Singular limits in thermodynamics of fluids

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Basic principle of mathematical modeling

Johann von Neumann
[1903-1957]

In mathematics you don't understand things. You just get used to them.

All pictures in the text thanks to wikipedia

Eulerian description of motion

Leonhard Paul Euler [1707-1783]

Physical space

- time $t \in [0, \infty)$
- position $\mathbf{x} \in \Omega \subset R^3$

Thermostatic variables

- mass density $\varrho = \varrho(t, x)$
- absolute temperature $\vartheta = \vartheta(t, x)$

Motion

- macroscopic velocity $\mathbf{u} = \mathbf{u}(t, x)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t, \mathbf{x}) = \mathbf{u}\Big(t, \mathbf{X}(t, \mathbf{x})\Big), \ \mathbf{X}(0, \mathbf{x}) = \mathbf{x}$$

Conservation (balance) laws

Integral form ("natural")

$$\int_B d(t_2, x) - d(t_1, x) \; \mathrm{d}x$$

$$= - \int_{t_1}^{t_2} \int_{\partial B} \mathbf{F}(t, x) \cdot \mathbf{n} \mathrm{dS}_x \; \mathrm{d}t + \int_{t_1}^{t_2} \int_B S(t, x) \; \mathrm{d}x\mathrm{d}t$$

- **density** $d = d(t, x)$ • **flux** $\mathbf{F} = \mathbf{F}(t, x)$ • **source** $S = S(t, x)$

Flux vector

$$\mathbf{F} = \boxed{d\mathbf{u}} + \mathbf{F}_\mathrm{d} \quad \text{convective flux} + \text{diffusive flux}$$

Differential form

$$\partial_t d(t, x) + \mathrm{div}_x \mathbf{F}(t, x) = S(t, x)$$

General structure

Field equations - linear

$$\partial_t d_i + \mathrm{div}_x \mathbf{F}_i = S_i \ \ i = 1, \dots, N$$

Constitutive relations - non-linear

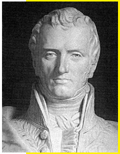
$$\mathbf{F}_i = \mathbf{F}_i(d_1, \dots, d_N), \ \ i = 1, \dots, N$$

$$S_i = S_i(d_1, \dots, d_N), \ \ i = 1, \dots, N$$

Constitutive relations - implicit

$$\mathcal{A}_j(d_1, \dots, d_N, \mathbf{F}_1, \dots, \mathbf{F}_N, S_1, \dots, S_N) = 0, \ \ j = 1, \dots, M$$

Navier-Stokes-Fourier system



Claude Louis Marie Henri Navier [1785-1836]

Mass conservation

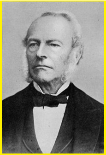
$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p = \mathrm{div}_x \mathbb{S} + \varrho \mathbf{f}$$

Internal energy balance

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u}$$

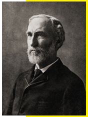


George Gabriel Stokes [1819-1903]

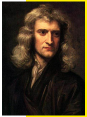
Total energy

$$E = \frac{1}{2} \varrho |\mathbf{u}|^2 + \varrho e$$

Constitutive relations



Willard Gibbs
[1839-1903]



Isaac Newton
[1643-1727]

Gibbs' equation

$$\vartheta D s(\varrho,\vartheta) = D e(\varrho,\vartheta) + p(\varrho,\vartheta) D\left(\frac{1}{\varrho}\right)$$

Newton's rheological law

$$\mathbb{S} = \mu\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta\,\mathrm{div}_x \mathbf{u}\mathbb{I}$$

Fourier's law

$$\mathbf{q} = -\kappa \nabla_x \vartheta$$

Joseph Fourier
[1768-1830]

Compressible Navier-Stokes system

Equation of continuity

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Equation of motion

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \varrho \omega \times \mathbf{u} + \nabla_x p(\varrho) = \mu \Delta_x \mathbf{u} + \lambda \nabla_x \mathrm{div}_x \mathbf{u} + \varrho \mathbf{f}$$

External forces

$\omega \parallel [0,0,1]$ axis of rotation

$$\mathbf{f} = \underbrace{\nabla_x G}_{\text{gravitational force}} + \underbrace{\nabla_x |\mathbf{x} \times \omega|^2}_{\text{centrifugal force}}, \ G \text{ gravitational potential}$$

