Global weak solutions to the Navier-Stokes-Fourier system on moving domains

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Setting of the problem

Let us start with the full Navier-Stokes-Fourier system written as a balance of mass, momentum and total energy

$$\begin{cases} \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0 \\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \\ \partial_t(\varrho E) + \mathrm{div}_x ((\varrho E + p(\varrho, \vartheta))\mathbf{u}) + \mathrm{div}_x \mathbf{q} = \mathrm{div}_x(\mathbb{S}(\nabla_x \mathbf{u})\mathbf{u}) \end{cases} \tag{1}$$

Here the total energy $E = e(\varrho, \vartheta) + \frac{1}{2} |\mathbf{u}|^2$, where $e(\varrho, \vartheta)$ is the specific internal energy.

Constitutive laws

We assume the following form of the pressure law

$$p = p(\varrho, \vartheta) = p_e(\varrho) + p_{th}(\varrho, \vartheta) = p_e(\varrho) + \vartheta p_\vartheta(\varrho). \qquad (2)$$

Accordingly, the specific internal energy takes the form of

$$e = e(\varrho, \vartheta) = P_e(\varrho) + Q(\vartheta), \qquad (3)$$

where the elastic potential $P_e(\varrho) = \int_1^\varrho \frac{p_e(z)}{z^2} \mathrm{d}z$ and the thermal energy is related to the specific heat at constant volume as follows

$$Q(\vartheta) = \int_0^\vartheta c_v(z) \mathrm{d}z \qquad (4)$$

Reformulation

Under some smoothness assumptions the original system (1) can be rewritten as

$$\begin{cases} \partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) \ = \ 0 \\ \partial_t(\varrho \mathbf{u}) + \mathrm{div}_x\left(\varrho \mathbf{u} \otimes \mathbf{u}\right) + \nabla_x p(\varrho, \vartheta) \ = \ \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) \\ \partial_t(\varrho Q(\vartheta)) + \mathrm{div}_x\left(\varrho Q(\vartheta)\mathbf{u}\right) + \mathrm{div}_x \mathbf{q} \\ \qquad = \ \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - \vartheta p_\vartheta(\varrho)\mathrm{div}_x \mathbf{u} \end{cases} \tag{5}$$

Here

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu(\nabla_x \mathbf{u} + \nabla_x^T \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}) + \eta\, \mathrm{div}_x \mathbf{u}\mathbb{I}, \tag{6}$$

with $\mu > 0$, $\eta \geq 0$. Finally we have the Fourier law for the heat flux

$$\mathbf{q} = -\kappa(\vartheta)\nabla_x \vartheta \tag{7}$$

Domain and boundary conditions

We study the equations (5) on time dependent domain Ω_t, where the motion of the domain is given. More specifically, let $\mathbf{V}$ be a smooth vector field, we solve

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{X}(t,\mathbf{x}) = \mathbf{V}(t,\mathbf{X}(t,\mathbf{x})), \quad t>0, \quad \mathbf{X}(0,\mathbf{x}) = \mathbf{x} \tag{8}$$

and set $\Omega_t = \mathbf{X}(t,\Omega_0)$ with $\Omega_0 \subset \mathbb{R}^3$ is a given domain.
We consider the following boundary conditions

$$\left.\begin{array}{rcl} (\mathbf{u}-\mathbf{V})\cdot\mathbf{n} & = & 0 \\ \mathbb{S}\mathbf{n}\times\mathbf{n} & = & 0 \\ \mathbf{q}\cdot\mathbf{n} & = & 0 \end{array}\right\} \qquad \text{on } \partial\Omega_t \tag{9}$$

Hypothesis

In quite vague terms we assume the following behavior of functions appearing in the system

- $p_e(\varrho) \sim \varrho^\gamma$, $\gamma > \frac{3}{2}$
- $p_\vartheta(\varrho) \sim \varrho^\Gamma$, $\Gamma = \frac{\gamma}{3}$
- $\kappa(\vartheta) \sim 1 + \vartheta^\alpha$, $\alpha \geq 4$, $\alpha \geq \frac{12(\gamma-1)}{\gamma}$
- $c_v(\vartheta) \sim 1 + \vartheta^{\frac{\alpha}{2}-1}$

Reformulation of the thermal energy eq.

