Continuum Mechanics, part 5

Stress_2

- Meaning of G.L. and 2nd PK for large rotations and small strains
- Strain and stress invariants
- Incremental and rate approach

The Green-Naghdi and Jaumann stress rates are commonly used for stress calculations with finite rotations and large deformations.

Corotational stress rate could be effectively used for objective evaluation of stresses in a corotational frame of reference

$$\dot{\Sigma}_{ij} = C_{ijkl}\, d_{kl}$$

$$\dot{\Sigma}_{ij} = C_{ijkl}\, d_{kl}$$

where

$$[d] = [R]^T\, [D]\, [R]$$

$$[d] = [R]^T [D] [R]$$

is a corotational rate of deformation, obtained from the velocity gradient [L] by

$$[D] = \tfrac{1}{2}([L] + [L]^T).$$

$$[D] = \frac{1}{2}\left([L] + [L]^T\right)$$

In this configuration the corotational stress could be updated for the next time step, regardless whether time variable is fictitious or real, by

$${}^{t+\Delta t}[\Sigma] = {}^{t}[\Sigma] + \Delta t\ [\dot{\Sigma}].$$

$${}^{t+\Delta t}\Sigma = {}^{t}\Sigma + \Delta t\,[\dot{\Sigma}]$$

Finally the Cauchy true stress at the new configuration is

$${}^{t+\Delta t}[\sigma] = [R]\ {}^{t+\Delta t}[\Sigma]\ [R]^{T}.$$

$${}^{t+\Delta t}[\sigma] = [R]\,{}^{t+\Delta t}[\Sigma][R]^{T}$$

Note

The evaluation of the rotation matrix [R] requires that we solve the standard eigenvalue problem numerically, i.e. that we find eigenvalues and eigenvectors of the right Cauchy-Green deformation gradient [C] = $[F]^T$ [F]. Hoger and Carlson (1984) suggested an elegant algorithm based on the application of the Cayley-Hamilton theorem which gives the [R] without looking for the eigenvalues and principal axes. The implementation of both approaches shows, however, that the number of floating point operations (flops) needed for the computation of [R] is in favour of analytical solutions for a 2D problem, while for 3D problems the number of flops is roughly the same. A closed form analytical solution for finding the rotation tensor can also be found in Malvern (1969).

INTERPRETATION OF STRESS AND STRAIN TENSORS IN CASE OF SMALL STRAINS AND LARGE ROTATIONS

LET'S SUMMARIZE

$$x_i = a_i + u_i$$

$$E_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial a_j} + \frac{\partial u_j}{\partial a_i}\right) + \frac{1}{2}\left(\frac{\partial u_k}{\partial a_i}\frac{\partial u_k}{\partial a_j}\right) = \varepsilon_{ij} + \eta_{ij}$$

NOW APPLY A FINITE ROTATION DEFINED BY ORTHOGONAL ROTATION TENSOR Q_{ij} so that A VECTOR $d\bar{a}$ TURNS INTO A NEW VECTOR $d\bar{a}'$

$$\{da'\} = [Q]\{da\}$$

AND ITS LENGTH IS UNCHANGED

$$\| \{da'\} \| = \| \{da\} \| = ds_0$$

a_i – initial coordinate system with unit vectors $\bar{e}_i$

a_i' – another coordinate system (initially coinciding with a_i, but rotating with the body) with unit vectors $\bar{e}_i'$

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NOW, THE BODY WILL BE GIVEN INFINITESIMAL DEFORMATION, SO THAT THE VECTOR $d\bar{a}'$ TURNS INTO $d\bar{x}$

$$d\bar{x} = dx_i e_i = dx_i' e_i'$$

ALSO THE DEFORMED VECTOR CAN BE EXPRESSED IN THE FORM

$$d\bar{x} = d\bar{a} + d\bar{u} = d\bar{a}' + d\bar{u}'$$

YOU SHOULD NOTICE THAT

$$\frac{du_i}{da_j} \text{ IS FINITE WHILE } \frac{du'_i}{da'_j} << 1$$

$$E'_{ij} = \underbrace{\frac{1}{2}\left(\frac{\partial u'_i}{\partial a'_j} + \frac{\partial u'_j}{\partial a'_i}\right.}_{\varepsilon'_{ij}} \left. + \frac{\partial u'_k}{\partial a'_i}\frac{\partial u'_k}{\partial a'_j} \right)$$