Scaled equations

Scaling

$$X \approx \frac{X}{X_{\rm char}}$$

Mass conservation

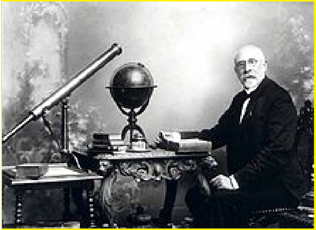
$$[{\rm Sr}]\partial_t \varrho + {\rm div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$[{\rm Sr}]\partial_t(\varrho \mathbf{u}) + {\rm div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{[{\rm Ro}]}\varrho \omega \times \mathbf{u} + \left[\frac{1}{{\rm Ma}^2}\right] \nabla_x p(\varrho)$$

$$= \left[\frac{1}{{\rm Re}}\right] (\Delta_x \mathbf{u} + \lambda \nabla_x {\rm div}_x \mathbf{u}) + (\text{external forces})$$

Characteristic numbers - Strouhal number



Čeněk Strouhal
[1850-1922]

Strouhal number

$$[\mathrm{Sr}] = \frac{\mathrm{length}_{\mathrm{char}}}{\mathrm{time}_{\mathrm{char}} \mathrm{velocity}_{\mathrm{char}}}$$

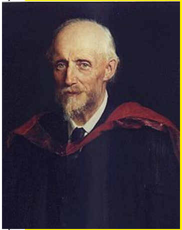
Scaling by means of Strouhal number is used in the study of the long-time behavior of the fluid system, where the characteristic time is large

Ernst Mach [1838-1916]

Mach number

$$[\mathrm{Ma}] = \frac{\mathrm{velocity}_{\mathrm{char}}}{\sqrt{\mathrm{pressure}_{\mathrm{char}} / \mathrm{density}_{\mathrm{char}}}}$$

Mach number is the ratio of the characteristic speed to the speed of sound in the fluid. Low Mach number limit, where, formally, the speed of sound is becoming infinite, characterizes incompressibility



Osborne Reynolds
[1842-1912]

Reynolds number

$$[\mathrm{Re}] = \frac{\mathrm{density}_{\mathrm{char}}\mathrm{velocity}_{\mathrm{char}}\mathrm{length}_{\mathrm{char}}}{\mathrm{viscosity}_{\mathrm{char}}}$$

High Reynolds number is attributed to turbulent flows, where the viscosity of the fluid is negligible

Rossby number

Carl Gustav Rossby
[1898-1957]

Rossby number

$$[\mathrm{Ro}] = \frac{\mathrm{velocity}_{\mathrm{char}}}{\omega_{\mathrm{char}} \mathrm{length}_{\mathrm{char}}}$$

Rossby number characterizes the speed of rotation of the fluid

Incompressible (low Mach number) limit

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon^2} \nabla_x p(\varrho)} = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Asymptotic incompressibility (formal)

$$\varepsilon \to 0 \Rightarrow p \to \mathrm{const} \Rightarrow \varrho \to \overline{\varrho}(\mathrm{const}) \Rightarrow \mathrm{div}_x \mathbf{u} = 0$$

Helmholtz decomposition

Helmholtz projection

$$\varrho \mathbf{u} = \underbrace{\mathbf{v}}_{\text{solenoidal component}} + \underbrace{\nabla_x \Phi}_{\text{acoustic potential}}, \ \mathrm{div}_x \mathbf{v} = 0$$

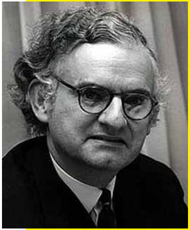
Incompressible (target) system

$$\partial_t \mathbf{v} + \mathbf{P}\Big[\varrho \mathbf{u} \otimes \mathbf{u} \Big] = \Delta \mathbf{v}$$

Lighthill's acoustic analogy

Pressure approximation

$$\frac{1}{\varepsilon}\nabla_x p(\varrho) = \nabla_x \frac{p(\varrho)-p(\overline{\varrho})}{\varepsilon} = p'(\overline{\varrho})\frac{\varrho-\overline{\varrho}}{\varepsilon} + \mathcal{O}(\varepsilon)$$



Michael James Lighthill
[1924-1998]

Acoustic equation

$$\varepsilon\partial_t\left[\frac{\varrho-\overline{\varrho}}{\varepsilon}\right] + \mathrm{div}_x(\varrho\mathbf{u}) = 0$$

$$\varepsilon\partial_t(\varrho\mathbf{u}) + p'(\overline{\varrho})\nabla_x\left[\frac{\varrho-\overline{\varrho}}{\varepsilon}\right] = \mathcal{O}(\varepsilon)$$

