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Commutative dilation theory

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Abstract

Dilation theory of single Hilbert space contractions is an important and very useful part of operator theory. By the main result of the theory, every Hilbert space contraction has the uniquely determined minimal unitary dilation. In many situations this enables to study instead of a general contraction its unitary dilation, which has much nicer properties.

The present paper gives a survey of dilation theory for commuting tuples of Hilbert space operators. The paper is organized as follows:

1. Introduction
2. Dilation theory of single contractions
3. Regular dilations
4. Ando's dilation and von Neumann inequality
5. Spherical dilations
6. Analytic models
7. Further examples
8. Concluding remarks

1 Introduction

The starting point of dilation theory is the following result of Sz.-Nagy (1953):

Theorem 1.1. *Let T be a contraction acting on a Hilbert space H. Then there exists a unitary dilation of T, i.e., there exist a Hilbert space $K \supset H$ and a unitary operator U acting on K such that*

$$T^k = P_H U^k|_H \qquad (k = 0, 1, \dots),$$

where P_H is the orthogonal projection onto H.

Moreover, under the minimality condition $K = \bigvee_{k\in\mathbb{Z}} U^k H$, the unitary dilation U is determined by T uniquely up to the unitary equivalence.

Since unitary operators have a simple and well-understood structure, in many situations dilation theory enables one to reduce problems for general contractions to simpler problems for unitary operators. So dilation theory became an important part of operator

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theory as an efficient tool for studying Hilbert space operators and as a subject of considerable independent interest. The most comprehensive reference for dilation theory is the monograph (Sz.-Nagy and Foiaş, 1970), or its recent new edition (Sz.-Nagy et al., 2010).

The goal of this paper is to give a brief survey of more recent extensions of dilation theory to the setting of n-tuples of mutually commuting operators.

The paper uses the standard multiindex notation. Denote by $\mathbb{Z}_+$ the set of all nonnegative integers. Let $\alpha = (\alpha_1, \dots, \alpha_n), \beta = (\beta_1, \dots, \beta_n) \in \mathbb{Z}_+^n$. Then write $|\alpha| = \sum_{i=1}^n \alpha_i$, $\operatorname{supp} \alpha = \{i : \alpha_i \neq 0\}$, $\alpha! = \alpha_1! \cdots \alpha_n!$, $\alpha + \beta = (\alpha_1 + \beta_1, \dots, \alpha_n + \beta_n)$. Write $\alpha \leq \beta$ if $\alpha_i \leq \beta_i$ for all $i = 1, \dots, n$. For $1 \leq j \leq n$ write $e_j = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 on the j-th position.

All spaces in this paper are complex Hilbert spaces. Denote by $B(H)$ the algebra of all bounded linear operators on a Hilbert space H. Let $T = (T_1, \dots, T_n) \in B(H)^n$ be an n-tuple of mutually commuting operators. For $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{Z}_+^n$ write $T^\alpha = \prod_{i=1}^n T_i^{\alpha_i}$.

Let H, K be Hilbert spaces, let $T = (T_1, \dots, T_n) \in B(H)^n$ and $S = (S_1, \dots, S_n) \in B(K)^n$ be n-tuples of operators. The n-tuples T and S are unitarily equivalent, for short $T \overset{u}{\sim} S$, if there exists an invertible isometry $V : H \to K$ such that $S_j = V^{-1} T_j V$ for $j = 1, \dots, n$.

Let H, K be Hilbert spaces, $H \subset K$, $n \geq 1$, let $T = (T_1, \dots, T_n) \in B(H)^n$ and $V = (V_, \dots, V_n) \in B(K)^n$ be n-tuples of commuting operators. Then V is called a dilation of T if

$$T^\alpha = P_H V^\alpha|_H$$

for all $\alpha \in \mathbb{Z}_+^n$.

Of special interest are dilations consisting of mutually commuting isometries or unitary operators. Both of these concepts are closely related because any n-tuple of commuting isometries can be extended to commuting unitaries, cf. (Sz.-Nagy and Foiaş, 1970, p.22).

Theorem 1.2. *Let $V = (V_1, \dots, V_n) \in B(H)^n$ be an n-tuple of commuting isometries. Then there exist a Hilbert space $K \supset H$ and commuting unitary operators $U_1, \dots, U_n \in B(K)$ such that $U_j H \subset H$ and $V_j = U_j|_H \quad (j = 1, \dots, n)$.*

Thus, if an n-tuple T has a dilation consisting of commuting isometries, then T has also a dilation consisting of commuting unitaries.

An n-tuple $V = (V_1, \dots, V_n) \in B(H)^n$ is called doubly commuting if $V_i V_j = V_j V_i$ and $V_i^* V_j = V_j V_i^*$ for all $i, j \in \{1, \dots, n\}$, $i \neq j$.

Recall that a commuting n-tuple of unitary operators is automatically doubly commuting by the Fuglede-Putnam theorem.

2 Dilation theory of single contractions

The multivariable dilation theory is inspired by the dilation theory of single contractions. The existence of a unitary dilation of a contraction can be proved in many ways, see (Sz.-Nagy and Foiaş, 1970; Sz.-Nagy et al., 2010). This section contains a simple geometrical approach which will be then generalized to the multivariable case in subsequent sections.

Let L be a Hilbert space. Denote by $\ell^2(\mathbb{Z}_+, L)$ the Hilbert space of all functions $f : \mathbb{Z}_+ \to L$ satisfying

$$\|f\|^2 := \sum_{i=0}^{\infty} \|f(i)\|^2 < \infty.$$

The *backward shift* (of multiplicity $\dim L$) is the operator S acting on $\ell^2(\mathbb{Z}_+, L)$ defined by $(Sf)(i) = f(i+1)$.

Equivalently, $\ell^2(\mathbb{Z}_+, L)$ may be interpreted as the Hardy space $H^2(\mathbb{D}, L)$ of all vector-valued analytic functions $\widetilde{f} : \mathbb{D} \to L$ defined on the open unit disc $\mathbb{D}$, $\widetilde{f}(z) = \sum_{i=0}^{\infty} f(i) z^i$ satisfying $\|\widetilde{f}\|^2 := \sum_{i=0}^{\infty} \|f(i)\|^2 = \lim_{r\to 1-} \frac{1}{2\pi} \int_0^{2\pi} \|\widetilde{f}(re^{it})\|^2 \mathrm{d}t < \infty$. Then S is the adjoint of the *multiplication* operator $M_z : H^2(\mathbb{D}, L) \to H^2(\mathbb{D}, L)$ defined by $(M_z\widetilde{f})(z) = z\widetilde{f}(z) \quad (\widetilde{f} \in H^2(\mathbb{D}, L), z \in \mathbb{D})$.

Consider the following simple question: which operators T are unitarily equivalent to the restriction of a backward shift of some multiplicity to an invariant subspace?

Certainly such an operator must be a contraction. So let H be a Hilbert space and $T \in B(H)$ a contraction. Let L be a Hilbert space and $S : \ell^2(\mathbb{Z}_+, L) \to \ell^2(\mathbb{Z}_+, L)$ the backward shift. It is necessary to find an isometry $V : H \to \ell^2(\mathbb{Z}_+, L)$ satisfying $VT = SV$. If such an isometry is found, its range VH will be the required subspace invariant for S and the corresponding restriction $S|_{VH}$ will be unitarily equivalent to T.

Suppose such a V exists. Let $V_0, V_1, \dots$ be the corresponding coordinate functions, $V_i h = (Vh)(i) \quad (i \in \mathbb{Z}_+, h \in H)$.

The condition $VT = SV$ can be then rewritten as $V_{i+1} = V_i T$ for all $i \geq 0$. Therefore by induction $V_i = V_0 T^i$. Thus it suffices to find only the operator $V_0 : H \to L$, the remaining operators $V_i \quad (i \geq 1)$ will be determined by the equations $V_i = V_0 T^i$.

Moreover, since V should be an isometry,

$$\|h\|^2 = \|Vh\|^2 = \sum_{i=0}^{\infty} \|V_i h\|^2 = \|V_0 h\|^2 + \|V_0 Th\|^2 + \|V_0 T^2 h\|^2 + \cdots$$

for all $h \in H$. In particular, for the vector $Th \in H$ this gives

$$\|Th\|^2 = \|VTh\|^2 = \|V_0 Th\|^2 + \|V_0 T^2 h\|^2 + \cdots$$

By subtracting, one gets $\|V_0 h\|^2 = \|h\|^2 - \|Th\|^2$ for all $h \in H$.

The latter inequality is satisfied by the *defect operator* $D_T = (I - T^*T)^{1/2} \in B(H)$ and this is essentially the only possible choice of V_0. In general, $V_0 = JD_T$ where $J : \overline{D_T H} \to L$ is any isometry. The simplest choice for L is $L = \mathcal{D}_T := \overline{D_T H}$ and $V_0 := D_T$.

Then for $h \in H$, one has

$$\|Vh\|^2 = \sum_{i=0}^{\infty} \|V_0 T^i h\|^2 = \sum_{i=0}^{\infty} (\|T^i h\|^2 - \|T^{i+1} h\|^2) = \|h\|^2 - \lim_{k\to\infty} \|T^k h\|^2.$$

Hence V is an isometry if and only if $T^k \to 0$ in the strong operator topology (SOT).

Thus it was proved:

Theorem 2.1. *Let $T \in B(H)$ be a contraction satisfying $T^k \to 0 \quad$ (SOT). Then T is unitarily equivalent to the restriction of the backward shift S of multiplicity* $\dim \mathcal{D}_T$ *to an invariant subspace.*

The condition $T^k \to 0 \quad$ (SOT) is clearly also necessary.

If the condition $T^k \to 0 \quad$ (SOT) is not satisfied, then V constructed above is not an isometry and satisfies only $\|Vh\|^2 = \|h\|^2 - \lim_{k\to\infty} \|T^k h\|^2 \quad (h \in H)$. In this case, it is possible to extend it to an isometry.

Theorem 2.2. *Let $T \in B(H)$ be a contraction. Then there exist a Hilbert space H', a unitary operator $W \in B(H')$ and a subspace $M \subset \ell^2(\mathbb{Z}_+, \mathcal{D}_T) \oplus H'$ invariant for $S \oplus W$ such that*

$$T \overset{u}{\sim} (S \oplus W)|_M. \tag{1}$$

Proof. By the previous construction, there exists an operator $V' : H \to \ell^2(\mathbb{Z}_+, \mathcal{D}_T)$ satisfying $V'T = SV'$ and

$$\|V'h\|^2 = \|h\|^2 - \lim_{k\to\infty} \|T^k h\|^2 \qquad (h \in H).$$

Define a new seminorm $|||\cdot|||$ on H by

$$|||h|||^2 = \lim_{k\to\infty} \|T^k h\|^2.$$

Let $N = \{x \in H : |||x||| = 0\}$. It is easy to see that N is a subspace invariant for T. Let $\widetilde{H}$ be the completion of $(H/N, |||\cdot|||)$. Then $\widetilde{H}$ is a Hilbert space. Define $\widetilde{T} : H/N \to H/N$ by $\widetilde{T}(h + N) = Th + N$. Then $\widetilde{T}$ is an isometry and can be extended uniquely to an isometry (denoted by the same symbol $\widetilde{T}$) on $\widetilde{H}$.

Since any isometry can be extended to a unitary operator, there exist a Hilbert space $H' \supset \widetilde{H}$ and a unitary operator $W \in B(H')$ such that $\widetilde{T} = W|_{\widetilde{H}}$.

