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The Robin problem for the scalar Oseen equation

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We study the Robin problem for the scalar Oseen equation in an open n-dimensional set with compact Ljapunov boundary. We prescribe two types of Robin boundary conditions, and prove the unique solvability of these problems as well as a representation formula for the solution in form of a scalar Oseen single layer potential. Moreover, we prove the maximum principle for the solution to the Robin problem of the scalar Oseen equation. Copyright © 2013 John Wiley & Sons, Ltd.

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1. Introduction

The Dirichlet, Neumann, and Robin boundary value problems for second order partial differential equations are important model problems in mathematical physics [1]. Traditionally, the Dirichlet and Neumann problems for the Laplace equation in domains with smooth boundary have been studied by the method of integral equations long time ago. Later, also the Robin problem for the Laplace equation in smooth and Lipschitz domains has been investigated by this method [2–7].

Recently, also the boundary value problems for the scalar Oseen equation

$$-\Delta u + 2\lambda\partial_1 u = 0 \quad \text{in } \Omega \subset \mathbb{R}^3, \quad \lambda \in \mathbb{R}, \tag{1}$$

have been studied by the method of integral equations [8, 9]. Here, the authors study the Dirichlet problem, that is, they prescribe the boundary condition

$$u = g \quad \text{on } \partial\Omega,$$

and the Oseen Neumann problem, prescribing the boundary condition

$$\frac{\partial u}{\partial n} - \lambda n_1 u = g \quad \text{on } \partial\Omega.$$

Here $n = n^\Omega$ is the outward (with respect to Ω) unit normal vector on $\partial\Omega$.

In the present paper, we study the Robin problem for the scalar Oseen equation in an open set $\Omega \subset \mathbb{R}^3$ with compact boundary $\partial\Omega$ of class $C^{1,\alpha}$, $\alpha > 0$, that is, the scalar Oseen problem (1) with prescribed boundary condition

$$\frac{\partial u}{\partial n} + hu = g \quad \text{on } \partial\Omega, \tag{2}$$

where h denotes a positive function, and the Robin problem corresponding to the Oseen Neumann condition studied in [8, 9], that is, we prescribe the boundary condition

$$\frac{\partial u}{\partial n} - \lambda n_1 u + hu = g \quad \text{on } \partial\Omega \tag{3}$$

with $h \geq 0$. We prove unique solvability of these problems, a representation of the solution in form of a scalar Oseen single layer potential, and the maximum principle for the solution of the Robin problem for the scalar Oseen equation.

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2. The maximum principle for the Robin problem

Let $\Omega \subset \mathbb{R}^3$ be an open set with boundary of class C^1. Denote by $\overline{\Omega}$ the closure of Ω, and by $n = n^{\Omega}(z)$ the outward (with respect to Ω) unit normal vector in $z \in \partial\Omega$. Let $g, h \in C^0(\partial\Omega)$ and $\lambda \in \mathbb{R}$ be given. Then we call u a classical solution of the Robin problem for the scalar Oseen Equations (1) and (2), if $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, if there exists $\partial u(z)/\partial n$ at each $z \in \partial\Omega$, and if Equations (1) and (2) are satisfied.

The following result holds true for general $2 \le m \in \mathbb{N}$.

Proposition 2.1
Let $G \subset \mathbb{R}^m$ be an open set with bounded boundary ∂G. Let g and $a \ge 0$ be functions defined on ∂G, and let $\lambda \in \mathbb{R}$. Let $u \in \mathcal{C}^2(G) \cap \mathcal{C}^0(\overline{G})$ be given with $-\Delta u + 2\lambda\partial_1 u = 0$ in G, and let $\theta = (\theta_1, \ldots, \theta_m)$ be a unit vector function on ∂G. Suppose that for $z \in \partial G$ with $a(z) \neq 0$ the function $\theta = \theta(z)$ satisfies $\{z + t\theta; -\delta < t < 0\} \subset \Omega$ and $\{z + t\theta; 0 < t < \delta\} \cap \Omega = \emptyset$ for some $\delta > 0$, and suppose that there exists

$$\frac{\partial u(z)}{\partial \theta} = \lim_{t \to 0_-} \frac{u(z + t\theta) - u(z)}{t}.$$