Following the book of E. Feireisl, instead of working with the thermal energy equation we rather work with two inequalities:

$$\partial_t(\varrho Q(\vartheta)) + \mathrm{div}_x\left(\varrho Q(\vartheta)\mathbf{u}\right) - \Delta\mathcal{K}(\vartheta) \\ \geq \mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \vartheta p_\vartheta(\varrho)\mathrm{div}_x\mathbf{u}, \tag{10}$$

where

$$\mathcal{K}(\vartheta) = \int_0^\vartheta \kappa(z)\mathrm{d}z, \tag{11}$$

together with the global total energy inequality, which on fixed domain would be simply

$$\int_\Omega \varrho\left(\frac{1}{2}|\mathbf{u}|^2 + P_e(\varrho) + Q(\vartheta)\right)(\tau)\mathrm{d}\mathbf{x} \\ \leq \int_\Omega \varrho\left(\frac{1}{2}|\mathbf{u}|^2 + P_e(\varrho) + Q(\vartheta)\right)(0)\mathrm{d}\mathbf{x}$$

Total energy inequality

However in the case of moving domain, this inequality is no longer this simple due to the presence of kinetic energy and the movement of the boundary. Therefore the total energy inequality reads as

$$\begin{aligned}
&\int_{\Omega_\tau} \varrho \left(\frac{1}{2} |\mathbf{u}|^2 + P_e(\varrho) + Q(\vartheta) \right) (\tau) \mathrm{d}\mathbf{x} \\
&\quad \leq \int_{\Omega_0} \varrho \left(\frac{1}{2} |\mathbf{u}|^2 + P_e(\varrho) + Q(\vartheta) \right) (0) \mathrm{d}\mathbf{x} \\
&\quad + \int_{\Omega_\tau} (\varrho \mathbf{u} \cdot \mathbf{V})(\tau) \mathrm{d}\mathbf{x} - \int_{\Omega_0} (\varrho \mathbf{u} \cdot \mathbf{V})(0) \mathrm{d}\mathbf{x} \\
&\quad + \int_0^\tau \int_{\Omega_t} \left(\mathbb{S} : \nabla_x \mathbf{V} - \varrho \mathbf{u} \cdot \partial_t \mathbf{V} - \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{V} - p \, \mathrm{div}_x \, \mathbf{V} \right) \mathrm{d}\mathbf{x} \mathrm{d}t
\end{aligned}$$

Let us illustrate the difficulty on the weak formulation of the momentum equation:

$$\int_{\Omega_\tau} \varrho \mathbf{u} \cdot \boldsymbol{\varphi}(\tau, \cdot) \mathrm{d}\mathbf{x} - \int_{\Omega_0} (\varrho \mathbf{u})_0 \cdot \boldsymbol{\varphi}(0, \cdot) \mathrm{d}\mathbf{x}$$
$$= \int_0^\tau \int_{\Omega_\tau} \varrho \mathbf{u} \cdot \partial_t \boldsymbol{\varphi} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \boldsymbol{\varphi} + p \operatorname{div}_x \boldsymbol{\varphi} - \mathbb{S} : \nabla_x \boldsymbol{\varphi} \mathrm{d}\mathbf{x}\mathrm{d}t$$

for any $\tau \in [0, T]$ and any test function $\boldsymbol{\varphi} \in C_c^\infty([0, T] \times \mathbb{R}^3)$ satisfying

$$\boldsymbol{\varphi} \cdot \mathbf{n} = 0 \tag{12}$$

on $\partial\Omega_\tau$ for any $\tau \in [0, T]$.
In particular $\mathbf{u}$ is not a proper test function due to the boundary condition $\mathbf{u} \cdot \mathbf{n} = \mathbf{V} \cdot \mathbf{n}$ and to derive the total energy inequality we have to test this equation by $\mathbf{u} - \mathbf{V}$.
Weak formulations of the other two equations are derived similarly.