FROM IT FOLLOWS THAT THE QUADRATIC PART OF E'_{ij} CAN BE NEGLECTED, SO

$$E'_{ij} = \varepsilon'_{ij}$$

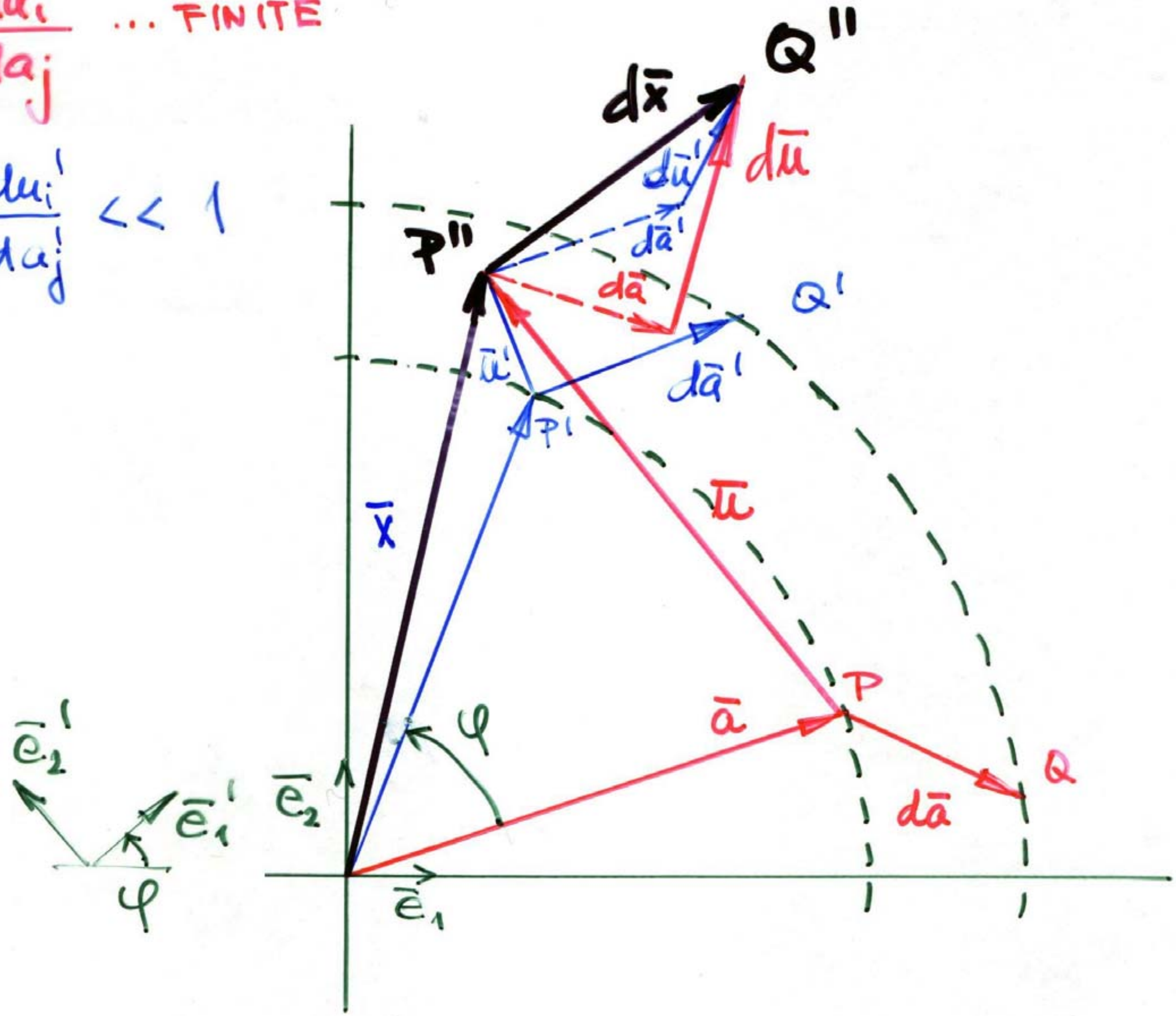
$$d\bar{x} = d\bar{a} + d\bar{u} = d\bar{a}' + d\bar{u}'$$

$$|d\bar{x}| = ds$$

$$|d\bar{a}| = |d\bar{a}'| = ds_0$$

$\dfrac{du_i}{da_j}$ ... FINITE

$\dfrac{du_i'}{da_j'} \ll 1$



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STRAIN INVARIANT IS EXPRESSED AS THE DIFFERENCE OF SQUARES OF LENGTHS

