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ASPLUND SPACES CHARACTERIZED BY RICH FAMILIES AND SEPARABLE REDUCTION OF FRÉCHET SUBDIFFERENTIABILITY

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Abstract. Asplund property of a Banach space X is characterized by the existence of a rich family, in the product $X \times X^*$, consisting of some carefully chosen separable subspaces. This structural result is then used to add a lot of precision and simplicity to the known separable reductions of Fréchet subdifferentials.

1. Introduction

A Banach space is called *Asplund* if every convex continuous function on it is Fréchet differentiable at a point (equivalently, at the points of a dense set, yet equivalently, at the points of a dense G_δ set). An important, and widely used, equivalent condition for the Asplund property of a Banach space is that *every separable subspace of it has separable dual*, see [Ph93, Theorem 2.34]. For several other characterizations of Asplund spaces see [F∼11, Theorem 11.8]. Asplund spaces occur quite frequently in the non-separable Banach space theory.

Section 2 offers a new structural result characterizing Asplund spaces— Theorem 2.3. This is done in the nowadays modern language of rich families. This instrument is strong enough to get almost immediately a vast bunch of linear projections in duals to Asplund spaces, in particular projectional resolutions of the identity, or a modern substitute of it— projectional skeletons.

The Asplund spaces are of particular importance in infinite-dimensional variational analysis, see [M06]. A reason for that is that *a Banach space is Asplund (if and) only if every lower semicontinuous function on it is somewhere Fréchet subdifferentiable* [F89]. Section 3 is devoted to the separable reduction, via a rich family, of a general, quite precise, assertion involving Fréchet subdifferentials— see, Theorem 3.1. It serves as a common tool for getting immediately separable reductions of several more concrete statements like non-emptiness of subdifferential, fuzzy calculus, extremal principle, ..., all in the sense of Fréchet. The proof, based on Theorem 2.3, brigs a novelty and more simplicity and precision when comparing with the so far existing technology, cf. [FI13, FI15].

2. Rich families in Asplund spaces

Let P be a set and let $\prec$ be a *partial order*, see [E77, page 21]. Assume moreover that P is *(up)-directed* by $\prec$, i.e., for every $t_1, t_2 \in P$ there is $t_3 \in P$ such that $t_1 \prec t_3$ and $t_2 \prec t_3$. A subset $R \subset P$ is called *cofinal* if for every $t \in P$ there is $r \in P$ such that $t \prec r$. R is called *σ-complete* if, whenever $r_1 \prec r_2 \prec \cdots$ is an increasing sequence in R, then there is $r \in R$ such that $r_i \prec r$ for every $i \in \mathbb{N}$ and $r \prec t$ whenever $t \in P$ and $r_i \prec t$ for every $i \in \mathbb{N}$. A set $R \subset P$ is called *rich* if it is both cofinal and σ-complete. Note that the whole P is rich if it is σ-complete.

Now, we are ready to provide a concrete example of the poset $(P, \prec)$ that emerges naturally in the framework of Banach spaces. Let Z be a (rather non-separable) Banach space. By $\mathcal{S}(Z)$ we

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denote the family of all separable closed subspaces of Z and we endow it by the partial order "$\subset$". Thus, we can consider *rich families* in the poset $(\mathcal{S}(Z),\subset)$. We then also say that they are *rich in* Z. This example of rich family was first articulated in a paper [BM00] by J.M. Borwein and W. Moors. The power of rich families is demonstrated by the following fundamental simple fact (see [BM00] and also [LPT12, page 37]).

Proposition 2.1. *The intersection of two (even of countably many) rich families of a given Banach space is (not only non-empty but again even) rich.*

Let $k\in\mathbb{N}$ be greater than 1, and let $X_1,\dots,X_k$ be Banach spaces. By a *block* we understand any product $Y_1\times\cdots\times Y_k$. Any family consisting of (some) blocks $Y_1\times\cdots\times Y_k$ where $Y_1\in\mathcal{S}(X_1),\ \dots,Y_k\in\mathcal{S}(X_k)$ is called a *block-family* in $\mathcal{S}(X_1\times\cdots\times X_k)$ or just in $X_1\times\cdots\times X_k$. If $k=2$, we speak about *rectangles* and *rectangle-families* and we denote by $\mathcal{S}_{\sqsubset\!\sqsupset}(X_1{\times}X_2)$ the maximal rectangle-family in $\mathcal{S}(X_1\times X_2)$. This $\mathcal{S}_{\sqsubset\!\sqsupset}(X_1\times X_2)$ is clearly a rich family in $\mathcal{S}(X_1{\times}X_2)$. For a Banach space X, with dual X^*, we will frequently work with rectangle-families in $\mathcal{S}_{\sqsubset\!\sqsupset}(X\times X^*)$.

It is fair to say a warning: If $\mathcal{R}$ is a rich rectangle-family in $\mathcal{S}_{\sqsubset\!\sqsupset}(X\times X^*)$ it may happen that the "projection" of it on, say, the second coordinate, that is, the family $\big\{Y:\ V\times Y\in\mathcal{R}$ for some $V\in\mathcal{S}(X)\big\}$, is `not` rich in $\mathcal{S}(X^*)$. Such a situation can be arranged easily even in Hilbert space. Fortunately, in one important case, the projection of $\mathcal{R}$ on the first coordinate is again rich; see Theorems 2.3 a 3.1 below.

Let X be a Banach space. For a set $A\subset X$ the symbols $\operatorname{sp}A$, $\overline{\operatorname{sp}}\,A$, and $\operatorname{sp}_{\mathbb{Q}}A$ mean the linear span of A, the norm-closed linear span of A and the set consisting of all finite linear combinations of elements in A with rational coefficients, respectively. For $A\subset X$ and $B\subset X^*$ we put $B|_A:=\big\{x^*|_A:\ x^*\in B\big\}$; hence, if A is a subspace of X, then $B|_A$ is a subset of the dual space A^*. The set of rational numbers is denoted by $\mathbb{Q}$ and we put $\mathbb{Q}^+:=\mathbb{Q}\cap(0,+\infty)$. For an infinite set M the symbol $[M]^{\leq\omega}$ means the family of all infinite countable subsets of M. Given a set M in a Banach space, then $\overline{M}$ always denotes the norm closure (not the weak neither weak* closure) of it.

Next, we introduce a concept which serves as a link between X and X^* (and exists right if and only if X is Asplund).

Definition 2.2. By an *Asplund generator* in a Banach space X we understand any correspondence $G:[X]^{\leq\omega}\longrightarrow[X^*]^{\leq\omega}$ such that

(a) $\big(\overline{\operatorname{sp}}\,C\big)^*=\overline{G(C)|_{\overline{\operatorname{sp}}\,C}}$ for every $C\in[X]^{\leq\omega}$;

(b) if $C_1,\ C_2,\ \dots$ is an increasing sequence in $[X]^{\leq\omega}$, then $G(C_1\cup C_2\cup\cdots)=G(C_1)\cup G(C_2)\cup\cdots$;

(c) $\bigcup\{G(C):\ C\in[X]^{\leq\omega}\}$ is a dense subset in X^*; and

(d) if $C_1,C_2\in[X]^{\leq\omega}$ are such that $\overline{\operatorname{sp}}\,C_1=\overline{\operatorname{sp}}\,C_2$, then $\overline{\operatorname{sp}}\,G(C_1)=\overline{\operatorname{sp}}\,G(C_2)$.

Now, consider a Banach space $(X,\|\cdot\|)$, not being Hilbert. Let V be a (closed) subspace of it. We focus on a question whether there exists a linear isometric extension operator from the dual V^* into X^*. (If there were so, then some arguments in the next section would become quite easy.) There do exist situations when this happens. For instance, if X is $c_0(\Gamma)$ or $\ell_p(\Gamma)$, $1\leq p<+\infty$, and V is $c_0(N)$ or $\ell_p(N)$, where $N\subset\Gamma$. However, we are afraid that for general X and V this may not be true. Fortunately, and this is the content of the next structural theorem: If X is Asplund, there are `plenty` of well behaving V's. We think that this statement actually elucidates what the Asplund property of a Banach space is. The proof of it gathers together ideas from several papers ranging over half a century, see in particular [L65, T70, JZ74, G79, FG88, St96, CF15] .

Theorem 2.3. (Main) *Let $(X,\|\cdot\|)$ be a (rather non-separable) Banach space. Then the following assertions are mutually equivalent.*
(i) X is an Asplund space.
(ii) X admits an Asplund generator.