If $a(z) = 0$ set $a(z)(\partial u(z)/\partial\theta) = 0$. Now assume that $a(\partial u/\partial\theta) + u = g$ on ∂G. If G is unbounded suppose moreover that $u(x) \to 0$ as $|x| \to \infty$. Then the following estimate holds true:

$$\inf_{x \in \partial G} g(x) \le \inf_{x \in G} u(x) \le \sup_{x \in G} u(x) \le \sup_{x \in \partial G} g(x).$$

Proof
The maximum principle ([10], Chapter 3, Theorem 3.1) gives that there exists $z \in \partial G$ such that $u \le u(z)$. If $a(z) \neq 0$ then $\partial u(z)/\partial\theta \ge 0$ because θ is an outward pointing vector and $u \le u(z)$. Thus

$$\sup_{x \in G} u(x) = u(z) \le a(z)\frac{\partial u(z)}{\partial \theta} + u(z) = g(z) \le \sup_{x \in \partial G} g(x).$$

Now, for $v = -u$, we have

$$\inf_{x \in \partial G} g(x) = -\sup_{x \in \partial G}(-g(x)) \le -\sup_{x \in G} v(x) = \inf_{x \in G} u(x).$$

$\square$

3. Potentials

Let $\lambda \in R$, $|x| := \sqrt{x_1^2 + x_2^2 + x_3^2}$, and let

$$E_{2\lambda}(x) = \frac{1}{4\pi|x|} e^{-(|\lambda x| - \lambda x_1)}$$

denote the fundamental solution of the scalar Oseen Equation (1). Note that $E_0(x) = 1/(4\pi|x|)$ is the fundamental solution of the Laplace equation $-\Delta u = 0$. Because $\Omega \subset \mathbb{R}^3$ is an open set with bounded boundary of class $C^{1,\alpha}$, $0 < \alpha$, and $\varphi \in C^0(\partial\Omega)$, then the scalar Oseen single layer potential

$$E_{2\lambda}^{\Omega}\varphi(x) = \int_{\partial\Omega} E_{2\lambda}(x - y)\varphi(y)\, \mathrm{d}\sigma_y$$

is well defined. Easy calculations yield $E_{2\lambda}^{\Omega}\varphi \in C^\infty(\mathbb{R}^3 \setminus \partial\Omega)$ and $-\Delta E_{2\lambda}^{\Omega}\varphi + 2\lambda\partial_1 u = 0$ in $\mathbb{R}^3 \setminus \partial\Omega$ [9]. Moreover, for $\lambda = 0$, we find

$$E_0^{\Omega}\varphi(x) = O(1/|x|), \quad |\nabla E_0^{\Omega}\varphi(x)| = O(1/|x|^2) \qquad \text{as } |x| \to \infty. \tag{4}$$

If $\lambda \neq 0$, then

$$|E_{2\lambda}^{\Omega}\varphi(x)| + |\nabla E_{2\lambda}^{\Omega}\varphi(x)| = O\left(e^{-(|\lambda x| - \lambda x_1)}/|x|\right) \qquad \text{as } |x| \to \infty. \tag{5}$$

Because $E_{2\lambda}^{\Omega}$ is an integral operator with weakly singular kernel, it is a compact linear operator on $C^0(\partial\Omega)$ (see, for example, [11]).

Lemma 3.1
Let $\Omega \subset \mathbb{R}^m$ be an open set with bounded boundary $\partial\Omega$ of class $C^{1,\alpha}, \alpha > 0$. Let $k(x, y)$ be defined for $[x, y] \in \mathbb{R}^m \times \partial\Omega; x \neq y$ and $k(x, y)| \le C|x - y|^{1-m+\beta}$ with positive constants C, β. Suppose that $k(x, \cdot)$ is measurable, and $k(\cdot, y)$ is continuous. Let $f \in L^\infty(\partial\Omega)$. Then

$$kf(x) = \int_{\partial\Omega} k(x, y) f(y)\, \mathrm{d}\sigma_y$$

is a continuous function in $\mathbb{R}^m$.

