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A rigorous justification of the Euler and Navier-Stokes equations with geometric effects

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Abstract

We derive the 1D isentropic Euler and Navier-Stokes equations describing the motion of a gas through a nozzle of variable cross section as the asymptotic limit of the 3D isentropic Navier-Stokes system in a cylinder, the diameter of which tends to zero. Our method is based on the relative energy inequality satisfied by any weak solution of the 3D Navier-Stokes system and a variant of Korn-Poincaré's inequality on thin channels that may be of independent interest.

Keywords: Isentropic Navier-Stokes system, isentropic Euler system, inviscid limit, Korn inequality, Poincaré inequality

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1 Introduction

A simple model of the flow of a compressible gas through a nozzle of variable cross section describes the evolution of the mass density $\varrho_E = \varrho_E(t,z)$ and the velocity $u_E = u_E(t,z)$ by means of the Euler system:

$$\partial_t(\varrho_E A) + \partial_z(\varrho_E u_E A) = 0, \tag{1.1}$$

$$\partial_t(\varrho_E u_E A) + \partial_z(\varrho_E u_E^2 A) + A\partial_z p(\varrho_E) = 0, \tag{1.2}$$

where $p = p(\varrho_E)$ is the pressure and $A = A(z)$ is the 2D measure of the cross section at the "vertical" position z, see e.g., LeFloch and Westdickenberg [10]. We also consider a similar model including the effect of viscosity with an additional drift term, namely:

$$\partial_t(\varrho_{NS} A) + \partial_z(\varrho_{NS} u_{NS} A) = 0, \tag{1.3}$$

$$\partial_t(\varrho_{NS} u_{NS} A) + \partial_z(\varrho_{NS} u_{NS}^2 A) + A\partial_z p(\varrho_{NS}) = A\Big(\frac{4\mu}{3} + \eta\Big)\partial_z^2 u_{NS} + A\left(\frac{\mu}{3} + \eta\right)\partial_z\Big(\frac{\partial_z A}{A} u_{NS}\Big). \tag{1.4}$$

The purpose of this paper is to show that (smooth) solutions of the above problems can be identified as the asymptotic limits of the 3D Navier-Stokes system:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.5}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \lambda \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.6}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu\left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}\right) + \eta \mathrm{div}_x \mathbf{u}\mathbb{I}, \qquad \mu > 0,\ \eta \geq 0, \tag{1.7}$$

considered in the physical domain:

$$\Omega_\varepsilon = \left\{ x = (x_1, x_2, z) \equiv (\mathbf{x}_h, z) \ \Big|\ z \in (0,1),\ \mathbf{x}_h \in \varepsilon\omega_h(z) \right\}, \tag{1.8}$$

under the slip boundary conditions:

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \qquad [\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}] \times \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \tag{1.9}$$

provided that $\varepsilon \to 0$. Here, $\{\omega_h(z)\}_{z\in[0,1]}$ is a family of sufficiently smooth open bounded simply connected subsets of $\mathbb{R}^2$, with fairly arbitrary geometry (see Section 2 for details), where we define:

$$A(z) := |\omega_h(z)|.$$

Our approach is based on the concept of *dissipative* weak solutions to the Navier-Stokes system and the associated relative energy inequality proved in [4], [6] (cf. also Germain [8]). This method provides an explicit rate of convergence in terms of the initial data and the parameters ε and λ. Namely, we show that the Euler system (1.1), (1.2) is obtained as the inviscid limit of (1.5–1.9) when both ε and the positive parameter λ in (1.6) tend to zero. Keeping $\lambda = 1$ we obtain the Navier-Stokes system (1.3), (1.4). Note that the dependence on the thin channels Ω_ε cross sections $\varepsilon\omega_h(z)$ in the residual equations (1.1)-(1.4), is manifested solely through the area $A(z)$, and it is independent of the curvature or other finer properties of the shape of the boundary. Strangely enough, the asymptotic analysis is more delicate for the Navier-Stokes limit, where certain quantities must be controlled by means of a variant of the celebrated *Korn-Poincaré inequality*:

$$\int_{\Omega_\varepsilon} |\mathbf{v}|^2 \, \mathrm{d}x \leq C_{KP} \int_{\Omega_\varepsilon} \left|\nabla_x \mathbf{v} + \nabla_x^t \mathbf{v}\right|^2 \, \mathrm{d}x \tag{1.10}$$

to be satisfied, with a constant C_{KP} independent of $\varepsilon \to 0$, for any vector field $\mathbf{v}$ such that:

$$\begin{aligned} \mathbf{v}(x) \cdot \mathbf{n} = 0 \qquad & \forall x = (\mathbf{x}_h, z) \in \partial\Omega_\varepsilon, \ z \in (0,1), \\ \mathbf{v}(x) = 0 \qquad & \forall x = (\mathbf{x}_h, z) \in \overline{\Omega}_\varepsilon, \ z \in \{0,1\}. \end{aligned}$$

Note that since we do not attempt to prove the *conformal* version of the Korn-Poincaré inequality, specifically:

$$\int_{\Omega_\varepsilon} |\mathbf{v}|^2 \, \mathrm{d}x \leq C_{\mathrm{CKP}} \int_{\Omega_\varepsilon} \left|\nabla_x \mathbf{v} + \nabla_x^t \mathbf{v} - \frac{2}{3}\mathrm{div}_x \mathbf{v}\mathbb{I}\right|^2 \, \mathrm{d}x, \tag{1.11}$$

we assume that the bulk viscosity η is strictly positive.

The paper is organized as follows. In Section 2, we recall the concept of dissipative weak solutions to the Navier-Stokes system (1.5–1.7), (1.9); state and explain the assumption on the channel-like domains Ω_ε and the pressure function p; and present the main results concerning the asymptotic limits. In Section 3, we introduce the relative entropy inequality and derive the necessary uniform bounds independent of the parameters ε and λ. The asymptotic limits are performed in Section 4. The paper is concluded by the proof of the Korn-Poincaré inequality (1.10) in Section 5, together with other related results and problems that may be of independent interest.

2 Preliminaries and statements of main results

Similarly to the notation $x = (\mathbf{x}_h, z)$, the subscript h used in the differential operators will refer to the horizontal variables. The pressure $p = p(\varrho)$ is assumed to be a function of the density, and to satisfy:

$$\begin{aligned} & p \in C[0,\infty) \cap C^3(0,\infty), \qquad p(0) = 0, \qquad p'(\varrho) > 0 \quad \forall \varrho > 0, \\ & \text{and} \quad \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0 \quad \text{for a certain } \gamma > \frac{3}{2}. \end{aligned} \tag{2.1}$$

Remark 2.1. The assumption for the pressure to be a strictly increasing function of ϱ is indispensable for our results. The growth restriction imposed through the value of γ is required by the available existence theory for the compressible Navier-Stokes system (1.5–1.7).

Next, we specify our requirements concerning the geometry of the spatial domains Ω_ε introduced in (1.8). As each Ω_ε is obtained via a simple scaling, it is convenient to formulate our hypotheses in terms of the basic domain:

$$\Omega = \left\{ x = (\mathbf{x}_h, z) \;\middle|\; z \in (0,1),\ \mathbf{x}_h \in \omega_h(z) \right\}.$$

Namely, we suppose there is a vector field $\mathbf{V}_h = \mathbf{V}_h(\mathbf{x}_h, z) : \overline{\Omega} \to \mathbb{R}^2$ such that:

$$\begin{aligned} &\nabla_h \mathrm{div}_h \mathbf{V}_h = 0 \quad \text{and} \quad \Delta_h \mathbf{V}_h = 0 \quad \text{in } \Omega; \\ &[\mathbf{V}_h(\mathbf{x}_h, z), 1] \in T_{(\mathbf{x}_h,z)}(\partial\Omega) \qquad \forall z \in (0,1),\ \mathbf{x}_h \in \partial\omega_h(z). \end{aligned} \tag{2.2}$$

The first condition above means that $\mathrm{div}_h \mathbf{V}_h$ depends only on the variable z, while the last condition states that the vector field $[\mathbf{V}_h, 1] \in \mathbb{R}^3$ is tangent to $\partial\Omega$ on the lateral boundary $\partial\Omega \cap \{0 < z < 1\}$.