(iii) There exists a rich rectangle-family $\mathcal{A} \subset \mathcal{S}_{\square}(X \times X^*)$ such that $Y_1 \subset Y_2$ whenever $V_1 \times Y_1,\ V_2 \times Y_2$ are in $\mathcal{A}$ and $V_1 \subset V_2$, and for every $V \times Y \in \mathcal{A}$ the assignment $Y \ni x^* \longmapsto x^*|_V \in V^*$ is a surjective isometry.
(iv) There exists a cofinal rectangle-family $\mathcal{A} \subset \mathcal{S}_{\square}(X \times X^*)$ such that for every $V \times Y \in \mathcal{A}$ the assignment $Y \ni x^* \longmapsto x^*|_V \in V^*$ is a surjection.

Proof. (i)$\Longrightarrow$(ii). In order not to get lost in the case of general Asplund space, assume first that the norm $\|\cdot\|$ on X is Fréchet smooth, or more generally, that there exist an open set $0 \in \Omega \subset X$ and a smooth function $f : \Omega \to \mathbb{R}$, with continuous derivative f' such that $f'(V \cap \Omega)|_V$ is dense in V^* for every subspace V of X; note that this easily implies that X is Asplund. Define then $G : [X]^{\le\omega} \longrightarrow [X^*]^{\le\omega}$ by

$$[X]^{\le\omega} \ni C \longmapsto f'\big(\mathrm{sp}_{\mathbb{Q}} C \cap \Omega\big) =: G(C) \in [X^*]^{\le\omega}.$$

It remains to verify the properties (a), (b), (c), and (d) in Definition 2.2. As regards (a), fix any $C \in [X]^{\le\omega}$ and any v^* in the dual $(\overline{\mathrm{sp}}\, C)^*$. Let any $\varepsilon > 0$ be given. The properties of f provide a $v \in \mathrm{sp}_{\mathbb{Q}} C \cap \Omega$ such that $\big\|v^* - f'(v)|_{\overline{\mathrm{sp}}\, C}\big\| < \varepsilon$. But $f'(v)$ belongs to $G(C)$. And, as $\varepsilon > 0$ was arbitrary, we get that v^* belongs $\overline{G(C)|_{\overline{\mathrm{sp}}\, C}}$. Thus (a) is verified. Concerning (b), let $C_1, C_2, \ldots$ be as in the premise. Because our G is "monotone", we have only to prove the inclusion "$\subset$". And for this it is enough to realize that $\mathrm{sp}_{\mathbb{Q}}(C_1 \cup C_2 \cup \cdots) = \mathrm{sp}_{\mathbb{Q}}(C_1) \cup \mathrm{sp}_{\mathbb{Q}}(C_2) \cup \cdots$. The claim (c) follows immediately from the fact that $f'(X)$ is dense in X^* and from the definition of G. The last property (d) is guaranteed by the continuity of f'.

If we are facing a general Asplund space (and we do not have at hand the function f as above), we have to work harder. Either, we use the information from [FG88] (where Simons' lemma is needed!), or we profit from [CF15], based on Ch. Stegall's ideas (and proved without use of Simons' lemma). More concretely, let S be any nonempty set in B_X. By $\mathcal{L}(S)$ we denote the family of all functions of the form

$$B_{X^*} \ni x^* \longmapsto 1 - \sum_{n=1}^{k} 2^{-n} \big|a_n - \langle x^*, s_n\rangle\big|,$$

where $k \in \mathbb{N},\ a_1, a_2, \ldots, a_k$ are rational numbers in $[-1, 1]$, and $s_1, s_2, \ldots, s_k$ are elements from S; note that $\#\mathcal{L}(S) = \#S + \aleph_0$. It is easy to check that each element of $\mathcal{L}(S)$ is weak* continuous and has the maximum norm at most equal to 1. Thus $\mathcal{L}(S)$ is a subset of (the closed unit ball of) the Banach space $C(B_{X^*})$ of all weak* continuous functions on the closed unit ball B_{X^*} in X^*. We can easily check that

$$\overline{\mathcal{L}(\overline{S})} = \overline{\mathcal{L}(S)}\,; \tag{2.1}$$

hence, the set above here is separable whenever S is separable.

Consider the multivalued mapping $\partial : C(B_{X^*}) \longrightarrow 2^{B_{X^*}}$ defined by

$$C(B_{X^*}) \ni f \longmapsto \partial(f) := \big\{x^* \in B_{X^*} :\ f(x^*) = \max f(B_{X^*})\big\}.$$

It is well known, and easy to check, that ∂ is norm to weak* upper semicontinuous and weak* compact-valued. Now, X being Asplund, its dual unit ball is weak* dentable, see [Ph93, Theorem 2.32], [F$\sim$11, Theorem 11.8]. Thus, we are ready to apply the selection theorem of Jayne and Rogers [JR85, Theorem 8], [F97, Theorem 8.1.2] to ∂ and get a sequence of norm to `norm` continuous mappings $\lambda_j : C(B_{X^*}) \longrightarrow X^*,\ j \in \mathbb{N}$, such that for every $f \in C(B_{X^*})$ the limit $\lim_{j\to\infty} \lambda_j(f) =: \lambda_0(f)$ exists in the `norm` topology of X^* and moreover $\lambda_0(f) \in \partial(f)$, that is, $f\big(\lambda_0(f)\big) = \max f(B_{X^*})$. Now, we define the multivalued mapping

$$C(B_{X^*}) \ni f \longmapsto \{\lambda_1(f), \lambda_2(f), \ldots\} =: \Lambda(f) \subset X^*;$$

thus $\Lambda : C(B_{X^*}) \longrightarrow 2^{X^*}$. The continuity of the mappings λ_j's and (2.1) then guarantee that

$$\overline{\Lambda\big(\overline{\mathcal{L}(\overline{S})}\big)} = \overline{\Lambda\big(\mathcal{L}(S)\big)}\,. \tag{2.2}$$

Hence, the set above is separable once S is separable.

Using the symbols $\mathcal{L}$ and Λ introduced above, define

$$[X]^{\leq\omega} \ni C \longmapsto\subset \mathrm{sp}_{\mathbb{Q}}\, \Lambda\big(\mathcal{L}\big(\mathrm{sp}_{\mathbb{Q}}\, C \cap B_X\big)\big) =: G(C) \in [X^*]^{\leq\omega}.$$

Let us check that G is an Asplund generator. As regards (a) in Definition 2.2, take any $C \in [X]^{\leq\omega}$. Put $V := \overline{\mathrm{sp}}\, C$. By [CF15, Proposition 1], $B_{V^*} \subset \overline{\Lambda\big(\mathcal{L}(B_V)\big)}\big|_V$. Hence, using (2.2)

$$\big(\overline{\mathrm{sp}}\, C\big)^* = V^* \subset \mathrm{sp}_{\mathbb{Q}}\, \overline{\Lambda\big(\mathcal{L}(B_V)\big)}\big|_V \subset \overline{G(C)}\big|_V \quad \big(\subset \big(\overline{\mathrm{sp}}\, C\big)^*\big).$$

(We actually got a stronger equality that $\big(\overline{\mathrm{sp}}\, C\big)^* = \overline{G(C)}\,\big|_{\overline{\mathrm{sp}}\, C}$.) (b) follows easily from the very definition of our G, the definition of Λ, $\mathcal{L}$, and from the monotonicity of the sequence $C_1, C_2, \ldots$ (c) follows immediately from [CF15, Proposition 1] saying also that $B_{X^*} \subset \overline{\Lambda\big(\mathcal{L}(B_X)\big)}$. (d) follows easily from (2.2) and from the definition of G. We thus proved that (ii) holds in a general Asplund space.

(ii)$\Longrightarrow$(iii). Let $G : [X]^{\leq\omega} \longrightarrow [X^*]^{\leq\omega}$ be an Asplund generator in X. Define $\mathcal{A} \subset \mathcal{S}_{\square}(X \times X^*)$ as the family consisting of all rectangles $\overline{\mathrm{sp}}\, C \times \overline{\mathrm{sp}}\, G(C)$, where $C \in [X]^{\leq\omega}$, such that the assignment

$$\overline{\mathrm{sp}}\, G(C) \ni x^* \longmapsto x^*|_{\overline{\mathrm{sp}}\, C} \in (\overline{\mathrm{sp}}\, C)^* \tag{2.3}$$

is a surjective isometry. We shall show that $\mathcal{A}$ is a rich family.