(See [12] or [13], Lemma 3.2)

Lemma 3.2
Let $\lambda \in \mathbb{R}$. Define $R_{2\lambda}(x) = E_{2\lambda}(x) - E_0(x)$. Then

$$R_{2\lambda}(x) = O(1), \quad |\nabla R_{2\lambda}(x)| = O(|x|^{-1}) \qquad |x| \to 0.$$

If $\Omega \subset \mathbb{R}^3$ is an open set with bounded boundary of class $C^{1,\alpha}$, $0 < \alpha < 1$, $\varphi \in C^0(\partial\Omega)$, then for

$$R^{\Omega}_{2\lambda}\varphi(x) = \int\limits_{\partial\Omega} R_{2\lambda}(x-y)\varphi(y)\, \mathrm{d}\sigma_y$$

we find $R^{\Omega}_{2\lambda}\varphi \in C^1(\mathbb{R}^3)$.

Proof
Put $f(t) = (e^t - 1)/t$ for $t \neq 0$, $f(0) = 1$. Then f is continuous. So, there is a constant C such that $|f(t)| \le C$ for $|t| \le 1$. If $0 < |t| \le 1$ then $|f'(t)| = |e^t/t - (e^t - 1)/t^2| \le (C + e)/t$. Clearly,

$$R_{2\lambda}(x)| = f(-(|\lambda x| - \lambda x_1))\frac{-(|\lambda x| - \lambda x_1)}{|x|}.$$

Thus, $|R_{2\lambda}(x)| = O(1)$ as $|x| \to 0$. Moreover,

$$|\nabla R_{2\lambda}(x)| \le |f'(-(|\lambda x| - \lambda x_1))| \frac{|8\lambda(|\lambda x| - \lambda x_1)|}{|x|} + |f(-(|\lambda x| - \lambda x_1))| O(1/|x|) = O(1/|x|), \qquad |x| \to 0.$$

Using Lemma 3.1 for $R^{\Omega}_{2\lambda}$ and $\partial_j R^{\Omega}_{2\lambda}$, we obtain $R^{\Omega}_{2\lambda}\varphi \in C^1(\mathbb{R}^3)$. □

Proposition 3.3
Let $\Omega \subset \mathbb{R}^3$ be an open set with bounded boundary of class $C^{1,\alpha}$, $\alpha > 0$, $\lambda \in R$. For $x, y \in \partial\Omega$, $x \neq y$ set

$$L^{\Omega}_{2\lambda}(x,y) = n^{\Omega}(x) \cdot \nabla_x E_{2\lambda}(x-y).$$

For $\varphi \in C^0(\partial\Omega)$ define

$$L^{\Omega}_{2\lambda}\varphi(x) = \int\limits_{\partial\Omega} L^{\Omega}_{2\lambda}(x,y)\varphi(y)\, \mathrm{d}\sigma_y.$$

Then $L^{\Omega}_{2\lambda}$ is a compact linear operator on $C^0(\partial\Omega)$.

Proof
It is well known that L^{Ω}_0 is a compact linear operator on $C^0(\partial\Omega)$ (see, for example, [11] or [14]). Because $L^{\Omega}_{2\lambda} - L^{\Omega}_0$ is an integral operator with weakly singular kernel (Lemma 3.2), it is a compact linear operator on $C^0(\partial\Omega)$ (see, for example, [11]). □

Proposition 3.4
Let $\Omega \subset \mathbb{R}^3$ be an open set with bounded boundary of class $C^{1,\alpha}$, $\alpha > 0$, $\lambda \in R$, $\varphi \in C^0(\partial\Omega)$. Then $E^{\Omega}_{2\lambda}\varphi \in C^0(\mathbb{R}^3)$. Put $u = E^{\Omega}_{2\lambda}\varphi$ in Ω. Then

$$\frac{\partial u(x)}{\partial n} = \frac{1}{2}\varphi(x) + L^{\Omega}_{2\lambda}\varphi(x).$$

Here $n = n(x)$ is the exterior unit normal with respect to Ω in $x \in \partial\Omega$.

Proof
The proposition is well known for $\lambda = 0$ (see, for example, [11] or [14].) By virtue of Lemma 3.2, we obtain the result for arbitrary λ. □

Corollary 3.5
Let $\Omega \subset \mathbb{R}^3$ be an open set with bounded boundary of class $C^{1,\alpha}$, $\alpha > 0$, $\lambda \in R$, $\varphi \in C^0(\partial\Omega)$. If $E^{\Omega}_{2\lambda}\varphi = 0$ on $\partial\Omega$, then $\varphi = 0$.