$$ds^2 - ds_0^2 = 2\,\{dx\}^T [E] \{dx\} = 2\,\{dx'\}^T [E'] \{dx'\}$$

INDEPENDENTLY OF THE CHOICE OF COORDINATE SYSTEM, SINCE

$$F'_{ij} = \frac{\partial x'_i}{\partial a_j} = \frac{\partial (Q_{ik}\, x_k)}{\partial a_j} = Q_{ik} \frac{\partial x_k}{\partial a_j} = Q_{ik} F_{kj}$$

$$[E'] = \frac{1}{2}\left([F']^T [F'] - [I] \right) = \frac{1}{2}\left([F]^T \underbrace{[Q]^T [Q]}_{[I]} [F] - [I] \right) = [E]$$

$$\Rightarrow \qquad E'_{ij} = E_{ij}$$ AND $$E_{ij} = \varepsilon'_{ij}$$

CONCLUSION

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IN CASE OF SMALL STRAINS AND LARGE ROTATIONS, THE GREEN-LAGRANGE STRAIN TENSOR COMPONENTS ARE EQUAL TO THE ENGINNERING STRAIN COMPONENTS IN A COORDINATE SYSTEM RIGIDLY ROTATED WITH THE BODY

see matlab program
lergrot1.m

SIMILARLY FOR STRESSES

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$$[F] = [R][U] \quad \ldots \text{ POLAR DECOMPOSITION}$$

$$[U] \doteq [I] + [\varepsilon] \quad \ldots \text{ FOR INFINITESIMAL STRAIN}$$

(SEE 27)

AND THEN THE DEFORMATION GRADIENT CAN BE APROXIMATED BY

$$[F] \doteq [R]$$

IN CASE OF INFINITESIMAL STRAIN ${}^{0}\rho / \rho = 1$ AND THE SECOND PIOLA KIRCHHOFF STRESS CAN BE EXPRESSED

$$[S] = [F]^{-1}[\sigma][F]^{-T} = [R]^{T}[\sigma][R]$$

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TRUE CAUCHY STRESS CAN BE EXPRESSED IN THE PRIMED (e_i') COORDINATE SYSTEM BY

$$[\sigma'] = [R]^T[\sigma][R]$$

COMPARING THE LAST TWO EQUATIONS: $[S]=[\sigma']$

CONCLUSION

IN CASE OF SMALL STRAINS AND LARGE ROTATIONS THE SECOND PIOLA KIRCHHOFF STRESS COMPONENTS ARE EQUAL TO THE CAUCHY, TRUE STRESS COMPONENTS IN A SYSTEM RIGIDLY ROTATING WITH THE PARTICLE

A SHORT SUMMARY

PRINCIPAL AXES AND INVARIANTS OF THE SECOND ORDER TENSOR

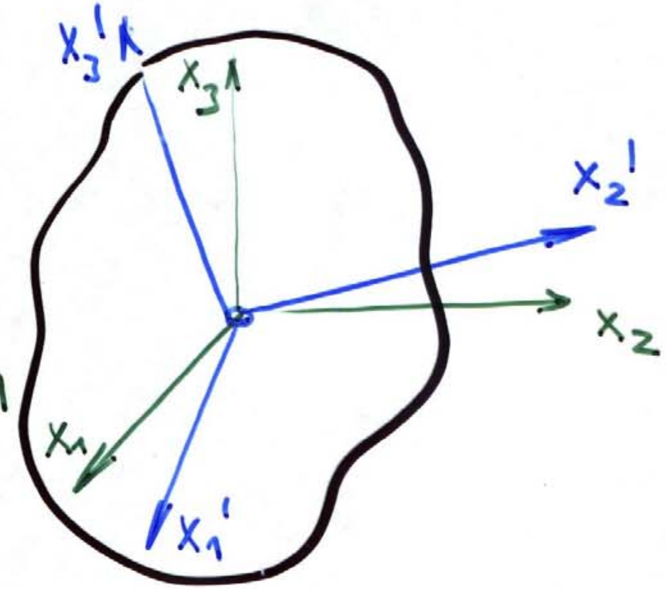
STRAIN INVARIANTS

TWO SETS OF INVARIANTS ARE DEFINED AND USED

1) ASSOCIATED WITH CHARACTERISTIC EQUATION OF TENSOR

STANDARD TRANSFORMATION

$$[\varepsilon'] = [A]\{\varepsilon\}[A]^T$$



WE ARE LOOKING FOR SUCH [A] WHICH GIVES $[\varepsilon']$ OF DIAGONAL FORM

STANDARD EIGENVALUE PROBLEM

$$\begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} ; \quad \left([\varepsilon] - \lambda [I] \right) \{a\} = \{\emptyset\}$$

