The theory of power spectrum break frequency in multi-flare accretion disc variability models

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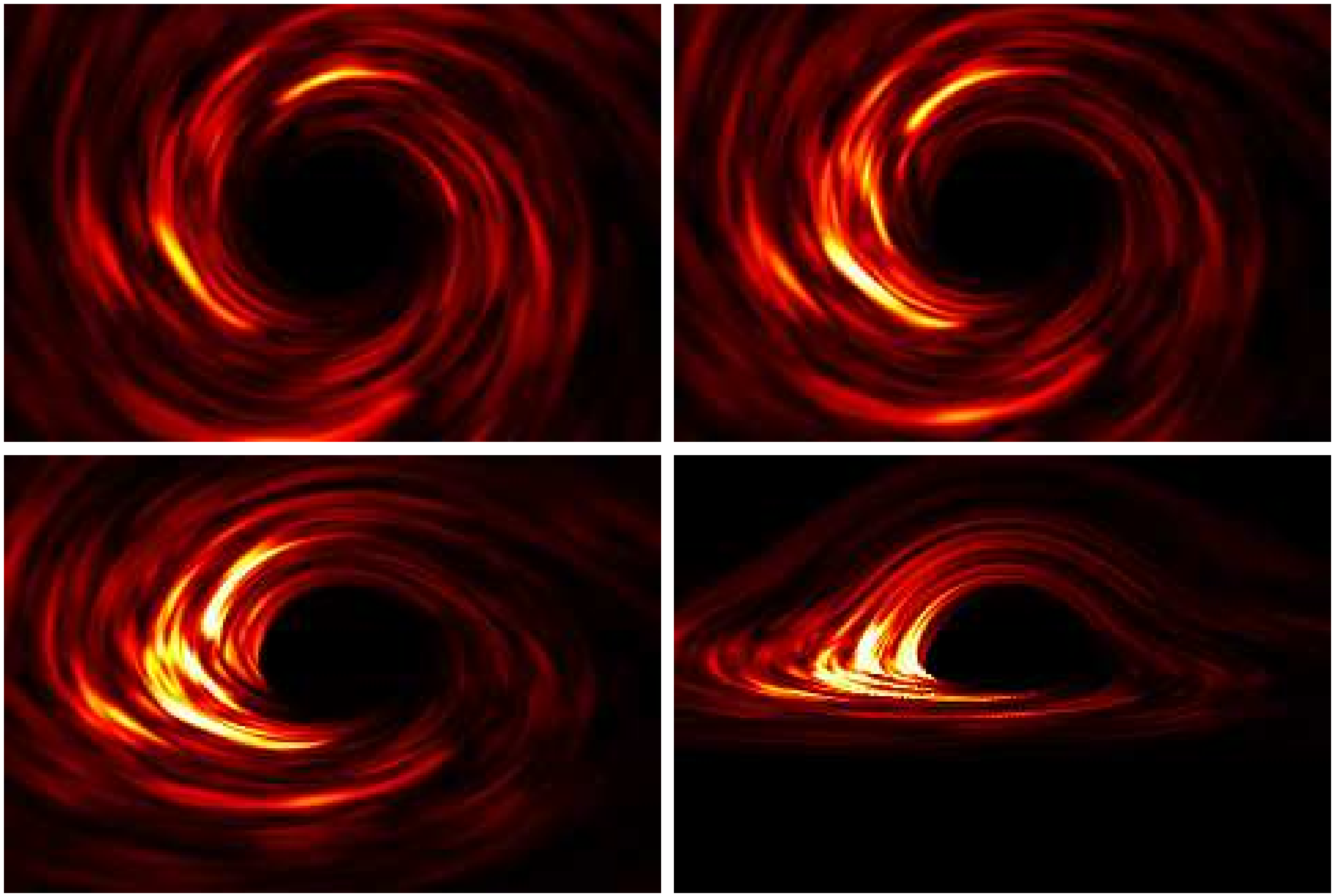
Motivation of the problem