Let $V'' : H \to H'$ be defined by $V''h = h + N \in H/N \subset \widetilde{H} \subset H'$. Let $V : H \to \ell^2(\mathbb{Z}_+, \mathcal{D}_T) \oplus H'$ be defined by $Vh = V'h \oplus V''h \quad (h \in H)$. Then $VT = (S \oplus W)V$ and $\|Vh\|^2 = \|h\|^2 - \lim_{k\to\infty} \|T^k h\|^2 + |||h|||^2 = \|h\|^2$. So V is an isometry and its range VH is the subspace invariant for $S \oplus W$ satisfying $T \overset{u}{\sim} (S \oplus W)|_{VH}$. $\square$

Remark 2.3. Note that the operator $S \oplus W$ is a coisometry. So every contraction has a coisometric extension.

Moreover, the backward shift S is a compression of the bilateral shift U acting on the space $\ell^2(\mathbb{Z}, \mathcal{D}_T)$ defined by $(Uf)(i) = f(i+1) \quad (f \in \ell^2(\mathbb{Z}, \mathcal{D}_T), i \in \mathbb{Z})$. Clearly $U \oplus W$ is the unitary dilation of T.

Of course, all the dilation theory can be formulated equivalently for T^* rather than for T. It is easy to see that (1) is equivalent to $T^* = P_H(M_z \oplus W^*)|_H$, where $M_z \oplus W^*$ is an isometrical dilation of T^*.

3 Regular dilations

The closest multivariable analogy of the single-contraction case are n-tuples of commuting contractions having a regular dilation.

The role of the backward shift is played by the backward multishift. Let L be a Hilbert space and let $n \in \mathbb{N}$ be fixed.

Let $\ell^2(\mathbb{Z}_+^n, L)$ denote the Hilbert space of all vector-valued functions $f : \mathbb{Z}_+^n \to L$ such that $\|f\|^2 := \sum_{\alpha\in\mathbb{Z}_+^n} \|f(\alpha)\|^2 < \infty$. Consider the operators $S_j : \ell^2(\mathbb{Z}_+^n, L) \to \ell^2(\mathbb{Z}_+^n, L) \quad (j = 1, \dots, n)$ defined by $(S_j f)(\alpha) = f(\alpha + e_j)$. Equivalently, $\ell^2(\mathbb{Z}_+^n, L)$ can be interpreted as the Hardy space $H^2(\mathbb{D}^n)$ of all analytic functions $\widetilde{f} : \mathbb{D}^n \to \mathbb{C}$, $\widetilde{f}(z) = \sum_{\alpha\in\mathbb{Z}_+^n} f(\alpha) z^\alpha \quad (z \in \mathbb{D}^n)$ satisfying $\|\widetilde{f}\| := \sum_{\alpha\in\mathbb{Z}_+^n} \|f(\alpha)\|^2 < \infty$. The operators

$S_j \quad (j = 1, \dots, n)$ are then the adjoints of the multiplication operators $M_{z_j} : H^2(\mathbb{D}^n) \to H^2(\mathbb{D}^n)$ defined by $M_{z_j}(\widetilde{f})(z) = z_j \widetilde{f}(z) \quad (z \in \mathbb{D}^n)$.

The n-tuple $S = (S_1, \dots, S_n)$ will be called the *backward multishift* (of multiplicity $\dim L$). Note that $S_1, \dots, S_n$ are doubly commuting coisometries.

Let H be a Hilbert space and $T = (T_1, \dots, T_n) \in B(H)^n$ a commuting n-tuple of contractions. As in the previous section, one tries to find out when T is unitarily equivalent to the restriction of the backward multishift to an invariant subspace.

Suppose that a Hilbert space L and an isometry $V : H \to \ell^2(\mathbb{Z}_+^n, L)$ satisfying

$$VT_j = S_j V \qquad (j = 1, \dots, n)$$

have been found. For $\alpha \in \mathbb{Z}_+^n$ let $V_\alpha : H \to H$ be the corresponding coordinate function, $V_\alpha h = (Vh)(\alpha)$.

The intertwining relations $VT_j = S_j V$ mean that for each $\alpha \in \mathbb{Z}_+^n$ and $h \in H$ one has

$$V_\alpha T_j h = (VT_j h)(\alpha) = (S_j V h)(\alpha) = V_{\alpha + e_j} h.$$

Hence $V_{\alpha+e_j} = V_\alpha T_j$ for all $\alpha \in \mathbb{Z}_+^n$ and $j = 1, \dots, n$. By induction, this gives

$$V_\alpha = V_{0,\dots,0} T^\alpha \tag{2}$$

for all $\alpha \in \mathbb{Z}_+^n$. Thus one can choose only $V_{0,\dots,0} : H \to H$, the remaining operators V_α are already given by (2).

Moreover, since V should be an isometry, it must satisfy

$$\|h\|^2 = \|Vh\|^2 = \sum_{\alpha \in \mathbb{Z}_+^n} \|V_\alpha h\|^2 = \sum_{\alpha \in \mathbb{Z}_+^n} \|V_{0,\dots,0} T^\alpha h\|^2 \qquad \text{for all } h \in H.$$

In particular, for each $F \subset \{1, \dots, n\}$ one has

$$\|T_F h\|^2 = \sum_{\alpha \in \mathbb{Z}_+^n} \|V_{0,\dots,0} T^\alpha T_F h\|^2,$$

where

$$T_F = \prod_{j \in F} T_j$$

(in particular, $T_\emptyset = I_H$). Denote by $|F|$ the cardinality of F. It follows from (2) that

$$\sum_{F \subset \{1,\dots,n\}} (-1)^{|F|} \|T_F h\|^2 = \sum_{F \subset \{1,\dots,n\}} (-1)^{|F|} \sum_{\alpha \in \mathbb{Z}_+^n} \|V_{0,\dots,0} T^\alpha T_F h\|^2$$
$$= \sum_{\beta \in \mathbb{Z}_+^n} \|V_{0,\dots,0} T^\beta h\|^2 \sum_{\substack{\alpha \le \beta \\ \max\{\beta_j - \alpha_j\} \le 1}} (-1)^{|\beta| - |\alpha|} = \|V_{0,\dots,0} h\|^2.$$

Hence T must satisfy

$$\sum_{F \subset \{1,\dots,n\}} (-1)^{|F|} \|T_F h\|^2 \ge 0 \tag{3}$$

for all $h \in H$, or equivalently,

$$\sum_{F \subset \{1,\dots,n\}} (-1)^{|F|} T_F^* T_F \ge 0. \tag{4}$$

If (4) is satisfied then set $D_T := \left(\sum_{F\subset\{1,\dots,n\}}(-1)^{|F|}T_F^*T_F\right)^{1/2}$ and $\mathcal{D}_T := \overline{D_T H}$.

The simplest choice for L is $L = \mathcal{D}_T$, $V_{0,\dots,0} = D_T$ and $V_\alpha = D_T T^\alpha \quad (\alpha \in \mathbb{Z}_+^n)$.

Then for each $h \in H$ one has

$$\begin{aligned}\|Vh\|^2 &= \sum_{\alpha\in\mathbb{Z}_+^n}\|V_\alpha h\|^2 = \lim_{k\to\infty}\sum_{\max\alpha_j\le k-1}\|V_{0,\dots,0}T^\alpha h\|^2\\ &= \lim_{k\to\infty}\sum_{\max\alpha_j\le k-1}\sum_{F\subset\{1,\dots,n\}}(-1)^{|F|}\|T^\alpha T_F h\|^2\\ &= \lim_{k\to\infty}\sum_{\max\beta_j\le k}\|T^\beta h\|^2\sum_{\substack{\alpha\le\beta,\max(\beta_j-\alpha_j)\le 1\\ \max\alpha_j\le k-1}}(-1)^{|\beta|-|\alpha|}.\end{aligned}$$

Note that

$$\sum_{\substack{\alpha\le\beta,\max(\beta_j-\alpha_j)\le 1\\ \max\alpha_j\le k-1}}(-1)^{|\beta|-|\alpha|} = 0$$

unless $\{\beta_1,\dots,\beta_n\}\subset\{0,k\}$. If $\{\beta_1,\dots,\beta_n\}\subset\{0,k\}$, then the sum is equal to $(-1)^{|\mathrm{supp}\,\beta|}$. So

$$\|Vh\|^2 = \lim_{k\to\infty}\sum_{F\subset\{1,\dots,n\}}(-1)^{|F|}\|T_F^k h\|^2 = \|h\|^2 + \lim_{k\to\infty}\sum_{\emptyset\ne F\subset\{1,\dots,n\}}(-1)^{|F|}\|T_F^k h\|^2.$$

So V will be an isometry if (SOT)$-\lim_{k\to\infty}T_j^k = 0$ for $j = 1,\dots,n$. Thus the following theorem was proved:

Theorem 3.1. *Let $T = (T_1,\dots,T_n)\in B(H)^n$ be a commuting n-tuple of contractions satisfying $\sum_{F\subset\{1,\dots,n\}}(-1)^{|F|}T_F^*T_F \ge 0$ and (SOT) $-\lim_{k\to\infty}T_j^k = 0$ for all $j = 1,\dots,n$. Then there exists a subspace $M\subset\ell^2(\mathbb{Z}_+^n,\mathcal{D}_T)$ invariant for all $S_1,\dots,S_n$ such that $T \overset{u}{\sim} S|_M$.*

If the conditions (SOT) $-\lim_{k\to\infty}T_j^k = 0 \quad (j = 1,\dots,n)$ are not satisfied then a weaker statement can be proved:

Theorem 3.2. *Let $T = (T_1,\dots,T_n)\in B(H)^n$ be a commuting n-tuple of contractions satisfying $\sum_{F\subset\{1,\dots,n\}}(-1)^{|F|}T_F^*T_F \ge 0$. Then there exists an operator $V : H \to \ell^2(\mathbb{Z}_+^n,\mathcal{D}_T)$ satisfying $VT_j = S_jV \quad (j = 1,\dots,n)$ and*

$$\|Vh\|^2 = \lim_{k\to\infty}\sum_{F\subset\{1,\dots,n\}}(-1)^{|F|}\|T_F^k h\|^2$$

for all $h\in H$.

As in the single variable case one would like to complete V to an isometry. This is little bit more complicated than before. The starting point is the following lemma.

Lemma 3.3. *Let $G\subset\{1,\dots,n\}$, $m = |G|$, $G = \{i_1,\dots,i_m\}$. Set $\overline{G} = \{1,\dots,n\}\setminus G$. Let $T = (T_1,\dots,T_n)\in B(H)^n$ be a commuting n-tuple of contractions satisfying $\sum_{F\subset G}(-1)^{|F|}T_F^*T_F \ge 0$ such that the operators T_j for $j\in\overline{G}$ are isometries. Then there* *exist a Hilbert space H_G, commuting unitary operators $W_{G,j} \in B(H_G) \quad (j \in \overline{G})$ and a mapping $V : H \to \ell^2(\mathbb{Z}_+^m, H_G)$ such that*

$$\|Vh\|^2 = \lim_{k\to\infty} \sum_{F\subset G} (-1)^{|F|} \|T_F^k h\|^2 \qquad (h \in H),$$

$$VT_{j_i} = S_i V \qquad (i = 1, \dots, m),$$

and

$$VT_j = W_{G,j} V \qquad (j \notin G),$$

where $W_{G,j}$ is lifted to $\ell^2(\mathbb{Z}_+^m, H_G)$ by $(W_{G,j} f)(\alpha) = W_{G,j} f(\alpha) \quad (\alpha \in \mathbb{Z}_+^m, f \in \ell^2(\mathbb{Z}_+^m, H_G))$.

Moreover, the n-tuple $S_1, \dots, S_m, W_{G,j} \quad (j \in \overline{G})$ is doubly commuting.

