Sr}{\mathcal{C}}\ \partial_t I + \vec{\omega}\cdot\nabla_x I = s = \mathcal{L}\sigma_a\,(B - I) + \mathcal{L}\mathcal{L}_s\sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I\ \mathrm{d}\vec{\omega} - I\right), \tag{3.1}$$

where we used the same notation B for the dimensionless Planck function $B(\nu,\vartheta) = \dfrac{\nu^3}{e^{\frac{\nu}{\vartheta}} - 1}$.

Denoting also by $E^R = \int_{\mathcal{S}^2}\int_0^\infty I\ \mathrm{d}\nu\ \mathrm{d}\vec{\omega}$, the (renormalized) radiative energy, by $\vec{F}^R = \int_{\mathcal{S}^2}\int_0^\infty \vec{\omega}\ I\ \mathrm{d}\nu\ \mathrm{d}\vec{\omega}$, the renormalized radiative momentum, by $s_E = \int_{\mathcal{S}^2}\int_0^\infty s\ \mathrm{d}\nu\ \mathrm{d}\vec{\omega}$, the renormalized radiative energy source, by $\vec{s}^R = -\int_0^\infty\int_{\mathcal{S}^2}\nu^2\left[n\log n - (n+1)\log(n+1)\right]\ d\vec{\omega}d\nu$, the renormalized radiative entropy with $n = n(I) = \frac{I}{\nu^3}$, by $\vec{q}^R = -\int_0^\infty\int_{\mathcal{S}^2}\nu^2\left[n\log n - (n+1)\log(n+1)\right]\vec{\omega}\ d\vec{\omega}d\nu$, the renormalized radiative entropy flux, and taking the first moment of (3.1) with respect to $\vec{\omega}$, we get first an equation for E^R

$$\frac{1}{\mathcal{C}}\ \partial_t E^R + \nabla_x \vec{F}^R = s_E. \tag{3.2}$$

The continuity equation is now

$$Sr\ \partial_t\varrho + \mathrm{div}_x(\varrho\vec{u}) = 0, \tag{3.3}$$

and the momentum equation

$$Sr\ \partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \frac{1}{Ma^2}\ \nabla_x p(\varrho,\vartheta) + \frac{1}{Ro}\varrho\vec{\chi}\times\vec{u} = \frac{1}{Re}\ \mathrm{div}_x\mathbb{S} + \frac{1}{Fr^2}\varrho\nabla\Psi - \frac{1}{Ro}\varrho\nabla_x|\vec{\chi}\times\vec{x}|^2 + \frac{1}{Al^2}\vec{j}\times\vec{B} \tag{3.4}$$

The balance of internal energy rewrites

$$Sr\ \partial_t\left(\varrho e+\mathcal{P}E^R\right)+\mathrm{div}_x\left(\varrho e\vec{u}+\mathcal{P}\mathcal{C}\vec{F}^R\right)+\frac{1}{Pe}\ \mathrm{div}_x\vec{q}=\frac{Ma^2}{Re}\ \mathbb{S}:\nabla_x\vec{u}-p\mathrm{div}_x\vec{u}+\frac{Ma^2}{Al^2}\vec{j}\cdot\vec{E},$$

and we get the balance of matter (fluid) entropy

$$Sr\partial_t\left(\varrho s\right)+\mathrm{div}_x\left(\varrho s\vec{u}\right)+\frac{1}{Pe}\ \mathrm{div}_x\left(\frac{\vec{q}}{\vartheta}\right)=\varsigma, \tag{3.5}$$

with

$$\varsigma=\frac{1}{\vartheta}\left(\frac{Ma^2}{Re}\mathbb{S}:\nabla_x\vec{u}-\frac{1}{Pe}\frac{\vec{q}\cdot\nabla_x\vartheta}{\vartheta}+\frac{\lambda}{\zeta}|\mathrm{curl}_x\vec{B}|^2\right)+\frac{S_E}{\vartheta},$$

and the balance of radiative entropy

$$\frac{Sr}{\mathcal{C}}\ \partial_t s^R+\mathrm{div}_x\vec{q}^R=\varsigma^R, \tag{3.6}$$

with

$$\varsigma^R=\mathcal{L}\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1}-\log\frac{n(B)}{n(B)+1}\right]\sigma_a(I-B)\ d\vec{\omega}d\nu$$

$$+\mathcal{L}\mathcal{L}_s\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\left[\log\frac{n(I)}{n(I)+1}-\log\frac{n(\tilde{I})}{n(\tilde{I})+1}\right]\sigma_s(I-\tilde{I})\ d\vec{\omega}d\nu+\frac{S_E}{\vartheta}.$$

The scaled equation for the electomagnetic field is

$$Sr\partial_t\vec{B}+\mathrm{curl}_x(\vec{B}\times\vec{u})+\mathrm{curl}_x(\lambda\ \mathrm{curl}_x\vec{B})=0. \tag{3.7}$$

The scaled equation for total energy gives finally the total energy balance

$$Sr\ \frac{d}{dt}\int_\Omega\left(\frac{Ma^2}{2}\ \varrho|\vec{u}|^2+\varrho e+\frac{1}{\mathcal{C}}\ E^R+\frac{Ma^2}{Al^2}\frac{1}{\zeta}|\vec{B}|^2-\frac{1}{2}\frac{Ma^2}{Fr^2}\varrho\Psi+\frac{1}{2}\frac{Ma^2}{Ro}\varrho|\vec{\chi}\times\vec{x}|^2\right)\ dx$$

$$+\mathcal{P}\int_0^\infty\int_{\Gamma_+}\vec{\omega}\cdot\vec{n}I\ d\Gamma_+d\nu=0. \tag{3.8}$$

In the sequel we analyze the asymptotic regime defined by

$$Ma=\varepsilon,\ Al=\varepsilon,\ Fr=\varepsilon^{1/2},\ \mathcal{C}=\varepsilon^{-1},$$

where $\varepsilon>0$ is small and we put $Sr=1$, $Pe=1$, $Re=1$, $Ro=1$, $\mathcal{P}=1$, $\mathcal{L}=\mathcal{L}_s=1$ in the previous system. Plugging this scaling into the previous system gives

$$\varepsilon\partial_t I+\vec{\omega}\cdot\nabla_x I=\sigma_a\left(B-I\right)+\sigma_s\left(\frac{1}{4\pi}\int_{\mathcal{S}^2}I\ \mathrm{d}\vec{\omega}-I\right), \tag{3.9}$$

$$\partial_t\varrho+\mathrm{div}_x(\varrho\vec{u})=0, \tag{3.10}$$

$$\partial_t(\varrho\vec{u})+\mathrm{div}_x(\varrho\vec{u}\otimes\vec{u})+\frac{1}{\varepsilon^2}\ \nabla_x p(\varrho,\vartheta)+\varrho\vec{\chi}\times\vec{u}=\ \mathrm{div}_x\mathbb{S}+\frac{1}{\varepsilon}\varrho\nabla\Psi-\varrho\nabla_x|\vec{\chi}\times\vec{x}|^2+\frac{1}{\varepsilon^2}\vec{j}\times\vec{B} \tag{3.11}$$

$$\partial_t \left(\varrho e + \varepsilon E^R\right) + \mathrm{div}_x \left(\varrho e \vec{u} + \vec{F}^R\right) + \mathrm{div}_x \vec{q} = \varepsilon^2 \mathbb{S} : \nabla_x \vec{u} - p \mathrm{div}_x \vec{u} + \vec{j} \cdot \vec{E} \tag{3.12}$$

$$\partial_t \left(\varrho s + \varepsilon s^R\right) + \mathrm{div}_x \left(\varrho s \vec{u} + \vec{q}^R\right) + \ \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta}\right) \geq \varsigma_\varepsilon, \tag{3.13}$$

with

$$\varsigma_\varepsilon = \frac{1}{\vartheta}\left(\varepsilon^2 \mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} + \frac{\lambda}{\zeta} |\mathrm{curl}_x \vec{B}|^2\right)$$
$$+ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(B)}{n(B)+1}\right] \sigma_a (I - B) \ d\vec{\omega} d\nu$$
$$+ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log \frac{n(I)}{n(I)+1} - \log \frac{n(\tilde{I})}{n(\tilde{I})+1}\right] \sigma_s (I - \tilde{I}) \ d\vec{\omega} d\nu,$$

$$\partial_t \vec{B} + \mathrm{curl}_x (\vec{B} \times \vec{u}) + \mathrm{curl}_x (\lambda \ \mathrm{curl}_x \vec{B}) = 0, \tag{3.14}$$

and finally

$$\frac{d}{dt} \int_\Omega \left(\frac{1}{2} \ \varepsilon^2 \ \varrho |\vec{u}|^2 + \varrho e + \varepsilon E^R + \frac{1}{2\zeta} |\vec{B}|^2 - \frac{1}{2} \varepsilon \varrho \Psi + \frac{1}{2} \varrho |\vec{\chi} \times \vec{x}|^2\right) \ dx$$
$$+ \int_0^\infty \int_{\Gamma_+} \vec{\omega} \cdot \vec{n} I \ d\Gamma_+ d\nu = 0 \tag{3.15}$$

where $\Gamma_+ = \{(x, \vec{\omega}) \in \partial\Omega \times \mathcal{S}^2 \ : \ \vec{\omega} \cdot \vec{n}_x > 0\}$

In order to compute the limit system, we consider now the formal expansions

$$(I, \varrho, \vec{u}, \vartheta, p, \vec{B}) = (I_0, \varrho_0, \vec{u}_0, \vartheta_0, p_0, \vec{B}_0) + \varepsilon (I_1, \varrho_1, \vec{u}_1, \vartheta_1, p_1, \vec{B}_1) + O(\varepsilon^2). \tag{3.16}$$

- We first observe from (3.11) that $\varrho_0 = Cte$ and $\vartheta_0 = Cte.$, moreover

$$\nabla_x p_1 = \varrho_0 \nabla_x \Psi(\varrho_0). \tag{3.17}$$

From (3.10) we derive the incompressibility condition

$$\mathrm{div}_x \vec{u}_0 = 0, \tag{3.18}$$

and

$$\partial_t \varrho_1 + \mathrm{div}_x \left(\varrho_0 \vec{u}_1 + \varrho_1 \vec{u}_0\right) = 0. \tag{3.19}$$

- From (3.9) we get now two stationary linear transport equations for the two moments I_0 and I_1

$$\vec{\omega} \cdot \nabla_x I_0 = \sigma_{a,0} \left(B_0 - I_0\right) + \sigma_{s,0} \left(\tilde{I}_0 - I_0\right), \tag{3.20}$$

$$\vec{\omega} \cdot \nabla_x I_1 = \sigma_{a,0} \left(\partial_\vartheta B_0 \vartheta_1 - I_1\right) + \partial_\vartheta \sigma_{a,0} \left(B_0 - I_0\right) \vartheta_1 + \partial_\vartheta \sigma_{s,0} \left(\tilde{I}_0 - I_0\right) \vartheta_1 + \sigma_{s,0} \left(\tilde{I}_1 - I_1\right), \tag{3.21}$$

where $\tilde{I} := \frac{1}{4\pi} \int_{\mathcal{S}^2} I \ \mathrm{d}\vec{\omega}$, $\sigma_{a,0} = \sigma_a(\nu, \vartheta_0)$, $\sigma_{s,0} = \sigma_s(\nu, \vartheta_0)$ and $B_0 = B(\nu, \vartheta_0)$.