Reynolds & Armitage 2003

Formulation of the problem

- Spot/flare with decaying emissivity on circular orbit.
- Each single spot/flare is described by
  - ◇ "time and place of birth": t_j & r_j, ϕ_j
  - ◇ Other shape determining parameters: $\vec{\xi}$, lifetime, emitted energy
- Observed signal is modulated by relativistic effects: (redshift, gravitational lensing, time delay)

Power spectrum of a random process

Process without relativistic effects ($F(t,r)=1$):

$$S(\omega) = m_1 \, \mathrm{E}[|\mathcal{F}[I](\omega)|^2] + S_{\mathrm{P}}(\omega) \; |\mathrm{E}[\mathcal{F}[I](\omega)]|^2$$

For stationary processes: $S_{\mathrm{P}}(\omega) = m_1^2 \mathcal{F}[\dot{c}(|t|)](\omega)$,

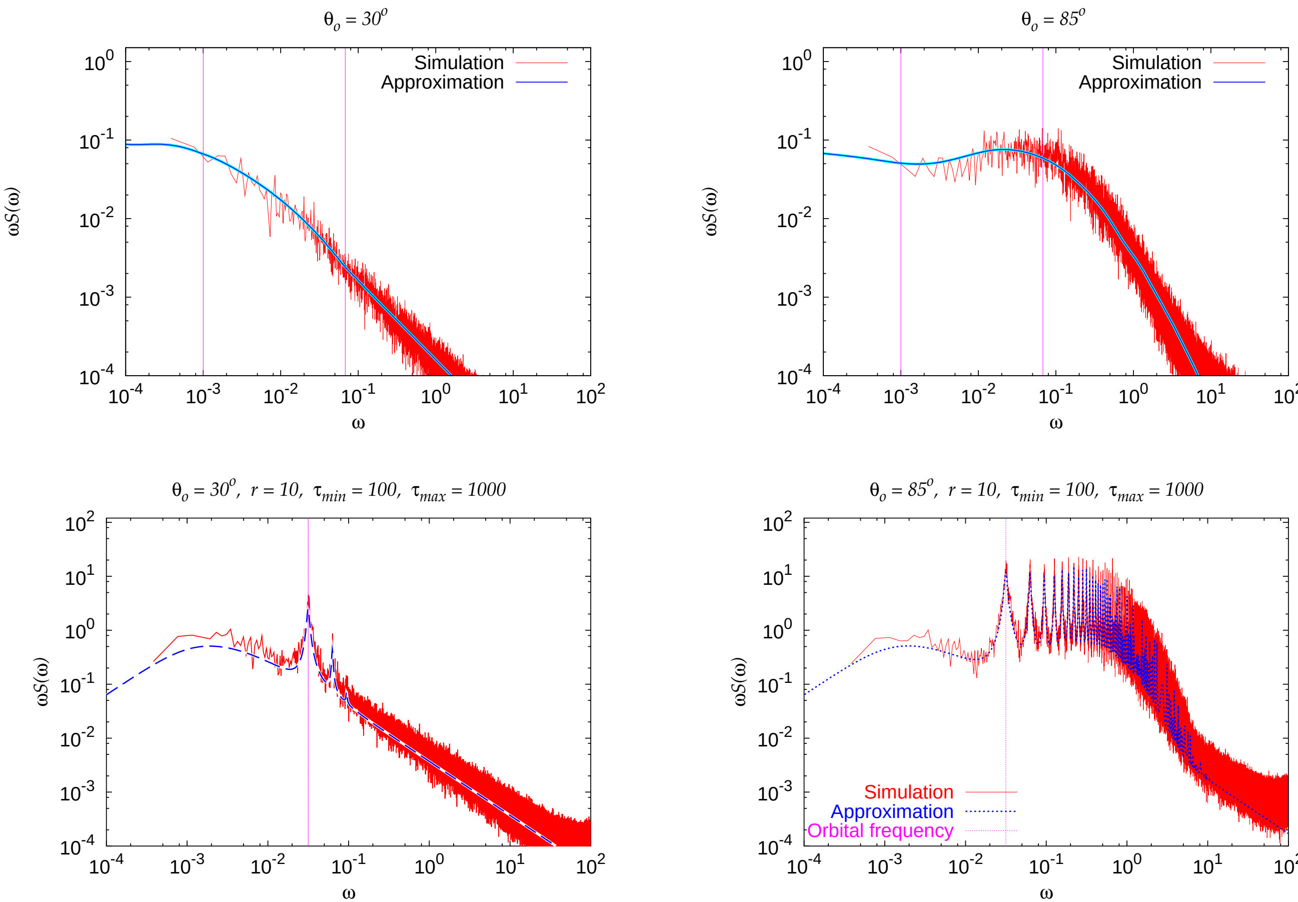
- $I(t,.)$ Signal of individual flare
- $c(t)$: Pair correlation function of flares