Proof. Let $D_G := \left(\sum_{F\subset G} (-1)^{|F|} T_F^* T_F\right)^{1/2}$. Clearly

$$N(D_G) = \Big\{ x \in H : \sum_{F\subset G} (-1)^{|F|} \|T_F x\|^2 = 0 \Big\}.$$

Let $j \in \overline{G}$. Since T_j is an isometry commuting with all $T_F \quad (F \subset G)$, one has $T_j N(D_G) \subset N(D_G)$. Define a mapping $\widetilde{T}_j : D_G H \to D_G H$ by $\widetilde{T}_j D_G h = D_G T_j h \quad (h \in H)$. Since $T_j N(D_G) \subset N(D_G)$, the definition is correct. Moreover,

$$\|\widetilde{T}_j D_G h\|^2 = \sum_{F\subset G} (-1)^{|F|} \|T_F T_j h\|^2 = \sum_{F\subset G} (-1)^{|F|} \|T_F h\|^2 = \|D_G h\|^2,$$

so $\widetilde{T}_j$ is an isometry and can be extended uniquely to an isometry (denoted also by $\widetilde{T}_j$) on the space $\mathcal{D}_G := \overline{D_G H}$.

Clearly $\widetilde{T}_j \quad (j \in \overline{G})$ are commuting isometries and can be extended to commuting unitary operators $W_{G,j} \quad (j \in \overline{G})$ acting on a Hilbert space $H_G \supset \mathcal{D}_G$.

Lift $W_{G,j} \quad (j \in \overline{G})$ to the space $\ell^2(\mathbb{Z}_+^m, H_G)$ by

$$(W_{G,j} f)(\alpha) = W_{G,j} f(\alpha).$$

Applying the previous theorem for the $|G|$-tuple $(T_j : j \in G)$, one concludes that there exists an operator $V : H \to \ell^2(\mathbb{Z}_+^m, \mathcal{D}_G) \subset \ell^2(\mathbb{Z}_+^m, H_G)$ such that $VT_{j_i} = S_i V \quad (i = 1, \dots, m)$ and

$$\|Vh\|^2 = \sum_{F\subset G} (-1)^{|F|} \lim_{k\to\infty} \|T_F^k h\|^2 \qquad (h \in H).$$

For $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{Z}_+^m$ define $\widetilde{\alpha} = (\widetilde{\alpha}_1, \dots \widetilde{\alpha}_n) \in \mathbb{Z}_+^n$ by $\widetilde{\alpha}_{j_i} = \alpha_i \quad (i = 1, \dots, m)$ and $\widetilde{\alpha}_j = 0 \quad (j \in \overline{G})$. For $j \in \overline{G}$ and $\alpha \in \mathbb{Z}_+^m$ one has

$$(W_{G,j} Vh)(\alpha) = W_{G,j}(Vh(\alpha)) = W_{G,j} V_\alpha h = W_{G,j} D_G T^{\widetilde{\alpha}} h = D_G T_j T^{\widetilde{\alpha}} h = V_\alpha T_j h = (VT_j h)(\alpha).$$

So $VT_j = W_{G,j} V \quad (j \notin G)$.

Clearly the n-tuple $S_1, \dots, S_m, W_{G,j} \quad (j \in \overline{G})$ is commuting and $S_i^* S_j = S_j S_i^* \quad (1 \le i, j \le m, i \ne j)$. By the Fuglede-Putnam theorem, the n-tuple $S_1, \dots, S_m, W_{G,j} \quad (j \in \overline{G})$ is doubly commuting. □

Corollary 3.4. *Let $G \subset \{1, \dots, n\}$, $G = \{i_1, \dots, i_m\}$. Let $\overline{G} = \{1, \dots, n\} \setminus G$. Let $T = (T_1, \dots, T_n) \in B(H)^n$ be a commuting tuple of contractions with $\sum_{F\subset G} (-1)^{|F|} T_F^* T_F \ge 0$.*

Then there exist a Hilbert space H_G, commuting unitary operators $W_{G,j} \in B(H_G) \quad (j \in \overline{G})$ and a mapping $V_G : H \to \ell^2(\mathbb{Z}_+^m, H_G)$ such that

$$\|V_G h\|^2 = \lim_{k\to\infty} \sum_{F\subset G} (-1)^{|F|} \|T_{\overline{G}}^k T_F^k h\|^2 \qquad (h \in H),$$

$$V_G T_{j_i} = S_i V_G \qquad (i = 1, \dots, m),$$

and

$$V_G T_j = W_{G,j} V_G \qquad (j \in \overline{G}),$$

where as above $W_{G,j}$ is lifted to $\ell^2(\mathbb{Z}_+^m, H_G)$. Moreover, the n-tuple $S_1, \dots, S_m, T_j \quad (j \in \overline{G})$ is doubly commuting.

Proof. Define a new seminorm $|||\cdot|||$ on H by $|||h||| = \lim_{k\to\infty} \|T_{\overline{G}}^k h\|^2$. Let $N = \{x \in H : |||x||| = 0\}$. Then N is a subspace invariant for all $T_j \quad (j = 1, \dots, n)$. Let L be the completion of H/N with the norm $|||\cdot|||$. Then L is a Hilbert space. For $j = 1, \dots, n$ define $\widetilde{T}_j : H/N \to H/N$ by $\widetilde{T}_j(h+N) = T_j h + N$. Then $\widetilde{T}_j$ extends uniquely to an operator acting on L. The operators $\widetilde{T}_1, \dots, \widetilde{T}_n$ are commuting contractions, $\widetilde{T}_j$ is an isometry for $j \in \overline{G}$ and the m-tuple $\widetilde{T}_j \quad (j \in G)$ satisfies the condition $\sum_{F\subset G} (-1)^{|F|} |||\widetilde{T}_F x|||^2 \geq 0 \quad (x \in L)$.

By Lemma 3.3, there exist a Hilbert space H_G, commuting unitary operators $W_{G,j} \in B(H_G) \quad (j \in \overline{G})$ and a mapping $V' : L \to \ell^2(\mathbb{Z}_+^m, H_G)$ satisfying

$$\begin{aligned} V'\widetilde{T}_{j_i} &= S_i V' \qquad (i = 1, \dots, m), \\ V'\widetilde{T}_j &= W_{G,j} V' \qquad (j \in \overline{G}), \end{aligned}$$

and

$$\|V'h\|^2 = \lim_{k\to\infty} \sum_{F\subset G} (-1)^{|F|} |||\widetilde{T}_F^k h|||^2.$$

Let $V'' : H \to L$ be defined be $V''h = h + N \in H/N \subset L$ and let $V_G := V'V'' : H \to \ell^2(\mathbb{Z}_+^m, H_G)$. Then

$$\begin{aligned} V_G T_{j_i} &= S_i V_G \qquad (i = 1, \dots, m), \\ V_G T_j &= W_{G,j} V_G \qquad (j \in \overline{G}) \end{aligned}$$

and

$$\|V_G h\|^2 = \lim_{k\to\infty} \sum_{F\subset G} (-1)^{|F|} \|T_{\overline{G}}^k T_F^k h\|^2.$$

Clearly the n-tuple $S_1, \dots, S_m, T_j \quad (j \in \overline{G})$ is doubly commuting. □

Let $T = (T_1, \dots, T_n) \in B(H)^n$ be commuting contractions. One says that T satisfies the *Brehmer conditions* if

$$\sum_{F\subset G} (-1)^{|F|} T_F^* T_F \geq 0 \tag{5}$$

for all $G \subset \{1, \dots, n\}$, see (Sz.-Nagy and Foiaş, 1970, Section I.9) or (Brehmer, 1961).

Theorem 3.5. *Let $T_1, \dots, T_n \in B(H)$ be commuting contractions satisfying (*5*).* Then for each $G \subset \{1, \dots, n\}$ there exist a Hilbert space X_G, doubly commuting coisometries $U_{G,1}, \dots, U_{G,n} \in B(X_G)$ and an isometry $V : H \to \bigoplus_G X_G$ such that

$$VT_j = \Big(\bigoplus_{G\subset\{1,\dots,n\}} U_{G,j} \Big) Vh \qquad (j = 1, \dots, n).$$

Moreover, for each $G \subset \{1,\dots,n\}$ the $|G|$-tuple $U_{G,j} \quad (j \in G)$ is the backward multishift and the operators $U_{G,j} \quad (j \notin G)$ are unitaries.

Proof. Let $X_G = \ell^2(\mathbb{Z}_+^{|G|}, H_G)$ and $V_G : H \to X_G$ be as in Corollary 3.4. Let $V = \bigoplus_{G\subset\{1,\dots,n\}} V_G$. Then

$$\begin{aligned}\|Vh\|^2 &= \sum_{G\subset\{1,\dots,n\}} \|V_G h\|^2 = \sum_{G\subset\{1,\dots,n\}} \sum_{F\subset G} (-1)^{|F|} \lim_{k\to\infty} \|T_G^k T_F^k h\|^2 \\ &= \sum_{A\subset\{1,\dots,n\}} \lim_{k\to\infty} \|T_A^k h\|^2 \sum_{F\subset A} (-1)^{|F|}.\end{aligned}$$

For each $A \neq \emptyset$ one has $\sum_{F\subset A}(-1)^{|F|} = 0$. Hence $\|Vh\|^2 = \|h\|^2$ for each $h \in H$, and so V is an isometry. $\square$

By Theorem 1.2, any n-tuple $T = (T_1,\dots,T_n) \in B(H)^n$ satisfying the Brehmer conditions has a dilation consisting of commuting unitaries. However, a stronger result is true. An n-tuple $U = (U_1,\dots,U_n) \in B(K)^n$ is called a regular dilation of T if

$$T^{*\alpha}T^\beta = P_H U^{*\alpha} U^\beta|_H$$

for all $\alpha, \beta \in \mathbb{Z}_+^n$ with $\operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset$. Equivalently, $\langle T^\alpha h, T^\beta h'\rangle = \langle U^\alpha h, U^\beta h'\rangle$ for all $\alpha,\beta \in \mathbb{Z}_+^n$, $\operatorname{supp}\alpha \cap \operatorname{supp}\beta = \emptyset$ and all $h, h' \in H$.

Theorem 3.6. *Let $T = (T_1,\dots,T_n) \in B(H)^n$ be commuting contractions. Then the following statements are equivalent:*

(i) T satisfies the Brehmer conditions (5)*;*

(ii) there exists an extension of T consisting of doubly commuting coisometries;

(iii) T has a regular unitary dilation.

Proof. (i)⇒(ii) was proved in the previous theorem.