- The limit momentum equation is

$$\varrho_0 \left(\partial_t \vec{u}_0 + \mathrm{div}_x (\vec{u}_0 \otimes \vec{u}_0)\right) + \nabla_x \Pi + \varrho_0 \vec{\chi} \times \vec{u}_0 = \mathrm{div}_x \mathbb{S}(\vec{u}_0) + \frac{1}{\zeta} \mathrm{curl}_x \vec{B}_1 \times \vec{B}_1 + \vec{F}, \tag{3.22}$$

where $\mu_0=\mu(\vartheta_0)$, $\vec{F}=\varrho_1\nabla_x\Psi_0+\varrho_0\nabla_x|\vec{\chi}\times\vec{x}|^2$ and Π is an effective pressure.

- The limit magnetic field $\vec{B}_1$ solves

$$\partial_t\vec{B}_1+\mathrm{curl}_x(\vec{B}_1\times\vec{u}_0)+\mathrm{curl}_x(\lambda_0\ \mathrm{curl}_x\vec{B}_1)=0, \tag{3.23}$$

for $\lambda_0=\lambda(\vartheta_0)$.

- At lowest order the energy equation gives

$$\varrho_0\partial_\vartheta e_0 D\vartheta_1+\left(\varrho_0\partial_\varrho e_0+e_0\right)D\varrho_1+\left(\varrho_0 e_0+p_0\right)\vec{u}_1-\mathrm{div}_x(\kappa_0\nabla_x\vartheta_1)=-S_{E1}, \tag{3.24}$$

where D is the transport operator $D:=\partial_t+\vec{u}_0\cdot\nabla_x$.

Observing that from (3.17) we have

$$\partial_\varrho p_0 D\varrho_1+\partial_\vartheta p_0 D\vartheta_1+\varrho_0\vec{u}_0\cdot\nabla_x\Phi_0=0, \tag{3.25}$$

where $D:=\partial_t+\vec{u}_0\cdot\nabla_x$, and from (3.17)

$$\varrho_0\mathrm{div}_x\vec{u}_1=-D\varrho_1,$$

and after (3.21)

$$S_{E1}=\int_0^\infty\int_{\mathcal{S}^2}\left\{\partial_\vartheta\sigma_{a,0}\left(B_0-I_0\right)\vartheta_1+\sigma_{s,0}\left(\partial_\vartheta B_0\vartheta_1-I_1\right)\right\}d\vec{\omega}\ d\nu,$$

we end with

$$\varrho_0\overline{c_P}\left(\partial_t\vartheta_1+\mathrm{div}_x(\vartheta_1\vec{u}_0)\right)-\mathrm{div}_x\left(\kappa_0\nabla_x\vartheta_1\right)=G,$$

where $\overline{c_P}=\partial_\vartheta e_0+\frac{\vartheta_0}{\varrho_0^2}\frac{\partial_\vartheta p_0^2}{\partial_\varrho p_0}$ and $G=-\frac{\varrho_0}{\partial_\vartheta p_0}\vec{u}_0\cdot\nabla_x\Psi(\varrho_0)-\int_0^\infty\int_{\mathcal{S}^2}\left\{\partial_\vartheta\sigma_{a,0}\left(B_0-I_0\right)\vartheta_1+\sigma_{s,0}\left(\partial_\vartheta B_0\vartheta_1-I_1\right)\right\}d\vec{\omega}\ d\nu$.

Putting

$$\vec{U}=\vec{u}_0,\ \Theta=\vartheta_1,\ \vec{B}=\vec{B}_1,\ \overline{\varrho}=\varrho_0,\ \overline{\vartheta}=\vartheta_0,\ \overline{\mu}=\mu(\vartheta_0),\ \overline{\lambda}=\lambda(\varrho_0),\ \sigma_a=\sigma_{a,0},\ \sigma_s=\sigma_{s,0},$$

$$B=B_0,\ \mathbb{D}(\vec{U})=\frac{1}{2}\left(\nabla\vec{u}_0+\nabla^T\vec{u}_0\right),\ \overline{\kappa}=\kappa_0,\quad \vec{F}=-\frac{\partial_\vartheta p(\overline{\varrho},\overline{\vartheta})}{\partial_\varrho p(\overline{\varrho},\overline{\vartheta})}\nabla_x\Psi(\overline{\varrho})\ \Theta,$$

and

$$G=-\frac{\overline{\varrho}}{\partial_\vartheta p(\overline{\varrho},\overline{\vartheta})}\vec{U}\cdot\nabla_x\Psi(\overline{\varrho})+\int_0^\infty\int_{\mathcal{S}^2}\sigma_{s,0}I_1\ d\vec{\omega}\ d\nu-\Theta\int_0^\infty\int_{\mathcal{S}^2}\Big(\partial_\vartheta\sigma_{a,0}\left(B_0-I_0\right)+\sigma_{s,0}\partial_\vartheta B_0\Big)\ d\vec{\omega}\ d\nu,$$

we obtain the limit system in $(0,T)\times\Omega$

$$\mathrm{div}_x\vec{U}=0, \tag{3.26}$$

$$\overline{\varrho}(\partial_t\vec{U}+\mathrm{div}_x(\vec{U}\otimes\vec{U}))+\ \nabla_x\Pi=\mathrm{div}_x(2\overline{\mu}\ \mathbb{D}(\vec{U}))+\frac{1}{\zeta}\mathrm{curl}_x\vec{B}\times\vec{B}+\vec{F} \tag{3.27}$$

$$\partial_t\vec{B}+\mathrm{curl}_x(\vec{B}\times\vec{U})+\mathrm{curl}_x(\overline{\lambda}\ \mathrm{curl}_x\vec{B})=0, \tag{3.28}$$

$$\mathrm{div}_x\vec{B}=0, \tag{3.29}$$

$$\overline{\varrho}\ \overline{c_P}(\partial_t\Theta+\mathrm{div}_x(\Theta\vec{U}))-\mathrm{div}_x(\overline{\kappa}\nabla\Theta)=G, \tag{3.30}$$

$$\vec{\omega} \cdot \nabla_x I_0 = \sigma_a (B - I_0) + \sigma_s \left(\tilde{I}_0 - I_0 \right), \tag{3.31}$$

$$\vec{\omega} \cdot \nabla_x I_1 = \Big(\sigma_a \partial_\vartheta B + \partial_\vartheta \sigma_a (B - I_0) + \partial_\vartheta \sigma_s (\tilde{I}_0 - I_0) \Big) \Theta - \sigma_a I_1 + \sigma_s \left(\tilde{I}_1 - I_1 \right), \tag{3.32}$$

together with the *Boussinesq relation* (3.17)

$$\partial_\vartheta p_0 \nabla_x \Theta + \partial_\varrho p_0 \nabla_x r = \varrho_0 \nabla_x \Psi(\varrho_0). \tag{3.33}$$

We finally consider the boundary conditions

$$\vec{U}|_{\partial\Omega} = 0, \ \nabla\Theta \cdot \vec{n}|_{\partial\Omega} = 0, \ \vec{B} \cdot \vec{n}|_{\partial\Omega} = 0, \ \mathrm{curl}_x \vec{B} \times \vec{n}|_{\partial\Omega} = 0 \tag{3.34}$$

for (3.26)-(3.30) and

$$I_0(x, \nu, \vec{\omega}) = 0 \text{ for } x \in \partial\Omega, \ \vec{\omega} \cdot \vec{n} \leq 0 \tag{3.35}$$

$$I_1(x, \nu, \vec{\omega}) = 0 \text{ for } x \in \partial\Omega, \ \vec{\omega} \cdot \vec{n} \leq 0 \tag{3.36}$$

for (3.31) and (3.32), and the initial conditions

$$\vec{U}|_{t=0} = \vec{U}_0, \quad \Theta|_{t=0} = \Theta_0, \quad \vec{B}|_{t=0} = \vec{B}_0, \quad I_0|_{t=0} = I_{0,0}, \quad I_1|_{t=0} = I_{1,0}. \tag{3.37}$$

For this system we have the following existence result (see the Appendix for a short proof)

Theorem 3.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain.*

For any $T > 0$ the initial-bounday value problem (3.26) - (3.37) has at least a weak solution $(\vec{U}, \Theta, \vec{B}, I_0, I_1)$ such that

1.

$$\vec{U} \in L^\infty(0, T; \mathcal{H}(\Omega)) \cap L^2(0, T; \mathcal{V}(\Omega)),$$

$$\vec{B} \in L^\infty(0, T; \mathcal{V}(\Omega)) \cap L^2(0, T; \mathcal{W}(\Omega)),$$

with $\mathcal{H}(\Omega) = \{\vec{U} \in L^2(\Omega; \mathbb{R}^3), \ \mathrm{div}_x \vec{U} = 0 \ in \ \Omega, \ \vec{U}\Big|_{\partial\Omega} = 0\}$, $\mathcal{U}(\Omega) = \mathcal{H}(\Omega) \cap W_0^{1,2}(\Omega; \mathbb{R}^3))$, $\mathcal{V}(\Omega) = \left\{ \vec{b} \in L^2(\Omega; \mathbb{R}^3) \ \mathrm{div}_x \vec{b} = 0, \ \vec{b} \cdot \vec{n}\Big|_{\partial\Omega} = 0 \right\}$ and $\mathcal{W}(\Omega) = \mathcal{V}(\Omega) \cap W_0^{1,2}(\Omega; \mathbb{R}^3)$,

2.

$$\Theta \in V_2^{1,1/2}((0, T) \times \Omega),$$

where $V_2^{1,1/2}$ is the energy space defined in [20] p.6,

3.

$$I_0, I_1 \in L^\infty((0, T) \times \Omega) \times \mathcal{S}^2 \times \mathbb{R}_+),$$

with

$$\vec{\omega} \cdot \nabla_x I_0, \ \vec{\omega} \cdot \nabla_x I_1 \in L^p((0, T) \times \Omega) \times \mathcal{S}^2 \times \mathbb{R}_+),$$

for any $p > 1$.