Relativistic effects:

$$\mathcal{F}[I](\omega) \rightarrow \sum_{k=-\infty}^{\infty} c_k(r) \mathcal{F}[I](\omega - k\Omega(r)),$$

where $F(t,r) = \sum_{k=-\infty}^{\infty} c_k(r) e^{ik\Omega(r)t}$

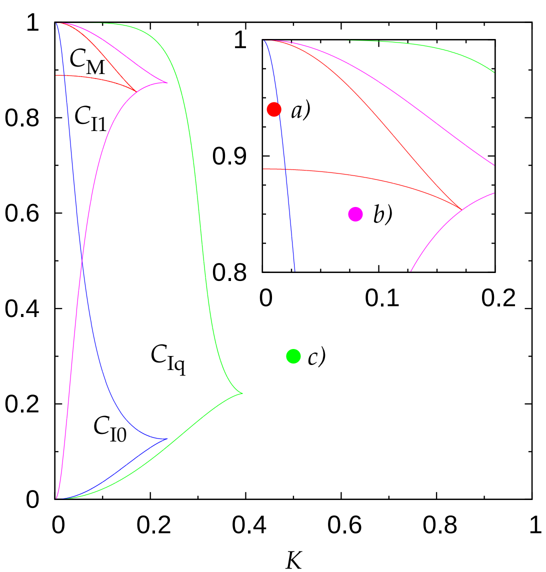
Poisson and Hawkes process + relativistic effects

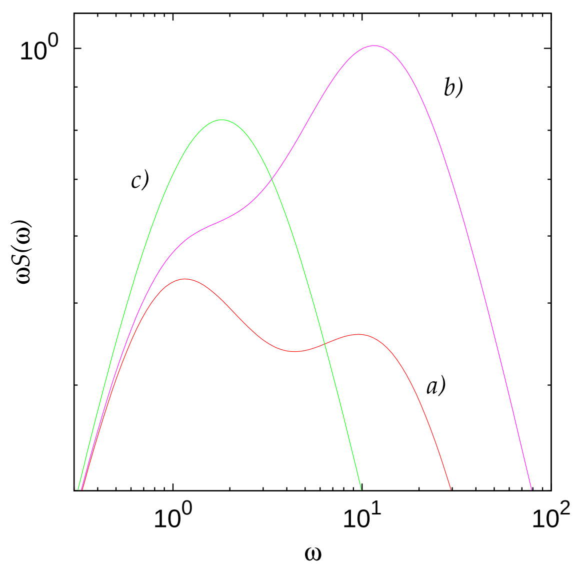


The problem of the broken power-spectra

The power-spectrum of a random process consisting of independent exponentials has the form,
$S(\omega) = \lambda \int \frac{\tau^2}{1+\omega^2\tau^2} I_0^2(\tau) p(\tau) \mathrm{d}\tau.$
A minimal non-trivial example,

$$S_{\mathrm{r}}(\omega) = \frac{\alpha}{1+\omega^2} + \frac{1-\alpha}{1+K^2\omega^2}.$$





Internal structure of the PSD:

- Extrema of $\omega S(\omega)$.
- Inflection point of $\omega S(\omega)$.
- Inflection point of $S(\omega)$.
- Inflection point of the local power-law index, $\frac{\mathrm{d}\ln(S(\omega))}{\mathrm{d}\ln(\omega)}$.