NONTRIVIAL SOLUTION ONLY IF

$$\det \left([\varepsilon] - \lambda [I] \right) = \emptyset$$

WHICH GIVES CHARACTERISTIC EQUATION OF STRAIN TENSOR

$$\lambda^3 - \bar{I}_1 \lambda^2 + \bar{I}_2 \lambda - \bar{I}_3 = \emptyset$$

WHERE λ_i are PRINCIPAL STRAINS and

$$\bar{I}_1 = \varepsilon_{ii}$$

$$\bar{I}_2 = \begin{vmatrix} \varepsilon_{11} & \varepsilon_{12} \\ \varepsilon_{12} & \varepsilon_{22} \end{vmatrix} + \begin{vmatrix} \varepsilon_{11} & \varepsilon_{13} \\ \varepsilon_{13} & \varepsilon_{33} \end{vmatrix} + \begin{vmatrix} \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{23} & \varepsilon_{33} \end{vmatrix}$$

$$\bar{I}_3 = \left| \varepsilon_{ij} \right|$$

ARE THE FIRST, SECOND AND THIRD STRAIN INVARIANTS

2) ASSOCIATED WITH THE TENSOR THROUGH ITS PROPERTY OF TRACE AND DIFFERENT ORDERS OF TRACE

$$I_1 = tr([\varepsilon])$$ tr... trace ... sum of diagonal t.

$$I_2 = tr([\varepsilon]^2) = \frac{1}{2}\varepsilon_{ij}\varepsilon_{ji}$$

$$I_3 = tr([\varepsilon]^3) = \frac{1}{3}\varepsilon_{ik}\varepsilon_{km}\varepsilon_{mi}$$

NOTE

- THESE INVARIANTS ARE INVARIANT WITH RESPECT TO THE CHOICE OF COORDINATE SYSTEM
- EACH PARTICLE HAS ITS OWN INVARIANTS
- A MULTIPLE OF INVARIANT IS AN INVARIANT AS WELL (SEE CHARACTERISTIC EQUATION)

STRESS INVARIANTS

SIMILARLY AS BEFORE FOR STRAINS

1) $\bar{J}_1 = \sigma_{ii}$

$$\bar{J}_2 = \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{13} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{23} & \sigma_{33} \end{vmatrix}$$

$$\bar{J}_3 = \left| \sigma_{ij} \right|$$

2) $J_1 = tr([\sigma]) = \sigma_{ii}$

$$J_2 = tr([\sigma]^2) = \frac{1}{2}\sigma_{ij}\sigma_{ji}$$

$$J_3 = tr([\sigma]^3) = \frac{1}{3}\sigma_{ik}\sigma_{km}\sigma_{mi}$$

DECOMPOSITION OF STRESS INTO VOLUMETRIC AND DEVIATORIC COMPONENTS

$$\sigma_{ij} = v_{ij} + s_{ij}$$

deviatoric part of stress responsible for changes of shape

volumetric part (sometimes mean) changes of volume only

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$$\sigma_{ij} = v_{ij} + s_{ij}$$

$$\sigma_{ij} = \begin{bmatrix} \sigma_m & & \\ & \sigma_m & \\ & & \sigma_m \end{bmatrix} + \begin{bmatrix} \sigma_{11}-\sigma_m & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22}-\sigma_m & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33}-\sigma_m \end{bmatrix}$$

$$\sigma_m = \frac{1}{3}\sigma_{ii}$$

volumetric stress tensor,
also: mean, hydrostatic, spherical
KULOVÝ TENSOR NAPJATOSTI

RECALL $J_1 = \sigma_{ii}$ – THE FIRST INVARIANT OF STRESS TENSOR

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_m\delta_{ij}$$ deviatoric stress

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INVARIANTS OF DEVIATORIC STRESS TENSOR

$$J_{D1} = s_{ii} = \sigma_{ii} - \frac{1}{3}\sigma_{kk}\delta_{ii} = 0 \quad \text{ALWAYS}$$

$$J_{D2} = \frac{1}{2} s_{ij} s_{ji} = \frac{1}{2} tr\left([s]^2 \right) = J_2 - J_1^2/6$$

$$J_{D3} = \frac{1}{3} s_{im} s_{mk} s_{ki} = \frac{1}{3} tr\left([s]^3 \right) = J_3 - \frac{2}{3} J_1 J_2 + \frac{2}{27} J_1^3$$