In the following we introduce the convergence result from the primitive system (1.1)-(1.13) to the incompressible limit (3.26)-(3.37).

4 Global existence for the primitive system and uniform estimates

Let us prepare initial data such that

$$\begin{cases} \varrho(0,\cdot) = \varrho_{0,\varepsilon} = \overline{\varrho} + \varepsilon\varrho^{(1)}_{0,\varepsilon}, \\ \vec{u}(0,\cdot) = \vec{u}_{0,\varepsilon}, \\ \vartheta(0,\cdot) = \vartheta_{0,\varepsilon} = \overline{\vartheta} + \varepsilon\vartheta^{(1)}_{0,\varepsilon}, \\ I(0,\cdot,\cdot,\cdot) = I_{0,\varepsilon} = \overline{I} + \varepsilon I^{(1)}_{0,\varepsilon}, \\ \vec{B}(0,\cdot) = B_{0,\varepsilon} = \varepsilon\vec{B}^{(1)}_{0,\varepsilon}, \end{cases} \tag{4.1}$$

where $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$, $\overline{I} > 0$ and $\int_\Omega \varrho^{(1)}_{0,\varepsilon}\ dx = 0$ for any $\varepsilon > 0$.

After [13], for any locally compact Hausdorff metric space X we denote by $\mathcal{M}(X)$ the set of signed Borel measures on X and by $\mathcal{M}^+(X)$ the cone of non-negative elements of $\mathcal{M}(X)$.

From Theorem 2.1 we get immediately (by combining the approximating schemes introduced in [10] and [5]) the existence of a weak solution $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon, \vec{B}_\varepsilon)$ to the radiative MHD system (1.1 - 1.11)

Theorem 4.1 *Let $\Omega \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1 - 2.6), and that the transport coefficients μ, λ, κ, σ_a, σ_s and the equilibrium function B comply with (2.7 - 2.11). Let the initial data $(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, I_{0,\varepsilon}, \vec{B}_{0,\varepsilon})$ be given by (4.1), where $(\varrho^{(1)}_{0,\varepsilon}, \vartheta^{(1)}_{0,\varepsilon}, I^{(1)}_{0,\varepsilon}, \vec{B}^{(1)}_{0,\varepsilon})$ are bounded measurable functions.*

Then for any $\varepsilon > 0$ small enough (in order to maintain positivity of $\varrho^{(1)}_{0,\varepsilon}$ and $\vartheta^{(1)}_{0,\varepsilon}$), there exits a weak solution $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon, \vec{B}_\varepsilon)$ to the radiative Navier-Stokes system (1.1 - 1.11) for $(t, x, \vec{\omega}, \nu) \in (0,T) \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$, supplemented with the boundary conditions (1.12 - 1.13) and the initial conditions (4.1).

More precisely we have

- $$\int_0^T \int_\Omega \varrho_\varepsilon b(\varrho_\varepsilon)\,(\partial_t\phi + \vec{u}_\varepsilon\cdot\nabla_x\phi)\ dx\ dt = \int_0^T \int_\Omega \beta(\varrho_\varepsilon)\mathrm{div}_x u_\varepsilon\ \phi\ dx\ dt - \int_\Omega \varrho_{0,\varepsilon} b(\varrho_{0,\varepsilon})\ \phi(0,\cdot)\ dx, \tag{4.2}$$

  for any β such that $\beta \in L^\infty \cap C[0,\infty)$, $b(\varrho) = b(1) + \int_1^\varrho \frac{\beta(z)}{z^2}\ dz$ and any $\phi \in C^\infty_c([0,T) \times \overline{\Omega})$,

- $$\int_0^T \int_\Omega \Big(\varrho_\varepsilon\vec{u}_\varepsilon\cdot\partial_t\phi + \varrho_\varepsilon\vec{u}_\varepsilon\otimes\vec{u}_\varepsilon : \nabla_x\phi + \frac{p_\varepsilon}{\varepsilon^2}\ \mathrm{div}_x\phi + \varrho_\varepsilon\vec{\chi}\times\vec{u}_\varepsilon\cdot\phi\Big)\ dx\ dt$$

  $$= \int_0^T \int_\Omega \Big(\mathbb{S}_\varepsilon : \nabla_x\phi - \frac{1}{\varepsilon}\varrho_\varepsilon\nabla_x\Psi_\varepsilon\cdot\varphi - \frac{1}{\varepsilon^2}(\vec{j}_\varepsilon\times\vec{B}_\varepsilon)\cdot\varphi - \varrho_\varepsilon\nabla_x|\vec{\chi}\times\vec{x}|^2\cdot\varphi\Big)\ dx\ dt - \int_\Omega \varrho_{0,\varepsilon}\vec{u}_{0,\varepsilon}\cdot\phi(0,\cdot)\ dx, \tag{4.3}$$

  for any $\phi \in C^\infty_c([0,T) \times \overline{\Omega}; \mathbb{R}^3)$ with $p_\varepsilon = p(\varrho_\varepsilon, \vartheta_\varepsilon)$, $\mathbb{S}_\varepsilon = \mathbb{S}(\vec{u}_\varepsilon, \vartheta_\varepsilon)$, and $\vec{j}_\varepsilon = \frac{1}{\zeta}\mathrm{curl}_x\vec{B}_\varepsilon$,

- $$\int_\Omega \left(\frac{\varepsilon^2}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon|^2 + \varrho_\varepsilon e_\varepsilon + \varepsilon E_\varepsilon^R + \frac{1}{2\zeta}|\vec{B}_\varepsilon|^2 - \frac{1}{2}\varepsilon \varrho_\varepsilon \Psi_\varepsilon + \frac{1}{2}\varrho_\varepsilon |\vec{\chi}\times\vec{x}|^2 \right)\ dx\ dt$$
$$+ \int_0^T \int_0^\infty \int_{\Gamma_+} \vec{\omega}\cdot\vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\ d\Gamma\ d\nu\ dt$$
$$= \int_\Omega \left(\frac{\varepsilon^2}{2}\ \varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon} e_{0,\varepsilon} + \varepsilon E_{0,\varepsilon}^R + \frac{1}{2\zeta}|\vec{B}_{0,\varepsilon}|^2 - \frac{1}{2}\varepsilon_{0,\varepsilon} \varrho \Psi_{0,\varepsilon} + \frac{1}{2}\varrho_{0,\varepsilon} |\vec{\chi}\times\vec{x}|^2 \right)\ dx, \tag{4.4}$$
for a.a. $t\in(0,T)$ *with* $\Gamma_+ = \{(x,\vec{\omega}) \in \partial\Omega\times\mathcal{S}^2\ :\ \vec{\omega}\cdot\vec{n}_x \geq 0\}$ *and with* $e_\varepsilon = e(\varrho_\varepsilon,\vartheta_\varepsilon)$, $\Psi_\varepsilon = \Psi(\varrho_\varepsilon)$, $\Psi_{0,\varepsilon} = \Psi(\varrho_{0,\varepsilon})$ *and* $E_\varepsilon^R(t,x) = \int_0^\infty \int_{\mathcal{S}^2} I_\varepsilon(t,x,\vec{\omega},\nu)\ d\vec{\omega}\ d\nu$

- $$\int_0^T \int_\Omega \left(\vec{B}_\varepsilon \cdot \partial_t \varphi - (\vec{B}_\varepsilon \times \vec{u}_\varepsilon + \lambda_\varepsilon \mathrm{curl}_x \vec{B}_\varepsilon)\cdot \mathrm{curl}_x \varphi \right)\ \mathrm{d}x\ \mathrm{d}t + \int_\Omega \vec{B}_{0,\varepsilon}\cdot\varphi(0,\cdot)\ \mathrm{d}x = 0, \tag{4.5}$$
for any vector field $\varphi \in \mathcal{D}([0,T)\times\mathbb{R}^3)$*, with* $\lambda_\varepsilon = \lambda(\vartheta_\varepsilon)$.

- $$\int_0^T \int_\Omega \left(\left(\varrho_\varepsilon s_\varepsilon + \varepsilon s_\varepsilon^R\right)\partial_t\varphi + \left(\varrho_\varepsilon s_\varepsilon \vec{u}_\varepsilon + \vec{q}_\varepsilon^R\right)\cdot\nabla_x\varphi \right)\ dx\ dt + \int_0^T \int_\Omega \frac{\vec{q}_\varepsilon}{\vartheta_\varepsilon}\cdot\nabla_x\varphi\ dx\ dt$$
$$+ \left\langle \varsigma_\varepsilon^m + \varsigma_\varepsilon^R;\varphi \right\rangle_{[\mathcal{M};C]([0,T)\times\overline{\Omega})} = - \int_\Omega \left((\varrho s_{0,\varepsilon} + s_{0,\varepsilon}^R)\varphi(0,\cdot) \right)\ dx, \tag{4.6}$$
where
$$\varsigma_\varepsilon^m \geq \frac{1}{\vartheta_\varepsilon}\left(\mathbb{S}_\varepsilon : \nabla_x \vec{u}_\varepsilon - \frac{\vec{q}_\varepsilon\cdot\nabla_x\vartheta_\varepsilon}{\vartheta_\varepsilon} + \frac{\lambda_\varepsilon}{\zeta}|\mathrm{curl}_x \vec{B}_\varepsilon|^2 \right),$$
and
$$\varsigma_\varepsilon^R \geq \frac{k}{h}\int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(B_\varepsilon)}{n(B_\varepsilon)+1} \right] \sigma_{a\varepsilon}(B_\varepsilon - I_\varepsilon)\ d\vec{\omega} d\nu$$
$$+ \int_0^\infty \int_{\mathcal{S}^2} \frac{1}{\nu}\left[\log\frac{n(I_\varepsilon)}{n(I_\varepsilon)+1} - \log\frac{n(\tilde{I}_\varepsilon)}{n(\tilde{I}_\varepsilon)+1} \right] \sigma_{s\varepsilon}(\tilde{I}_\varepsilon - I_\varepsilon)\ d\vec{\omega} d\nu,$$
for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$ *with* $\varsigma_\varepsilon^m \in \mathcal{M}^+([0,T)\times\overline{\Omega})$ *and* $\varsigma_\varepsilon^R \in \mathcal{M}^+([0,T)\times\overline{\Omega})$*, and with* $\sigma_{a\varepsilon} = \sigma_a(\nu,\vartheta_\varepsilon)$, $\sigma_{s\varepsilon} = \sigma_s(\nu,\vartheta_\varepsilon)$, $B_\varepsilon = B(\nu,\vartheta_\varepsilon)$, $\vec{q}_\varepsilon = \kappa(\varrho_\varepsilon,\vartheta_\varepsilon)\nabla_x\vartheta_\varepsilon$, , $s_\varepsilon = s(\varrho_\varepsilon,\vartheta_\varepsilon)$, $s_\varepsilon^R = s^R(I_\varepsilon)$, $\vec{q}_\varepsilon^R = \vec{q}^R(I_\varepsilon)$ *and* $\tilde{I}_\varepsilon := \frac{1}{4\pi}\int_{\mathcal{S}^2} I_\varepsilon(t,x,\nu,\vec{\omega})\ \mathrm{d}\vec{\omega}$,