Main theorem

Theorem 1

Let $\Omega_0 \subset \mathbb{R}^3$ *be a bounded domain of class* $C^{2+\nu}$ *and let* $\mathbf{V} \in C^1([0,T]; C_c^3(\mathbb{R}^3))$. *Let the initial data satisfy* $\varrho_0 \in L^\gamma$, $\varrho \geq 0$, $\varrho \not\equiv 0$, $(\varrho \mathbf{u})_0 = 0$ *a.e. on the set* $\{\varrho_0 = 0\}$, $\int_{\Omega_0} \frac{1}{\varrho_0} |(\varrho \mathbf{u})_0|^2 \, \mathrm{d}\mathbf{x} < \infty$, $\vartheta_0 \in L^\infty$, $\vartheta_0 \geq \underline{\vartheta} > 0$. *Then the problem admits a weak solution on any time interval* $(0,T)$ *with the following properties*

- $\varrho \in L^\infty(0,T; L^\gamma(\mathbb{R}^3)) \cap C([0,T]; L^1(\mathbb{R}^3))$ *and* $\varrho = 0$ *in* $\mathbb{R}^3 \setminus \overline{\Omega_\tau}$ *for a.a.* τ *in* $(0,T)$
- $\mathbf{u} \in L^2(0,T; W^{1,2}(\mathbb{R}^3))$ *with* $(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n} = 0$ *on* $\partial\Omega_\tau$ *for a.a.* τ *in* $(0,T)$
- $\varrho Q(\vartheta) \in L^\infty(0,T; L^1(\mathbb{R}^3))$, $\mathcal{K}(\vartheta) \in L^1((0,T) \times \Omega_\tau)$.

Strategy of the proof

The proof is based on the penalization method and a series of limiting processes. More specifically:

- We formulate the approximate problem on a fixed domain B such that $\bigcup_\tau \Omega_\tau \subset B$
- To the momentum equation we introduce a term penalizing flux through the physical boundary

$$\frac{1}{\varepsilon}\int_0^\tau \int_{\Gamma_\tau} (\mathbf{u}-\mathbf{V})\cdot\mathbf{n}\varphi\cdot\mathbf{n}\,\mathrm{d}S_x\mathrm{d}t \tag{13}$$

  Note that the physical boundaries $\Gamma_\tau = \partial\Omega_\tau$ are known interfaces since $\mathbf{V}$ is given.
- We introduce variable viscosity coefficients $\mu = \mu_\omega$, where μ_ω is constant in the fluid region and vanishes in the solid region as $\omega \to 0$.

Strategy of the proof II

- We need also variable heat conductivity coefficient $\kappa_{\nu,\zeta}(\vartheta)$. Here we first consider discontinuous function $\kappa_\zeta(\vartheta)$ such that $\kappa_\zeta(\vartheta) = \kappa(\vartheta)$ on the fluid part and $\kappa_\zeta(\vartheta) = \zeta\kappa(\vartheta) \to 0$ as $\zeta \to 0$ on the solid part. The parameter ν then denotes the mollification of such discontinuous function.
- We introduce the artificial pressure $p_\delta(\varrho, \vartheta) = p(\varrho, \vartheta) + \delta\varrho^\beta$ for some $\beta \geq 2$.

The thermal energy equation in the weak formulation at the approximate level is moreover satisfied in the renormalized sense (multiplied by $h(\vartheta)$ for $h \in C^\infty([0, \infty))$ with h having some properties...)
The formulation of the approximate problem on the fixed domain B is done in the spirit of works of E. Feireisl on this topic, in particular, we use his result to prove the existence of weak solutions to the approximate problem.
Then we have to handle four limiting processes.

Limiting processes

After we derive a priori estimates independent of ε, ν, ω but some depending on δ, we proceed in the following order

(i) Penalization limit $\varepsilon \to 0$: After this limit we recover something, which can be described as a two fluids system separated by a membrane (at least concerning momentum equations).

In this step we use the following lemma proved in the paper of Feireisl, K., Nečasová, Neustupa, Stebel on global existence of weak solutions in the barotropic case to show that the density on the solid part is in fact zero.