Wave equation

$$\frac{\varrho-\overline{\varrho}}{\varepsilon} = Z,\ \varepsilon\partial_t Z + \Delta_x\Phi = 0,\ \varepsilon\partial_t\Phi + p'(\overline{\varrho})Z = 0$$

Duhamel's formula

Acoustic potential

$$\Phi(t,\cdot) = \frac{1}{2}\exp\left(\mathrm{i}\sqrt{-\Delta}\frac{t}{\varepsilon}\right)\left[\Phi_0 - \frac{\mathrm{i}}{\sqrt{-\Delta}}Z_0\right]$$
$$+\frac{1}{2}\exp\left(-\mathrm{i}\sqrt{-\Delta}\frac{t}{\varepsilon}\right)\left[\Phi_0 + \frac{\mathrm{i}}{\sqrt{-\Delta}}Z_0\right]$$

Jean-Marie Constant Duhamel
[1797-1872]

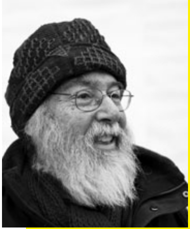
Time derivative

$$Z(t,\cdot) = \frac{1}{2}\exp\left(\mathrm{i}\sqrt{-\Delta}\frac{t}{\varepsilon}\right)\left[\mathrm{i}\sqrt{-\Delta}[\Phi_0] + Z_0\right]$$
$$+\frac{1}{2}\exp\left(-\mathrm{i}\sqrt{-\Delta}\frac{t}{\varepsilon}\right)\left[-\mathrm{i}\sqrt{-\Delta}[\Phi_0] + Z_0\right]$$

Oscillations vs. dispersion

Bounded domain - Fourier modes

$$\exp\left(\pm \mathrm{i}\sqrt{-\Delta}\frac{t}{\varepsilon}\right)[h] = \sum_k \exp\left(\pm \mathrm{i}\sqrt{\lambda_k}\frac{t}{\varepsilon}\right)\langle h, e_k\rangle e_k$$



Robert S. Strichartz

Strichartz estimates

$$\int_{-T}^{T} \left\| \exp\left(\pm \mathrm{i}\sqrt{-\Delta}\frac{t}{\varepsilon}\right)[h]\right\|^p_{L^q(R^3)} \ \mathrm{d}t \leq \varepsilon \|h\|^p_{H^{1,2}(R^3)}$$

$$\frac{1}{2} = \frac{1}{p} + \frac{3}{q}, \ q < \infty$$

Problems on large domains

Acoustic equation

$$\partial^2_{t,t}\Phi - \frac{1}{\varepsilon^2}\Delta_x\Phi = 0$$

Finite speed of propagation

$$\mathrm{supp}[\Phi(t,\cdot)] \subset \left\{ x \ \Big| \ \mathrm{dist}\Big(x; \mathrm{supp}[\Phi(0,\cdot)]\Big) \leq \frac{1}{\varepsilon} \right\}$$

Large domains

$$\Omega \approx r(\varepsilon)\mathcal{O}, \ \ r(\varepsilon) >> \frac{1}{\varepsilon}$$

Preparing the initial data

Ill prepared initial data

$$\varrho(0,\cdot) = \overline{\varrho} + \varepsilon \varrho^{(1)}_{0,\varepsilon},\ \mathbf{u}(0,\cdot) = \mathbf{u}_{0,\varepsilon}$$

$$\left\{ \varrho^{(1)}_{0,\varepsilon} \right\}_{\varepsilon > 0} \ \text{bounded in } L^2 \cap L^\infty$$

$$\left\{ \mathbf{u}_{0,\varepsilon} \right\}_{\varepsilon > 0} \ \text{bounded in } L^2$$

Well prepared initial data

$$\varrho^{(1)}_{0,\varepsilon} \to 0 \text{ in } L^2 \text{ as } \varepsilon \to 0$$

$$\mathbf{u}_{0,\varepsilon} \to \mathbf{u}_0 \text{ in } L^2 \text{ as } \varepsilon \to 0,\ \mathrm{div}_x \mathbf{u}_0 = 0$$

Rotating (incompressible) fluids

Incompressibility

$$\mathrm{div}_x \mathbf{u} = 0$$

Momentum equation

$$\partial_t \mathbf{u} + \mathrm{div}_x(\mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon}\omega \times \mathbf{u}} + \nabla_x p = \Delta \mathbf{u}$$