- $$\int_0^T \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} (\varepsilon\partial_t\psi + \vec{\omega}\cdot\nabla_x\psi)\, I_\varepsilon\ d\vec{\omega}\ d\nu\ dx\ dt$$
$$+ \int_0^T \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \left[\sigma_{a\varepsilon}\,(B_\varepsilon - I_\varepsilon) + \sigma_{s\varepsilon}\left(\tilde{I}_\varepsilon - I_\varepsilon\right) \right] \psi\ d\vec{\omega}\ d\nu\ dx\ dt,$$
$$= \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \varepsilon I_{0,\varepsilon}\psi(0,x,\vec{\omega},\nu)\ d\vec{\omega}\ d\nu\ dx + \int_0^T \int_{\Gamma_+} \int_0^\infty \vec{\omega}\cdot\vec{n}_x I_\varepsilon \psi\ d\Gamma\ d\nu\ dt, \tag{4.7}$$
for any $\psi \in C_c^\infty([0,T)\times\overline{\Omega}\times\mathcal{S}^2\times\mathbb{R}_+)$.

4.1 Uniform estimates

We recall from [13] the necessary definitions in the formalism of essential and residual sets (see [11]).

Given three numbers $\overline{\varrho} \in \mathbb{R}_+$, $\overline{\vartheta} \in \mathbb{R}_+$ and $\overline{E} \in \mathbb{R}_+$ we define $\mathcal{O}^H_{ess}$ the set of hydrodynamical essential values

$$\mathcal{O}^H_{ess} := \left\{ (\varrho, \vartheta) \in \mathbb{R}^2 \ : \ \frac{\overline{\varrho}}{2} < \varrho < 2\overline{\varrho},\ \frac{\overline{\vartheta}}{2} < \vartheta < 2\overline{\vartheta} \right\}, \tag{4.8}$$

and $\mathcal{O}^R_{ess}$ the set of radiative essential values

$$\mathcal{O}^R_{ess} := \left\{ E^R \in \mathbb{R} \ : \ \frac{\overline{E}}{2} < E^R < 2\overline{E} \right\}, \tag{4.9}$$

with $\mathcal{O}_{ess} := \mathcal{O}^H_{ess} \cup \mathcal{O}^R_{ess}$, and their residual counterparts

$$\mathcal{O}^H_{res} := (\mathbb{R}_+)^2 \backslash \mathcal{O}^H_{ess}, \quad \mathcal{O}^R_{res} := \mathbb{R}_+ \backslash \mathcal{O}^R_{ess}, \quad \mathcal{O}_{res} := (\mathbb{R}_+)^3 \backslash \mathcal{O}_{ess}. \tag{4.10}$$

Let $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, I_\varepsilon\}_{\varepsilon>0})$ a family of solutions of the scaled radiative Navier-Stokes system given in Theorem 4.1. We call $\mathcal{M}^\varepsilon_{ess} \subset (0,T) \times \Omega$ the set

$$\mathcal{M}^\varepsilon_{ess} = \left\{ (t,x) \in (0,T) \times \Omega \ : \ (\varrho_\varepsilon(t,x), \vartheta_\varepsilon(t,x), E^R_\varepsilon(t,x)) \in \mathcal{O}_{ess} \right\},$$

and $\mathcal{M}^\varepsilon_{res} = (0,T) \times \Omega \backslash \mathcal{M}^\varepsilon_{ess}$ the corresponding residual set.

To any measurable function h we associate its decomposition into essential and residual parts

$$h = [h]_{ess} + [h]_{res},$$

where $[h]_{ess} = h \cdot \mathbb{I}_{\mathcal{M}^\varepsilon_{ess}}$ and $[h]_{res} = h \cdot \mathbb{I}_{\mathcal{M}^\varepsilon_{res}}$.

Denoting by $H_{\overline{\vartheta}}$ the Helmholtz function for matter

$$H_{\overline{\vartheta}}(\varrho, \vartheta) = \varrho e - \overline{\vartheta}\ \varrho s,$$

and

$$H^R{}_{\overline{\vartheta}}(I) = E^R - \overline{\vartheta}\ s^R,$$

the corresponding radiative function and using (4.6) we rewrite (4.4) as

$$\int_\Omega \left(\frac{\varepsilon^2}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon|^2 + H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) + \varepsilon H^R{}_{\overline{\vartheta}}(I_\varepsilon) + \frac{1}{2\zeta}|\vec{B}_\varepsilon|^2 - \frac{1}{2}\varepsilon\varrho_\varepsilon\Psi_\varepsilon + \frac{1}{2}\varepsilon\varrho_\varepsilon|\vec{\chi} \times \vec{x}|^2 \right)\ dx$$

$$+ \int_0^T \int_{\Gamma_+} \vec{\omega} \cdot \vec{n}_x I_\varepsilon(t,x,\vec{\omega},\nu)\ d\Gamma\ d\nu\ dt + \overline{\vartheta}\left(\varsigma^m_\varepsilon + \varsigma^R_\varepsilon\right)\left[[0,t] \times \overline{\Omega}\right]$$

$$= \int_\Omega \left(\frac{\varepsilon^2}{2}\ \varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon}|^2 + \varrho_{0,\varepsilon} e_{0,\varepsilon} + \varepsilon E^R_{0,\varepsilon} + \frac{1}{2\zeta}|\vec{B}_{0,\varepsilon}|^2 - \frac{1}{2}\varepsilon_{0,\varepsilon}\varrho\Psi_{0,\varepsilon} + \frac{1}{2}\varepsilon\varrho_{0,\varepsilon}|\vec{\chi} \times \vec{x}|^2 \right)\ dx.$$

Observing that the total mass is a constant of motion $M = \int_\Omega \varrho_\varepsilon\ dx = \overline{\varrho}|\Omega|$ and using Hardy-Littlewood-Sobolev inequality, we get $\frac{\varepsilon}{2}\int_\Omega \varrho_\varepsilon \Psi_\varepsilon\ dx \leq \frac{G\varepsilon}{2} C M^{2/3} \|\varrho_\varepsilon\|^{4/3}_{L^{4/3}(\Omega)}$. After (2.1) and (2.5) we have also $\varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) \geq a\vartheta^4_\varepsilon + \frac{3p_\infty}{2}\varrho^{5/3}_\varepsilon$, so we have the lower bound

$$\int_\Omega H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - \frac{1}{2}\varepsilon\varrho_\varepsilon\Psi_\varepsilon\ dx \geq c \int_\Omega H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon)\ dx,$$

for ε small and a $c(\varepsilon) < 1$ and we deduce finally the energy-entropy inequality

$$\int_\Omega \left(\frac{\varepsilon^2}{2}\ \varrho_\varepsilon |\vec{u}_\varepsilon|^2 + H_{\overline{\vartheta}}(\varrho_\varepsilon, \vartheta_\varepsilon) - (\varrho_\varepsilon - \overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) + \frac{1}{2\zeta}|\vec{B}_\varepsilon|^2 + \varepsilon {H^R}_{\overline{\vartheta}}(I_\varepsilon) \right) dx$$

$$+ \int_0^T \int_{\Gamma_+} \vec{\omega} \cdot \vec{n}_x I_\varepsilon(t, x, \vec{\omega}, \nu)\ d\Gamma\ d\nu\ dt +\ \overline{\vartheta}\left(\varsigma^m_\varepsilon + \varsigma^R_\varepsilon\right) [[0, t] \times \overline{\Omega}]$$

$$\leq C \int_\Omega \left(\frac{\varepsilon^2}{2}\ \varrho_{0,\varepsilon} |\vec{u}_{0,\varepsilon}|^2 + \left(H_{\overline{\vartheta}}(\varrho_{0,\varepsilon}, \vartheta_{0,\varepsilon}) - (\varrho_{0,\varepsilon} - \overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})\right) + \frac{1}{2\zeta}|\vec{B}_{0,\varepsilon}|^2 + \varepsilon {H^R}_{\overline{\vartheta}}(I_{0,\varepsilon}) \right)\ dx. \tag{4.11}$$

Now, after Lemma 4.1 in [11] (see [13]) we have the following properties for matter and radiative Helmholtz functions

Lemma 4.1 *Let $\overline{\varrho} > 0$ and $\overline{\vartheta} > 0$ two given constants and let*

$$H_{\overline{\vartheta}}(\varrho, \vartheta) = \varrho e - \overline{\vartheta}\ \varrho s,$$

and

$$ {H^R}_{\overline{\vartheta}}(I) = E^R - \overline{\vartheta}\ s^R.$$

Let $\mathcal{O}_{ess}$ and $\mathcal{O}_{res}$ be the sets of essential and residual values introduced in (4.8- 4.10).