Lemma 2.1. *Assume that the lateral boundary of Ω is of class $C^{r,\alpha}$ with $r \geq 2, \alpha \in (0,1)$. Then:*

(i) There exists a vector field $\mathbf{V}_h \in C^{r-1,\alpha}(\overline{\Omega}; \mathbb{R}^2)$ satisfying (2.2).

(ii) Let ϕ be the flow of $\mathbf{V}_h$, namely: $\frac{d}{dt}\phi(\cdot,t) = \mathbf{V}_h(\phi(\cdot,t),t)$ and $\phi(\cdot,0) = id_{\omega_h(0)}$. Then:

$$\omega_h(z) = \left\{ \phi(\mathbf{x}_h, z) \;\middle|\; \mathbf{x}_h \in \omega_h(0) \right\} \qquad \forall z \in [0,1].$$

(iii) Recalling that $A(z) = |\omega_h(z)|$, there holds

$$A(z)\mathrm{div}_h \mathbf{V}_h(z) = \partial_z A(z). \tag{2.3}$$

Proof. **1.** To prove (i), we first define a vector field $\mathbf{w}_h \in \mathbb{R}^2$ on the lateral boundary of Ω, through the following two conditions:

$\mathbf{w}_h(\mathbf{x}_h, z)$ is parallel to the normal vector $\mathbf{n}_h$ to $\omega_h(z)$ at $\mathbf{x}_h \in \partial\omega_h(z)$;
the vector $[\mathbf{w}_h(\mathbf{x}_h, z), 1]$ is tangent to $\partial\Omega$ at $(\mathbf{x}_h, z)$.

Let now $\mathbf{w}_h = \mathbf{w}_h(\mathbf{x}_h, z) \in \mathbb{R}^2$ be any extension of $\mathbf{w}_h$ on $\overline{\Omega}$, of regularity $C^{r-1,\alpha}$, and denote $\tilde{\mathbf{X}} = [\mathbf{w}_h, 1] \in \mathbb{R}^3$ the vector field on $\overline{\Omega}$, whose flow $\tilde{\Phi}$ describes the evolution of the cross sections $z \mapsto \omega_h(z)$. Namely:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\Phi}(\cdot,t) = \tilde{\mathbf{X}}(\tilde{\Phi}(\cdot,t),t), \qquad \tilde{\Phi}(\cdot,0) = id_{\omega_h(0)}$$

and we have:

$$\Big\{\tilde{\Phi}(\mathbf{x}_h, z) \ \Big| \ \mathbf{x}_h \in \omega_h(0)\Big\} = \Big\{\tilde{\phi}(\mathbf{x}_h, z) \ \Big| \ \mathbf{x}_h \in \omega_h(0)\Big\} \times \{z\} = \omega_h(z) \times \{z\}, \tag{2.4}$$

where $\tilde{\phi}$ is the flow of $\mathbf{w}_h$, so that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\phi}(\cdot, t) = \mathbf{w}_h(\tilde{\phi}(\cdot, t), t), \qquad \tilde{\phi}(\cdot, 0) = id_{\omega_h(0)}.$$

By a change of variables, we now obtain:

$$\begin{aligned}
\partial_z A(z) &= \partial_z\Big(\int_{\omega_h(0)} \det\nabla_h\tilde{\phi}(\mathbf{x}_h, z)\ \mathrm{d}\mathbf{x}_h\Big) = \int_{\omega_h(0)} \partial_z\big(\det\nabla_h\tilde{\phi}(\mathbf{x}_h, z)\big)\ \mathrm{d}\mathbf{x}_h \\
&= \int_{\omega_h(0)} \big(\det\nabla_h\tilde{\phi}(\mathbf{x}_h, z)\big)\Big(\mathrm{div}_h\mathbf{w}_h(\tilde{\phi}(\mathbf{x}_h, z), z)\Big)\ \mathrm{d}\mathbf{x}_h \\
&= \int_{\omega_h(z)} \mathrm{div}_h\mathbf{w}_h(\mathbf{x}_h, z)\ \mathrm{d}\mathbf{x}_h = \int_{\partial\omega_h(z)} \mathbf{w}_h \cdot \mathbf{n}_h.
\end{aligned} \tag{2.5}$$