PSDs of exponential processes

$I(t,\tau) = I_0 \exp(-t/\tau)\theta(t)$. Local maximum of $\omega S(\omega)$:

$$\frac{\mathrm{d}}{\mathrm{d}\omega}\left[\omega S(\omega)\right] = \int \frac{1-\omega^2\tau^2}{(1+\omega^2\tau^2)^2}\,\tilde{p}(\tau)\mathrm{d}\tau$$

$$= \mathrm{E}\left[\frac{1-\omega^2\tau^2}{(1+\omega^2\tau^2)^2}\right].$$

Linearity of the averaging operator:

$$\mathrm{E}\left[\left(1+\omega_{\mathrm{M}}^2\tau^2\right)^{-2}\right] = \omega_{\mathrm{M}}^2 \mathrm{E}\left[\tau^2\left(1+\omega_{\mathrm{M}}^2\tau^2\right)^{-2}\right].$$

We substitute $x = \omega_{\mathrm{M}}^2$ and define function $g(x)$ as

$$g(x) = \frac{\mathrm{E}\left[\left(1+\tau^2 x\right)^{-2}\right]}{\mathrm{E}\left[\tau^2\left(1+\tau^2 x\right)^{-2}\right]}.$$

- Local extremes of the spectra are given by $g(x) = x$.
- $g(x)$ is non-decreasing and constrained by

$$(\mathrm{E}[\tau^2])^{-1} = g(0) \leq g(x) \leq g(\infty) = \frac{\mathrm{E}[\tau^{-4}]}{\mathrm{E}[\tau^{-2}]}$$

- Iterations of $g(x)$: $g^{(1)}(x) = g(x)$, $g^{(n)}(x) = g(g^{(n-1)}(x))$.
- (Almost)contracting map: $g^{(n)}(0) \geq g^{(n-1)}(0)$, $g^{(n)}(\infty) \leq g^{(n-1)}(\infty)$
- Iff $g'(x) < 1$ for all x, the local extreme is unique.

The spectra of exponential processes

- It works for an arbitrary distribution $p(\tau)$.
- Analogical calculation can be done for position of inflection point.
- Less straightforward but possible for profiles of the form $I(t,\tau) = P(t)\exp(-t/\tau)\theta(t)$.
- Similar analysis for the relativistic modulation with arbitrary distribution $p(\tau, r)$.

Applications in modeling (Starting from the distribution $p(\tau)$):

- The position of PSD break is constrained by $g(0)$ and $g(\infty)$
- Let $\mathcal{C}_{\mathrm{M}}$ be the set of all spectra with more than one break. We can find its boundaries $\mathcal{C}_{\mathrm{M}}^{\mathrm{S}} \subseteq \mathcal{C}_{\mathrm{M}} \subseteq \mathcal{C}_{\mathrm{M}}^{\mathrm{N}}$ at the cost of calculation a few mean values $\mathrm{E}[\tau^k]$.