(ii)⇒(iii): Let $V = (V_1,\dots,V_n)$ be doubly commuting coisometric extensions of T and let $U^* = (U_1^*,\dots,U_n^*)$ be the unitary extension of V^*. For $\alpha,\beta \in \mathbb{Z}_+^n$ with disjoint supports and $h, h' \in H$ one has

$$\begin{aligned}\langle U^{*\alpha}U^\beta h, h'\rangle &= \langle U^{*\alpha}h, U^{*\beta}h'\rangle = \langle V^{*\alpha}h, V^{*\beta}h'\rangle = \langle V^{*\alpha}V^\beta h, h'\rangle \\ &= \langle V^\beta h, V^\alpha h'\rangle = \langle T^\beta h, T^\alpha h'\rangle = \langle T^{*\alpha}T^\beta h, h'\rangle.\end{aligned}$$

(iii)⇒(i): Let $U = (U_1,\dots,U_n)$ be a regular dilation of T, i.e., $\langle U^\alpha h, U^\beta h'\rangle = \langle T^\alpha h, T^\beta h'\rangle$ for all $\alpha,\beta \in \mathbb{Z}_+^n$, $\operatorname{supp}\alpha\cap\operatorname{supp}\beta = \{0\}$, $h, h' \in H$. Let $G \subset \{1,\dots,n\}$ and $x \in H$. Then

$$\Big\|\sum_{F\subset G}(-1)^{|F|}U_{G\setminus F}T_F x\Big\|^2 \geq 0. \tag{6}$$

The left hand side of (6) is equal to

$$\sum_{F,F'\subset G}(-1)^{|F|+|F'|}\langle U_{G\setminus F}T_F x, U_{G\setminus F'}T_{F'}x\rangle = \sum_{F,F'\subset G}(-1)^{|F|+|F'|}\langle U_{(F\cup F')\setminus F}T_F x, U_{(F\cup F')\setminus F'}T_{F'}x\rangle$$

$$= \sum_{F,F'\subset G} (-1)^{|F|+|F'|}\|T_{F\cup F'}x\|^2 = \sum_{L\subset G} c(L)\|T_L x\|^2,$$

where

$$c(L) = \sum_{F,F'\subset L, F\cup F'=L} (-1)^{|F|+|F'|} = \sum_{M\subset L} \sum_{\substack{F,F':M=F\cap F'\\ F\cup F'=L}} (-1)^{|F\setminus M|+|F'\setminus M|}.$$

Let $M\subset L\subset\{1,\dots,n\}$, $|M|=m$, $|L|=l$. Then

$$\sum_{\substack{F,F':M=F\cap F'\\ F\cup F'=L}} (-1)^{|F\setminus M|+|F'\setminus M|} = (-1)^{l-m}2^{l-m},$$

and so

$$c(L) = \sum_{m=0}^{l} \binom{l}{m}(-1)^{l-m}2^{l-m} = (-1)^l(2-1)^l = (-1)^l.$$

This together with (6) gives the Brehmer conditions. □

The fact that Brehmer's conditions imply the existence of a regular dilation is already classical, cf. (Sz.-Nagy and Foiaş, 1970, p. 32). The structure of the regular dilation was studied in more details in (Curto and Vasilescu, 1993; 1995; Gaspar and Suciu, 1997). Theorem 3.5 was formulated explicitly in (Timotin, 1998).

Examples 3.7. Let $T=(T_1,\dots,T_n)\in B(H)^n$ be a commuting n-tuple of contractions. The Brehmer conditions are satisfied, and so T has a regular unitary dilation, in particular in the following cases:

(i) the operators $T_1,\dots,T_n$ are doubly commuting. Indeed, in this case

$$\Big\langle \sum_{F\subset G}(-1)^{|F|}T_F^*T_Fx,x\Big\rangle = \Big\langle \prod_{i\in G}(I-T_i^*T_i)x,x\Big\rangle = \Big\|\prod_{i\in G} D_{T_i}x\Big\|^2 \ge 0.$$

(ii) the operators $T_1,\dots,T_n$ are isometries. Indeed, $\sum_{F\subset G}(-1)^{|F|}\|T_Fx\|^2 = 0$ for all $G\subset\{1,\dots,n\}$ and $x\in H$. (In fact, it is sufficient to assume that all operators T_j but one are isometries.)

(iii) Suppose that $\sum_{i=1}^n \|T_ix\|^2 \le \|x\|^2$ for all $x\in H$ (such n-tuples are called spherical contractions). Then it is easy to show that

$$\sum_{F\subset G,|F|=k+1} \|T_Fx\|^2 \le \sum_{F\subset G,|F|=k} \|T_Fx\|^2$$

for all $G\subset\{1,\dots,n\}$, $k=0,\dots,|G|-1$ and $x\in H$. Consequently, T satisfies the Brehmer conditions.

Remark 3.8. In general it is not sufficient to assume only that (3), namely

$$\sum_{F\subset\{1,\dots,n\}} (-1)^{|F|}\|T_Fx\|^2 \ge 0,$$

holds for all $x\in H$. Indeed, this condition is automatically satisfied if one of the operators $T_1,\dots,T_n$ is an isometry, but the Brehmer conditions may be false.

On the other hand, if additional conditions $(\mathrm{SOT})-\lim_{k\to\infty} T_j^k = 0$ for $j=1,\dots,n$ are assumed, then T has a regular dilation by Theorem 3.1, and so T satisfies Brehmer's conditions (5).

4 The Ando dilation and von Neumann inequality

For $n = 2$ the situation is better. Any pair of commuting contractions has a unitary dilation, cf. (Sz.-Nagy and Foiaş, 1970, p. 20).

Theorem 4.1. *(Ando) Let $T_1, T_2 \in B(H)$ be commuting contractions. Then there exist a Hilbert space $K \supset H$ and commuting isometries $V_1, V_2 \in B(K)$ such that*

$$T_1^i T_2^j = P_H V_1^i V_2^j|_H$$

for all $i, j \in \mathbb{Z}_+$.

Proof. Let $K = \bigoplus_{i=0}^{\infty} H$. Define operators $W_i : K \to K \quad (i = 1, 2)$ by

$$W_i(h_0, h_1, \dots) = (T_i h_0, D_{T_i} h_0, 0, h_1, h_2, \dots) \qquad (i = 1, 2).$$

Clearly W_1, W_2 are isometries since $\|T_i h_0\|^2 + \|D_{T_i} h_0\|^2 = \|h_0\|^2$. However, in general W_1 and W_2 do not commute. In order to obtain commuting isometries, one can modify the operators W_1, W_2 in the following way.

Set $L = H \oplus H \oplus H \oplus H$ and identify K with $H \oplus \bigoplus_{i=1}^{\infty} L$ by the natural identification

$$(h_0, h_1, h_2, \dots) = \big(h_0, (h_1, h_2, h_3, h_4), (h_5, h_6, h_7, h_8), \dots\big).$$

Let $G : L \to L$ be a unitary operator. Define $\widehat{G} : K \to K$ by

$$\widehat{G}(h_0, h_1, h_2, \dots) = \big(h_0, G(h_1, h_2, h_3, h_4), G(h_5, h_6, h_7, h_8), \dots\big).$$

Clearly $\widehat{G}$ is also unitary and $V_1 := \widehat{G} W_1$ and $V_2 := W_2 \widehat{G}^{-1}$ are isometries. One is looking for G such that V_1, V_2 will be commuting. A direct calculation gives

$$V_1 V_2(h_0, h_1, h_2, \dots) = \big(T_1 T_2 h_0, G(D_{T_1} T_2 h_0, 0, T_2 h_0, 0), (h_1, h_2, h_3, h_4), (h_5, h_6, h_7, h_8), \dots\big)$$

and

$$V_2 V_1(h_0, h_1, h_2, \dots) = \big(T_2 T_1 h_0, (D_{T_2} T_1 h_0, 0, T_1 h_0, 0), (h_1, h_2, h_3, h_4), (h_5, h_6, h_7, h_8), \dots\big).$$

Since $T_1 T_2 = T_2 T_1$, the isometries V_1 and V_2 will be commuting if and only if

$$G(D_{T_1} T_2 h_0, 0, T_2 h_0, 0) = (D_{T_2} T_1 h_0, 0, T_1 h_0, 0) \qquad (h_0 \in H). \tag{7}$$

Let $L_1 = \{(D_{T_1} T_2 h_0, 0, T_2 h_0, 0) : h_0 \in H\}^-$ and $L_2 = \{(D_{T_2} T_1 h_0, 0, T_1 h_0, 0) : h_0 \in H\}^-$. It is easy to verify that $\|D_{T_1} T_2 h_0\|^2 + \|T_2 h_0\|^2 = \|D_{T_2} T_1 h_0\|^2 + \|T_1 h_0\|^2$ and $\dim L \ominus L_1 = \dim L \ominus L_2$, so it is possible to define a unitary operator $G : L \to L$ satisfying (7).

Define $V_1 = \widehat{G} W_1$ and $V_2 = W_2 \widehat{G}^{-1}$. It is easy to see that the pair (V_1, V_2) is a commuting isometric dilation of (T_1, T_2). $\square$

By Theorem 1.2, the isometries V_1, V_2 can be extended to commuting unitary operators, so any pair of commuting contractions has a unitary dilation. This implies immediately the following important von Neumann type inequality.

Corollary 4.2. *Let $T_1, T_2 \in B(H)$ be commuting contractions. Then*

$$\|p(T_1, T_2)\| \le \|p\|_{\mathbb{D}^2} := \sup\{|p(z_1, z_2)| : (z_1, z_2) \in \mathbb{D}^2\}$$

for all polynomials p in two variables.

Proof. By Theorem 4.1, there exist a Hilbert space $K \supset H$ and commuting unitary operators $U_1, U_2 \in B(K)$ dilating T. So

$$\|p(T)\| = \|P_H p(U)|_H\| \le \|p(U)\| = r(p(U)) \le \sup\{|p(z_1, z_2)| : (z_1, z_2) \in \mathbb{D}^2\}.$$

□

An alternative proof follows from (Agler and McCarthy, 2005), where it was shown that any pair of commuting contractive matrices has a commutative coisometric-pair extension.

For 3 or more commuting contractions the above von Neumann type inequality is not true in general. The following example was constructed by (Crabb and Davie, 1975).

Example 4.3. Let H be the 8-dimensional Hilbert space with an orthonormal basis $e, f_1, f_2, f_3, g_1, g_2, g_3, h$. Define operators $T_1, T_2, T_3 \in B(H)$ by

$$T_j e = f_j, T_j f_j = -g_j, T_i f_j = g_k \quad (\text{for } i, j, k \text{ all different}), T_j g_i = \delta_{i,j} h, T_j h = 0.$$

It is easy to verify that the operators T_1, T_2, T_3 are mutually commuting and that they are contractions (in fact they are partial isometries). Let $p(z_1, z_2, z_3) = z_1^3 + z_2^3 + z_3^3 - z_1 z_2 z_3$. Then

$$\|p(T_1, T_2, T_3)\| \ge |\langle p(T)e, h\rangle| = 4.$$

However, $\|p\|_{\mathbb{D}^3} < 4$. Indeed, clearly $\|p\|_{\mathbb{D}^3} \le 4$. If $\|p\|_{\mathbb{D}^3} = 4$ then there are $\eta_1, \eta_2, \eta_3 \in \partial\mathbb{D}$ such that $\eta_1^3 = \eta_2^3 = \eta_3^3 = -\eta_1\eta_2\eta_3$. Thus $(\eta_1\eta_2\eta_3)^3 = -(\eta_1\eta_2\eta_3)^3$ and so $1 = -1$, a contradiction. Hence $\|p\|_{\mathbb{D}^3} < 4$.

A similar example was given by (Varopoulos, 1974). He constructed 3 commuting contractions on a 5-dimensional Hilbert space and a homogeneous polynomial of degree 2 such that the von Neumann inequality is not true.

The following central problem is still open.

Problem 4.4. Let $n \ge 3$. Does there exist a constant $K(n)$ such that

$$\|p(T_1, \dots, T_n)\| \le K(n) \cdot \|p\|_{\mathbb{D}^n}$$

for all n-tuples of commuting contractions $T_1, \dots, T_n$ and all polynomials p in n variables?

Let c_n be the supremum of the norms $\|p(T_1, \dots, T_n)\|$ taken over all commuting n-tuples $(T_1, \dots, T_n)$ of contractions and all polynomials p in n variables with $\|p\|_{\mathbb{D}^n} = 1$.

Clearly $c_1 \le c_2 \le \cdots$. The dilation theory for a single contraction gives $c_1 = 1$. The Ando dilation gives $c_2 = 1$. Not much is known about the values of c_n for $n \ge 3$. The above example gives $c_3 > 1$ but it is not known even whether $c_3 < \infty$. It is known (Varopoulos, 1974) that $\lim_{n\to\infty} c_n = \infty$. Moreover, by (Dixon, 1976), c_n grows asymptotically faster than any power of n. The best explicit estimate seems to be $c_n > \frac{1}{11}\sqrt{n}$ for all n, see (Dixon, 1976).