As regards the cofinality of $\mathcal{A}$, fix any $V \times Y \in \mathcal{S}_{\square}(X \times X^*)$. Since G is an Asplund generator, the condition (c) guarantees that there is $C_0 \in [X]^{\leq\omega}$ so big that $\overline{C_0} \supset V$ and $\overline{G(C_0)} \supset Y$. Assume that for some $m \in \mathbb{N}$ we already found countable sets $C_0 \subset C_1 \subset \cdots \subset C_{m-1} \subset X$. Realizing that $\mathrm{sp}_{\mathbb{Q}} G(C_{m-1})$ is countable, we find $C_m \in [X]^{\leq\omega}$ so big that $C_m \supset C_{m-1}$ and that $\|x^*\| = \sup \langle x^*, C_m \cap B_X\rangle$ for every $x^* \in \mathrm{sp}_{\mathbb{Q}} G(C_{m-1})$. Do so subsequently for every $m \in \mathbb{N}$ and put finally $C := C_0 \cup C_1 \cup \cdots$. Clearly $C \in [X]^{\leq\omega}$ and also $\overline{\mathrm{sp}}\, C \times \overline{\mathrm{sp}}\, G(C) \supset V \times Y$. It remains to show that the assignment (2.3), with our just constructed C, is a surjective isometry.

From the above we have that for every $m \in \mathbb{N}$ and for every $x^* \in \mathrm{sp}_{\mathbb{Q}} G(C_{m-1})$

$$\|x^*\| = \sup \langle x^*, C_m \cap B_X\rangle \leq \sup\langle x^*, \overline{\mathrm{sp}}\, C \cap B_X\rangle = \big\|x^*|_{\overline{\mathrm{sp}}\, C}\big\| \leq \|x^*\|.$$

Hence, by (b) in Definition 2.3, we get that $\big\|x^*|_{\overline{\mathrm{sp}}\, C}\big\| = \|x^*\|$ holds for every $x^* \in \overline{\mathrm{sp}}\, G(C)$. We proved that the assignment (2.3) with our C is isometrical; denote it for a moment by R. Then, using (a) from Definition 2.3 and the isometric property of R we have

$$\big(\overline{\mathrm{sp}}\, C\big)^* \subset \overline{G(C)|_{\overline{\mathrm{sp}}\, C}} \subset \overline{R\big(\overline{\mathrm{sp}}\, G(C)\big)} = R\big(\overline{\mathrm{sp}}\, G(C)\big) = \overline{\mathrm{sp}}\, G(C)|_{\overline{\mathrm{sp}}\, C}.$$

This shows the surjectivity of the assignment (2.3) with our C. We thus proved that $\overline{\mathrm{sp}}\, C \times \overline{\mathrm{sp}}\, G(C)$ belongs to $\mathcal{A}$, and hence, the family $\mathcal{A}$ is cofinal.

For checking the σ-completeness of $\mathcal{A}$, consider any increasing sequence $V_1 \times Y_1,\ V_2 \times Y_2,\ \ldots$ of elements in $\mathcal{A}$. Then, clearly, $\overline{V_1 \times Y_1 \cup V_2 \times Y_2 \cup \cdots}$ is of form $V \times Y$ and this is an element of $\mathcal{S}_{\square}(X \times X^*)$. Also, clearly, $V = \overline{V_1 \cup V_2 \cup \cdots}$ and $Y = \overline{Y_1 \cup Y_2 \cup \cdots}$. From the definition of $\mathcal{A}$, for every $i \in \mathbb{N}$ find $C_i \in [X]^{\leq\omega}$ such that $V_i = \overline{\mathrm{sp}}\, C_i$ and $Y_i = \overline{\mathrm{sp}}\, G(C_i)$. Put $C := C_1 \cup C_2 \cup \cdots$; then $C \in [X]^{\leq\omega}$ and $\overline{\mathrm{sp}}\, C = V$. Since $V_1 \subset V_2 \subset \cdots$ for every $i \in \mathbb{N}$ we have $\overline{\mathrm{sp}}\, C_i = \overline{\mathrm{sp}}\,(C_1 \cup \cdots \cup C_i)$, and hence by (d) $Y_i = \overline{\mathrm{sp}}\, G(C_1 \cup \cdots \cup C_i)$. Then

$$Y = \overline{Y_1 \cup Y_2 \cup \cdots} = \overline{\mathrm{sp}}\,\big(G(C_1) \cup G(C_1 \cup C_2) \cup \cdots\big) \overset{(b)}{=} \overline{\mathrm{sp}}\, G(C_1 \cup C_2 \cup \cdots) = \overline{\mathrm{sp}}\, G(C)$$

Now, by (a), $V^* \subset \overline{Y|_V}\,$.

Further, we will verify that the assignment $Y \ni x^* \longmapsto x^*|_V \in V^*$ is a surjective isometry. As regards the isometric property, we recall that for every $i \in \mathbb{N}$ the rectangle $V_i \times Y_i$ belongs to $\mathcal{A}$, and so for every $x^* \in Y_i$ we have

$$\|x^*\| = \big\|x^*|_{V_i}\big\| \le \big\|x^*|_V\big\| \le \|x^*\|.$$

It then follows, using the density of $Y_1 \cup Y_2 \cup \cdots$ in Y, that $\|x^*\| = \big\|x^*|_V\big\|$ for every $x^* \in Y$. Now, once having the information just proved, we have that $\overline{Y|_V} = Y|_V$ $(\subset V^*)$, and hence $V^* = Y|_V$. Therefore, summarizing all the above, we are sure that our $\mathcal{A}$ is a rich family.

Finally, consider any $V_1 \times Y_1,\ V_2 \times Y_2$ in $\mathcal{A}$ such that $V_1 \subset V_2$. From the very definition of $\mathcal{A}$ we find $C_1, C_2 \in [X]^{\le\omega}$ such that $\overline{\mathrm{sp}}\, C_1 = V_1$ and $\overline{\mathrm{sp}}\, C_2 = V_2$. Then

$$C_2 \subset C_1 \cup C_2 \subset \overline{\mathrm{sp}}\, C_1 \cup \overline{\mathrm{sp}}\, C_2 = V_1 \cup V_2 = V_2 = \overline{\mathrm{sp}}\, C_2,$$

and so $\overline{\mathrm{sp}}\, C_2 \subset \overline{\mathrm{sp}}\, (C_1 \cup C_2) \subset \overline{\mathrm{sp}}\, C_2$. Now (d) in Definition 2.2 gives that $\overline{\mathrm{sp}}\, G(C_2) = \overline{\mathrm{sp}}\, G(C_1 \cup C_2)$, and so

$$Y_2 = \overline{\mathrm{sp}}\, G(C_1 \cup C_2) \overset{(b)}{=} \overline{\mathrm{sp}}\, \big(G(C_1) \cup G(C_1 \cup C_2)\big) \supset \overline{\mathrm{sp}}\, G(C_1) = Y_1.$$

We completely proved (iii).

(iii)$\Longrightarrow$(iv) is trivial.

(iv)$\Longrightarrow$(i). Assume (iv) holds. Let $Z \in \mathcal{S}(X)$ be arbitrary. From the cofinality of $\mathcal{A}$, find $V \times Y \in \mathcal{A}$ such that $V \times Y \supset Z \times \{0\}$. Then V^*, being the image of Y $(\in \mathcal{S}(X^*))$, is itself separable. It then follows that Z^*, the quotient of V^*, must be also separable. Now it remains to use the aforementioned characterization of the Asplund property, and thus (i) follows. $\square$

Remark 2.4. Assume that the norm $\|\cdot\|$ on X is Fréchet smooth and define $f := \|\cdot\|^2$. Then for every subspace $V \subset X$ we get that $V^* \subset \overline{f'(V)|_V}$ but not $V^* \subset \overline{f'(V)}\,|_V$. Indeed, this stronger inclusion seems to be a privilege of only some V's; we can find such subspaces by playing a suitable "volleyball" with countably many moves, see the proof of (ii)$\Rightarrow$(iii) above. (Fortunately, these "selected/better" V's form a rich family in $\mathcal{S}(X)$.) From this, and from the proof of implication (i)$\Rightarrow$(ii) above, it follows that using the Stegall's approach here is somehow stronger and simpler, see [CF15, Proposition 1]. Likewise, the whole Stegall's approach [St96] is stronger and simpler than that from [FG88], see [CF15, Remark 2].

It can be useful to extend Theorem 2.3 to the following statement.