The spectra of exponential processes

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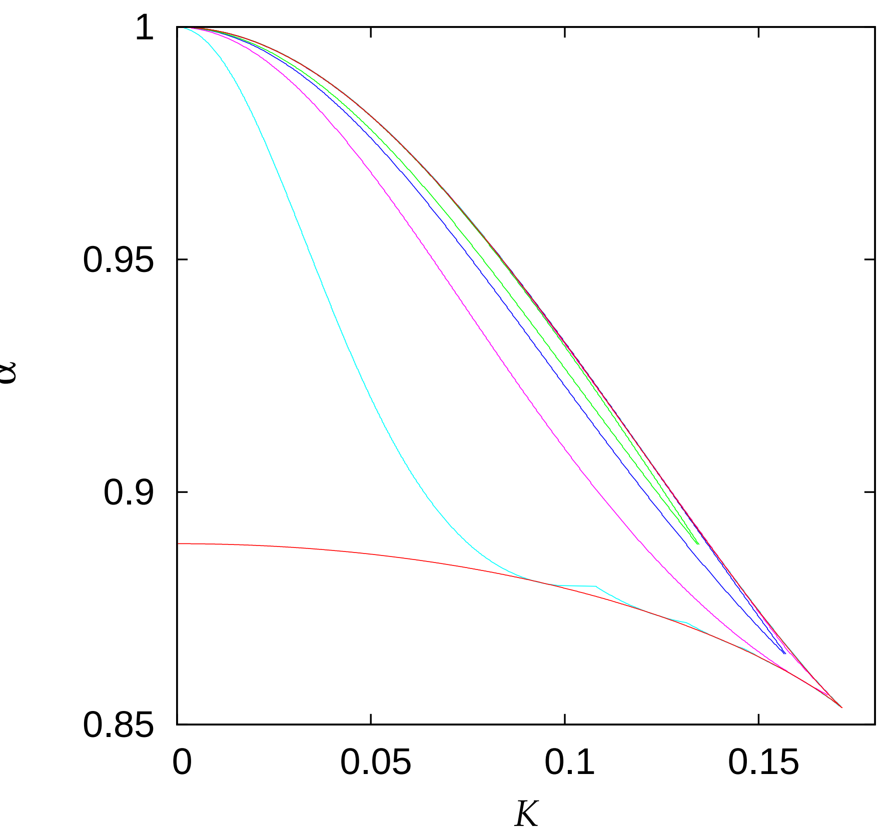
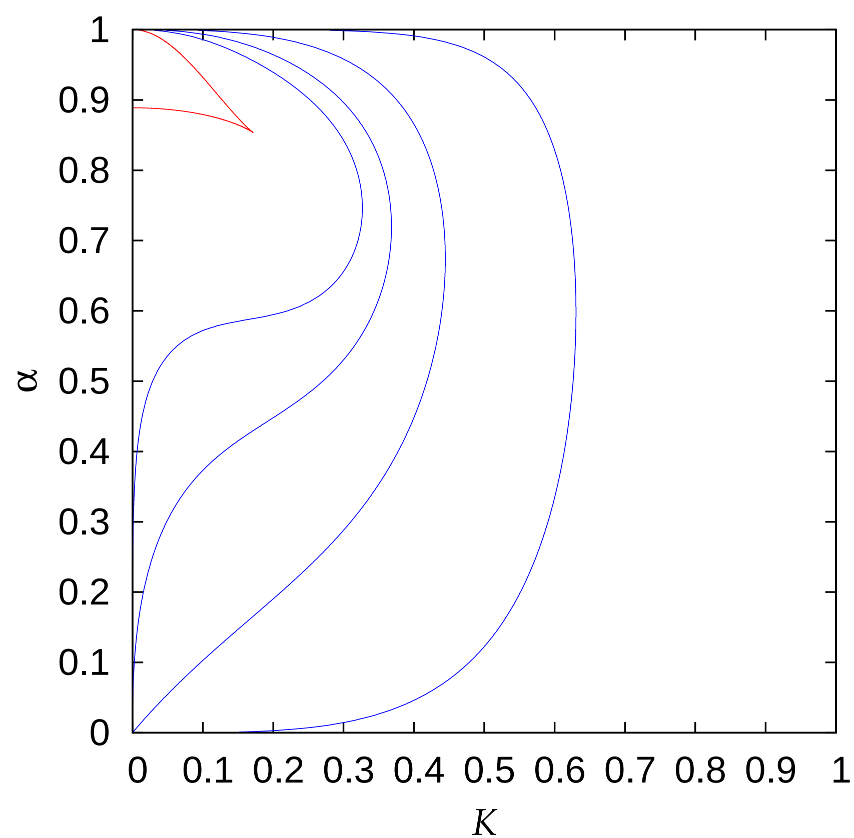
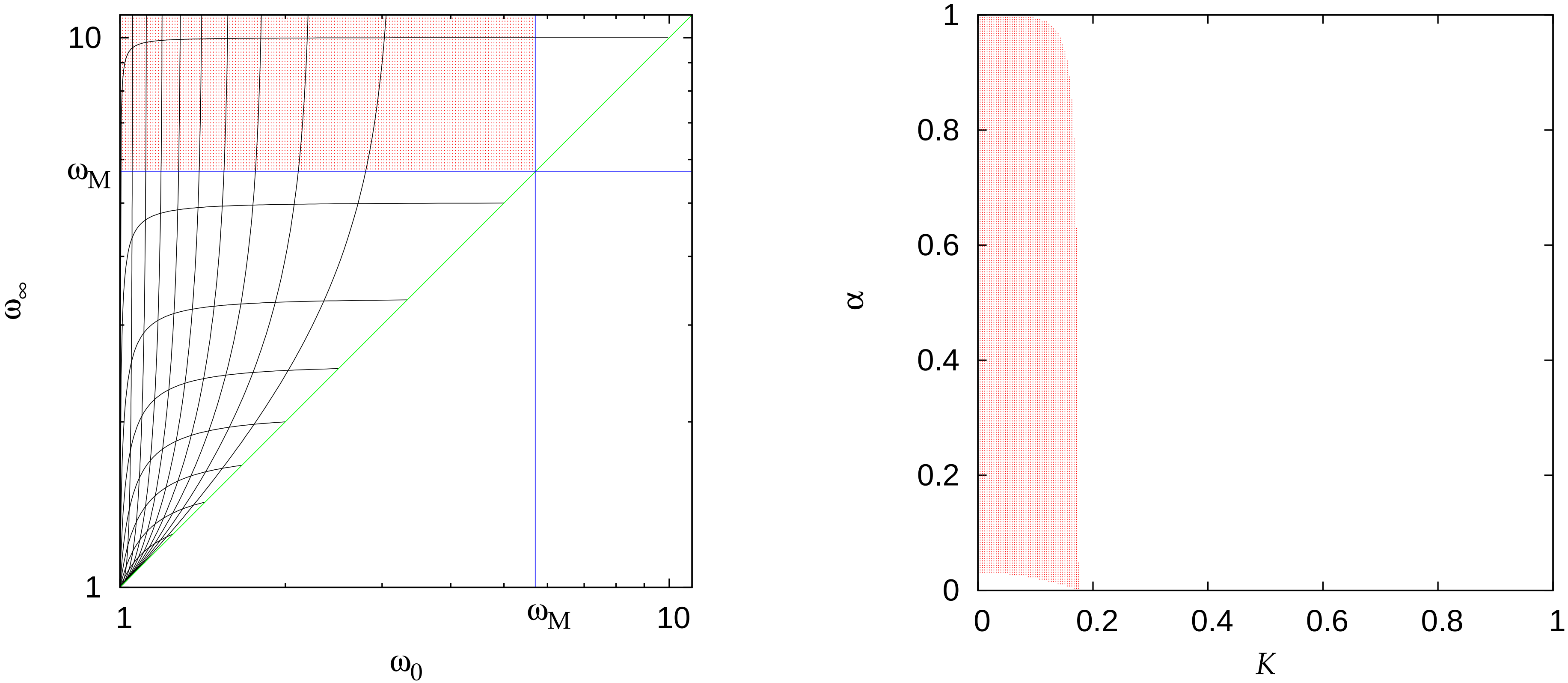