Of course, if $T = (T_1, \dots, T_n) \in B(H)^n$ is a commuting n-tuple of operators satisfying the Brehmer conditions, then T has a regular dilation, and so the von Neumann inequality $\|p(T)\| \le \|p\|_{\mathbb{D}^n}$ is satisfied for all polynomials p.

5 Spherical dilations

Regular dilations considered in Section 3 are closely connected with the polydisc $\mathbb{D}^n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : |z_j| < 1 \quad (j = 1, \dots, n)\}$. This section outlines a parallel theory connected with the unit ball $\mathbb{B}_n = \{(z_1, \dots, z_n) \in \mathbb{C}^n : \sum_{j=1}^n |z_j|^2 < 1\}$. For more details see (Müller and Vasilescu, 1993).

For an operator $A \in B(H)$ denote by $M_A : B(H) \to B(H)$ the operator defined by $M_A(X) = A^*XA \quad (X \in B(H))$. Let $T = (T_1, \dots, T_n) \in B(H)^n$ be a commuting n-tuple of operators. The Brehmer conditions can be rewritten in this notation as

$$\prod_{j \in G} \big(I_{B(H)} - M_{T_j}\big)(I_H) \geq 0$$

for all $G \subset \{1, \dots, n\}$.

For spherical dilations one can consider the operator $M_T = \sum_{j=1}^n M_{T_j} : B(H) \to B(H)$, i.e., $M_T(X) = \sum_{j=1}^n T_j^* X T_j \quad (X \in B(H))$. Note that

$$M_T^m(X) = \sum_{\substack{\alpha \in \mathbb{Z}_+^n \\ |\alpha| \leq m}} \frac{m!}{\alpha!} T^{\alpha *} X T^\alpha$$

for each $m \in \mathbb{N}$.

A commuting n-tuple of operators $T = (T_1, \dots, T_n) \in B(H)^n$ is called a *spherical contraction* if $M_T(I_H) \leq I_H$. Equivalently, $T_1^*T_1 + \cdots + T_n^*T_n \leq I_H$, or $\sum_{i=1}^n \|T_i x\|^2 \leq \|x\|^2$ for all $x \in H$. Write $\Delta_T^{(1)} = I_H - T_1^*T_1 - \cdots - T_n^*T_n$. So T is a spherical contraction if and only if $\Delta_T^{(1)} \geq 0$. If $\Delta_T^{(1)} = 0$, i.e., if $T_1^*T_1 + \cdots + T_n^*T_n = I_H$, then T is called a *spherical isometry.*

More generally, it is possible to consider also conditions $\Delta_T^{(m)} \geq 0$, where

$$\Delta_T^{(m)} := (I_{B(H)} - M_T)^m (I_H) = \sum_{\substack{\alpha \in \mathbb{Z}_+^n \\ |\alpha| \leq m}} (-1)^{|\alpha|} \frac{m!}{\alpha!(m - |\alpha|)!} T^{\alpha *} T^\alpha.$$

The most interesting are the cases where $m = 1$ and $m = n$.

The role of the backward multishift for regular dilations will be played by certain weighted backward multishifts.

As in the previous sections let $\ell^2(\mathbb{Z}_+^n, H)$ be the Hilbert space of all functions $f : \mathbb{Z}_+^n \to H$ satisfying $\|f\|^2 := \sum_\alpha \|f(\alpha)\|^2 < \infty$. Let $m \in \mathbb{N}$ be a fixed parameter. For $\alpha \in \mathbb{Z}_+^n$ write

$$\rho_m(\alpha) = \frac{(m + |\alpha| - 1)!}{\alpha!(m-1)!}.$$

The *weighted multishift* $S^{(m)} = (S_1^{(m)}, \dots, S_n^{(m)})$ acting on the space $\ell^2(\mathbb{Z}_+^n, H)$ is defined by

$$(S_j^{(m)} f)(\alpha) = \Big(\frac{\rho_m(\alpha)}{\rho_m(\alpha + e_j)}\Big)^{1/2} f(\alpha + e_j).$$

Following the same way as in Section 3 it is possible to prove

Theorem 5.1. *Let $T = (T_1, \dots, T_n) \in B(H)^n$ be a commuting n-tuple of operators and $m \in \mathbb{N}$. Then the following conditions are equivalent:*

(i) $\Delta_T^{(m)} \geq 0$ *and (SOT)* $- \lim_{k\to\infty} M_T^k(I_H) = 0$*;*

(ii) T *is unitarily equivalent to the restriction of* $S^{(m)}$ *to an invariant subspace.*

It is interesting to note that $\Delta_T^{(m)} \geq 0$ and $M_T^k(I_H) \to 0$ (SOT) implies $\Delta_T^{(1)} \geq 0$. Moreover, $\Delta_T^{(1)} \geq 0$ and $\Delta_T^{(m)} \geq 0$ implies that $\Delta_T^{(s)} \geq 0$ for all $s, 1 \leq s \leq m$, see (Müller and Vasilescu, 1993).

Again if the condition (SOT) $- \lim_{k\to\infty} M_T^k(I_H) = 0$ is not satisfied it is possible to complete the model.

Theorem 5.2. *Let* $T = (T_1, \dots, T_n) \in B(H)^n$ *be a commuting* n*-tuple of operators and* $m \in \mathbb{N}$*. Then the following conditions are equivalent:*

(i) $\Delta_T^{(1)} \geq 0$ *and* $\Delta_T^{(m)} \geq 0$*;*

(ii) T *is unitarily equivalent to the restriction of* $S^{(m)} \oplus W$ *to an invariant subspace, where* W *is a spherical isometry on some Hilbert space* H'*;*

(iii) T *is unitarily equivalent to the restriction of* $S^{(m)} \oplus N$ *to an invariant subspace, where* $N = (N_1, \dots, N_n)$ *is an* n*-tuple of commuting normal operators on some Hilbert space* H'' *satisfying* $N_1^* N_1 + \cdots + N_n^* N_n = I_{H''}$*.*

Implication (ii)$\Rightarrow$(iii) is based on the following result of (Athavale, 1990): any spherical isometry can be extended to an n-tuple of commuting normal operators $N = (N_1, \dots, N_n)$ satisfying $N_1^* N_1 + \cdots + N_n^* N_n = I$. Note that such an n-tuple of normal operators satisfies $\sigma(N) \subset \partial\mathbb{B}_n$.

As mentioned above, the most interesting case is the case $m = n$. Then the weighted multishift $S^{(n)}$ has an additional property that $\Delta_{S^{(n)*}}^{(1)} = 0$, i.e., $S^{(n)*}$ is a spherical isometry. So $S^{(n)*}$ can be extended to commuting normal operators. Thus one has

Theorem 5.3. *Let* $T = (T_1, \dots, T_n) \in B(H)^n$ *be a commuting* n*-tuple of operators satisfying* $\Delta_T^{(1)} \geq 0$ *and* $\Delta_T^{(n)} \geq 0$*. Then there exist a Hilbert space* $K \supset H$ *and commuting normal operators* $N = (N_1, \dots, N_n) \in B(K)^n$ *such that* $N_1^* N_1 + \cdots + N_n^* N_n = I_K$ *and*

$$T^\alpha = P_H N^\alpha |_H$$

for all $\alpha \in \mathbb{Z}_+^n$*.*

So in this case there is a complete analogy with the dilation theory of single contractions.

Corollary 5.4. *Let* $T = (T_1, \dots, T_n) \in B(H)^n$ *be a commuting* n*-tuple of operators satisfying* $\Delta_T^{(1)} \geq 0$ *and* $\Delta_T^{(n)} \geq 0$*. Then*

$$\|p(T)\| \leq \|p\|_{\mathbb{B}_n}$$

for any polynomial p *in* n *variables.*

Let $T = (T_1, \dots, T_n)$ be a spherical contraction, i.e., $\sum_{i=1}^n T_i^* T_i \leq I$. Then T satisfies the Brehmer conditions and so it has the regular dilation. Thus the von Neumann inequality $\|p(T)\| \leq \|p\|_{\mathbb{D}^n}$ is satisfied for all polynomials p. However, for spherical contractions it is more natural to consider the ball norm $\|p\|_{\mathbb{B}_n}$ instead of the polydisc norm $\|p\|_{\mathbb{D}^n}$.

By Theorem 5.2, any spherical contraction $T = (T_1, \ldots, T_n)$ is a restriction of $S^{(1)} \oplus N$ to an invariant subspace, where $S^{(1)} = (S_1^{(1)}, \ldots, S_n^{(1)})$ is the backward weighted multishift defined above and $N = (N_1, \ldots, N_n)$ is an n tuple of commuting normal operators with $\sigma(N) \subset \partial\mathbb{B}_n$. This implies the following result, see (Drury, 1978):

Theorem 5.5. *Let $T = (T_1, \ldots, T_n) \in B(H)^n$ be a spherical contraction. Then*

$$\|p(T)\| \le \|p(S^{(1)})\|$$

for each polynomial p in n variables.

However, $\|p(S^{(1)})\|$ may be bigger than $\|p\|_{\mathbb{B}_n}$. By (Varopoulos, 1974),

Theorem 5.6. *For each $K > 0$ there exist $n \in \mathbb{N}$ and a commuting n-tuple of operators $T = (T_1, \ldots, T_n)$ satisfying $\sum_{j=1}^n \|T_j\|^2 \le 1$ and a polynomial p of degree 3 such that*

$$\|p(T)\| > K\|p\|_{\mathbb{B}_n}.$$

(Consequently, $\|p(S^{(1)})\| > K\|p\|_{\mathbb{B}_n}$).

6 Analytic models

In previous sections one always started with a nice n-tuple of operators - (weighted) backward multishifts - and found a class of operators for which this multishift served as a model. The model multishift was closely related with some Hardy/Bergman space of analytic functions.

Another approach is presented in this section. One starts with a nice space of analytic functions with reproducing kernel and builds a dilation theory connected with this function space. Only an outline of the proofs is given, for details see (Ambrozie et al. 2002).

Let D be an open domain in $\mathbb{C}^n$. A Hilbert space $\mathcal{H}$ of functions analytic on D is called a *D-space* if conditions (i) – (iii) below are satisfied:

(i) $\mathcal{H}$ is invariant under the operators Z_i, $i = 1, \ldots, n$ of multiplication by the coordinate functions,

$$(Z_i f)(z) := z_i f(z), \ f \in \mathcal{H}, \ z = (z_1, \ldots, z_n) \in D.$$

It follows from the next assumption and the close graph theorem that the operators Z_j are, in fact, bounded.

(ii) For each $z \in D$, the evaluation functional $f \mapsto f(z)$ is continuous on $\mathcal{H}$. By the Riesz theorem there exists a vector $C_z \in \mathcal{H}$ such that $f(z) = \langle f, C_z \rangle$ for all $f \in \mathcal{H}$. Let $D' = \{\bar{z} : z \in D\}$. Define the function $C(z, w) := C_{\bar{w}}(z)$ for $z \in D$, $w \in D'$. (The function $C(z, \bar{w})$ is known as the reproducing kernel of $\mathcal{H}$). It is easy to see that C is analytic on $D \times D'$.

(iii) $C(z, w) \ne 0$ for all $z \in D$, $w \in D'$.

Let $\mathcal{H}$ be a D-space and H an abstract Hilbert space. Denote by $\mathcal{H} \otimes H$ the (completed) Hilbertian tensor product. Elements of $\mathcal{H} \otimes H$ can be viewed upon as H-valued functions analytic on D.

Consider the multiplication operators M_{z_j} on $\mathcal{H} \otimes H$ defined by $M_{z_j} = Z_j \otimes I_H \quad (j = 1, \ldots, n)$ and write $M_z = (M_{z_1}, \ldots, M_{z_n})$.