Theorem 2.5. *Let $(Z, \|\cdot\|)$ be a Banach space, $(X, \|\cdot\|)$ an* `Asplund` *space, $T : Z \to X$ a bounded linear operator, and let $z^* \in Z^*$ be given. Then there exists a rich block-family $\mathcal{A}_T$ in $Z \times X \times X^*$ such that $Y_1 \subset Y_2$ whenever $U_1 \times V_1 \times Y_1,\ U_2 \times V_2 \times Y_2 \in \mathcal{A}_T$ and $V_1 \subset V_2$, and that for every $U \times V \times Y$ in $\mathcal{A}_T$ we have $T(U) \subset V$, the restriction assignment $Y \ni x^* \longmapsto x^*|_V \in V^*$ is a surjective isometry, and $\|T^*x^* - z^*\| = \big\|(T|_U)^*(x^*|_V) - (z^*|_U)\big\|$ for every $x^* \in Y$.*

Proof. It is easy (and left to a reader) to check that the rectangle-family $\mathcal{R}_T$ consisting of all $U \times V \in \mathcal{S}_\square(Z \times X)$ such that $T(U) \subset V$ is rich in $Z \times X$. Denote

$$\mathcal{R}_1 := \big\{U \times V \times Y :\ U \times V \in \mathcal{R}_T \ \text{ and } \ Y \in \mathcal{S}(X^*)\big\},$$

$$\mathcal{R}_2 := \big\{U \times V \times Y :\ U \in \mathcal{S}(Z) \ \text{ and } \ V \times Y \in \mathcal{A}\big\}$$

where $\mathcal{A}$ is from Theorem 2.3. Clearly, both these families are rich, and therefore $\mathcal{R} := \mathcal{R}_1 \cap \mathcal{R}_2$ is a rich block-family in $Z \times X \times X^*$. Clearly, every triple $U \times V \times Y$ in $\mathcal{R}$ possesses the first two properties from the conclusion of our theorem. Now, define the family

$$\mathcal{A}_T := \big\{U \times V \times Y \in \mathcal{R} :\ \|T^*x^* - z^*\| = \big\|(T|_U)^*(x^*|_V) - (z^*|_U)\big\| \ \text{ for every } \ x^* \in Y\big\}.$$

Clearly, $\mathcal{A}_T$ has all the three required properties. Thus, it remains to check that $\mathcal{A}_T$ is rich.

As regards the cofinality of $\mathcal{A}_T$, consider any $M \in \mathcal{S}(Z \times X \times X^*)$. From the cofinality of $\mathcal{R}$, find $U_0 \times V_0 \times Y_0$ in $\mathcal{R}$ such that $U_0 \times V_0 \times Y_0 \supset M$. We shall construct an increasing sequence

$U_m \times V_m \times Y_m,\ m \in \mathbb{N}$, in $\mathcal{R}$ as follows. Let $m \in \mathbb{N}$ and assume that we have already found $U_{m-1} \times V_{m-1} \times Y_{m-1}$. Using the separability of Y_{m-1} find $C_{m-1} \in [Z]^{\leq\omega}$ such that $\overline{C_{m-1}} \supset U_{m-1}$ and $\|T^*x^* - z^*\| = \sup\langle T^*x^* - z^*, C_{m-1} \cap B_Z\rangle$ for every $x^* \in Y_{m-1}$. Find $U_m \times V_m \times Y_m$ in $\mathcal{R}$ so big that it contains $(U_{m-1} \cup C_{m-1}) \times V_{m-1} \times Y_{m-1}$. Doing so for every $m \in \mathbb{N}$, put finally $U := \overline{\bigcup U_m}$, $V := \overline{\bigcup V_m}$, and $Y := \overline{\bigcup Y_m}$. Clearly, $U \times V \times Y = \overline{\bigcup U_m \times V_m \times Y_m} \supset M$. The σ-completeness of $\mathcal{R}$ guarantees that $U \times V \times Y$ lies in $\mathcal{R}$. Now fix any $m \in \mathbb{N}$ and any $x^* \in Y_{m-1}$. We can estimate

$$\begin{aligned} \|T^*x^* - z^*\| &= \sup\big\langle T^*x^* - z^*, C_{m-1} \cap B_Z\big\rangle \leq \sup\big\langle T^*x^* - z^*, B_U\big\rangle \\ &= \sup\big\{\big\langle x^*|_V, (T|_U)u\big\rangle - \langle z^*|_U, u\rangle :\ u \in B_U\big\} \\ &= \big\|(T|_U)^*(x^*|_V) - z^*|_U\big\| \leq \|T^*x^* - (z^*)\|\,. \end{aligned}$$

Thus $\|T^*x^* - z^*\| = \big\|(T|_U)^*(x^*|_V) - (z^*)|_U\big\|$ for every x^* from $\bigcup Y_m$, and finally, for every x^* from Y. We verified that $U \times V \times Y \in \mathcal{A}_T$, and hence $\mathcal{A}_T$ is cofinal.

As regards the σ-completeness of $\mathcal{A}_T$, consider any increasing sequence $U_1{\times}V_1{\times}Y_1,\ U_2{\times}V_2{\times}Y_2, \ldots$ in $\mathcal{A}_T$. Put $U := \overline{\bigcup U_i}$, $V := \overline{\bigcup V_i}$, and $Y := \overline{\bigcup Y_i}$. Clearly, $U \times V \times Y = \overline{\bigcup U_i \times V_i \times Y_i}$. As $\mathcal{R}$ was rich, our $U{\times}V{\times}Y$ belongs to it. Take any $i \in \mathbb{N}$ and any $x^* \in Y_i$. Since $U_i \times V_i \times Y_i \in \mathcal{A}_T$, we have that $\|T^*x^* - z^*\| = \big\|(T|_{U_i})^*(x^*|_{V_i}) - (z^*|_{U_i})\big\|$. But we can easily verify the following monotonicity

$$\big\|(T|_{U_i})^*(x^*|_{V_i}) - (z^*|_{U_i})\big\| \leq \big\|(T|_U)^*(x^*|_V) - (z^*|_U)\big\| \leq \|T^*x^* - z^*\|.$$

Thus $\|T^*x^* - z^*\| = \big\|(T|_U)^*(x^*|_V) - (z^*|_U)\big\|$ holds for every x^* from $\bigcup Y_i$, and hence for every x^* from Y. We proved that $U \times V \times Y$ belongs to $\mathcal{A}_T$, and therefore this family is σ-complete.

$\square$

Remark 2.6. Of course, Theorem 2.5 can be easily extended to several spaces $Z_1, \ldots, Z_k$, to $z_i^* \in Z_i^*$, and to operators $T_i : Z_i \to X,\ i = 1, \ldots, k$.

3. Separable reduction for statements with Fréchet subdifferentials

Separable reductions for Fréchet (sub)differentiability originated some three-four decades ago in works by D. Gregory [Ph93, pages 23, 24, 37], D. Preiss [Pr84], and M. Fabian, N. V. Zhivkov [FZ85]. In all these papers and in many subsequent ones until the recent contributions —see [P10], [I11], [FI13], [FI15]— there was a common belief that for a successful performing separable reductions of statements involving Fréchet subdifferentiability (like non-emptiness of subdifferential, fuzzy calculus, etc.), it is necessary to first translate such statements completely into terms of the Banach space X in question (with no use of its dual X^*). The present approach below destroys this longstanding taboo in the case when we restrict to the framework of Asplund spaces.[1] Indeed, once we have at hand the deeper structural characterization of the Asplund property (see Theorem 2.3), we can work with the original definition of Fréchet (sub)differentiability ("... there exists an element of `the dual` X^* such that ..."). This way, the separable reductions in Asplund spaces can be substantially simplified and obtained results become exact (see Theorem 3.1); for comparison see [FI15].

Let $(X, \|\cdot\|)$ be a Banach space, let $f : X \longrightarrow (-\infty, +\infty]$ be any proper function, i.e. $f \not\equiv +\infty$, and let $x \in X$ be a point where $f(x) < +\infty$. The *Fréchet subdifferential* $\partial_F f(x)$ of f at x is the (possibly empty) set consisting of all $x^* \in X^*$ such that $f(x+h) - f(x) - \langle x^*, h\rangle > -o(\|h\|)$ for all $0 \neq h \in X$ where $o : (0, +\infty) \longrightarrow [0, +\infty]$ is a suitable function with the property that $\frac{o(t)}{t} \to 0$ as $t \downarrow 0$; or in other words, if for every $\varepsilon > 0$ there is $\delta > 0$ such that $\frac{1}{\|h\|}\big(f(x+h) - f(x) - \langle x^*, h\rangle\big) > -\varepsilon$ whenever $h \in X$ and $0 < \|h\| < \delta$.

[1]Separable reduction of Fréchet differentiability realized without any translation into terms of X was presented recently in [LPT12, Sections 3.5 and 3.6].