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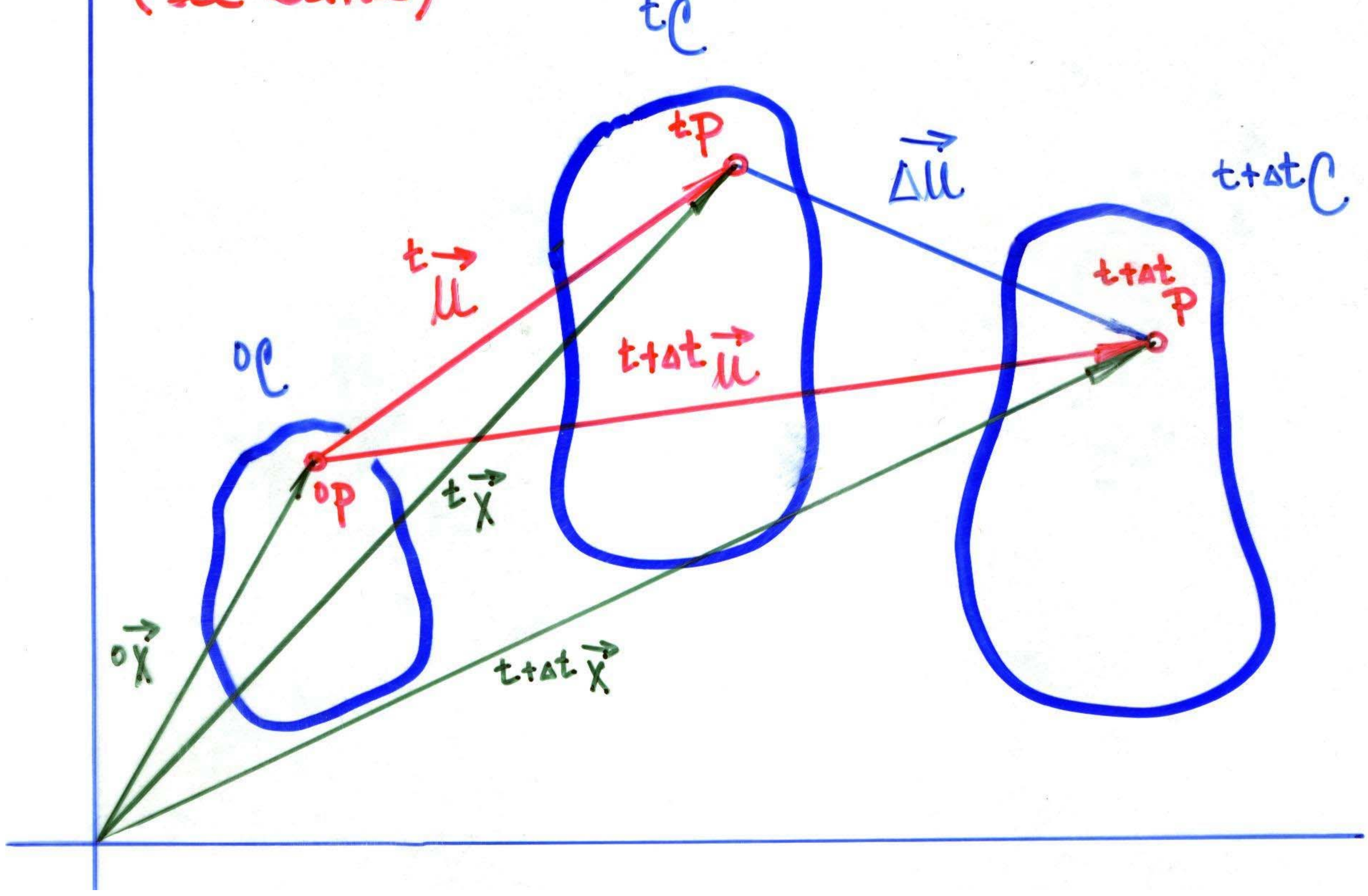
NOTATION for INCREMENTAL APPROACH
(see Bathe)



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LEFT SUPERSCRIPT REFERS TO THE CONFIGURATION

Displacement at ${}^{t}C$: $\quad {}^{t}u_i = {}^{t}x_i - {}^{0}x_i$

at ${}^{t+\Delta t}C$: $\quad {}^{t+\Delta t}u_i = {}^{t+\Delta t}x_i - {}^{0}x_i$

Increment of displacement: $\quad \Delta u_i = {}^{t+\Delta t}u_i - {}^{t}u_i =$

SOMETIMES WRITTEN WITHOUT $\quad = {}^{t+\Delta t}x_i - {}^{t}x_i$

We will concentrate our effort on the motion of the body between configurations ${}^{t}C \to {}^{t+\Delta t}C$

Incremental quantities play an essential role in constitutive equations written in incremental form.