Let us study commuting n-tuples $T = (T_1, \ldots, T_n) \in B(H)^n$ for which M_z^* serves as a model. The first step is to consider the n-tuples T having the joint Taylor spectrum $\sigma(T)$ contained in D'. Then one deals, under slightly stronger assumptions on $\mathcal{H}$, with n-tuples whose spectrum lies only in $\overline{D'}$.

The basic prototype of a D-space is the Hardy space H^2 on the open unit disc $\mathbb{D} \subset \mathbb{C}$. In this case $C(z, w) = (1 - zw)^{-1}$ and M_z^* is the backward shift.

Let $T = (T_1, \ldots, T_n)$ be a commuting n-tuple of operators on an abstract Hilbert space H with the Taylor joint spectrum $\sigma(T) \subset D'$. Consider the $(2n)$-tuple $(L_{T^*}, R_T) = (L_{T_1^*}, \ldots, L_{T_n^*}, R_{T_1}, \ldots, R_{T_n})$ of operators acting on $B(H)$ defined by $L_{T_j^*}(X) = T_j^* X$, $R_{T_j}(X) = XT_j \quad (X \in B(H), j = 1, \ldots, n)$. It is easy to see that (L_T^*, R_T) is a commuting $(2n)$-tuple. Moreover, it is possible to show that the joint Taylor spectrum of this $(2n)$-tuple satisfies

$$\sigma(L_T^*, R_T) \subset \sigma(L_T^*) \times \sigma(R_T) = \sigma(T^*) \times \sigma(T) \subset D \times D'.$$

The hereditary functional calculus $f \mapsto f(T^*, T)$ defined for functions analytic on a neighborhood of $\sigma(L_T^*, R_T)$ by $f(T^*, T) = f(L_T^*, R_T)(I_H)$ is a generalization of the Taylor functional calculus, see (Agler, 1988). Note that if $p(x, y) = \sum_{\alpha, \beta \in \mathbb{Z}_+^n} c_{\alpha,\beta} x^\alpha y^\beta$ is a polynomial (or a power series) then $p(T^*, T) = \sum_{\alpha,\beta \in \mathbb{Z}_+^n} c_{\alpha,\beta} T^{*\alpha} T^\beta$, where $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ are commuting n-tuples, but the variables x_i and y_j do not commute.

In the situation considered above, one can define $\frac{1}{C}(T^*, T) = \frac{1}{C}(L_T^*, R_T)(I_H)$. Assume that $\frac{1}{C}(T^*, T) \geq 0$ and let the *defect operator* of T be defined by

$$D_T = \Big(\frac{1}{C}(T^*, T)\Big)^{1/2}.$$

Note that in the basic example $\mathcal{H} = H^2$ the positivity condition $\frac{1}{C}(T^*, T) \geq 0$ reduces to $I - T^*T \geq 0$, i.e., T is a contraction, and this notation agrees with the classical one, $D_T = (I - T^*T)^{1/2}$.

Also, define below a bounded linear operator $C_{T^*} : H \to \mathcal{H} \otimes H$. To this aim, let $k_u(w)$ denote the Martinelli kernel in n variables (Kordula and Müller, 1995; Vasilescu, 1978), that is a differential form of degree $n-1$ in $d\overline{w}_1, \ldots, d\overline{w}_n$ and degree n in $dw_1, \ldots, dw_n$. The definition of C_{T^*} is motivated by the reproducing kernel property of the Martinelli kernel, for use in the proof of (8) below. That is, for every analytic function f and point u in its domain one has $f(u) = \int_{\partial\Delta} f(w) k_{\overline{u}}(w)$ where Δ is a bounded open domain with smooth boundary, such that $\overline{\Delta}$ is included in the domain of f. For $\overline{\Delta} \subset D'$ and $\sigma(T) \subset \Delta$, set

$$C_{T^*} h := \int_{\partial\Delta} C_{\overline{w}} \otimes k_T(w) h, \qquad (h \in H)$$

where k_T is the operator version of the Martinelli kernel, with coefficients in $B(H)$, the exact form of which can be found in (Kordula and Müller, 1995), for Hilbert space operators see (Vasilescu, 1978). Following the lines of the formal identities $f(T^*) = \int_{\partial\Delta} f(w) k_T(w)$ and

$$C_{T^*}(z) = C(z, T) = \int_{\partial\Delta} C(z, w) k_T(w) = \int_{\partial\Delta} C_{\overline{w}}(z) k_T(w),$$

after some computation using also the reproducing kernel property $f(w) = \langle f, C_w \rangle$ of C, one can obtain the equality

$$\langle C_{T^*}h, f \otimes h' \rangle = \langle h, f(T^*)h' \rangle \tag{8}$$

for all h, $h' \in H$ and $f \in \mathcal{H}$. This shows in particular that C_{T^*} does not depend on the choice of Δ. Moreover, C_{T^*} is a bounded operator.

Define a mapping $V : H \to \mathcal{H} \otimes H$ by the formula

$$V = (I_{\mathcal{H}} \otimes D_T)C_{T^*}. \tag{9}$$

Within this context it is possible to prove the following result.

Theorem 6.1. *Let $\mathcal{H}$ be a D-space and T a commuting n-tuple of operators on a Hilbert space H such that $\sigma(T) \subset D'$ and $\frac{1}{C}(T^*, T) \geq 0$. Then the mapping $V : H \to \mathcal{H} \otimes H$ defined by (9) is an isometry and*

$$VT_j = M_{z_j}^* V \qquad (j = 1, \ldots, n).$$

Hence T is unitarily equivalent to the restriction of M_z^* to an invariant subspace, namely $T_j \overset{u}{\sim} M_{z_j}^*|_{VH} \quad (j = 1, \ldots, n)$.

In many interesting cases D is bounded and $\mathcal{H}$ is isometrically embedded into a space $L^2(m)$, where m is a finite nonnegative Borel measure on $\overline{D}$. Then $Z = (Z_1, \ldots, Z_n)$ is a subnormal n-tuple. More precisely, it extends to the (bounded and normal) n-tuple M of multiplications by the coordinate functions on $L^2(m)$. Note that $\sigma(M) = \operatorname{supp} m \subset \overline{D}$.

Corollary 6.2. *Let $\mathcal{H}$ be a D-space, T a commuting n-tuple of operators on a Hilbert space H such that $\sigma(T) \subset D'$ and $\frac{1}{C}(T^*, T) \geq 0$. Suppose that $\mathcal{H} \subset L^2(m)$ isometrically, where $m \geq 0$ is a finite Borel measure with support $\operatorname{supp} m \subset \overline{D}$.*

Then T^* has a normal dilation N with spectrum $\sigma(N) \subset \operatorname{supp} m$.

Proof. Indeed, $T_j \overset{u}{\sim} M_{z_j}^*|_{VH}$, $j = 1, \ldots, n$, whence $T^\alpha \overset{u}{\sim} M_z^{*\alpha}|_{VH}$, and so $T^{*\alpha} \overset{u}{\sim} P_{VH} M_z^\alpha|_{VH}$ for any multiindex α, where all the operators M_{z_j} considered in the space $L^2(m) \otimes H$ are normal. □

Corollary 6.3. *Let $\mathcal{H}$ be a D-space and T a commuting n-tuple of operators on a Hilbert space H such that $\sigma(T) \subset D'$ and $\frac{1}{C}(T^*, T) \geq 0$. Suppose that $\mathcal{H} \subset L^2(m)$ isometrically, where $m \geq 0$ is a finite Borel measure with $\operatorname{supp} m \subset \overline{D}$. Then von Neumann's inequality*

$$\|p(T)\| \leq \sup_{z \in D'} |p(z)|$$

holds for all polynomials p in n variables.

Consider now n-tuples T such that $\sigma(T) \subset \overline{D'}$. To this aim, assume the following additional hypotheses:

(iv) The polynomials are dense in $\mathcal{H}$, and the function $\frac{1}{C}$ is a polynomial.

Arrange the monomials $z^\alpha \in \mathcal{H}$, $\alpha \in \mathbb{Z}_+^n$ in some order. By the Gram-Schmidt orthogonalization one can find an orthonormal sequence of polynomials $(\psi_k(z))_{k\geq 1}$. Define the polynomial functions

$$f_m(z,u) = 1 - \sum_{k=1}^{m-1} \psi_k(z) \frac{1}{C}(z,u) \overline{\psi_k(\overline{u})}.$$

Note that in the case where $n = 1$ and $\mathcal{H}$ is the Hardy space $H^2(\mathbb{D})$ on the unit disc, then $f_m(z,u) = z^m u^m$.

Let $T = (T_1, \ldots, T_n)$ be a commuting n-tuple of operators such that $\frac{1}{C}(T^*, T) \geq 0$ and $\sup_{m\geq 1} f_m(T^*, T) < \infty$. Define now $V : H \to \mathcal{H} \otimes H$ by

$$Vh = \sum_{k\geq 1} \psi_k \otimes D_T \psi_k(T^*)^* h. \tag{10}$$

Then V turns out to be well-defined, bounded and moreover this definition agrees with the previous definition (9) of V, as is shown by the following proposition.

Proposition 6.4. *Let D, $\mathcal{H}$ and T be as above. Let V be given by (10) and $h \in H$. Then $I = f_0(T^*, T) \geq f_1(T^*, T) \geq f_2(T^*, T) \geq \cdots$ and*

$$\|Vh\|^2 = \|h\|^2 - \lim_{m\to\infty} \langle f_m(T^*, T)h, h\rangle.$$

Moreover, for any $g \in H$ and polynomial $f \in \mathcal{H}$,

$$V^*(f \otimes g) = f(T^*) D_T g.$$

Hence definitions (9) and (10) agree.

Proof. For any $k \geq 1$ one has

$$\Big\langle \psi_k(T^*) \frac{1}{C}(T^*, T) \psi_k(T^*)^* h, h \Big\rangle = \Big\langle \frac{1}{C}(T^*, T) \psi_k(T^*)^* h, \psi_k(T^*)^* h \Big\rangle \geq 0,$$

and so $f_m(T^*, T) \geq f_{m+1}(T^*, T)$ for every m. Then the limit $\lim_{m\to\infty} \langle f_m(T^*, T)h, h\rangle$ exists. Moreover for any j, m, with $j < m$,

$$\begin{aligned} \Big\| \sum_{k=j}^{m-1} \psi_k \otimes D_T \psi_k(T^*)^* h \Big\|^2 &= \sum_{k=j}^{m-1} \|D_T \psi_k(T^*)^* h\|^2 \\ = \quad \sum_{k=j}^{m-1} \Big\langle \psi_k(T^*) \frac{1}{C}(T^*, T) \psi_k(T^*)^* h, h \Big\rangle &= \langle (f_j - f_m)(T^*, T)h, h\rangle, \end{aligned}$$

and hence the partial sums of the right hand side of (10) form a Cauchy sequence, and so convergent, sequence. Thus V given by (10) is well-defined. Letting $j = 0$ and $m \to \infty$ above one obtains, since $f_0(T^*, T) = I$, that

$$\|Vh\|^2 = \lim_m \langle (f_0 - f_m)(T^*, T)h, h\rangle = \|h\|^2 - \lim_m \langle f_m(T^*, T)h, h\rangle.$$

Since the functions ψ_j were obtained by Gram-Schmidt orthogonalization, any polynomial is a linear combination of such functions, and so it is sufficient to verify the second equality for $f = \psi_j$ and then use (8) to check that (9) and (10) agree. For any $h \in H$,

$$\langle h, V^*(\psi_j \otimes g)\rangle = \langle Vh, \psi_j \otimes g\rangle = \Big\langle \sum_k \psi_k \otimes D_T \psi_k(T^*)^* h, \psi_j \otimes g \Big\rangle$$

$$= \langle D_T \psi_j(T^*)^* h, g\rangle = \langle h, \psi_j(T^*) D_T g\rangle \overset{(8)}{=} \langle C_{T^*} h, \psi_j \otimes D_T g\rangle = \langle (I_{\mathcal{H}} \otimes D_T) C_{T^*} h, \psi_j \otimes g\rangle.$$

Hence definitions (9) and (10) agree. $\square$

Following these lines, the following results can be obtained.