Target system

$$\mathbf{P}\left[\omega \times \mathbf{u}\right] = 0 \Leftrightarrow \omega \times \mathbf{u} = \nabla_x \Phi \Leftrightarrow -u_2 = \partial_{x_1}\Phi,\ u_1 = \partial_{x_2}\Phi,\ \partial_{x_3}\Phi = 0$$

$$\Rightarrow$$

$$u_j = u_j(t, x_h),\ j = 1, 2,\ x_h = (x_1, x_2),\ \mathrm{div}_h \mathbf{u} = 0 \Rightarrow u_3 = u_3(t, x_h)$$

Incompressible limit + fast rotation

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon}\varrho \omega \times \mathbf{u}} + \boxed{\frac{1}{\varepsilon^{2m}} \nabla_x p(\varrho)} = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Path dependence

$$m > 1$$

Oscillatory component - Poincaré waves

Equation of continuity

$$\varepsilon^m \partial_t \left[\frac{\varrho - \overline{\varrho}}{\varepsilon^m} \right] + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equation

$$\varepsilon^m \partial_t(\varrho \mathbf{u}) + \varepsilon^{m-1} \omega \times (\varrho \mathbf{u}) + p'(\overline{\varrho}) \nabla_x \left[\frac{\varrho - \overline{\varrho}}{\varepsilon^m} \right] = \mathcal{O}(\varepsilon^m)$$

Critical case

$$m = 1$$

A triple singular limit

Mass conservation

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

Momentum balance

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \boxed{\frac{1}{\varepsilon} \varrho \omega \times \mathbf{u}} + \boxed{\frac{1}{\varepsilon^{2m}} \nabla_x p(\varrho)} = \boxed{\varepsilon^{\alpha}} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Path dependence

$$m > 1, \ \alpha > 0$$

Target system

Incompressible limit

Low Mach number $\Rightarrow$ compressible $\rightarrow$ incompressible

Fast rotation

Low Rossby number $\Rightarrow$ 3D motion $\rightarrow$ 2D motion

Inviscid limit

High Reynolds number $\Rightarrow$ viscous flow $\rightarrow$ inviscid flow

Conclusion

3D compressible Navier-Stokes system $\rightarrow$ 2D incompressible Euler system

Target system

Incompressibility

$$\mathrm{div}_x \mathbf{u} = 0$$

Inviscid motion

$$\partial_t \mathbf{u} + \mathrm{div}_x (\mathbf{u} \otimes \mathbf{u}) + \nabla_x p = 0$$

Fundamental issues

Solvability of the primitive system

The primitive system should admit (global) in time solutions for any choice of the scaling parameters and any admissible initial data

Solvability of the target system

The target system should admit solutions, at least locally in time; the solutions are regular

Stability

The family of solutions to the primitive system should be stable with respect to the scaling parameters

Control of the "oscillatory" component of solutions

The component of solutions to the primitive system that "disappears" in the singular limit must be controlled

Analysis of singular limits

Primitive system

$$\partial_t U + \frac{1}{\varepsilon}\mathcal{A}[U] + \mathcal{B}[U] + \varepsilon\mathcal{C}[U] = 0, \ U(0,\cdot) = U_0$$

- Existence of solutions on a time interval $(0, T)$, T independent of ε

Identifying the limit system

$$\mathcal{A}[U] = 0, \ U_{\rm limit} \in {\rm Ker}[\mathcal{A}], \ U_{\rm osc} \in {\rm Range}[\mathcal{A}], \ U = U_{\rm osc} + U_{\rm limit}$$

Uniform bounds

- Find uniform bounds $\|U_\varepsilon\|_X < c$ independent of $\varepsilon \to 0$, prepared initial data

Equations for the limit and oscillatory components

Compactness of the "limit" component

$$\partial_t U_{\rm lim} + \mathcal{B}[U_{\rm lim}] = 0$$

- Convergence via standard compactness arguments or "stability" of the system

Equation for the oscillatory component

$$\varepsilon \partial_t U_{\rm osc} + \mathcal{A}[U_{\rm osc}] \approx 0, \ \ U_{\rm osc} \approx V\left(\frac{t}{\varepsilon}\right), \ \ \partial_t V + \mathcal{A}[V] = 0$$

- Goal is to show

$$U_{\rm osc} \to 0 \text{ in some sense}$$

- Convergence via dispersive estimates