There exist positive constants $C_j = C_j(\overline{\varrho}, \overline{\vartheta})$ for $j = 1, \cdots, 8$ such that

1.

$$C_1\left(|\varrho - \overline{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2\right) \leq H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})$$
$$\leq C_2\left(|\varrho - \overline{\varrho}|^2 + |\vartheta - \overline{\vartheta}|^2\right), \tag{4.12}$$

for all $(\varrho, \vartheta) \in \mathcal{O}^H_{ess}$,

2.

$$H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta})$$
$$\geq \inf_{\tilde{\varrho}, \tilde{\vartheta} \in \mathcal{O}_{res}} \left\{ H_{\overline{\vartheta}}(\tilde{\varrho}, \tilde{\vartheta}) - (\tilde{\varrho} - \overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \right\} = C_3, \tag{4.13}$$

for all $(\varrho, \vartheta) \in \mathcal{O}^H_{res}$,

3.

$$H_{\overline{\vartheta}}(\varrho, \vartheta) - (\varrho - \overline{\varrho})\partial_\varrho H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) - H_{\overline{\vartheta}}(\overline{\varrho}, \overline{\vartheta}) \geq C_4\left(\varrho e(\varrho, \vartheta) + \varrho |s(\varrho, \vartheta)|\right), \tag{4.14}$$

for all $(\varrho, \vartheta) \in \mathcal{O}^H_{res}$,

4.

$$C_5 |E^R - \overline{E}|^2 \leq E^R(I) - \overline{\vartheta}\ s^R(I) \leq C_6 |E^R - \overline{E}|^2, \tag{4.15}$$

for all $E \in \mathcal{O}^R_{ess}$,

5.
$$E^R(I) - \overline{\vartheta}\ s^R(I) \geq \inf_{\tilde{I}\in\mathcal{O}_{res}} E^R(\tilde{I}) - \overline{\vartheta}\ s^R(\tilde{I}) = C_7, \tag{4.16}$$

for all $E \in \mathcal{O}^R_{res}$,

6.
$$E^R(I) - \overline{\vartheta}\ s^R(I) \geq C_8 \left(E^R(I) + |s^R(I)|\right) \tag{4.17}$$

for all $E \in \mathcal{O}^R_{res}$

Using (4.11) and Lemma 4.1, we get the following energy estimates

Lemma 4.2 *Suppose that the initial data satisfy*

$$\|[\varrho_{0,\varepsilon} - \overline{\varrho}]\|^2_{L^2(\Omega)} \leq C\varepsilon^2,\ \|[\vartheta_{0,\varepsilon} - \overline{\vartheta}]\|^2_{L^2(\Omega)} \leq C\varepsilon^2,\ \|E^R_{0,\varepsilon} - \overline{E}\|^2_{L^2(\Omega)} \leq C\varepsilon^2,\ \|\vec{B}_{0,\varepsilon}\|^2_{L^2(\Omega;\mathbb{R}^3)} \leq C\varepsilon^2,$$

and
$$\|\sqrt{\varrho_{0,\varepsilon}}\ \vec{u}_{0,\varepsilon}\|_{L^2(\Omega;\mathbb{R}^3)} \leq C,$$

the following estimates hold
$$\operatorname{ess}\sup_{t\in(0,T)} |\mathcal{M}^\varepsilon_{res}(t)| \leq C\varepsilon^2. \tag{4.18}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \|[\varrho_\varepsilon - \overline{\varrho}]_{ess}(t)\|^2_{L^2(\Omega)} \leq C\varepsilon^2, \tag{4.19}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \|[\vartheta_\varepsilon - \overline{\vartheta}]_{ess}(t)\|^2_{L^2(\Omega)} \leq C\varepsilon^2, \tag{4.20}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \|[E^R_\varepsilon - \overline{E}]_{ess}(t)\|^2_{L^2(\Omega)} \leq C\varepsilon, \tag{4.21}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \|[\varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \leq C\varepsilon^2, \tag{4.22}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \|[\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \leq C\varepsilon^2, \tag{4.23}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \|[E^R(I_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \leq C\varepsilon, \tag{4.24}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \|[s^R(I_\varepsilon)]_{res}(t)\|_{L^1(\Omega)} \leq C\varepsilon. \tag{4.25}$$

$$\left(\varsigma^m_\varepsilon + \varsigma^R_\varepsilon\right)\left[[0,t]\times\overline{\Omega}\right] \leq C\varepsilon^2, \tag{4.26}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \left\|\frac{\vec{B}_\varepsilon(t)}{\varepsilon}\right\|_{L^2(\Omega;\mathbb{R}^3)} \leq C, \tag{4.27}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\ \vec{u}_\varepsilon(t)\|_{L^2(\Omega;\mathbb{R}^3)} \leq C. \tag{4.28}$$

$$\operatorname{ess}\sup_{t\in(0,T)} \int_\Omega \left([\varrho_\varepsilon]^{\frac{5}{3}}]_{res} + [\vartheta_\varepsilon]^4]_{res}\right)(t)\ dx \leq C\varepsilon^2, \tag{4.29}$$

$$\int_0^T \|\vec{u}_\varepsilon(t)\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)}\ dt \le C, \tag{4.30}$$

$$\int_0^T \left\| \frac{\vartheta_\varepsilon - \overline{\vartheta}}{\varepsilon}(t) \right\|^2_{W^{1,2}(\Omega)}\ dt \le C, \tag{4.31}$$

$$\int_0^T \left\| \frac{\log(\vartheta_\varepsilon) - \log(\overline{\vartheta})}{\varepsilon}(t) \right\|^2_{W^{1,2}(\Omega)}\ dt \le C, \tag{4.32}$$

$$\int_0^T \left\| \frac{\vec{B}_\varepsilon(t)}{\varepsilon} \right\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)}\ dt \le C. \tag{4.33}$$

Proof: Estimate (4.18) follow after (4.13). Bounds (4.19),(4.20) and (4.24) follow after (4.12) and (4.15). Bounds (4.22) and (4.23) follow after (4.14) Bounds (4.24) and (4.25) follow after (4.17). Bounds (4.26), (4.27) and (4.28) follow after energy inequality (4.11). Bound (4.29) follows after (4.22) and the expression (2.5) of e.

From (4.26) we see that

$$\int_0^T \|\nabla_x \vec{u}_\varepsilon + \nabla_x^t \vec{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x \vec{u}_\varepsilon \mathbb{I}\|^2_{L^2\left(\Omega;\mathbb{R}^{3\times 3}\right)} dt \le C. \tag{4.34}$$

From (4.18), (4.28) and (4.34) we get (4.30). From (4.26) we get

$$\int_0^T \left\|\nabla_x \left(\frac{\vartheta_\varepsilon}{\varepsilon}\right)\right\|^2_{L^2(\Omega)} + \left\|\nabla_x \left(\frac{\log \vartheta_\varepsilon}{\varepsilon}\right)\right\|^2_{L^2(\Omega)} dt \le C,$$

which, using (4.19) and (4.20) gives (4.31) and (4.32).

Finally after (4.26) one gets

$$\left\| \frac{\mathrm{curl}_x \vec{B}_\varepsilon}{\varepsilon} \right\|^2_{L^2(\Omega;\mathbb{R}^3)} \le C,$$

and (4.33) follows by using Theorem 6.1 in [12].

Our goal in the next Section will be to prove that the incompressible system (3.26)-(3.37) is the limit of the primitive system (4.2)-(4.7) in the following sense

Theorem 4.2 *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1 - 2.6) with $P \in C^1[0,\infty) \cap C^2(0,\infty)$, and that the transport coefficients μ, η, κ, λ, σ_a, σ_s and the equilibrium function B comply with (2.7 - 2.11).*

Let $(\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, \vec{B}_\varepsilon, I_\varepsilon)$ be a weak solution of the scaled system (1.1 - 1.11) for $(t, x, \vec{\omega}, \nu) \in [0,T] \times \Omega \times \mathcal{S}^2 \times \mathbb{R}_+$, supplemented with the boundary conditions (1.12 - 1.13) and initial conditions $(\varrho_{0,\varepsilon}, \vec{u}_{0,\varepsilon}, \vartheta_{0,\varepsilon}, \vec{B}_{0,\varepsilon}, I_{0,\varepsilon})$ given by

$$\varrho_\varepsilon(0,\cdot) = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \ \vec{u}_\varepsilon(0,\cdot) = \vec{u}_{0,\varepsilon},\ \ \vartheta_\varepsilon(0,\cdot) = \overline{\vartheta} + \varepsilon \vartheta^{(1)}_{0,\varepsilon},\ \ I_\varepsilon(0,\cdot) = \overline{I} + \varepsilon I^{(1)}_{0,\varepsilon},\ \ \vec{B}_\varepsilon(0,\cdot) = \varepsilon \vec{B}^{(1)}_{0,\varepsilon},$$

where $\overline{\varrho} > 0$, $\overline{\vartheta} > 0$, $\overline{I} > 0$ are constants and

$$\int_\Omega \varrho^{(1)}_{0,\varepsilon}\ dx = 0,\ \int_\Omega \vartheta^{(1)}_{0,\varepsilon}\ dx = 0,\ \int_\Omega I^{(1)}_{0,\varepsilon}\ dx = 0,\ \int_\Omega \vec{B}^{(1)}_{0,\varepsilon}\ dx = 0 \quad \textit{for all } \varepsilon > 0.$$

Assume that

$$\begin{cases} \varrho_{0,\varepsilon}^{(1)} \to \varrho_0^{(1)} \ \ weakly-(*) \ in \ L^\infty(\Omega), \\ \vec{u}_{0,\varepsilon}^{(1)} \to \vec{U}_0 \ \ weakly-(*) \ in \ L^\infty(\Omega;\mathbb{R}^3), \\ \vartheta_{0,\varepsilon}^{(1)} \to \vartheta_0^{(1)} \ \ weakly-(*) \ in \ L^\infty(\Omega), \\ I_{0,\varepsilon}^{(1)} \to I_0^{(1)} \ \ weakly-(*) \ in \ L^\infty(\Omega\times\mathcal{S}^2\times\mathbb{R}_+), \\ \vec{B}_{0,\varepsilon}^{(1)} \to \vec{B}_0^{(1)} \ \ weakly-(*) \ in \ L^\infty(\Omega;\mathbb{R}^3), \end{cases}$$

Then

$$\operatorname{ess}\sup_{t\in(0,T)} \|\varrho_\varepsilon(t)-\overline{\varrho}\|_{L^{\frac{4}{3}}(\Omega)} \le C\varepsilon, \tag{4.35}$$

and up to subsequences

$$\vec{u}_\varepsilon \to \vec{U} \ weakly-(*) \ in \ L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \tag{4.36}$$

$$\frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon} = \vartheta^{(1)} \to \Theta \ \ weakly-(*) \ in \ L^2(0,T;W^{1,2}(\Omega)), \tag{4.37}$$

$$I_\varepsilon \to I_0 \ \ weakly-(*) \ in \ L^2(0,T;L^2(\Omega\times\mathcal{S}^2\times\mathbb{R}_+)), \tag{4.38}$$

$$\frac{\vec{B}_\varepsilon}{\varepsilon} = \vec{B}^{(1)} \to \vec{B} \ weakly-(*) \ in \ L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)), \tag{4.39}$$

and

$$\frac{I_\varepsilon-\overline{I}}{\varepsilon} = I^{(1)} \to I_1 \ \ weakly-(*) \ in \ L^2(0,T;L^2(\Omega\times\mathcal{S}^2\times\mathbb{R}_+)), \tag{4.40}$$

where $(\vec{U},\Theta,\vec{B},I_0,I_1)$ *solves the system (3.26)-(3.32).*

5 Proof of Theorem 4.2

Let us first quote the following result of [11] (see [13])