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LEFT SUPERCRIPT ALSO REFERS TO OTHER QUANTITIES WHOSE VALUES ARE CHANGING CONTINUOUSLY. EXAMPLES: ${}^{0}\rho$, ${}^{t}\rho$, ${}^{t+\Delta t}\rho$

${}^{0}A$, ${}^{t}A$, ${}^{t+\Delta t}A$

${}^{0}V$, ${}^{t}V$, ${}^{t+\Delta t}V$

LEFT SUBSRIPT INDICATES WITH RESPECT TO WHICH CONFIGURATION THE QUANTITY IS CONSIDERED

EXAMPLES :

deformation gradient $\,{}^t_0F_{ij} = \dfrac{\partial\, {}^t x_i}{\partial\, {}^0 x_j}$,

its inverse $\,{}^t_0F^{-1}_{ij} = {}^0_tF_{ij}$,

green-Lagrange strain tensor

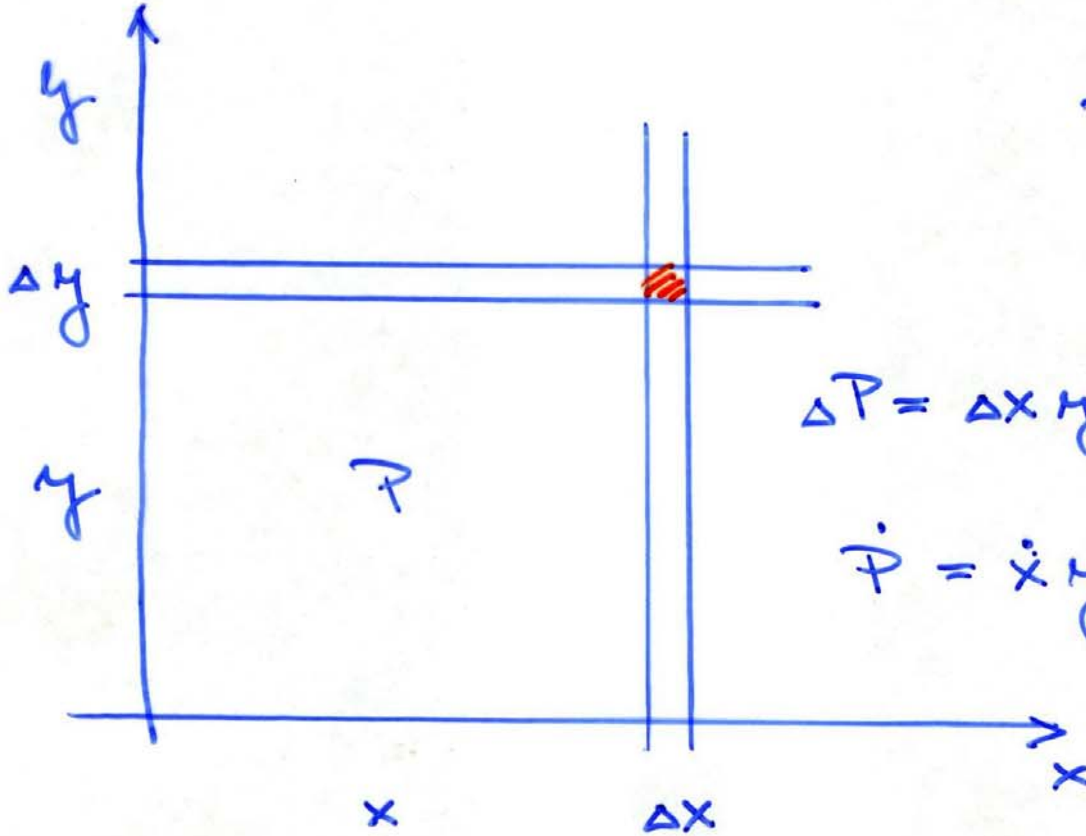
$${}^t_0E_{ij} = \frac{1}{2}\left({}^t_0Z_{ij} + {}^t_0Z_{ji} + {}^t_0Z_{ki}\, {}^t_0Z_{kj} \right),$$

$${}^t_0Z_{ij} = \frac{\partial\, {}^t u_i}{\partial\, {}^0 x_j}$$

etc.