Proof

Put $u = E^{\Omega}_{2\lambda}\varphi$ in Ω, $v = E^{\Omega}_{2\lambda}\varphi$ in $G = \mathbb{R}^3 \setminus \overline{\Omega}$. Proposition 3.4 gives that $E^{\Omega}_{2\lambda}\varphi \in C^0(\mathbb{R}^3)$. Moreover, $E^{\Omega}_{2\lambda}\varphi$ is a solution of the scalar Oseen equation in $\mathbb{R}^3 \setminus \partial\Omega$ and $E^{\Omega}_{2\lambda}\varphi(x) \to 0$ as $x \to \infty$. The maximum principle ([10], Chapter 3, Theorem 3.1) gives that $E^{\Omega}_{2\lambda}\varphi = 0$ in $\mathbb{R}^3$. Fix $x \in \partial\Omega$. Let n be the unit outward normal of Ω at x. According to Proposition 3.4, we have

$$0 = \frac{\partial u(x)}{\partial n} + \frac{\partial v(x)}{\partial(-n)} = \frac{1}{2}\varphi(x) + L^{\Omega}_{2\lambda}\varphi(x) + \frac{1}{2}\varphi(x) + L^{G}_{2\lambda}\varphi(x) = \varphi(x).$$

□

Proposition 3.6

Let $\Omega \subset \mathbb{R}^3$ be an open set with bounded boundary of class $C^{1,\alpha}$, $\alpha > 0$, $\lambda \in R$, $\varphi \in C^0(\partial\Omega)$. Then

$$\int_{\partial\Omega} (E^{\Omega}_{2\lambda}\varphi)\left(\frac{1}{2}\varphi + L^{\Omega}_{2\lambda}\varphi - \lambda n_1 E^{\Omega}_{2\lambda}\varphi\right) \mathrm{d}\sigma_y = \int_{\Omega} |\nabla E^{\Omega}_{2\lambda}\varphi|^2 \,\mathrm{d}y.$$

Proof

If $\lambda = 0$, [15]. Let now $\lambda \neq 0$. Suppose first that Ω is bounded. We know that $|\nabla E^{\Omega}_{0}\varphi| \in L^2(\Omega)$. Lemma 3.2 gives that $R^{\Omega}_{2\lambda}\varphi \in C^1(\mathbb{R}^3)$. Thus $|\nabla E^{\Omega}_{2\lambda}\varphi| \in L^2(\Omega)$. For $\epsilon > 0$ denote $\Gamma_\epsilon = \left\{x - \epsilon n^{\Omega}(x); x \in \partial\Omega\right\}$, and by Ω_ϵ an open set such that $\partial\Omega_\epsilon = \Gamma_\epsilon$, $\Omega_\epsilon \subset \Omega$. We know that $E^{\Omega}_{2\lambda}\varphi$ is continuous in $\mathbb{R}^3$. Because $\partial E^{\Omega}_{0}\varphi\left(x - \epsilon n^{\Omega}(x)\right)/\partial n^{\Omega}(x) \to \varphi(x)/2 + L^{\Omega}_{0}\varphi$ as $\epsilon \to 0$ uniformly with respect to $\partial\Omega$ ([16], Chapter XV, §3 or [11], §28 or [14], §1.2, Theorem 2), we deduce that $\partial E^{\Omega}_{2\lambda}\varphi\left(x - \epsilon n^{\Omega}(x)\right)/\partial n - \lambda n_1 E^{\Omega}_{2\lambda})^2\left(x - \epsilon n^{\Omega}(x)\right) \to \varphi(x)/2 + L^{\Omega}_{2\lambda}\varphi(x) - \lambda n_1 E^{\Omega}_{2\lambda}\varphi(x)$ as $\epsilon \to 0$ uniformly with respect to $\partial\Omega$. By virtue of Green's Formula, we find by the same way like for the Neumann problem for the Laplace equation (compare [11,12,16])