Theorem 6.5. *Let $\mathcal{H}$ be a D-space such that the polynomials are dense and $\frac{1}{C}$ is a polynomial. Let T be a tuple of commuting operators on a Hilbert space H. The following statements are equivalent:*

(i) T is unitarily equivalent to the restriction of M_z^* to an invariant subspace;

(ii) $\frac{1}{C}(T^*, T) \geq 0$ and $\lim_{m\to\infty} f_m(T^*, T)h = 0$ for each $h \in H$.

Corollary 6.6. *Let T and $\mathcal{H}$ satisfy the previous hypotheses and condition (ii) from Theorem 6.5. Suppose moreover that $\mathcal{H} \subset L^2(m)$ where $m \geq 0$ is a finite Borel measure with $\operatorname{supp} m \subset \overline{D}$. Then the tuple $M_z = (M_{z_1}, \ldots, M_{z_n})$ of multiplications by coordinate functions on $L^2(m, H)$ is a normal dilation for T^* and*

$$\|p(T)\| \leq \sup_{z\in D'} |p(z)|$$

for any polynomial p.

7 Further examples

This section lists several cases (**1 - 8**) of analytic models for commuting tuples of Hilbert space operators, providing the existence of normal dilations and von Neumann's inequality.

1. The basic example is that of contractions $T : H \to H$, $\|T\| \leq 1$ (Sz.-Nagy and Foiaş, 1970; S.-Nagy et al., 2010), the model for which is the multiplication by the variable on the Hardy space $H^2(\mathbb{D})$. In this case $C_{T^*}h = \sum_j z^j T^j h \quad (h \in H)$, and $V : H \to H^2(\mathbb{D}) \otimes H$ is given by

$$Vh = \sum_{j\geq 0} z^j (I - T^*T)^{1/2} T^j h.$$

Then it is possible to recover the well-known fact that V is an isometry if and only if $T^m h \to 0$ for all $h \in H$, in which case $T \overset{u}{\sim} M_z^*|_{VH}$, cf. Section 2.

2. Another example is the case when $\mathcal{H}$ is the Bergman space on the unit disc $\mathbb{D}$. This space consists of those analytic functions that are square-integrable with respect to the planar Lebesgue measure, and its reproducing kernel is $C(z, \overline{w}) = (1 - z\overline{w})^{-2}$. Then $\frac{1}{C}(T^*, T) = I - 2T^*T + T^{*2}T^2$ and $f_m(T^*, T) = (m + 1)T^{*m}T^m - mT^{*m+1}T^{m+1}$. The condition $f_m(T^*, T) \to 0 \quad$ (SOT) turns out to be equivalent to $T^m \to 0 \quad$ (SOT). Hence if $I - 2T^*T + T^{*2}T^2 \geq 0$ and $T^m \to 0 \quad$ (SOT), then T is equivalent to the restriction of M_z^* to an invariant subspace, where M_z is the shift on the Bergman space, see (Agler, 1985).

More generally, let $\mathcal{H}$ be the k-Bergman space, in which case the reproducing kernel is $C(z, \overline{w}) = (1 - z\overline{w})^{-k}$. Models for operators T satisfying the positivity condition

$$\frac{1}{C}(T^*, T) = \sum_{j=0}^{k} (-1)^j \binom{k}{j} T^{*j} T^j \geq 0 \tag{11}$$

were introduced in (Agler, 1985). Contractions satisfying (11) are called *k-hypercontractions*.

3. If $D := \mathbb{B}_n$ is the Euclidean unit ball in $\mathbb{C}^n$ and $\mathcal{H}$ its Hardy space, then $C(z,\overline{w}) = (1 - z_1\overline{w}_1 - \cdots - z_n\overline{w}_n)^{-n}$ and $\mathcal{H} \subset L^2(\partial D, m)$ isometrically, where m is the rotation invariant probability measure on the unit sphere ∂D.

More generally, let $\mathcal{H}$ be the k-Bergman space, where $k > n$ and m is the Lebesgue measure on D. In this case the k-Bergman space is isometrically contained in $L^2((1 - |z|^2)^{k-n+1}m)$, $C(z,\overline{w}) = (1 - z_1\overline{w}_1 - \cdots - z_n\overline{w}_n)^{-k}$ and the results in Sections 5 and 6 apply.

If $1 \leq k < n$ then the k-Bergman space is also a D-space and the results of Section 6 apply. Thus one obtains a model for n-tuples T satisfying the corresponding positivity condition, see also Section 5. However, in this case the k-Bergman space is not of L^2-type, so it is not possible to obtain the von Neumann inequality.

4. Also, if $D := \mathbb{D}^n$ is the unit polydisc and $\mathcal{H}$ its Hardy space, then $C(z,\overline{w}) = \prod_{i=1}^n (1 - z_i\overline{w}_i)^{-1}$ and $\mathcal{H} \subset L^2(\partial_0 D, m)$ isometrically; here m is the normalized Lebesgue measure on the Shilov boundary $\partial_0 D = \{z = (z_1, \ldots, z_n) : |z_1| = \cdots = |z_n| = 1\}$ of D. Note that the existence of a regular dilation of an n-tuple T is equivalent to Brehmer's conditions

$$\sum_{0 \leq \alpha \leq \beta} (-1)^{|\alpha|} T^{*\alpha} T^{\alpha} \geq 0$$

for all $\beta \leq (1, \ldots, 1)$, and the inequality $\frac{1}{C}(T^*, T) \geq 0$ is Brehmer's condition of maximal degree for T, see Section 3.

A more general notion of γ-contractions for $\gamma \in \mathbb{Z}_+^n$ was studied in (Curto and Vasilescu, 1993; 1995).

5. One can similarly consider also more general models, over certain domains D given by inequalities of the form

$$D = \Big\{ (z_1, \ldots, z_n) \in \mathbb{C}^n : \sum_j c_{i,j} |z_j|^2 < 1, i = 1, \ldots, l \Big\},$$

where all $c_{i,j} \geq 0$ (Vasilescu, 1995), or of the form $p(|z_1|^2, \ldots, |z_n|^2) < 1$ where p is a polynomial with nonnegative coefficients (Pott, 1999).

6. Another interesting case is when D is a Cartan domain, and $\mathcal{H}$ a generalized Bergman space. For the definitions and general properties of such spaces see (Hua, 1979; Upmeier, 1996). The unit disc in $\mathbb{C}$ and unit ball in $\mathbb{C}^n$ are particular cases of such domains. Other examples are provided by the operator matrix balls $\mathbb{D}_{p,q}$ consisting of all $p \times q$ $(p \leq q)$ complex matrices $z = (z_{i,j})_{1 \leq i \leq p, 1 \leq j \leq q} : \mathbb{C}^q \to \mathbb{C}^p$ of supremum operator norm $\|z\| = \sup_{h \in \mathbb{C}^q, \|h\| < 1} \|z \cdot h\| < 1$, endowed with suitable Hilbert spaces $\mathcal{H}$ of analytic functions. For instance if $D = \mathbb{D}_{2,2} = \left\{ z = (z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : \left\| \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \right\| < 1 \right\}$, its Hardy space $\mathcal{H}_2$ has the reproducing kernel

$$C(z,\overline{w}) = \det \left(I_2 - \begin{pmatrix} z_1 & z_2 \\ z_3 & z_4 \end{pmatrix} \cdot \begin{pmatrix} \overline{w}_1 & \overline{w}_3 \\ \overline{w}_2 & \overline{w}_4 \end{pmatrix} \right)^{-2},$$

see (Hua, 1979; Upmeier, 1996). This can provide models (Ambrozie et al., 2002) for certain commuting 4-tuples $T = (T_{i,j})_{i,j=1}^2$ such that the operator matrix $\begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}$ is contractive on $H \oplus H$.

In the more general case of $D = \mathbb{D}_{p,q}$, for any $\lambda \geq q$ the λ-Bergman space $H_\lambda^2(\mathbb{D}_{p,q})$ is a D-space with reproducing kernel

$$C(z, \overline{w}) = \det\left(1_p - zw^*\right)^{-\lambda}.$$

Moreover, there exists a probability measure $\nu = \nu_\lambda$ on $\overline{\mathbb{D}}_{p,q}$ such that $\mathcal{H}_\lambda(\mathbb{D}_{p,q}) \subset L^2(\nu)$ isometrically, and the pq-tuple $M_z = (M_{z_{i,j}})_{i,j}$ is subnormal, see (Bagchi and Misra, 1996; Hua, 1979; Upmeier, 1995). Subsequent developments of such models over general Cartan domains are given in (Arazy and Engliš, 2003), where the generalized Bergman spaces $\mathcal{H} = \mathcal{H}_\lambda$ are considered for the parameter λ in the continuous Wallach set, and moreover the multiplication M_z is replaced by $M_z \oplus W$, W being a certain generalization of a unitary tuple. This provides a multivariable generalization of the well-known result by Sz.-Nagy and Foiaş stating that T is a contraction if and only if T is unitarily equivalent to the restriction to an invariant subspace of the direct sum $S \oplus W$ of a backward shift S of infinite multiplicity and a unitary operator W, see Theorem 2.2.

7. If $\mathcal{H}$ is a regular coanalytic model of (Agler, 1982), then it is a D-space in the present sense, where D is the unit disc in $\mathbb{C}$. Thus the model described in Section 6 generalizes the results in (Agler, 1982).

8. A model for commuting row contractions $T : \sum_{i=1}^n T_i T_i^* \leq I$ was given by (Arveson, 1998), by means of the Hilbert space $\mathcal{H} := H^2(\mathbb{B}_n)$ of analytic functions on the unit ball $\mathbb{B}_n$ the reproducing kernel of which is $C(z, \overline{w}) = (1 - \sum_{i=1}^n z_i \overline{w}_i)^{-1}$ $(z,\, w \in \mathbb{B}_n)$. Let $\mathcal{T} \subset B(\mathcal{H})$ denote the Toeplitz C^*-algebra generated by operators $M_{z_1}, \ldots, M_{z_n}$ of the multishift $M_z = (M_{z_1}, \ldots, M_{z_n})$ and $\mathcal{A} \subset \mathcal{T}$ be the subalgebra of $\mathcal{T}$ generated by M_z. Then by (Arveson, 1998, Theorem 6.2) for every such T on a Hilbert space H there exists a unique completely positive unital map $\phi : \mathcal{T} \to B(H)$ such that $\phi(M_{z_i}) = T_i$ for all $i = 1, \ldots, n$ and $\phi(AX) = \phi(A)\phi(X)$ for all $A \in \mathcal{A}$ and $X \in \mathcal{T}$. A map $\phi : \mathcal{T} \to B(H)$ on a C^*-algebra $\mathcal{T}$ is m-positive if $I_m \otimes \phi : M_m \otimes \mathcal{T} \to M_m \otimes B(H)$ is positive, namely $(I_m \otimes \phi)x \geq 0$ for every $x \geq 0$, where M_m is the set of all complex $m \times m$ matrices. ϕ is completely positive if it is m-positive for all $m \geq 1$. Then (Arveson, 1998, Theorem 8.1) for every polynomial p in n variables one has von Neumann's inequality $\|p(T)\| \leq \|p\|_{\mathcal{M}}$ where $\|\cdot\|_{\mathcal{M}}$ denotes the multiplier operator norm on $\mathcal{H}$.