Theorem 3.1. (Main) *Let $(X,\|\cdot\|)$ be a (rather non-separable)* `Asplund` *space and let $f : X \longrightarrow (-\infty,+\infty]$ be any proper function. Then there exists a rich rectangle-family $\mathcal{R} \subset \mathcal{S}_{\square}(X \times X^*)$ such that $Y_1 \subset Y_2$ whenever $V_1 \times Y_1,\ V_2 \times Y_2 \in \mathcal{R}$ and $V_1 \subset V_2$, with further properties that for every $V \times Y \in \mathcal{R}$ the assignment $Y \ni x^* \longmapsto x^*|_V \in V^*$ is an* `isometry` *from $Y|_V$ onto V^* and for every $v \in V$ we have that*

$$\big(\partial_F f(v) \cap Y\big)|_V = \big(\partial_F f(v)\big)|_V = \partial_F (f|_V)(v);$$

that is, in more detail, if $v^* \in \partial_F(f|_V)(v)$, there exists a `unique` *$x^* \in \partial_F f(v) \cap Y$* such that $x^*|_V = v^*$ and $\|x^*\| = \|v^*\|$.

Remark 3.2. It is worth to compare the theorem above with what was proved in [FZ85, F89]: *In a general (possibly non-Asplund) Banach space X, there is a cofinal family $\mathcal{C}$ in $\mathcal{S}(X)$ such that for every $V \in \mathcal{C}$ and for every $v \in V$ the non-emptiness of $\partial_F(f|_V)(v)$ implies the non-emptiness of $\partial_F f(v)$.*

Proof. We obviously have that

$$\big(\partial_F f(v) \cap Y\big)|_V \subset \big(\partial_F f(v)\big)|_V \subset \partial_F(f|_V)(v) \tag{3.1}$$

for every $V \times Y \in \mathcal{S}_{\square}(X \times X^*)$ and every $v \in V$.

Further for $x \in X,\ x^* \in X^*,\ r \in \mathbb{R},\ 0 < \delta_1 < \delta_2$, and $V \subset X$ we define

$$I_V(x,x^*,r,\delta_1,\delta_2) := \inf\left\{ \frac{1}{\|h\|}\big(f(x+h) - r - \langle x^*, h\rangle\big) :\ h \in V \ \text{ and } \ \delta_1 < \|h\| < \delta_2 \right\}.$$

(A novelty here is that we operate also with x^*, an element of the dual X^*, which was "forbidden" for three decades, and that $f(x)$ is replaced by $r \in \mathbb{R}$.) Further for each "cortege" $x, x^*, r, \delta_1, \delta_2$ as above and for each $\gamma > 0$ we find a vector $v(x,x^*,r,\delta_1,\delta_2,\gamma) \in X$ such that $\delta_1 < \|v(x,x^*,r,\delta_1,\delta_2,\gamma)\| < \delta_2$ and

$$\frac{1}{\|v(x,x^*,r,\delta_1,\delta_2,\gamma)\|}\big(f(x+v(x,x^*,r,\delta_1,\delta_2,\gamma)) - r - \langle x^*, v(x,x^*,r,\delta_1,\delta_2,\gamma)\rangle\big) < I_X(x,x^*,r,\delta_1,\delta_2) + \gamma \tag{3.2}$$

if $I_X(x,x^*,r,\delta_1,\delta_2) > -\infty$, and

$$\frac{1}{\|v(x,x^*,r,\delta_1,\delta_2,\gamma)\|}\big(f(x+v(x,x^*,r,\delta_1,\delta_2,\gamma)) - r - \langle x^*, v(x,x^*,r,\delta_1,\delta_2,\gamma)\rangle\big) < -\frac{1}{\gamma} \tag{3.3}$$

if $I_X(x,x^*,r,\delta_1,\delta_2) = -\infty$.

Let $\mathcal{A} \subset \mathcal{S}_{\square}(X \times X^*)$ be the rich family found in Theorem 2.3. We define the family $\mathcal{R}$ as that consisting of all $V \times Y \in \mathcal{A}$ satisfying

$$I_X(x,x^*,r,\delta_1,\delta_2) = I_V(x,x^*,r,\delta_1,\delta_2) \ \text{ whenever } \ x \in V,\ x^* \in Y,\ r \in \mathbb{R}, \ \text{ and } \ 0 < \delta_1 < \delta_2. \tag{3.4}$$

We shall prove that $\mathcal{R}$ is cofinal in $\mathcal{S}(X \times X^*)$. Fix any $Z \in \mathcal{S}(X \times X^*)$. Since $\mathcal{A}$ is rich, there is $V_0 \times Y_0 \in \mathcal{A}$ such that $V_0 \times Y_0 \supset Z$. Find countable sets C_0, D_0 contained and dense in V_0 and Y_0, respectively. We shall construct increasing sequences $Y_0 \times V_0,\ V_1 \times Y_1,\ V_2 \times Y_2,\ \ldots$ in $\mathcal{A}$, and $C_0 \times D_0,\ C_1 \times D_1,\ C_2 \times D_2,\ \ldots$ in $[X \times X^*]^{\leq\omega}$ such that $\overline{C_i} = V_i,\ \overline{D_i} = Y_i$ for every $i \in \mathbb{N}$, and having some extra properties described below. Let $m \in \mathbb{N}$ be arbitrary and assume that we have already found $V_{m-1}, Y_{m-1}, C_{m-1}, D_{m-1}$. From the cofinality of $\mathcal{A}$ we find $V_m \times Y_m \in \mathcal{A}$ such that V_m contains the (countable) set

$$\widetilde{C} := C_{m-1} \cup \big\{ v(x,x^*,q,\delta_1,\delta_2,\gamma) :\ x \in C_{m-1},\ x^* \in D_{m-1},\ q \in \mathbb{Q},\ \delta_1,\delta_2,\gamma \in \mathbb{Q}_+, \text{ and } \delta_1 < \delta_2 \big\}$$

and Y_m contains Y_{m-1}. Find then a countable set $\widetilde{C} \subset C_m \subset V_m$ such that $\overline{C_m} = V_m$ and a countable set $D_{m-1} \subset D_m \subset Y_m$ such that $\overline{D_m} = Y_m$. Do so subsequently for every $m \in \mathbb{N}$. Put $V := \overline{V_0 \cup V_1 \cup V_2 \cup \cdots}$ and $Y := \overline{Y_0 \cup Y_1 \cup Y_2 \cup \cdots}$. The σ-completeness of $\mathcal{A}$ guarantees that $V \times Y$ belongs to $\mathcal{A}$.

We shall show that $V\times Y\in\mathcal{R}$. This means that we have to verify (3.4). So, fix any cortege $x,x^*,r,\delta_1,\delta_2$ as there. Consider any fixed $h\in X$ such that $\delta_1<\|h\|<\delta_2$. We have to show that $\frac{1}{\|h\|}\big(f(x+h)-r-\langle x^*,h\rangle\big)\geq I_V(x,x^*,r,\delta_1,\delta_2)$. Pick some $\delta_1',\delta_2'\in\mathbb{Q}$ such that $\delta_1<\delta_1'<\|h\|<\delta_2'<\delta_2$. It is easy to check that $V=\overline{C_0\cup C_1\cup\cdots}$ and $Y=\overline{D_0\cup D_1\cup\cdots}$. Find $x_0\in C_0,\ x_1\in C_1,\ \dots$ and $x_0^*\in D_0,\ x_1^*\in D_1,\ \dots$ such that $\|x_i-x\|\longrightarrow 0$ and $\|x_i^*-x^*\|\longrightarrow 0$ as $i\to\infty$. Consider any fixed $\gamma\in\mathbb{Q}_+$. Pick $q\in\mathbb{Q}$ such that $|r-q|<\gamma\|h\|$. Denote $N_1:=\big\{i\in\mathbb{N}:\ \|x_i-x\|<\min\{\delta_1'-\delta_1,\delta_2-\delta_2'\}\big\}$; this is a co-finite set in $\mathbb{N}$.