$$\begin{aligned}
\int_{\partial\Omega} \left(E^{\Omega}_{2\lambda}\varphi\right)\left(\frac{1}{2}\varphi + L^{\Omega}_{2\lambda}\varphi - \lambda n_1 E^{\Omega}_{2\lambda}\varphi\right) \mathrm{d}\sigma_y &= \lim_{\epsilon\to 0}\int_{\Gamma_\epsilon} \left\{\left(E^{\Omega}_{2\lambda}\varphi\right)\frac{\partial E^{\Omega}_{2\lambda}\varphi}{\partial n} - \lambda n_1\left(E^{\Omega}_{2\lambda}\varphi\right)^2\right\} \mathrm{d}\sigma_y \\
&= \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} \left\{\left(E^{\Omega}_{2\lambda}\varphi\right)\left(\Delta E^{\Omega}_{2\lambda}\varphi\right) + |\nabla E^{\Omega}_{2\lambda}\varphi|^2 - 2\lambda\left(E^{\Omega}_{2\lambda}\varphi\right)\partial_1 E^{\Omega}_{2\lambda}\varphi\right)\right\} \mathrm{d}y \\
&= \lim_{\epsilon\to 0}\int_{\Omega_\epsilon} \left\{|\nabla E^{\Omega}_{2\lambda}\varphi|^2 + \left(E^{\Omega}_{2\lambda}\varphi\right)\left(\Delta E^{\Omega}_{2\lambda}\varphi - 2\lambda\partial_1 E^{\Omega}_{2\lambda}\varphi\right)\right\} \mathrm{d}y = \int_{\Omega} |\nabla E^{\Omega}_{2\lambda}\varphi|^2 \,\mathrm{d}y.
\end{aligned}$$

(We can also prove this equality by approximating Ω from inside by [17], Theorem 1.12.)

Now, let Ω be unbounded. Set $G(R) = \{x \in \Omega; |x| < R\}$, and define $\varphi = 0$ outside of $\partial\Omega$. Then we obtain

$$\begin{aligned}
\int_{\Omega} |\nabla E^{\Omega}_{2\lambda}\varphi|^2 \,\mathrm{d}y &= \lim_{R\to\infty}\int_{G(R)} |\nabla E^{\Omega}_{2\lambda}\varphi|^2 \,\mathrm{d}y \\
&= \lim_{R\to\infty}\int_{\partial G(R)} \left(E^{G(R)}_{2\lambda}\varphi\right)\left(\frac{1}{2}\varphi + L^{G(R)}_{2\lambda}\varphi - \lambda n_1 E^{G(R)}_{2\lambda}\varphi\right) \mathrm{d}\sigma_y \\
&= \int_{\partial\Omega} \left(E^{\Omega}_{2\lambda}\varphi\right)\left(\frac{1}{2}\varphi + L^{\Omega}_{2\lambda}\varphi - \lambda n_1 E^{\Omega}_{2\lambda}\varphi\right) \mathrm{d}\sigma_y + \lim_{R\to\infty}\int_{\{|x|=R\}} \left\{\left(E^{\Omega}_{2\lambda}\varphi\right)\frac{\partial E^{\Omega}_{2\lambda}\varphi}{\partial n} - \lambda n_1\left(E^{\Omega}_{2\lambda}\varphi\right)^2\right\} \mathrm{d}\sigma_y.
\end{aligned}$$

With help of Equations (5) and (4) and Lebesque's Lemma, this implies

$$\begin{aligned}
\left|\int_{\{|x|=R\}} \left\{(E^{\Omega}_{2\lambda}\varphi)\frac{\partial E^{\Omega}_{2\lambda}\varphi}{\partial n} - \lambda n_1 (E^{\Omega}_{2\lambda}\varphi)^2\right\} \mathrm{d}\sigma_y\right| &\leq \int_{\{|x|=R\}} Ce^{-2(|\lambda x| - \lambda x_1)}|x|^{-2} \,\mathrm{d}\sigma_y \\
&= \int_{\{|x|=1\}} Ce^{-2R(|\lambda x| - \lambda x_1)} \,\mathrm{d}\sigma_y \to 0
\end{aligned}$$

as $R \to \infty$. Thus the proposition is proved. □

4. The boundary condition (2) with $h > 0$

We shall look for a solution of the problems (1) and (2) in the form of a single layer potential $E^{\Omega}_{2\lambda}\varphi$ with $\varphi \in C^0(\partial\Omega)$. According to Proposition 3.4, the function $E^{\Omega}_{2\lambda}\varphi$ is a classical solution of the problems (1) and (2) if and only if

$$\frac{1}{2}\varphi + L^{\Omega}_{2\lambda}\varphi + hE^{\Omega}_{2\lambda}\varphi = g \quad \text{on } \partial\Omega. \tag{6}$$

Theorem 4.1
Let $\Omega \subset \mathbb{R}^3$ be an open set with bounded boundary of class $C^{1,\alpha}$, $\alpha > 0$, $\lambda \in R$. If $h \in C^0(\partial\Omega)$, $h > 0$, then $T = (1/2)I + L^{\Omega}_{2\lambda} + hE^{\Omega}_{2\lambda}$ is a continuously invertible operator in $C^0(\partial\Omega)$. Assume $g \in C^0(\partial\Omega)$.