Almansi $\,{}^t_tA_{ij} = \;\ldots\; {}^t_t\overline{Z}_{ij} = \dfrac{\partial\, {}^t u_i}{\partial\, {}^t x_j}$

A quantity, its increment and its rate



$$P = x y$$

$$\Delta P = \Delta x \, y + x \, \Delta y + \Delta x \, \Delta y$$

$$\dot{P} = \dot{x} \, y + x \, \dot{y}$$

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Incremental G.-L. strain can be obtained from

$$\left[{}^{t}_{0}E\right] = \frac{1}{2}\left(\left[{}^{t}_{0}Z\right] + \left[{}^{t}_{0}Z\right]^{T} + \left[{}^{t}_{0}Z\right]^{T}\left[{}^{t}_{0}Z\right]\right)$$

where $\left[{}^{t}_{0}Z\right] = \dfrac{\partial\, {}^{t}u_i}{\partial\, {}^{0}x_j}$

LET'S DENOTE

$$\left[\Delta Z\right] = \left[{}^{t+\Delta t}_{0}Z\right] - \left[{}^{t}_{0}Z\right] = \frac{\partial\, {}^{t+\Delta t}u_i - \partial\, {}^{t}u_i}{\partial\, {}^{0}x_j} = \frac{\partial\, \Delta u_i}{\partial\, {}^{0}x_j}$$

DIFFERENTIANTING WE GET REQUIRED INCREMENTS

$$\left[\Delta E\right] = \left[{}^{t+\Delta t}_{0}E\right] - \left[{}^{t}_{0}E\right] =$$

$$= \frac{1}{2}\left(\left[\Delta Z\right] + \left[\Delta Z\right]^{T} + \left[\Delta Z\right]^{T}\left[Z\right] + \left[Z\right]^{T}\left[\Delta Z\right]\right) + \frac{1}{2}\left(\left[\Delta Z\right]^{T}\left[\Delta Z\right]\right)$$

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linear quadratic

parts of increment

$$\text{so}: \quad [\Delta E] = [\Delta E_{LIN}] + [\Delta E_{NL}] = [\Delta e] + [\Delta \eta]$$

SOMETIMES

IN INCIDENTAL NOTATION WE GET

$$\Delta e_{ij} = (\Delta E_{LIN})_{ij} = \frac{1}{2}\left(\frac{\partial \Delta u_i}{\partial {}^0x_j} + \frac{\partial \Delta u_j}{\partial {}^0x_i} + \frac{\partial \Delta u_k}{\partial {}^0x_i}\frac{\partial {}^t u_k}{\partial {}^0x_j} + \frac{\partial {}^t u_k}{\partial {}^0x_i}\frac{\partial \Delta u_k}{\partial {}^0x_j}\right)$$

$$\Delta \eta_{ij} = (\Delta E_{NL})_{ij} = \frac{1}{2}\left(\frac{\partial \Delta u_k}{\partial {}^0x_j}\frac{\partial \Delta u_k}{\partial {}^0x_i}\right)$$

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the change of configurations between t and $t+\Delta t$ is usually small. You could notice that increments of displacements play the role of displacements. For the change ${}^{t}C \rightarrow {}^{t+\Delta t}C$ the infinitesimal theory could be used. Cauchy strain (infinitesimal) is defined with respect to the original configuration But in this process it is just ${}^{t}C$.

So the increment of infinitesimal strain tensor in the neighbourhood of ${}^{t}C$ can be expressed as

$$\Delta \varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial \Delta u_i}{\partial {}^{t}x_j} + \frac{\partial \Delta u_j}{\partial {}^{t}x_i}\right)$$

So the invariants expressing the square of length of two 'close' particles can be written.

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$$\Delta E_{ij}\, d^{0}x_i\, d^{0}x_j = \Delta \varepsilon_{ij}\, d^{t}x_i\, d^{t}x_j$$

$$\Delta E_{ij} \qquad = \Delta \varepsilon_{mn} \frac{d^{t}x_m}{d^{0}x_i} \frac{d^{t}x_n}{d^{0}x_j}\, d^{0}x_i\, d^{0}x_j$$

This must be valid for any $d^{0}x_i\ d^{0}x_j$, so

$$\Delta E_{ij} = {}^{t}_{0}F_{mi}\ \Delta \varepsilon_{mn}\ {}^{t}_{0}F_{nj}$$

or

$$[\Delta E] = [{}^{t}_{0}F]^{T} [\Delta \varepsilon] [{}^{t}_{0}F]$$

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$$\Delta E_{ij}\, d^0x_i\, d^0x_j = \Delta\varepsilon_{ij}\, d^tx_i\, d^tx_j$$

$$\left\{d^0x\right\}^T [\Delta E] \left\{d^0x\right\} = \left\{d^tx\right\}^T [\Delta\varepsilon] \left\{d^tx\right\}$$

$$\left\{d^tx\right\} = [F]\left\{d^0x\right\}$$

$$\left\{d^0x\right\}^T [\Delta E] \left\{d^0x\right\} = \left\{d^0x\right\}^T [F]^T [\Delta\varepsilon] [F] \left\{d^0x\right\}$$

$$[\Delta E] = [F]^T [\Delta\varepsilon] [F]$$

Once more, since it is an important conclusion

$$\left[\Delta E\right] = \left[{}^{t}_{0}F\right]^{T}\left[\Delta\varepsilon\right]\left[{}^{t}_{0}F\right]$$