The spectra of exponential processes

Applications in adaptation of the model (Starting from a measured break frequency ω_{M}):

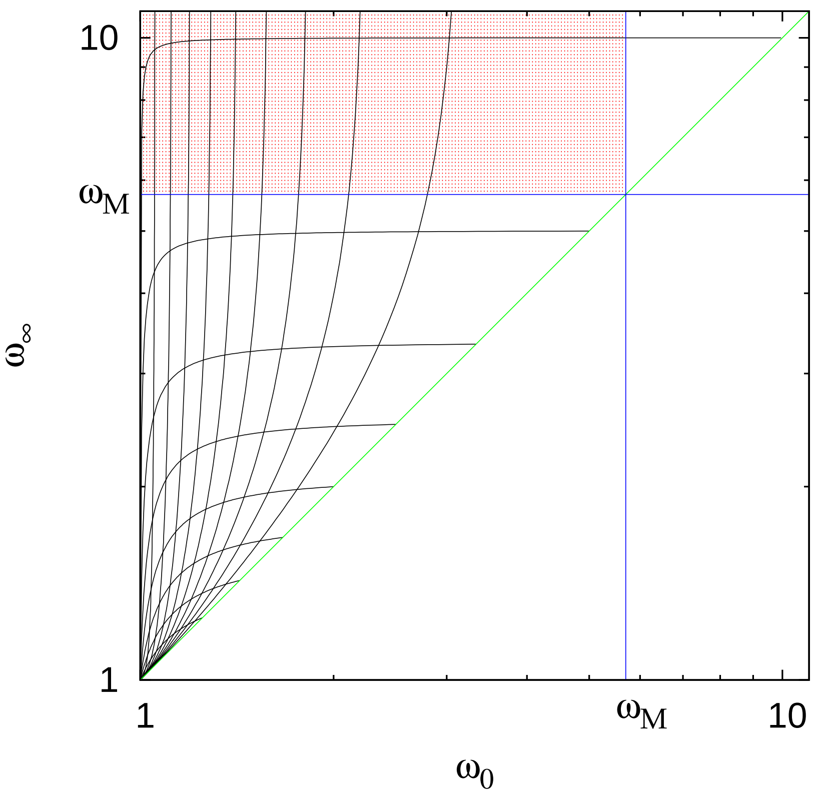
$$\omega_0 = \sqrt{g(0)} \leq \omega_{\mathrm{M}} \leq \sqrt{g(\infty)} = \omega_\infty$$