Proposition 5.1 *Let* $\{\varrho_\varepsilon\}_{\varepsilon>0}, \{\vartheta_\varepsilon\}_{\varepsilon>0}\{I_\varepsilon\}_{\varepsilon>0}$ *be three sequences of non-negative measurable functions such that*

$$\left[\varrho_\varepsilon^{(1)}\right]_{ess} \to \varrho^{(1)} \ weakly-(*) \ in \ L^\infty(0,T;L^2(\Omega)),$$

$$\left[\vartheta_\varepsilon^{(1)}\right]_{ess} \to \vartheta^{(1)} \ weakly-(*) \ in \ L^\infty(0,T;L^2(\Omega)),$$

$$\left[I_\varepsilon^{(1)}\right]_{ess} \to I^{(1)} \ weakly-(*) \ in \ L^\infty(0,T;L^2(\Omega)), \ a.e. \ in \ \mathcal{S}^2\times\mathbb{R}_+,$$

where

$$\varrho_\varepsilon^{(1)} = \frac{\varrho_\varepsilon-\overline{\varrho}}{\varepsilon}, \ \vartheta_\varepsilon^{(1)} = \frac{\vartheta_\varepsilon-\overline{\vartheta}}{\varepsilon}, \ I_\varepsilon^{(1)} = \frac{I_\varepsilon-\overline{I}}{\varepsilon}.$$

Suppose that

$$\operatorname{ess}\sup_{t\in(0,T)} |\mathcal{M}^\varepsilon_{res}(t)| \le C\varepsilon^2. \tag{5.1}$$

Let $G, G^R \in C^1(\overline{\mathcal{O}_{ess}})$ *be given functions. Then*

$$\frac{[G(\varrho_\varepsilon,\vartheta_\varepsilon)]_{ess}-G(\overline{\varrho},\overline{\vartheta})}{\varepsilon} \to \frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial\varrho}\ \varrho^{(1)} + \frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial\vartheta}\ \vartheta^{(1)},$$

weakly $-(*)$ *in* $L^\infty(0,T;L^2(\Omega))$, *and if we note*

$$\big[G^R(I_\varepsilon)\big]_{ess} := \big[G^R(I_\varepsilon(\cdot,\cdot,\vec{\omega},\nu))\big]_{ess} = G^R(I_\varepsilon)\cdot \mathbb{I}_{\mathcal{M}^\varepsilon_{ess}},\ \ for\ a.a.\ (\vec{\omega},\nu)\in \mathcal{S}^2\times\mathbb{R}_+,$$

we have

$$\frac{\big[G^R(I_\varepsilon)\big]_{ess} - G^R(\overline{I})}{\varepsilon} \to \frac{\partial G(\overline{I})}{\partial I}\ I^{(1)},$$

weakly $-(*)$ *in* $L^\infty(0,T;L^2(\Omega))$, *a.e. in* $\mathcal{S}^2\times\mathbb{R}_+$.

Moreover if $G, G^R \in C^2(\overline{\mathcal{O}_{ess}})$ *then*

$$\left\| \frac{[G(\varrho_\varepsilon,\vartheta_\varepsilon)]_{ess} - G(\overline{\varrho},\overline{\vartheta})}{\varepsilon} - \frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial \varrho}\ \Big[\varrho^{(1)}\Big]_{ess} - \frac{\partial G(\overline{\varrho},\overline{\vartheta})}{\partial \vartheta}\ \Big[\vartheta^{(1)}\Big]_{ess} \right\|_{L^\infty(0,T;L^1(\Omega))} \le C\varepsilon,$$

and

$$\left\| \frac{\big[G^R(I_\varepsilon)\big]_{ess} - G^R(\overline{I})}{\varepsilon} - \frac{\partial G(\overline{I})}{\partial I}\ \Big[I^{(1)}\Big]_{ess} \right\|_{L^\infty(0,T;L^1(\Omega))} \le C\varepsilon,$$

for a.a. $(\vec{\omega},\nu)\in\mathcal{S}^2\times\mathbb{R}_+$.

Clearly, this result provides us with the convergence properties (4.35-4.40).

To conclude the proof of Theorem 4.2, let us prove that the limit quantities $(\vec{U},\Theta,\vec{B},I_0,I_1)$ solve the target system (3.26)-(3.32).

As number of terms in the equations of our model are similar to those of the radiative Navier-Stokes-Fourier analyzed in [11] we only focus on the new contributions.

5.1 Continuity and Momentum equations

For the continuity equation, one expects that in the low Mach number limit, it reduces to the incompressibility constraint. In fact after Lemma 4.2 we know that $\int_0^T \|\vec{u}_\varepsilon(t)\|^2_{W^{1,2}(\Omega;\mathbb{R}^3)}\ dt \le C$ so passing to the limit after possible extraction of a subsequence, we deduce that

$$\vec{u}_\varepsilon \to \vec{U},\ \text{weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)). \tag{5.2}$$

In the same stroke $\varrho_\varepsilon \to \overline{\varrho}$, weakly in $L^\infty(0,T;L^{5/3}(\Omega;\mathbb{R}^3))$. So we can pass to the limit in the weak continuity equation (4.2) which gives $\int_0^T\int_\Omega \vec{U}\ \nabla_x\phi\ dx\ dt = 0$ for all $\phi\in\mathcal{D}((0,T)\times\overline{\Omega})$, which rewrites

$$\mathrm{div}_x\vec{U} = 0,\ \ \text{a.e. in } (0,T)\times\Omega,\quad \vec{U}\Big|_{\partial\Omega} = 0,$$

provided $\partial\Omega$ is regular.

For the momentum equation one knows that due to possible strong time oscillations of the gradient component of velocity, one has only $\varrho_\varepsilon\vec{u}_\varepsilon\otimes\vec{u}_\varepsilon \to \overline{\varrho\vec{U}\otimes\vec{U}}$ weakly in $L^2(0,T;L^{\frac{30}{29}}(\Omega;\mathbb{R}^3))$. However one can show after the analysis in [13] that one can pass to the limit in the convective term and obtain

$$\int_0^T\int_\Omega \overline{\varrho\,\vec{U}\otimes\vec{U}} : \nabla_x\phi\ dx\ dt \to \int_0^T\int_\Omega \overline{\varrho}\,\vec{U}\otimes\vec{U} : \nabla_x\phi\ dx\ dt.$$

Moreover after the hypotheses on the pressure law, the temperature ϑ_ε is bounded in $L^\infty((0,T);L^4(\Omega))\cap L^2(0,T;L^6(\Omega))$, which implies that $\mathbb{S}_\varepsilon \to \mu(\overline{\vartheta})(\nabla_x \vec{U} + \nabla_x^t \vec{U})$ weakly in $L^q(0,T;L^q(\Omega;\mathbb{R}^3))$ for a $q>1$.

So taking a divergence free test vector field ϕ in (4.3), we have

$$\int_0^T \int_\Omega (\varrho_\varepsilon \vec{u}_\varepsilon \cdot \partial_t \phi + \varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon : \nabla_x \phi + \varrho_\varepsilon \vec{\chi} \times \vec{u}_\varepsilon \cdot \phi)\ dx\ dt$$

$$= \int_0^T \int_\Omega \Big(\mathbb{S}_\varepsilon : \nabla_x \phi - \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \nabla_x \Psi_\varepsilon \cdot \varphi - \frac{1}{\zeta} \frac{\mathrm{curl}_x \vec{B}_\varepsilon}{\varepsilon} \times \frac{\vec{B}_\varepsilon}{\varepsilon} \cdot \varphi - \varrho_\varepsilon \nabla_x |\vec{\chi} \times \vec{x}|^2 \cdot \varphi \Big)\ dx\ dt - \int_\Omega \varrho_{0,\varepsilon} \vec{u}_{0,\varepsilon} \cdot \phi(0,\cdot)\ dx. \tag{5.3}$$

Moreover, using (2.15) together with estimates (4.27), (4.33) and Lions-Aubin lemma we get

$$\frac{\vec{B}_\varepsilon}{\varepsilon} \to \vec{B} \text{ weakly in } L^2(0,T;W^{1,2}(\Omega;\mathbb{R}^3)) \text{ and strongly in } L^2((0,T)\times\Omega;\mathbb{R}^3), \tag{5.4}$$

$$\frac{1}{\zeta} \frac{\mathrm{curl}_x \vec{B}_\varepsilon}{\varepsilon} \times \frac{\vec{B}_\varepsilon}{\varepsilon} \to \frac{1}{\zeta} \mathrm{curl}_x \vec{B} \times \vec{B} \text{ weakly in } L^q((0,T)\times\Omega;\mathbb{R}^3),$$

for a certain $q > 1$.

Then passing to the limit and using (4.36)-(4.40), we get

$$\int_0^T \int_\Omega \Big(\overline{\varrho} \vec{U} \cdot \partial_t \phi + \overline{\varrho} \vec{U} \otimes \vec{U} : \nabla_x \phi + \overline{\varrho} \vec{\chi} \times \vec{U} \cdot \phi \Big)\ dx\ dt$$

$$= \int_0^T \int_\Omega \Big(\mu(\overline{\vartheta})(\nabla_x \vec{U} + \nabla_x^t \vec{U}) : \nabla_x \phi - \varrho_1 \nabla_x \Psi(\overline{\varrho}) \cdot \phi - \frac{1}{\zeta} \mathrm{curl}_x \vec{B} \times \vec{B} \cdot \phi - \overline{\varrho} \nabla_x |\vec{\chi} \times \vec{x}|^2 \cdot \phi \Big)\ dx\ dt - \int_\Omega \overline{\varrho} \vec{U}_0 \cdot \phi\ dx,$$

provided that $\vec{u}_{0,\varepsilon} \to \vec{U}_0$ weakly $*$ in $L^\infty(\Omega;\mathbb{R}^3)$.