This case can be recovered by the model described in Example 3 in this section. The map ϕ is given by $\phi(X) = V^* X V$ for $X \in \mathcal{T}$, where V is the isometry provided by Theorem 6.1 (for T^*). For another approach see Section 5. Note also (Arveson, 1998) that $\|p\|_{\mathcal{M}} = \|p(S)\|$, where S is the n-tuple of symmetric left creation operators on the symmetric Fock space. Similar results appeared in (Müller and Vasilescu, 1993), and a result for the tuples rT with $0 < r < 1$ has been established by (Drury, 1978). This approach is related to the matrix-valued von Neumann inequality, namely, by the general dilation theorem of Arveson, an n-tuple T of commuting contractions on a Hilbert space H has a unitary dilation if and only if for any $k \geq 1$ and matrix-valued polynomial $p = \sum_{\alpha \in \mathbb{Z}_+^n} c_\alpha z^\alpha$ with $c_\alpha \in M_k$ one has $\|p(T)\| \leq \sup_{z \in \mathbb{D}^n} \|p(z)\|$, where $p(T) = \sum_\alpha c_\alpha \otimes T^\alpha \in M_k \otimes B(H) \equiv B(H^k)$, see also (Kalyuzhnyi, 1998).

The results for commuting row contractions can also be recovered from the non-commuting setting by "letting the variables commute", see (Popescu, 1989b).

8 Concluding remarks

1. There are other directions of generalizing Sz. Nagy-Foiaş' dilation theory to systems of linear operators. One of them is Agler's approach to model theory (Agler, 1988), see also (Athavale, 1987), providing models by means of liftings to algebra representations. The general idea is that two classes $\mathcal{B} \subset \mathcal{F}$ of bounded linear operators on Hilbert spaces are given, where the set $\mathcal{B}$ is closed with respect to direct sums and unital representations. As an example, take the case when $\mathcal{F}$ consists of all contractions while $\mathcal{B}$ is the set of all coisometries. A typical conclusion of a *lifting theorem* has the following form: if $T \in \mathcal{F}$, then there exist an operator $S \in \mathcal{B}$, $S \in B(H)$ and a closed linear subspace L of H, invariant under S, such that $T \overset{u}{\sim} S|_L$. In interesting cases, there exists a universal element a in a unital C^*–algebra A such that $B \in \mathcal{B} \cap B(H)$ if and only if there is a unital representation $\pi : A \to B(H)$ with $\pi(a) = B$.

Denote by $\mathcal{P}$ the set of all *hereditary polynomials* $p = \sum_{\alpha,\beta} c_{\alpha,\beta} y^\alpha x^\beta$ with complex coefficients $c_{\alpha,\beta}$, where α, $\beta \in \mathbb{Z}_+^n$ are multiindices $\alpha = (\alpha_1, \ldots, \alpha_n)$, $\beta = (\beta_1, \ldots, \beta_n)$ and $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ are n-tuples, each of them commuting but not commuting to each other. Note for later use that for any integer $k \geq 1$, one may consider as well the set $M_k \otimes \mathcal{P}$ of all $k \times k$ matrix-valued hereditary polynomials $p = (p_{s,t})_{s,t=1}^k$ where each $p_{s,t} \in \mathcal{P}$. Set also $p(T) = \sum_{\alpha,\beta} c_{\alpha,\beta} T^{*\alpha} T^\beta$ for $T = (T_1, \ldots, T_n)$. Similarly, this makes sense as well for matrix-valued hereditary polynomials $p = (p_{s,t})_{s,t=1}^k$, by setting $p(T) = (p_{s,t}(T))_{s,t=1}^k \quad (k \geq 1)$. Then results of the following form can be proved (for simplicity we state below Theorem 1.1 of (Agler, 1988), concerning the case $n = 1$).

Theorem 8.1. *Let $T \in B(H)$, let A be a unital C^*-algebra and $a \in A$. The following statements are equivalent:*

(i) T is unitarily equivalent to $\pi(a)|_L$, where $\pi : A \to B(K)$ is a unital representation on some Hilbert space K and $L \subset K$ is invariant under $\pi(a)$;

(ii) $p(T) \geq 0$ for every matrix-valued hereditary polynomial $p \in \bigcup_{k=1}^\infty M_k \otimes \mathcal{P}$ such that $p(a) \geq 0$.

In order to make better use of such a statement, note, for instance, that given a *hereditary homomorphism* $\phi : \{p(a)\}_{p \in \mathcal{P}} \to L(H)$, namely a linear map ϕ such that $(\phi(p(a)) = p(\phi(a))$ for all $p \in \mathcal{P}$, if $\phi(a)$ is n-cyclic, then the complete positivity of ϕ is equivalent to its n-positivity (Agler, 1988, Theorem 1.20). In this case ϕ stands for the hereditary homomorphism given (and uniquely determined) by $\phi(a) := T$, and (ii) from above is indeed a complete positivity condition. The definitions and various applications of such lifting theorems can be found in (Agler, 1988).

2. An important result in the dilation theory of single contractions is the commutant lifting theorem, see (Sz.-Nagy and Foiaş, 1970) and for more details (Foiaş and Frazho, 1990):

Theorem 8.2. *Let $T \in B(H)$ be a contraction, $V \in B(K)$ its minimal isometric dilation, let $X \in B(H)$ such that $TX = XT$. Then there exists $Y \in B(K)$ such that $YV = VY$, $\|Y\| = \|X\|$ and $X = P_H Y|_H$.*

A similar result for n-tuples of contractions is not true even for n-tuples having a regular unitary dilation, see (Müller, 1994).

Nevertheless, there is a multivariable version of this result, called the functional commutant lifting theorem. Let L be a Hilbert space, $H^2(\mathbb{D}, L)$ the Hardy space, M_z the operator of multiplication by the variable acting on this space, let $H \subset H^2(\mathbb{D}, L)$ be a subspace invariant for M_z^* and let $T = P_H M_z|_H$ (by Theorem 2.2, such a representation is possible if and only if $T^{*k} \to 0 \quad \text{(SOT)}$). The commutant lifting theorem in this particular situation states that if $X \in B(H)$, $XT = TX$ then there exists a bounded analytic function $f : \mathbb{D} \to B(L)$ such that $\|f\|_{\mathbb{D}} = \|X\|$ and $X = P_H M_f|_H$, where M_f if the operator of multiplication by f.

The situation is more complicated for n arbitrary, as is shown below. Note that for $n \geq 3$ it is necessary to replace the usual norm $\|f\|_{\mathbb{D}^n}$ of f by the (Agler-)Schur norm $\|f\|_S := \sup_Z \|f(Z)\|$, where the supremum is taken over all commuting n-tuples $Z = (Z_1, \ldots, Z_n)$ of strict contractions (i.e., $\|Z_j\| < 1$). The result also provides the characterization of the functions f from the unit ball $\|f\|_S \leq 1$ as transfer functions of linear systems with the unitary connection matrix (Ball et al., 1999).

Theorem 8.3. *Let E be a Hilbert space and $H \subset H^2(\mathbb{D}^n) \otimes E$ be a M_z^*-invariant closed linear subspace. For $j = 1, \ldots, n$, set $T_j = P_H M_{z_j}|_H$ and let $T = (T_1, \ldots, T_n)$. Let $X \in B(H)$ satisfy $XT_j = T_j X$ for all $j = 1, \ldots, n$. The following two statements are equivalent:*

(i) There exists an analytic function $f : \mathbb{D}^n \to B(E)$ such that $X = P_H M_f|_H$ and $\|f\|_S \leq 1$;

(ii) There exist positive semidefinite operators $G_i \in B(H)$ such that $I - XX^* = \sum_{j=1}^n G_j$ and $\prod_{j \neq i}(I_{B(H)} - M_{T_j^*})\,(\,G_i\,) \geq 0$ for each $i = 1, \ldots, n$.

In this case, there exist a Hilbert space K and a unitary operator

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} : \begin{matrix} K^n \\ \oplus \\ E \end{matrix} \to \begin{matrix} K^n \\ \oplus \\ E \end{matrix}$$

such that $f(z) = d + c\,(I - \widetilde{z}\,a)^{-1}\,\widetilde{z}\,b$ where $\widetilde{z} : K^n \to K^n$ is the diagonal operator $\widetilde{z} = diag\,(z_1, \ldots, z_n)$ of multiplications by the coordinate functions.

For $n = 2$ this reduces to $G_1 - T_2 G_1 T_2^* \geq 0$ and $G_2 - T_1 G_2 T_1^* \geq 0$, and $\|f\|_S = \|f\|_\infty$.

For further generalizations of the commutant lifting theorem see for instance (Ball et al., 1997; Davidson and Le, 2010; Ambrozie and Eschmeier, 2005).

We mention the following version of the commutant lifting theorem for spherical contractions (Davidson and Le, 2010), see also the previous work (Ball et al., 1997) concerned only with the case of the weighted multishift (when $N = 0$):

Theorem 8.4. *Let $T = (T_1, \ldots, T_n) \in B(H)^n$ be a (commuting) spherical contraction, $X \in B(H)$, $XT_j = T_j X$ for all j. Let $S^{(1)} \oplus N$ be the extension described in Section 5 (i.e., $S^{(1)}$ is the weighted multishift described there and $N = (N_1, \ldots, N_n)$ is an n-tuple of commuting normal operators with $\sigma(N) \subset \partial\mathbb{B}_n$). Then there exists a dilation Y of X commuting with $S_j^{(1)} \oplus N_j$ for all j such that $\|Y\| = \|X\|$.*

3. The multivariate functional commutant lifting has applications to H^∞-type interpolation for various problems of Nevanlinna-Pick and Carathéodory-Fejér type on domains in $\mathbb{C}^n$, see (Sarason, 1967) $(n = 1)$ and, for instance, (Ball et al., 1999) $(n \geq 1)$, as well as its various subsequent developments.

4. Let $T = (T_1, \ldots, T_n)$ be an n-tuple of commuting contractions. If T satisfies the Brehmer conditions then it has a (regular) unitary dilation and therefore it satisfies the von Neumann inequality $\|p(T)\| \leq \|p\|_{\mathbb{D}^n}$ for all polynomials p in n variables.

In (Grinshpan et al., 2009) Brehmer's conditions were relaxed. If $\|T_j\| < 1$ for all j and there are two subsets $M_1, M_2 \subset \{1, \ldots, n\}$ with $|M_1| = |M_2| = n - 1$ such that $\sum_{F \subset M_j} (-1)^{|F|} T_F^* T_F \geq 0 \quad (j = 1, 2)$, then T has a (in general non-regular) unitary dilation, and so T satisfies the von Neumann inequality. So this result is a generalization of the Ando dilation theorem.

5. Another related direction is the study of joint subnormality for commuting n-tuples of operators on Hilbert spaces, starting from the Bram-Halmos criterion for $n = 1$, see for instance (Itô, 1958).

6. The characteristic function, an important tool for studying completely nonunitary contractions on Hilbert spaces (Sz.-Nagy and Foiaş, 1970; Sz.-Nagy et al., 2010), was also generalized to the case of completely non-coisometric row contractions of commuting operators on a Hilbert space (Bhattacharyya et al., 2006). For generalization to the non-completely non-coisometric case see (Ball and Bolotnikov, 2012).

The characteristic function for non-commuting row contractions was studied in (Popescu, 1989a).

7. Some of the results of dilation theory can be generalized also to infinite families of commuting operators, see e.g. (Sz.-Nagy and Foiaş, 1970) and construction of regular dilations.

8. Several other directions exist in the theory of the multivariable dilations, like for instance Agler's multipliers (Agler, 1982), Arveson's curvature invariant (Arveson, 2000), Douglas' Hilbert modules (Douglas et al., 2012), Popescu's non-commutative theory (Popescu, 1998), and so on.

For these interesting topics, that are beyond the aim of the present survey concerned with commutative dilation theory on Hilbert spaces, see the cited papers.

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