- If Ω is bounded, then there exists a unique classical solution u of the problems (1) and (2).
- If Ω is unbounded, then there exists a unique classical solution u of the problems (1) and (2) such that $u(x) \to 0$ as $|x| \to \infty$.

Moreover, $u = E^{\Omega}_{2\lambda}T^{-1}g$ and

$$\inf_{x\in\partial\Omega}\frac{g(x)}{h(x)} \le \inf_{x\in\Omega} u(x) \le \sup_{x\in\Omega} u(x) \le \sup_{x\in\partial\Omega}\frac{g(x)}{h(x)}. \tag{7}$$

Proof
If u is a solution of problems (1) and (2), then u satisfies the boundary condition $h^{-1}(\partial u/\partial n) + u = g/h$ on $\partial\Omega$. Proposition 2.1 gives uniqueness and the estimate (7).

Now, let $\varphi \in C^0(\partial\Omega)$ satisfy $T\varphi = 0$. Then $u = E^{\Omega}_{2\lambda}\varphi$ is a classical solution of problems (1) and (2) with $g = 0$. We have proved that $E^{\Omega}_{2\lambda}\varphi = u = 0$ on $\overline{\Omega}$. Corollary 3.5 gives that $\varphi = 0$.

$L^{\Omega}_{2\lambda}$ is a compact linear operator on $C^0(\partial\Omega)$ by Proposition 3.3. Because $E^{\Omega}_{2\lambda}$ is an integral operator with weakly singular kernel, it is a compact operator on $C^0(\partial\Omega)$, see, for example, [11]. Thus, the operator $T - (1/2)I$ is compact, too. Because T is one to one, the Riesz–Schauder theory gives that T is a continuously invertible operator on $C^0(\partial\Omega)$. Hence, $u = E^{\Omega}_{2\lambda}T^{-1}g$ is a classical solution of the Robin problems (1) and (2). □

Corollary 4.2
Let $\Omega \subset \mathbb{R}^3$ be an open set with bounded boundary of class $C^{1,\alpha}$, $\alpha > 0$, $\lambda \in R$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ such that there exists $\partial u(z)/\partial n$ at each $z \in \partial\Omega$, $\partial u/\partial n \in C^0(\partial\Omega)$, and Equation (1) holds true. If Ω is unbounded suppose in addition that $u(x) \to 0$ as $|x| \to \infty$. Then there exists $\varphi \in C^0(\partial\Omega)$ such that $u = E^{\Omega}_{2\lambda}\varphi$. Moreover, we find

$$\int_{\partial\Omega} u\left(\frac{\partial u}{\partial n} - \lambda n_1 u\right) \, d\sigma_y = \int_{\Omega} |\nabla u|^2 \, dy < \infty. \tag{8}$$

Proof
Set $h = 1, g = \partial u/\partial n + u$. Then u is a classical solution of the Robin problems (1) and (2). Theorem 4.1 gives that there exists $\varphi \in C^0(\partial\Omega)$ such that $u = E^{\Omega}_{2\lambda}\varphi$. Then Proposition 3.4 and Proposition 3.6 prove Equation (8). □

5. The boundary condition (3) with $h \ge 0$

In this section, we shall look for a solution of problems (1) and (3) in the form of a single layer potential $E^{\Omega}_{2\lambda}\varphi$ with $\varphi \in C^0(\partial\Omega)$. According to Proposition 3.4, the function $E^{\Omega}_{2\lambda}\varphi$ is a classical solution of the problems (1) and (3), if and only if

$$\frac{1}{2}\varphi + L^{\Omega}_{2\lambda}\varphi - \lambda n_1 E^{\Omega}_{2\lambda}\varphi + hE^{\Omega}_{2\lambda}\varphi = g \quad \text{on } \partial\Omega. \tag{9}$$