As in [13], the formal relation between $\varrho^{(1)}$ and $\vartheta^{(1)}$ is recovered by multiplying the momentum equation by ε. One gets, using Proposition 5.1 and passing to the limit

$$\int_0^T \int_\Omega \Big(\nabla_x p^{(1)} - \overline{\varrho} \nabla_x \Psi(\overline{\varrho}) \Big) \cdot \phi\ dx\ dt = 0, \tag{5.5}$$

which is the weak formulation of

$$\partial_\varrho p(\overline{\varrho}, \overline{\vartheta}) \nabla_x \varrho^{(1)} + \partial_\vartheta p(\overline{\varrho}, \overline{\vartheta}) \nabla_x \vartheta^{(1)} - \overline{\varrho} \nabla_x \Psi(\overline{\varrho}) = 0. \tag{5.6}$$

5.2 Radiative transfer equation

Using the L^∞ bound shown in the previous sections for I_ε, it is clear that $I_\varepsilon \to I_0$ weakly in $L^2((0,T)\times\Omega\times\mathcal{S}^2\times\mathbb{R}_+)$, and we have also after Lemma 4.2 $\vartheta_\varepsilon \to \overline{\vartheta}$ weakly in $L^2(0,T;W^{1,2}(\Omega))$.

Using the cut-off hypotheses (2.9)(2.11), we can pass to the limit which gives

$$\int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \vec{\omega} \cdot \nabla_x \psi\ I_0\ d\vec{\omega}\ d\nu\ dx + \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \Big[\sigma_a(\nu,\overline{\vartheta}) \left(B(\nu,\overline{\vartheta}) - I_0 \right) + \sigma_s(\nu,\overline{\vartheta}) \Big(\tilde{I}_0 - I_0 \Big) \Big] \psi\ d\vec{\omega}\ d\nu\ dx$$

$$= \int_{\Gamma_+} \int_0^\infty \vec{\omega} \cdot \vec{n}_x I_0\ \psi\ d\Gamma\ d\nu,$$

using the same notation for any time-independent test function $\psi \in C_c^\infty(\overline{\Omega} \times \mathcal{S}^2 \times \mathbb{R}_+)$, which is the weak formulation of the stationary problem

$$\vec{\omega} \cdot \nabla_x I_0 = S_0, \tag{5.7}$$

with the boundary condition

$$I_0 = 0 \quad \text{on } \Gamma_+, \tag{5.8}$$

where $S_0 = \sigma_a(\nu, \overline{\vartheta}) \left(B(\nu, \overline{\vartheta}) - I_0 \right) + \sigma_s(\nu, \overline{\vartheta}) \left(\tilde{I}_0 - I_0 \right)$.

Now from (4.7)

$$\int_0^T \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} (\varepsilon \partial_t \psi + \vec{\omega} \cdot \nabla_x \psi) \; \frac{I_\varepsilon - I_0}{\varepsilon} \; d\vec{\omega} \; d\nu \; dx \; dt + \int_0^T \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \left[\frac{S_\varepsilon - S_0}{\varepsilon} \right] \psi \; d\vec{\omega} \; d\nu \; dx \; dt,$$

$$= \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \varepsilon \frac{I_{0,\varepsilon} - I_0}{\varepsilon} \; \psi(0, x, \vec{\omega}, \nu) \; d\vec{\omega} \; d\nu \; dx + \int_0^T \int_{\Gamma_+} \int_0^\infty \vec{\omega} \cdot \vec{n}_x \; \frac{I_\varepsilon - I_0}{\varepsilon} \psi \; d\Gamma \; d\nu \; dt,$$

for any $\psi \in C_c^\infty([0,T] \times \overline{\Omega} \times \mathcal{S}^2 \times \mathbb{R}_+)$, with $S_\varepsilon - S_0 = S(I_\varepsilon) - S(I_0)$. From Proposition 5.1, we get

$$\frac{S_\varepsilon - S_0}{\varepsilon} \to S_1 := \partial_\vartheta(\sigma_a B)(\nu, \overline{\vartheta}) \vartheta^{(1)} - \partial_\vartheta \sigma_a(\nu, \overline{\vartheta}) \vartheta^{(1)} I_0 - \sigma_a(\nu, \overline{\vartheta}) I_1$$

$$+ \partial_\vartheta \sigma_s(\nu, \overline{\vartheta}) \vartheta^{(1)} \tilde{I}_0 + \sigma_s(\nu, \overline{\vartheta}) \tilde{I}_1 - \partial_\vartheta \sigma_s(\nu, \overline{\vartheta}) \vartheta^{(1)} I_0 - \sigma_s(\nu, \overline{\vartheta}) I_1,$$

weakly in $L^\infty((0,T); L^2(\Omega \times \mathcal{S}^2 \times \mathbb{R}_+))$ with $I_1 := I^{(1)}$.

Passing to the limit we find the limit equation

$$\int_\Omega \int_0^\infty \int_{\mathcal{S}^2} \vec{\omega} \cdot \nabla_x \psi \; I_1 \; d\vec{\omega} \; d\nu \; dx + \int_\Omega \int_0^\infty \int_{\mathcal{S}^2} S_1 \psi \; d\vec{\omega} \; d\nu \; dx, = \int_{\Gamma_+} \int_0^\infty \vec{\omega} \cdot \vec{n}_x \; I_1 \psi \; d\Gamma \; d\nu, \tag{5.9}$$

using the same notation for any time-independent test function $\psi \in C_c^\infty(\overline{\Omega} \times \mathcal{S}^2 \times \mathbb{R}_+)$ which is the weak formulation of the stationary problem

$$\vec{\omega} \cdot \nabla_x I_1 = S_1, \tag{5.10}$$

with the boundary condition

$$I_1 = 0 \quad \text{on } \Gamma_+. \tag{5.11}$$

5.3 Entropy balance

We rewrite equation (4.6) as

$$\int_0^T \int_\Omega \left\{ \varrho_\varepsilon \; \frac{s_\varepsilon - \overline{s}}{\varepsilon} \left(\partial_t \varphi + \vec{u}_\varepsilon \cdot \nabla_x \varphi \right) + \frac{s_\varepsilon^R - \overline{s^R}}{\varepsilon} \; \varepsilon \partial_t \varphi + \frac{\vec{q}_\varepsilon^R - \overline{\vec{q}^R}}{\varepsilon} \cdot \nabla_x \varphi \right\} \; dx \; dt$$

$$+ \int_0^T \int_\Omega \frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon} \nabla_x \left(\frac{\vartheta_\varepsilon}{\varepsilon} \right) \cdot \nabla_x \varphi \, dx \, dt + \frac{1}{\varepsilon} \left\langle \varsigma_\varepsilon^m + \varsigma_\varepsilon^R; \varphi \right\rangle_{[\mathcal{M};C]([0,T \times \overline{\Omega})} = - \int_\Omega \left\{ \left(\varrho_{0,\varepsilon} \frac{s_{0,\varepsilon} - \overline{s}}{\varepsilon} + \varepsilon \frac{s_\varepsilon^R - \overline{s^R}}{\varepsilon} \right) \varphi(0, \cdot) \right\} \; dx,$$

Similarly to [13], using Proposition 5.1 and energy estimates, we see that

$$\varrho_\varepsilon \; \frac{s_\varepsilon - \overline{s}}{\varepsilon} \to \overline{\varrho} \left(\partial_\varrho s(\overline{\varrho}, \overline{\vartheta}) \varrho^{(1)} + \partial_\vartheta s(\overline{\varrho}, \overline{\vartheta}) \vartheta^{(1)} \right),$$

weakly $*$ in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$,

$$\frac{\kappa(\vartheta_\varepsilon)}{\vartheta_\varepsilon}\nabla_x\Big(\frac{\vartheta_\varepsilon}{\varepsilon}\Big) \to \frac{\kappa(\overline{\vartheta})}{\overline{\vartheta}}\ \nabla_x \vartheta^{(1)},$$

weakly $*$ in $L^2(0,T;L^2(\Omega;\mathbb{R}^3))$ and

$$\frac{1}{\varepsilon}\left\langle \varsigma_\varepsilon^m + \varsigma_\varepsilon^R;\phi\right\rangle_{[\mathcal{M};C]([0,T\times\overline{\Omega})} \to 0.$$

Moreover

$$\varrho_\varepsilon\ \frac{s_\varepsilon - \overline{s}}{\varepsilon}\cdot \vec{u}_\varepsilon \to \overline{\varrho}\left(\partial_\varrho s(\overline{\varrho},\overline{\vartheta})\varrho^{(1)} + \partial_\vartheta s(\overline{\varrho},\overline{\vartheta})\vartheta^{(1)}\right)\cdot \vec{U},$$

weakly $*$ in $L^2(0,T;L^{3/2}(\Omega;\mathbb{R}^3))$. Now applying Proposition 5.1 in the same stroke, we get

$$\varepsilon\frac{s_\varepsilon^R - \overline{s^R}}{\varepsilon} \to 0,$$

weakly $*$ in $L^\infty(0,T;L^2(\Omega;\mathbb{R}^3))$.

Let us compute the limit of $\frac{\vec{q}_\varepsilon^R - \overline{\vec{q}^R}}{\varepsilon}$. We have

$$\vec{q}_\varepsilon^R = \vec{q}^R(I_\varepsilon) = -\int_0^\infty\int_{\mathcal{S}^2}\nu^2\left\{n_\varepsilon \log n_\varepsilon - (n_\varepsilon+1)\log(n_\varepsilon+1)\right\}\ d\vec{\Omega}\ d\nu,$$

with $n_\varepsilon = n(I_\varepsilon) = \frac{I_\varepsilon}{\nu^3}$.