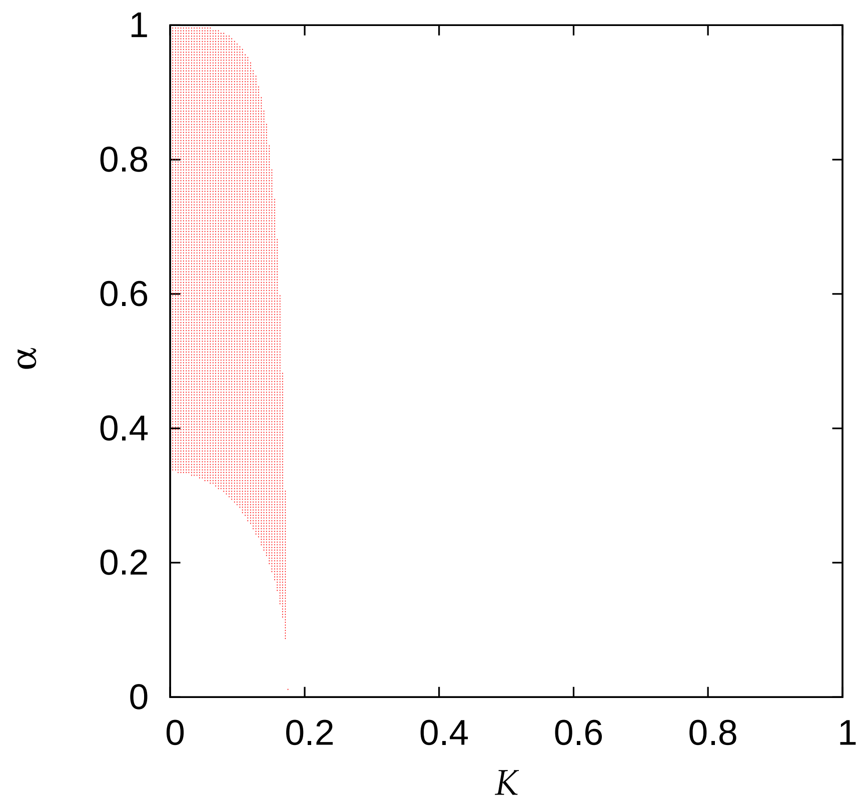


The spectra of exponential processes

Applications in adaptation of the model (Starting from a measured break frequency ω_{M}):

$$\omega_0 = \sqrt{g(0)} \leq \sqrt{g^{(2)}(0)} \leq \omega_{\mathrm{M}} \leq \sqrt{g^{(2)}(\infty)} \leq \sqrt{g(\infty)} = \omega_\infty$$