ALSO: $\frac{\Delta E}{\Delta t} = \dot{E}$ $\quad \frac{\Delta \varepsilon}{\Delta t} = \dot{\varepsilon} = D$

$$\Rightarrow \dot{E} = F^{T} D F$$

the increment of G.-L. strain tensor can be expressed by means of the increment of the Cauchy infinitesimal strain tensor written for ${}^{t}C$.

We have already seen that the velocity gradient

$${}^{t}_{t}L_{ij} = \frac{\partial\, {}^{t}v_i}{\partial\, {}^{t}x_j}$$

very often the indices are omitted – it is an instantaneous variable

can be decomposed into symmetric and antisym. parts

$$L_{ij} = D_{ij} + W_{ij}$$

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and for infinitesimal elasticity we have shown that

$D_{ij} = \dot{\varepsilon}_{ij}$ RATE OF DEFORMATION TENSOR

$W_{ij} = \dot{\Omega}_{ij}$ SPIN TENSOR or RATE OF INFINITESIMAL ROTATION T.

Also $\left[{}^{t}_{t}L\right] = \left[{}^{t}_{0}\dot{F}\right]\left[{}^{t}_{0}F\right]^{-1}$ FOR ${}^{0}C \rightarrow {}^{t}C$

for small deformation theory simplifies into

$[L] = [\dot{F}]$ since $[F] = [F]^{-1} = [I]$

VELOCITY GRADIENT

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$$ {}^{t}_{t}L_{ij} = \frac{\partial\, {}^{t}v_i}{\partial\, {}^{t}x_j} $$

$$ {}^{t}_{t}L_{ij}\,\Delta t = \frac{\partial({}^{t}v_i\,\Delta t)}{\partial\, {}^{t}x_j} = \frac{\partial(\Delta u_i)}{\partial\, {}^{t}x_j} = \overline{Z}_{ij} $$

SPATIAL DISPLAC. GRADIENT

$$ {}^{t}_{t}L_{ij} = \frac{\overline{Z}_{ij}}{\Delta t} = \underbrace{\dot{\overline{Z}}_{ij} = \dot{Z}_{ij}}_{\text{small changes between } {}^{t}C \text{ and } {}^{t+\Delta t}C} $$

but $F_{ij} = Z_{ij} + \delta_{ij}$ and $\dot{F}_{ij} = \dot{Z}_{ij}$ and finally

$$ {}^{t}_{t}L_{ij} = \dot{F}_{ij} \qquad \dot{F}_{ij} = \frac{{}^{t+\Delta t}_{0}F_{ij} - {}^{t}_{0}F_{ij}}{\Delta t} $$

THAT'S WHY L_{ij} IS SOMETIMES CALLED INCREMENTAL DEFORMATION GRADIENT

VELOCITY GRADIENT

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$$ {}^t_t L_{ij} = \frac{\partial\, {}^t v_i}{\partial\, {}^t x_j} $$

$$ {}^t_t L_{ij}\, \Delta t = \frac{\partial ({}^t v_i\, \Delta t)}{\partial\, {}^t x_j} = \frac{\partial (\Delta u_i)}{\partial\, {}^t x_j} = \overline{Z}_{ij} $$

SPATIAL DISPLAC. GRADIENT

$$ {}^t_t L_{ij} = \frac{\overline{Z}_{ij}}{\Delta t} = \underbrace{\dot{\overline{Z}}_{ij} = \dot{Z}_{ij}}_{\text{small changes between } {}^tC \text{ and } {}^{t+\Delta t}C} $$

but $F_{ij} = Z_{ij} + \delta_{ij}$ and $\dot{F}_{ij} = \dot{Z}_{ij}$ and finally

$$ {}^t_t L_{ij} = \dot{F}_{ij} \qquad \dot{F}_{ij} = \frac{{}^{t+\Delta t}_{0}F_{ij} - {}^{t}_{0}F_{ij}}{\Delta t} $$

THAT'S WHY L_{ij} IS SOMETIMES CALLED INCREMENTAL DEFORMATION GRADIENT