Applying once more Proposition 5.1 with $G^R(I) = n(I)\log n(I) - (n(I)+1)\log(n(I)+1)$ and integrating on $\mathcal{S}^2\times\mathbb{R}_+$, we find

$$\frac{\vec{q}_\varepsilon^R - \overline{\vec{q}^R}}{\varepsilon} \to \int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\nu}\ \log\left(\frac{n(\overline{I})+1}{n(\overline{I})}\right)\vec{\omega}\ I^{(1)}\ d\vec{\omega}\ d\nu,$$

and as $\frac{n(\overline{I})+1}{n(\overline{I})} = \frac{\nu}{\vartheta}$, we have

$$\frac{\vec{q}_\varepsilon^R - \overline{\vec{q}^R}}{\varepsilon} \to \frac{1}{\overline{\vartheta}}\ \vec{F}^R(I^{(1)}),$$

with the radiative momentum $\vec{F}^R(I^{(1)}) = \int_0^\infty\int_{\mathcal{S}^2}\vec{\omega}\ I^{(1)}\ d\vec{\omega}\ d\nu$. So

$$\int_0^T\int_\Omega\left(\frac{\vec{q}_\varepsilon^R - \overline{\vec{q}^R}}{\varepsilon}\right)\cdot\nabla_x\varphi\ dx\ dt \to -\int_0^T\int_\Omega\frac{\mathrm{div}_x\vec{F}^R(I^{(1)})}{\overline{\vartheta}}\ \phi\ dx\ dt.$$

As we have, from (5.10)

$$\mathrm{div}_x\vec{F}^R = \int_0^\infty\int_{\mathcal{S}^2}\Big[\partial_\vartheta\sigma_a(\nu,\overline{\vartheta})\left(B(\nu,\overline{\vartheta}) - I_0\right)\vartheta^{(1)} + \sigma_a(\nu,\overline{\vartheta})\left(\partial_\vartheta B(\nu,\overline{\vartheta})\vartheta^{(1)} - I_1\right)\Big]\ d\vec{\omega}\ d\nu,$$

the limit contribution in the right-hand side becomes

$$-\int_0^T\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\overline{\vartheta}}\Big[\partial_\vartheta\sigma_a(\nu,\overline{\vartheta})\left(B(\nu,\overline{\vartheta}) - I_0\right)\vartheta^{(1)} + \sigma_a(\nu,\overline{\vartheta})\left(\partial_\vartheta B(\nu,\overline{\vartheta})\vartheta^{(1)} - I_1\right)\Big]\phi\ \ d\vec{\omega}\ d\nu\ \ dx\ dt,$$

Gathering all of these terms, we find the limit equation for entropy

$$-\int_0^T\int_\Omega \overline{\varrho}\left(\partial_\varrho s(\overline{\varrho},\overline{\vartheta})\varrho^{(1)}+\partial_\vartheta s(\overline{\varrho},\overline{\vartheta})\vartheta^{(1)}\right)\left(\partial_t\phi+\vec{U}\cdot\nabla_x\phi\right)\,dx\,dt-\int_0^T\int_\Omega\frac{\kappa(\overline{\varrho},\overline{\vartheta})}{\overline{\vartheta}}\,\nabla_x\vartheta^{(1)}\,\nabla_x\phi\,dx\,dt$$

$$+\int_0^T\int_\Omega\int_0^\infty\int_{\mathcal{S}^2}\frac{1}{\overline{\vartheta}}\left[\partial_\vartheta\sigma_a(\nu,\overline{\vartheta})\left(B(\nu,\overline{\vartheta})-I_0\right)\vartheta^{(1)}+\sigma_a(\nu,\overline{\vartheta})\left(\partial_\vartheta B(\nu,\overline{\vartheta})\vartheta^{(1)}-I_1\right)\right]\phi\;\,d\vec{\omega}\,d\nu\;\,dx\,dt$$

$$=-\int_\Omega\overline{\varrho}\left(\partial_\varrho s(\overline{\varrho},\overline{\vartheta})\varrho_0^{(1)}+\partial_\vartheta s(\overline{\varrho},\overline{\vartheta})\vartheta_0^{(1)}\right)\phi(0,\cdot)\,dx.$$

Using (5.6), it is routine to check that we finally obtain the thermal equation (3.30).

5.4 Maxwell equation

From (5.2) and (5.4) we get

$$\frac{\vec{B}_\varepsilon}{\varepsilon}\times\vec{u}\to\vec{B}\times\vec{U}\text{ weakly in }L^q(0,T;L^q(\Omega,\mathbb{R}^3))\text{ for }q>1,$$

and

$$\lambda\mathrm{curl}_x\frac{\vec{B}_\varepsilon}{\varepsilon}\to\lambda\mathrm{curl}_x\vec{B}\text{ weakly in }L^2(0,T,L^2(\Omega,\mathbb{R}^3)).$$

Then it is easy to pass to the limit in (4.5).

Appendix: Proof of Theorem 3.1

1. The stationary radiative problem (3.31),(3.35) has a weak solution $I_0\in L^\infty(\Omega\times\mathcal{S}^2\times\mathbb{R}_+)$ such that $\vec{\omega}\cdot\nabla_x I_0\in L^p(\Omega\times\mathcal{S}^2\times\mathbb{R}_+)$ for any $p>1$, after Theorem 1 and Proposition 2 of [2].

2. Consider now the linearly coupled problem for the remaining equations

$$\mathrm{div}_x\vec{U}=0, \tag{.12}$$

$$\partial_t\vec{U}+(\vec{U}\cdot\nabla_x)\vec{U}+\;\nabla_x\Pi-\overline{\mu}\Delta\vec{U}+\frac{1}{\zeta}\nabla_x\left(\frac{\vec{B}^2}{2}\right)-\frac{1}{\zeta}(\vec{B}\cdot\nabla_x)\vec{B}=\vec{\alpha}\Theta, \tag{.13}$$

$$\partial_t\vec{B}+(\vec{U}\cdot\nabla_x)\vec{B}+(\vec{B}\cdot\nabla_x)\vec{U}-\overline{\lambda}\Delta\vec{B}=0, \tag{.14}$$

$$\mathrm{div}_x\vec{B}=0, \tag{.15}$$

$$\partial_t\Theta+(\vec{U}\cdot\nabla_x)\Theta-\mathrm{div}_x(\overline{K}\nabla\Theta)=\vec{\beta}\cdot\vec{U}+\eta\Theta+\int_0^\infty\int_{\mathcal{S}^2}\sigma_s(\nu,\overline{\vartheta})I_1(x,\nu,\vec{\omega})\,d\vec{\omega}\,d\nu, \tag{.16}$$

$$\vec{\omega}\cdot\nabla_x I_1+\sigma_a I_1-\sigma_s\left(\tilde{I}_1-I_1\right)=\xi\Theta, \tag{.17}$$

where $\vec{\alpha}\in(C^\infty(\Omega))^3$, $\vec{\beta}\in(L^\infty(\Omega))^3$, $\eta,\xi\in L^\infty(\Omega)$, together with the boundary conditions

$$\vec{U}|_{\partial\Omega}=0,\;\nabla\Theta\cdot\vec{n}|_{\partial\Omega}=0,\;\vec{B}\cdot\vec{n}|_{\partial\Omega}=0,\;\mathrm{curl}_x\vec{B}\times\vec{n}|_{\partial\Omega}=0 \tag{.18}$$

for (.12)-(.14) and

$$I_1(x,\nu,\vec{\omega}) = 0 \text{ for } x \in \partial\Omega,\ \vec{\omega}\cdot\vec{n} \le 0 \tag{.19}$$

for (.15), and the initial conditions

$$\vec{U}|_{t=0} = \vec{U}_0,\quad \Theta|_{t=0} = \Theta_0,\quad \vec{B}|_{t=0} = \vec{B}_0,\quad I_1|_{t=0} = I_{1,0}. \tag{.20}$$

In order to apply Schauder's fixed point method used by Nečas and Roubíček [23] (see [28] Chap. XII.2) we first consider, for Θ given, the solution $(\vec{U},\vec{B},I_1)$ of the"radiative-MHD problem"

$$\mathrm{div}_x\vec{U} = 0, \tag{.21}$$

$$\partial_t\vec{U} + (\vec{U}\cdot\nabla_x)\vec{U} + \ \nabla_x\Pi - \overline{\mu}\Delta\vec{U} = -\frac{1}{\zeta}\mathrm{curl}_x\vec{B}\times\vec{B} - \vec{\alpha}\Theta, \tag{.22}$$

$$\partial_t\vec{B} + (\vec{U}\cdot\nabla_x)\vec{B} - \overline{\lambda}\Delta\vec{B} = 0, \tag{.23}$$

$$\mathrm{div}_x\vec{B} = 0, \tag{.24}$$

$$\vec{\omega}\cdot\nabla_x I_1 + \sigma_a I_1 - \sigma_s\left(\tilde{I}_1 - I_1\right) = \xi\Theta, \tag{.25}$$

with

$$\vec{U}|_{\partial\Omega} = 0,\ \vec{B}\cdot\vec{n}|_{\partial\Omega} = 0,\ \mathrm{curl}_x\vec{B}\times\vec{n}|_{\partial\Omega} = 0,$$

and

$$\vec{U}|_{t=0} = \vec{U}_0,\quad \vec{B}|_{t=0} = \vec{B}_0,\quad I_1|_{t=0} = I_{1,0}.$$

The mhd part has a weak solution $\vec{U}\in L^2(0,T;\mathcal{U}(\Omega))$, $\vec{B}\in L^2(0,T;\mathcal{W}(\Omega))$ after an extension of the Leray-Hopf Theorem (see [29]). Moreover the inhomogeneous stationary radiative equation (.25) also has a weak solution $I_1\in L^2((0,T)\times\Omega)\times\mathcal{S}^2\times\mathbb{R}_+)$ after Theorem 1 and Proposition 2 of [2]. Consequently the mapping

$$\mathcal{A}:\Theta\to(\vec{U},\vec{B},I_1):\ L^2(0,T;W^{-1,2}(\Omega))\to L^2(0,T;\mathcal{U}(\Omega))\times L^2(0,T;\mathcal{W}(\Omega))\times L^2((0,T)\times\Omega)\times\mathcal{S}^2\times\mathbb{R}_+),$$

is continuous.

Then we consider the solution Θ of the transport-diffusion equation

$$\partial_t\Theta + (\vec{V}\cdot\nabla_x)\Theta - \mathrm{div}_x(\overline{K}\nabla\Theta) - \eta\Theta = \vec{\beta}\cdot\vec{U} + \int_0^\infty\int_{\mathcal{S}^2}\sigma_s(\nu,\overline{\vartheta})I_1(x,\nu,\vec{\omega})\ d\vec{\omega}\ d\nu, \tag{.26}$$

with

$$\nabla\Theta\cdot\vec{n}|_{\partial\Omega} = 0 \ \text{ and } \ \Theta|_{t=0} = \Theta_0.$$

It has a weak solution $\Theta\in V_2^{1,1/2}((0,T)\times\Omega)$ after Theorem 5.1 in [20] Chapter III, moreover $\Theta\in L^2(0,T;W^{-1,2}(\Omega))$ and the mapping

$$\mathcal{B}:(\vec{U},I_1)\to\Theta:\ L^2(0,T;\mathcal{U}(\Omega))\times\Omega)\times\mathcal{S}^2\times\mathbb{R}_+)\to L^2(0,T;W^{-1,2}(\Omega)),$$

is also continuous.

So we can follow verbatim the scheme of proof of Proposition 12.6 in [23] to conclude.

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