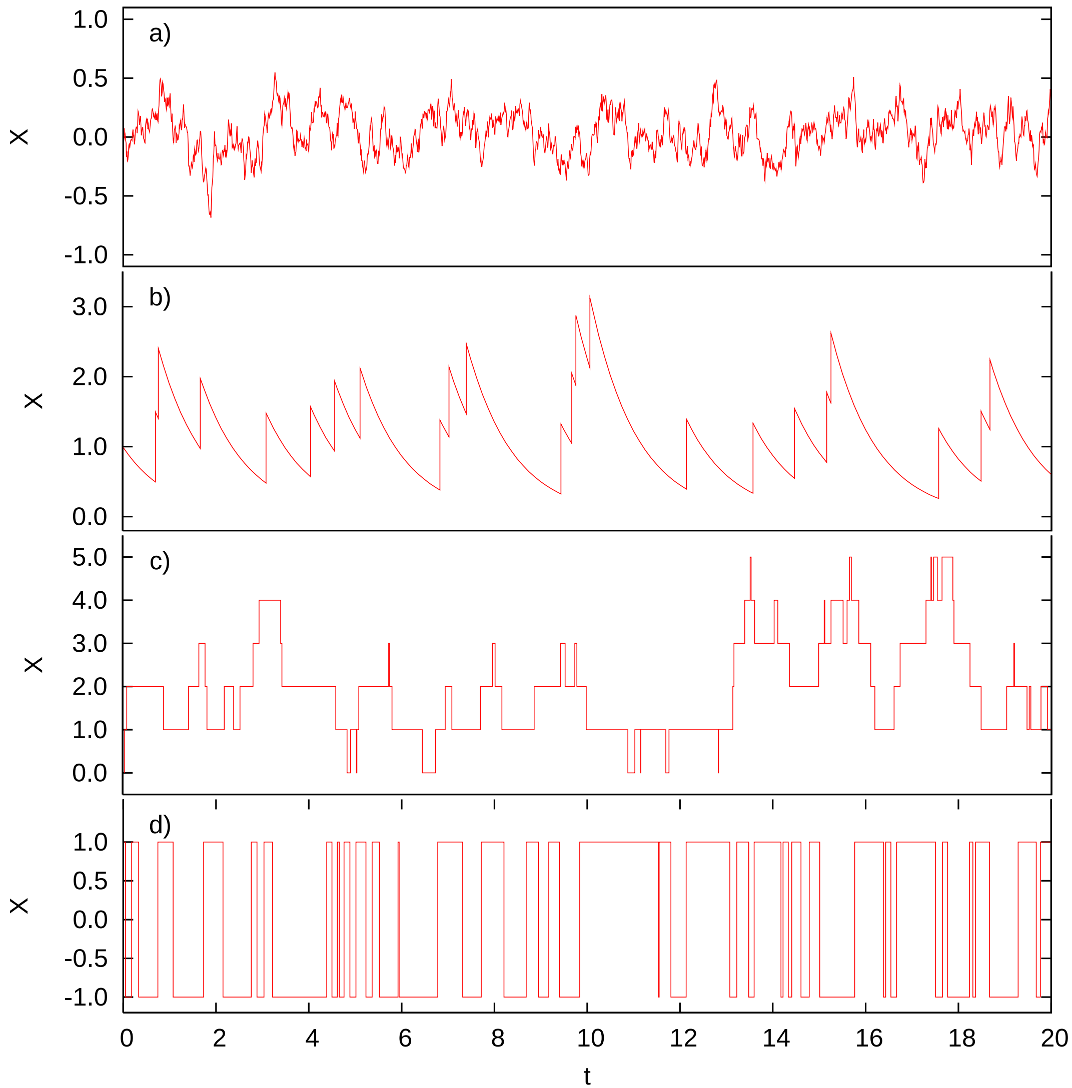


Conclusions

- Given a physical model, it is easy to calculate $S(\omega)$.
- Due to overwhelming degeneracy of the problem it is difficult to use to data interpretation.
- General robust constrains on the shape of $S(\omega)$ are good tool for ruling out of models.
- Evaluation of $g(x)$ is less complex than calculation of PSD $S(\omega)$.

Questions & Answers

Appendix



Different Processes:

a) Gaussian noise

b) Mutually independent exponential "flares"

c) & d) Continuous time Markov chains with discrete states.

Identical PSD, $S(\omega) = 1/(1+\omega^2)$.

The PSD is related to variance. Break frequency constrains only moments of $p(\tau)$.

Questions & Answers