Key lemma

Lemma 2

Let $\varrho \in L^\infty(0, T; L^2(B))$*,* $\varrho \geq 0$*,* $\mathbf{u} \in L^2(0, T; W_0^{1,2}(B))$ *be a weak solution of the equation of continuity on* B*. Let moreover* $(\mathbf{u} - \mathbf{V}) \cdot \mathbf{n} = 0$ *on* Γ_τ *for a.a.* $\tau \in (0, T)$*. Finally let* $\varrho_0 \in L^2(\mathbb{R}^3)$*,* $\varrho_0 \geq 0$*,* $\varrho_0 = 0$ *on* $B \setminus \Omega_0$*. Then* $\varrho(\tau, \cdot) = 0$ *on* $B \setminus \Omega_\tau$ *for a.a.* $\tau \in [0, T]$*.*

In this limit as well as in the following ones we use heavily the theory started in the works of P.-L. Lions and then further developed by E. Feireisl (based on the effective viscous pressure identity and the oscillation defect measure) to prove strong convergence of the densities. In particular, since we use local pressure estimates, the momentum equation in the limit $\varepsilon \to 0$ is satisfied only for test functions having also the property

$$\mathrm{supp}[\mathrm{div}_x\, \boldsymbol{\varphi}(\tau, \cdot)] \cap \Gamma_\tau = \emptyset \tag{14}$$

Extending the class of test functions

On the other hand, it was also shown in the previous work of Feireisl, K., Nečasová, Neustupa, Stebel, that such class of test function can be extended by density arguments to avoid the condition (14).
As a consequence of the Key lemma, all quantities which are multiplied by ϱ vanish outside of the fluid part. In particular, in the momentum equation, the only surviving term supported on the solid part is the stress tensor

$$\int_0^t \int_{B \setminus \Omega_\tau} \mathbb{S}_\omega(\nabla_x \mathbf{u}) : \nabla_x \varphi \mathrm{d}\mathbf{x} \mathrm{d}t \tag{15}$$

Discontinuous heat conductivity

We proceed next with the limit

(ii) Discontinuous heat conductivity $\nu \to 0$: This limit does not need any additional theory in comparison to the previous one. It basically justifies the choice of the test functions in the thermal energy inequality to satisfy the condition $\nabla_x \varphi \cdot \mathbf{n} = 0$ on Γ_τ.

Vanishing viscosity

The third limit in the process is

(iii) Vanishing viscosity $\omega \to 0$: We get rid of the only remaining term in the momentum equation integrated over the solid part. Again, no new contributions to the theory we are using are needed. In the thermal energy equation (actually inequality) and the total energy inequality we use positivity of terms of the type $\mathbb{S}_\omega : \nabla_x \mathbf{u}$ to pass to the limit.

Vanishing artificial pressure

Finally, we set $\zeta = \delta^2$ and

(iv) pass with δ to zero.

Here we need to be more careful and derive new a priori estimates independent of δ. This is where the condition $\alpha \geq \frac{12(\gamma-1)}{\gamma}$ appears.

In this final step we also proceed in the spirit of E. Feireisl to handle the limit in the thermal energy inequality and switch from the renormalized formulation to the variational formulation to handle the limit of sequence $\mathcal{K}(\vartheta_\delta)$ which is just bounded in L_1.

Concluding remarks and open problems

- We can handle also the case of nonconstant viscosities $\mu(\vartheta)$, $\eta(\vartheta)$ (bounded, globally Lipschitz)
- The same result can be obtained also in the case of Navier slip boundary condition
- The case of Dirichlet boundary condition for the velocity should be treated similarly (Brinkman type penalization) (to be checked)

Open problems:

- Main problems are caused by temperature ϑ on the vacuum regions $\varrho = 0$. This is the reason why we (at the moment) can not work (for example) with the radiation pressure $a\vartheta^4$
- Having global existence of weak solutions, plenty of other problems may be attacked in the future - relative entropy inequality, weak-strong uniqueness, singular limits etc...
- The problem on moving domain with prescribed **V** is also a first step towards problems of fluid-structure interaction

Thank you

Thank you for your attention.