Now, take any $k\in V$, with $\delta_1'<\|k\|<\delta_2'$. For $i\in N_1$ we have $\delta_1<\|x_i-x+k\|<\delta_2$ and then we can estimate

$$\begin{aligned}
&\frac{1}{\|k\|}\big(f(x_i+k)-q-\langle x_i^*,k\rangle\big)\\
&=\frac{\|k+x_i-x\|}{\|k\|}\cdot\frac{1}{\|k+x_i-x\|}\big(f(x+(x_i-x+k))-r-\langle x^*,x_i-x+k\rangle\big)\\
&\quad+\frac{1}{\|k\|}\big(\langle x^*,x_i-x+k\rangle-\langle x_i^*,k\rangle\big)+\frac{r-q}{\|k\|}\\
&\geq\Big(1+s_i\frac{\|x_i-x\|}{\|k\|}\Big)I_V(x,x^*,r,\delta_1,\delta_2)-\frac{1}{\delta_1}\big(\|x^*\|\|x_i-x\|+\delta_2\|x^*-x_i^*\|\big)-\gamma\frac{\delta_2}{\delta_1'}
\end{aligned}\tag{3.5}$$

where $s_i=1$ if $I_V(x,x^*,r,\delta_1,\delta_2)\leq 0$ and $s_i=-1$ otherwise. It then follows that

$$\begin{aligned}
&I_V(x_i,x_i^*,q,\delta_1',\delta_2')\\
&\geq\Big(1+s_i\frac{\|x_i-x\|}{\delta_1}\Big)I_V(x,x^*,r,\delta_1,\delta_2)-\frac{1}{\delta_1}\big(\|x^*\|\|x_i-x\|+\delta_2\|x^*-x_i^*\|\big)-\gamma\frac{\delta_2}{\delta_1'}
\end{aligned}\tag{3.6}$$

holds for every $i\in N_1$.

Now, put

$$N_2:=\big\{i\in N_1:\ \delta_1'<\|h+x-x_i\|<\delta_2' \text{ and } \langle x_i^*,x-x_i\rangle+\langle x_i^*-x^*,h\rangle>-\|h\|\gamma\big\}$$

—this is still a co-finite set in $\mathbb{N}$— and then put $N_3:=\big\{i\in N_2:\ I_X(x_i,x_i^*,q,\delta_1',\delta_2')=-\infty\big\}$.

First, assume that N_3 is finite. Using (3.2), for every $i\in N_2\setminus N_3$ we can estimate

$$\begin{aligned}
&\frac{1}{\|h\|}\big(f(x+h)-r-\langle x^*,h\rangle\big)\\
=&\frac{\|x-x_i+h\|}{\|h\|}\cdot\frac{1}{\|x-x_i+h\|}\big(f(x_i+(x-x_i+h))-q-\langle x_i^*,x-x_i+h\rangle\big)\\
&+\frac{1}{\|h\|}\big(\langle x_i^*,x-x_i\rangle+\langle x_i^*-x^*,h\rangle\big)+\frac{q-r}{\|h\|}\\
>&\frac{\|x-x_i+h\|}{\|h\|}I_X(x_i,x_i^*,q,\delta_1',\delta_2')-\gamma-\gamma\\
>&\frac{\|x-x_i+h\|}{\|h\|}\Big[\frac{1}{\|v(x_i,x_i^*,q,\delta_1',\delta_2',\gamma)\|}\big(f\big(x_i+v(-)\big)-q-\big\langle x_i^*,v(-)\big\rangle\big)-\gamma\Big]-2\gamma\\
\geq&\frac{\|x-x_i+h\|}{\|h\|}\big[I_V\big(x_i,x_i^*,q,\delta_1',\delta_2',\big)-\gamma\big]-2\gamma;
\end{aligned}\tag{3.7}$$

here $v(-)$ meant $v(x_i,x_i^*,q,\delta_1',\delta_2',\gamma)$ $(\in V)$. Now, plugging here (3.6), and then letting $N_2\setminus N_3\ni i\to\infty$, we get that

$$\frac{1}{\|h\|}\big(f(x+h)-r-\langle x^*,h\rangle\big)\geq I_V(x,x^*,r,\delta_1,\delta_2)-3\gamma-\gamma\frac{\delta_2}{\delta_1'}\,.$$

And, realizing that $\gamma \in \mathbb{Q}_+$ could be arbitrarily small, we get that $\frac{1}{\|h\|}\big(f(x+h) - r - \langle x^*, h\rangle\big) \geq I_V(x, x^*, r, \delta_1, \delta_2)$. This, of course, implies that $I_X(x, x^*, r, \delta_1, \delta_2) \geq I_V(x, x^*, r, \delta_1, \delta_2)$.

Second, assume that the set N_3 is infinite. Take any $\gamma' \in \mathbb{Q}_+$. For $i \in N_3$ we have from (3.3) that

$$\begin{aligned}
-\frac{1}{\gamma'} &> \frac{1}{\|v(x_i, x_i^*, q, \delta_1', \delta_2', \gamma')\|}\big(f(x_i + v(x_i, x_i^*, q, \delta_1', \delta_2', \gamma')) - q - \langle x_i^*, v(x_i, x_i^*, q, \delta_1', \delta_2', \gamma')\rangle\big) \\
&= \frac{\|x_i - x + v(-)\|}{\|v(-)\|} \cdot \frac{1}{\|x_i - x + v(-)\|}\big(f(x + (x_i - x + v(-))) - r - \langle x^*, x_i - x + v(-)\rangle\big) \\
&\quad + \frac{r-q}{\|v(-)\|} + \frac{1}{\|v(-)\|}\big(\langle x^*, x_i - x\rangle + \langle x^* - x_i^*, v(-)\rangle\big) \\
&\geq \frac{\|x_i - x + v(-)\|}{\|v(-)\|} I_V(x, x^*, r, \delta_1, \delta_2) - \frac{|r-q|}{\delta_1'} - \frac{1}{\delta_1'}\big|\langle x^*, x_i - x\rangle + \langle x^* - x_i^*, v(-)\rangle\big|;
\end{aligned}$$

here $v(-)$ always meant $v(x_i, x_i^*, q, \delta_1', \delta_2', \gamma')$ $(\in V)$. And letting $N_3 \ni i \to \infty$, we get that $-\frac{1}{\gamma'} \geq I_V(x, x^*, r, \delta_1, \delta_2) - \frac{|r-q|}{\delta_1}$. Here $\gamma' \in \mathbb{Q}_+$ could be arbitrarily small. Therefore $I_V(x, x^*, r, \delta_1, \delta_2) = -\infty$, and hence for sure $I_X(x, x^*, r, \delta_1, \delta_2) \geq I_V(x, x^*, r, \delta_1, \delta_2)$. Therefore $V \times Y \in \mathcal{R}$.

The proof of σ-completeness of $\mathcal{R}$ is similar to (but a bit different from) the proof of cofinality. Let $V_1, \times Y_1,\ V_2 \times Y_2,\ \ldots$ be an increasing sequence of elements in our $\mathcal{R}$. We have to verify that $\overline{V_1 \times Y_1 \cup V_2 \times Y_2 \cup \cdots}$ also belongs to $\mathcal{R}$. Clearly, this set is of form $V \times Y$. As $\mathcal{A}$ is σ-complete, $V \times Y \in \mathcal{A}$. It remains to verify (3.4). So, fix any cortege $x, x^*, r, \delta_1, \delta_2$ as there. Consider any $h \in X$ such that $\delta_1 < \|h\| < \delta_2$. We have to show that $\frac{1}{\|h\|}(f(x+h) - r - \langle x^*, h\rangle) \geq I_V(x, x^*, r, \delta_1, \delta_2)$. Pick some $\delta_1', \delta_2' \in \mathbb{R}$ such that $\delta_1 < \delta_1' < \|h\| < \delta_2' < \delta_2$. It is easy to check that $V = \overline{V_1 \cup V_2 \cup \cdots}$ and $Y = \overline{Y_1 \cup Y_2 \cup \cdots}$. Find $x_1 \in V_1,\ x_2 \in V_2,\ \ldots$ and $x_1^* \in Y_1,\ x_2^* \in Y_2,\ \ldots$ such that $\|x_i - x\| \longrightarrow 0$ and $\|x_i^* - x^*\| \longrightarrow 0$ as $i \to \infty$. Denote $M_1 := \big\{i \in \mathbb{N} :\ \|x_i - x\| < \min\{\delta_1' - \delta_1, \delta_2 - \delta_2'\}\big\}$; this is a co-finite set in $\mathbb{N}$.