Theorem 5.1
Let $\Omega \subset \mathbb{R}^3$ be an open set with bounded boundary of class $C^{1,\alpha}$, $\alpha > 0$, $\lambda \in R \setminus \{0\}$. If $h \in C^0(\partial\Omega)$, $h \ge 0$, then $T = (1/2)I + L^{\Omega}_{2\lambda} - \lambda n_1 E^{\Omega}_{2\lambda}\varphi + hE^{\Omega}_{2\lambda}$ is a continuously invertible operator in $C^0(\partial\Omega)$. Fix $g \in C^0(\partial\Omega)$.

- If Ω is bounded, then there exists a unique classical solution u of the problems (1) and (3).
- If Ω is unbounded, then there exists a unique classical solution u of the problems (1) and (3) such that $u(x) \to 0$ as $|x| \to \infty$.

Moreover, $u = E^{\Omega}_{2\lambda}T^{-1}g$ and

$$\sup_{x\in\Omega}|u(x)| \le C \sup_{x\in\partial\Omega}|g(x)|,$$

where the constant C depends only on Ω and λ.

Proof

Suppose first that u is a classical solution of the problems (1) and (3) with $g = 0$. According to Corollary 4.2

$$0 = \int_{\partial\Omega} u \left(\frac{\partial u}{\partial n} - \lambda n_1 u + hu \right) \, d\sigma_y = \int_{\Omega} |\nabla u|^2 \, dy + \int_{\partial\Omega} hu^2 \, d\sigma_y.$$

Thus, $\nabla u = 0$ in Ω and $hu = 0$ on $\partial\Omega$. Because $\nabla u = 0$ in Ω, the function u is constant on each component of Ω and $0 = \partial u/\partial n - \lambda n_1 u + hu = -\lambda n_1 u$ on $\partial\Omega$. Because u is constant on each component of $\partial\Omega$, we infer that $u = 0$ on $\partial\Omega$. Because u is constant on each component of $\overline{\Omega}$, we deduce that $u \equiv 0$.

If $\varphi \in C^0(\partial\Omega)$, $T\varphi = 0$, then $E^{\Omega}_{2\lambda}\varphi$ is a classical solution of problems (1) and (3) with $g = 0$. We have proved that $E^{\Omega}_{2\lambda}\varphi = 0$ on $\partial\Omega$. Corollary 3.5 gives that $\varphi = 0$.

The operator $L^{\Omega}_{2\lambda}$ is a compact linear operator on $C^0(\partial\Omega)$ by Proposition 3.3. Because $E^{\Omega}_{2\lambda}$ is an integral operator with weakly singular kernel, it is a compact operator on $C^0(\partial\Omega)$ (see, for example, [11]). Thus, the operator $T - (1/2)I$ is compact. Because T is one to one, the Riesz–Schauder theory gives that T is a continuously invertible operator in $C^0(\partial\Omega)$. If $g \in C^0(\partial\Omega)$ then $u = E^{\Omega}_{2\lambda}T^{-1}g$ is a classical solution of the Robin problems (1) and (3).

The operator $E^{\Omega}_{2\lambda}T^{-1}$ is a linear operator from $C^0(\partial\Omega)$ to $C^0(\overline{\Omega})$. Suppose that $\varphi_n \to \varphi$ in $C^0(\partial\Omega)$, $E^{\Omega}_{2\lambda}T^{-1}\varphi_n \to \psi$ in $C^0(\overline{\Omega})$. If $x \in \Omega$, an easy calculation gives that $E^{\Omega}_{2\lambda}T^{-1}\varphi_n(x) \to E^{\Omega}_{2\lambda}T^{-1}\varphi(x)$. Hence, $E^{\Omega}_{2\lambda}T^{-1}\varphi = \psi$, and the operator $E^{\Omega}_{2\lambda}T^{-1}$ is closed. The closed graph theorem ([18], Theorem II.1.9) gives that the operator $E^{\Omega}_{2\lambda}T^{-1}$ is bounded. So, there exists a constant C such that

$$\sup_{x\in\Omega} \left| E^{\Omega}_{2\lambda}T^{-1}g(x) \right| \leq C \sup_{x\in\partial\Omega} |g(x)|$$

for each $g \in C^0(\partial\Omega)$. □

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