Now, take any $k \in V$, with $\delta_1' < \|k\| < \delta_2'$. For $i \in M_1$ we have $\delta_1 < \|x_i - x + k\| < \delta_2$ and then we can estimate (This chain is a bit simpler than (3.5) since we now do not need r replaced by $q \in \mathbb{Q}$.)

$$\begin{aligned}
& \frac{1}{\|k\|}\big(f(x_i + k) - r - \langle x_i^*, k\rangle\big) \\
= \quad & \frac{\|k + x_i - x\|}{\|k\|} \cdot \frac{1}{\|k + x_i - x\|}\big(f(x + (x_i - x + k)) - r - \langle x^*, x_i - x + k\rangle\big) \\
& + \frac{1}{\|k\|}\big(\langle x^*, x_i - x + k\rangle - \langle x_i^*, k\rangle\big) \\
\geq \quad & \Big(1 + s_i \frac{\|x_i - x\|}{\|k\|}\Big) I_V(x, x^*, r, \delta_1, \delta_2) - \frac{1}{\delta_1}\big(\|x^*\|\|x_i - x\| + \delta_2\|x^* - x_i^*\|\big)
\end{aligned}$$

where $s_i = 1$ if $I_V(x, x^*, r, \delta_1, \delta_2) \leq 0$ and $s_i = -1$ otherwise. It then follows that (This is a bit simpler than (3.6).)

$$\begin{aligned}
& I_V(x_i, x_i^*, r, \delta_1', \delta_2') \\
& \geq \Big(1 + s_i \frac{\|x_i - x\|}{\delta_1}\Big) I_V(x, x^*, r, \delta_1, \delta_2) - \frac{1}{\delta_1}\big(\|x^*\|\|x_i - x\| + \delta_2\|x^* - x_i^*\|\big)
\end{aligned} \tag{3.8}$$

holds for every $i \in M_1$.

Now, consider any $\gamma > 0$ and put (This M_2 is defined exactly as N_2 above.)

$$M_2 := \big\{i \in M_1 :\ \delta_1' < \|h + x - x_i\| < \delta_2' \text{ and } \langle x_i^*, x - x_i\rangle + \langle x_i^* - x^*, h\rangle > -\gamma\|h\|\big\};$$

this is still a co-finite set in $\mathbb{N}$. Using (3.8), for every $i \in M_2$ we can estimate (The following chain is different from (3.7).)

$$\begin{aligned}
& \frac{1}{\|h\|}\big(f(x+h) - r - \langle x^*, h\rangle\big) \\
= \ & \frac{\|x-x_i+h\|}{\|h\|} \cdot \frac{1}{\|x-x_i+h\|}\big(f(x_i + (x-x_i+h)) - r - \langle x_i^*, x-x_i+h\rangle\big) \\
& + \frac{1}{\|h\|}\big(\langle x_i^*, x-x_i\rangle + \langle x_i^* - x^*, h\rangle\big) \\
\geq \ & \frac{\|x-x_i+h\|}{\|h\|} I_X(x_i, x_i^*, r, \delta_1', \delta_2') - \gamma \\
= \ & \frac{\|x-x_i+h\|}{\|h\|} I_{V_i}(x_i, x_i^*, r, \delta_1', \delta_2') - \gamma \quad \text{(as } (x_i, x_i^*) \in V_i \times Y_i \in \mathcal{R} \text{ and (3.4) holds)} \\
\geq \ & \frac{\|x-x_i+h\|}{\|h\|} I_V(x_i, x_i^*, r, \delta_1', \delta_2') - \gamma.
\end{aligned}$$

Now, plugging here (3.8), and then letting $M_2 \ni i \to \infty$, we get that

$$\frac{1}{\|h\|}\big(f(x+h) - r - \langle x^*, h\rangle\big) \geq I_V(x, x^*, r, \delta_1, \delta_2) - \gamma\,,$$

Finally, realizing that $\gamma > 0$ could be arbitrarily small, we get that $\frac{1}{\|h\|}\big(f(x+h) - r - \langle x^*, h\rangle\big) \geq I_V(x, x^*, r, \delta_1, \delta_2)$. This, of course, implies that $I_X(x, x^*, r, \delta_1, \delta_2) \geq I_V(x, x^*, r, \delta_1, \delta_2)$, and so $V \times Y \in \mathcal{R}$. We proved that $\mathcal{R}$ is σ-complete, and therefore $\mathcal{R}$ is a rich rectangle-family in $X \times X^*$.

That $Y_1 \subset Y_2$ whenever $V_1 \times Y_1,\ V_2 \times Y_2 \in \mathcal{R}$ and $V_1 \subset V_2$, follows immediately from the same property shared by $\mathcal{A}$.

It remains to prove that our $\mathcal{R}$ "works", that is , that $\partial_F(f|_V)(v) \subset \big(\partial_F f(v) \cap Y\big)|_V$ whenever $V \times Y \in \mathcal{R}$ and $v \in V$. So, pick any such $V \times Y$. We know from Theorem 2.3 that $Y \ni x^* \longmapsto x^*|_V \in V^*$ is an isometry onto. Fix any $v \in V$. Assume there is v^* in $\partial_F(f|_V)(v)$. Find (a unique) $x^* \in Y$ such that $x^*|_V = v^*$. We shall show that $x^* \in \partial_F f(v)$. So, fix any $\varepsilon > 0$. Find $\delta > 0$ such that $f(v+k) - f(v) - \langle v^*, k\rangle > -\varepsilon\|k\|$ whenever $k \in V$ and $0 < \|k\| < \delta$; then $I_V(v, x^*, f(v), \delta_1, \delta) \geq -\varepsilon$ for every $\delta_1 \in (0, \delta)$. Now, let $h \in X$ be any vector such that $0 < \|h\| < \delta$. Pick $\delta_1 \in (0, \|h\|)$. Then we have

$$\frac{1}{\|h\|}\big(f(v+h) - f(v) - \langle x^*, h\rangle\big) \geq I_X(v, x^*, f(v), \delta_1, \delta) = I_V(v, v^*, f(v), \delta_1, \delta) \geq -\varepsilon$$

by (3.4). We proved that x^* belongs to $\partial f(v) \cap Y$, and so v^* belongs to $\big(\partial_F f(v) \cap Y\big)|_V$. Therefore $\partial_F(f|_V)(v) \subset \big(\partial_F f(v) \cap Y\big)|_V$. This together with (3.1) completes the proof. □

Corollary 3.3. *Let $(X, \|\cdot\|)$ be a (rather non-separable) Asplund space and let $f : X \longrightarrow (-\infty, +\infty]$ be any proper function. Then there exists a rich family $\mathcal{R} \subset \mathcal{S}(X)$ such that for every $V \in \mathcal{R}$ and for every $v \in V$ we have:*
(i) $\partial_F f(v) \neq \emptyset$ if (and only if) $\partial_F(f|_V)(v) \neq \emptyset$ (cf. [F89]*).*
(ii) For $t \geq 0$ we have $\partial_F f(v) \setminus tB_{X^*} \neq \emptyset$ if (and only if) $\partial_F(f|_V)(v) \setminus tB_{V^*} \neq \emptyset$ (cf. [FM02]*).*
(iii) f is Fréchet differentiable at v if (and only if) $f|_V$ is Fréchet differentiable at v; and in this case $\|f'(v)\| = \big\|(f|_V)'(v)\big\|$ (cf. [Pr84, Z12]*).*

Proof. Let $\mathcal{R}_1$ and $\mathcal{R}_2$ be rich rectangle-families found in Theorem 3.1 for the functions f and $-f$, respectively. Let $\mathcal{R}$ be the "projection of $\mathcal{R}_1 \cap \mathcal{R}_2$ on the first coordinate", that is, put

$$\mathcal{R} := \big\{V \in \mathcal{S}(X) :\ V \times Y \in \mathcal{R}_1 \cap \mathcal{R}_2 \ \text{ for some } \mathrm{Y} \in \mathcal{S}(\mathrm{X}^*)\big\}.$$

It is easy check that $\mathcal{R}$ is rich. It works. Indeed, (i) and (ii) follow immediately from Theorem 3.1. As regards (iii), take any $V \in \mathcal{R}$ and any $v \in V$. Find $Y \in \mathcal{S}(X^*)$ so that $V \times Y$ is in $\mathcal{R}_1 \cap \mathcal{R}_2$. Then (i) and (ii) immediately follow from Theorem 3.1. Further, assume that $f|_V$ is Fréchet differentiable at v and put $v^* := (f|_V)'(v)$. This implies that $v^* \in \partial_F(f|_V)(v)$ and $-v^* \in \partial_F((-f)|_V)(v)$. Find the unique $x^* \in Y$ such that $x^*|_V = v^*$ and $\|x^*\| = \|v^*\|$; then $(-x^*)|_V = -v^*$. Now, by Theorem 3.1, $x^* \in \partial_F f(v)$ and $-x^* \in \partial_F(-f)(v)$. It then follows that f is Fréchet differentiable at v, with $f'(v) = x^*$ and $\|f'(v)\| = \|x^*\| = \|v^*\| = \|(f|_V)'(v)\|$. □

Corollary 3.4. *Let $(X, \|\cdot\|)$ be an Asplund space, let $f : X \longrightarrow (-\infty, +\infty]$ be a lower semicontinuous function, and $g : X \longrightarrow (-\infty, +\infty]$ be a function uniformly continuous in a vicinity of a certain $\overline{x} \in X$. Then:*
*(i) The set $\{x \in X : \ \partial_F f(x) \neq \emptyset\}$ is dense in the domain of f (*see [F89]*).*
*(ii) If $x^* \in \partial_F(f+g)(\overline{x})$, then for every $\varepsilon > 0$ there are $x_1, x_2 \in X$, $x_1^* \in \partial_F f(x_1)$, and $x_2^* \in \partial_F g(x_2)$ such that $\|x_1 - \overline{x}\| < \varepsilon$, $\|x_2 - \overline{x}\| < \varepsilon$, and $\|x_1^* + x_2^* - x^*\| < \varepsilon$ (*see [F89] modulo [I83]*).*

Proof. Assume first that X is separable. Find an equivalent Fréchet smooth norm, see e.g. [DGZ93, pages 48, 43] arbitrarily close to $\|\cdot\|$. Then (i) can be easily obtained using Borwein-Preiss or Deville-Godefroy-Zizler smooth variational principles [Ph93, Section 4]. As regards (ii), proceed as in [I83], using the same smooth principles.

Second, assume that X is non-separable. As regards (i), combine the just proved separable statement with Corollary 3.3 (i). To prove (ii), assume that $x^* \in \partial_F(f+g)(\overline{x})$ and let $\varepsilon > 0$ be given. By Theorem 3.1, find rich families $\mathcal{R}_1, \ \mathcal{R}_2$ corresponding to f, g, respectively, and put $\mathcal{R} := \mathcal{R}_1 \cap \mathcal{R}_2$. Find $V \times Y \in \mathcal{R}$ so that it contains $(\overline{x}, x^*)$. Using the validity of the separable statement, find $x_1, x_2 \in V$, $v_1^* \in \partial_F(f|_V)(x_1)$, and $v_2^* \in \partial_F(g|_V)(x_2)$ such that $\|x_1 - \overline{x}\| < \varepsilon$, $\|x_2 - \overline{x}\| < \varepsilon$, and $\|v_1^* + v_2^* - x^*|_V\| < \varepsilon$. Now, the conclusion of Theorem 3.1 provides unique $x_1^* \in \partial_F f(x_1) \cap Y$ and $x_2^* \in \partial_F g(x_2) \cap Y$ such that $x_i^*|_V = v_i^*$, $i = 1, 2$. Hence, using the isometric property of the restriction mapping $Y \ni \xi \longmapsto \xi|_V$, we conclude that $\|x_1^* + x_2^* - x^*\| = \|v_1^* + v_2^* - x^*|_V\| < \varepsilon$. □

Let $(X, \|\cdot\|)$ be a Banach space, let $\Omega \subset X$, and let $\overline{x} \in \Omega$. The *Fréchet normal cone* $N_F(\overline{x}, \Omega)$ of Ω at $\overline{x}$ is defined as the Fréchet subdifferential of the indicator function ι_Ω of Ω at $\overline{x}$; note that $N_F(\overline{x}, \Omega)$ always contains 0. Let Ω_1, Ω_2 be two subsets of X, with non-empty intersection, and fix an $\overline{x}$ in it. If there are $\varepsilon > 0$ and sequences (a_n^1), (a_n^2) in X satisfying that $(a_n^1 + \Omega_1) \cap (a_n^2 + \Omega_2) \cap (\overline{x} + \varepsilon B_X) = \emptyset$ for every $n \in \mathbb{N}$, then we say that $\overline{x}$ is a *local extremal point* of $\Omega_1 \cap \Omega_2$ and the triple $\{\Omega_1, \Omega_2, \overline{x}\}$ is called an *extremal system* in X, see [M06, page 172].

Corollary 3.5. *Let $(X, \|\cdot\|)$ be an Asplund space. Then:*
(i) For every nonempty closed set $\Omega \subset X$ the set $\{x \in X : \ N_F(x, \Omega) \neq \{0\}\}$ is dense in the boundary of Ω (see [FM99] *).*
(ii) For every extremal system $(\Omega_1, \Omega_2, \overline{x})$ in X the "Fréchet" extremal principle holds, that is, for every $\varepsilon > 0$ there are $x_1, x_2 \in X$ such that $\|x_1 - \overline{x}\| < \varepsilon$, $\|x_2 - \overline{x}\| < \varepsilon$ and there are $x_i^* \in N_F(x_i, \Omega_i) + \varepsilon B_{X^*}$, $i = 1, 2$, such that $\|x_1^*\| + \|x_2^*\| = 1$, and $x_1^* + x_2^* = 0$ (see [MS96]*).*

The proof is very similar to that of Corollary 3.4, once we have at hand the "separable" statements [MS96].

Remark 3.6. Of course, (ii) in Corollary 3.5 is usually derived from (ii) in Corollary 3.4 with help of Ekeland's principle. We thank the referee for this comment. It should be also noted that (ii) in Corollary 3.4 and 3.5 were obtained in [FM02] via a much more complicated separable reduction.

We finish by deriving easily a strengthening of the main result of the paper [FI15] from Theorems 2.5 and 3.1 in the framework of Asplund spaces.

Theorem 3.7. *Let $k \in \mathbb{N}$, let X be a non-separable* `Asplund` *space, let $Z_1, \ldots, Z_k$ be Banach spaces, let $z_1^* \in Z_1^*, \ \ldots, z_k^* \in Z_k^*$, let $T_i : Z_i \to X$, $i = 1, \ldots, k$, be bounded linear operators,*

and let $f : X \longrightarrow (-\infty, +\infty]$ be a proper function. Then there exists a rich block-family $\mathcal{R}$ in $Z_1 \times \cdots \times Z_k \times X$ such that, for every $U_1 \times \cdots \times U_k \times V \in \mathcal{R}$ we have $T_1(U_1) \subset V, \ \ldots, \ T_k(U_k) \subset V$ and there is $Y \in \mathcal{S}(X^*)$ such that:
(i) The assignment $Y \ni x^* \longmapsto x^*|_V \in V^*$ is an `isometry onto` *V^*;*
(ii) $\big(\partial_F f(v) \cap Y\big)|_V = \big(\partial_F f(v)\big)|_V = \partial_F (f|_V)(v)$ for every $v \in V$; and
(iii) $\big\|T_i^* x^* - z_i^*\big\| = \big\|\big(T_i|_{U_i}\big)^*(x^*|_V) - (z_i^*|_{U_i})\big\|$ for every $x^* \in Y$ and $i = 1, \ldots, k$

Proof. Putting together Theorem 2.5 and Remark 2.6, we find a rich block-family $\mathcal{A}_{T_1,\ldots,T_k}$ in $Z_1 \times \cdots \times Z_k \times X \times X^*$ with similar properties as the family $\mathcal{A}_T$ in Theorem 2.5 has. Let $\mathcal{R}'$ be the rich family in $X \times X^*$ found in Theorem 3.1. Define

$$\mathcal{R} := \big\{U_1 \times \cdots \times U_k \times V : \ U_1 \times \cdots \times U_k \times V \times Y \in \mathcal{A}_{T_1,\ldots,T_k} \text{ and } V \times Y \in \mathcal{R}' \text{ for some } Y \in \mathcal{S}(X^*)\big\}.$$

Clearly, $\mathcal{R}$ is cofinal. And from the "monotonicity" property of $\mathcal{A}_{T_1,\ldots,T_k}$ we easily get that $\mathcal{R}$ is σ-complete. That (i) and (iii) are true follows directly from Theorem 2.5. (ii) comes immediately from Theorem 3.1. $\square$

Remark 3.8. From the theorem above we immediately get a strengthening of [FI15, Corollary 4.5] and afterwards [FI15, Theorem 4.6], provided we are in the framework of Asplund spaces.

Remark 3.9. There are at least two methods of proving separable reduction theorems. One is "the method via rich families" presented above. An alternative to this is a set-theoretical approach called "the method of suitable models". On one hand, proofs using the latter method, require some knowledge of set theory or logic, on the other hand it seems that this approach is more powerful and less technical. All the statements from this paper can be formulated and proved in the language of "suitable models". The proofs would be shorter; however, for a reader not familiar with set theory or logic, less readable. This is why we have chosen to present the proofs using the first mentioned method. We refer a reader interested in the second mentioned method to the paper [C12] where basics of this alternative way is explained, to [CK14] where the relation between the both methods is investigated, and to a forthcoming paper [C16] where it will be proved (among other things) that in Asplund spaces the both methods are in some sense equivalent.

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