2. Next, we define $U_h = U_h(\mathbf{x}_h, z) \in \mathbb{R}$ to be the unique solution of the Neumann problem:

$$\Delta_h U_h(\mathbf{x}_h, z) = \frac{\partial_z A(z)}{A(z)} \quad \text{in } \omega_h(z), \qquad \nabla_x U_h(\mathbf{x}_h, z) \cdot \mathbf{n}_h = \mathbf{w}_h(\mathbf{x}_h, z) \cdot \mathbf{n}_h \quad \text{on } \partial\omega_h(z). \tag{2.6}$$

This problem has a solution $U_h \in C^{r-1,\alpha}$ enjoying "horizontal" regularity $U_h \in C^{r,\alpha}(\omega_h(z))$ because of the compatibility in: $\int_{\omega_h(z)} \frac{\partial_z A(z)}{A(z)}\ \mathrm{d}\mathbf{x}_h = \partial_z A(z) = \int_{\partial\omega_h(z)} \mathbf{w}_h \cdot \mathbf{n}_h$, valid in view of (2.5). The desired vector field $\mathbf{V}_h$ can then be taken as:

$$\mathbf{V}_h(\mathbf{x}_h, z) = \nabla_h U_h(\mathbf{x}_h, z) \quad \text{ in } \Omega.$$

Clearly, $\mathrm{div}_h\mathbf{V}_h = \Delta_h U_h$ is constant in $\omega_h(z)$ and $\Delta_h\mathbf{V}_h = \nabla_h\Delta_h U_h = 0$ by (2.6). Moreover, on the lateral boundary of Ω, the vector fields $\mathbf{V}_h$ and $\mathbf{w}_h$ differ by a vector tangent to $\partial\omega_h(z)$. Therefore $\mathbf{V}_h$ satisfies (2.2), which achieves (i). We also automatically obtain (ii), by the same reasoning as in (2.4). Finally, applying (2.5) where ϕ replaces $\tilde{\phi}$ and $\mathbf{V}_h$ replaces $\mathbf{w}_h$, we get (iii):

$$\partial_z A(z) = \int_{\omega_h(z)} \mathrm{div}_h\mathbf{V}_h(\mathbf{x}_h, z)\ \mathrm{d}\mathbf{x}_h = A(z)\mathrm{div}_h\mathbf{V}_h(z).$$

3. To finish the proof, we establish regularity of the field $\mathbf{V}_h(\mathbf{x}_h, z)$ with respect to the "vertical" variable z. To this end, we pull back the boundary problem (2.6) to the fixed domain $\omega_h(0)$:

$$\begin{aligned}
\mathrm{div}_h\Big(\mathbb{B}(\mathbf{x}_h, z)\nabla_h\tilde{U}_h(\mathbf{x}_h, z)\Big) &= \Big(\det\nabla_h\tilde{\phi}(\mathbf{x}_h, z)\Big)\frac{\partial_z A(z)}{A(z)} \qquad \text{in } \omega_h(0), \\
\nabla_h\tilde{U}_h(\mathbf{x}_h, z) \cdot \tilde{\mathbf{n}}_h(\mathbf{x}_h, z) &= \tilde{\mathbf{w}}_h(\mathbf{x}_h, z) \cdot \mathbf{n}_h(\tilde{\phi}(\mathbf{x}_h, z)) \qquad \text{on } \partial\omega_h(0),
\end{aligned}$$

where:

$$\begin{aligned}
\tilde{U}_h(\mathbf{x}_h, z) &= U_h(\tilde{\phi}(\mathbf{x}_h, z), z), \qquad \tilde{\mathbf{w}}_h(\mathbf{x}_h, z) = \mathbf{w}_h(\tilde{\phi}(\mathbf{x}_h, z), z) \\
\mathbb{B}(\mathbf{x}_h, z) &= \left[\mathrm{cof}\ \nabla_h \tilde{\phi}(\mathbf{x}_h, z)\right]^t \left[(\nabla_h \tilde{\phi})^{-1}(\mathbf{x}_h, z)\right]^t \\
&= \left(\det \nabla_h \tilde{\phi}(\mathbf{x}_h, z)\right) \left[(\nabla_h \tilde{\phi})^{-1}(\mathbf{x}_h, z)\right] \left[(\nabla_h \tilde{\phi})^{-1}(\mathbf{x}_h, z)\right]^t \\
\tilde{\mathbf{n}}_h(\mathbf{x}_h, z) &= \left[(\nabla_h \tilde{\phi})^{-1}(\mathbf{x}_h, z)\right] \mathbf{n}_h(\tilde{\phi}(\mathbf{x}_h, z), z).
\end{aligned}$$

Thus, differentiating with respect to z and using the standard elliptic estimates we obtain the desired regularity in z. This ends the proof of Lemma 2.1. □

Example 2.2. A typical example of a thin channel that we have in mind is:

$$\Omega_\varepsilon = \left\{ x = (x_1, x_2, z) \equiv (\mathbf{x}_h, z) \;\middle|\; z \in (0,1),\ |\mathbf{x}_h - \varepsilon X(z)|^2 < R^2(z) \right\},$$

where $X : [0,1] \to \mathbb{R}^2$ and $R : [0,1] \to (0, +\infty)$ are two given smooth functions, to the effect that each cross section $\omega_h(z)$ is simply a circle $B(X(z), R(z)) \subset \mathbb{R}^2$. Note that we can then take:

$$\mathbf{V}_h(\mathbf{x}_h, z) = \frac{\partial_z R(z)}{R(z)}(\mathbf{x}_h - X(z)) + \partial_z X(z).$$

We also check directly that $A(z)\mathrm{div}_h \mathbf{V}_h(z) = \pi R(z)^2 \cdot 2\frac{\partial_z R(z)}{R(z)} = 2\pi R(z)\partial_z R(z) = \partial_z A(z)$.

2.1 Dissipative weak solutions to the compressible Navier-Stokes system

Definition 2.1. *We say that $[\varrho, \mathbf{u}]$ is a (weak)* dissipative solution *to the Navier-Stokes system (1.5–1.7) in the space-time cylinder $(0,T) \times \Omega_\varepsilon$ with the boundary conditions (1.9) if and only if:*

- $\varrho \in C_{\mathrm{weak}}([0,T]; L^\gamma(\Omega_\varepsilon))$, $\varrho\mathbf{u} \in C_{\mathrm{weak}}([0,T]; L^\gamma(\Omega_\varepsilon; \mathbb{R}^3))$, $\mathbf{u} \in L^2(0,T; W^{1,2}(\Omega_\varepsilon; \mathbb{R}^3))$, *and* $\varrho \geq 0$ *a.e. in* $(0,T) \times \Omega_\varepsilon$, $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0$;
- *For any test function $\varphi \in C^\infty([0,T] \times \overline{\Omega}_\varepsilon)$ there holds:*

$$\left[\int_{\Omega_\varepsilon} \varrho\varphi\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega_\varepsilon} (\varrho\partial_t\varphi + \varrho\mathbf{u} \cdot \nabla_x\varphi)\ \mathrm{d}x\ \mathrm{d}t; \tag{2.7}$$

- *For any test function $\varphi \in C^\infty([0,T] \times \overline{\Omega}_\varepsilon; \mathbb{R}^3)$, $\varphi \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0$ there holds:*

$$\left[\int_{\Omega_\varepsilon} \varrho\mathbf{u} \cdot \varphi\ \mathrm{d}x\right]_{t=0}^{t=\tau} = \int_0^\tau \int_{\Omega_\varepsilon} (\varrho\mathbf{u} \cdot \partial_t\varphi + \varrho\mathbf{u} \otimes \mathbf{u} : \nabla_x\varphi + p(\varrho)\mathrm{div}_x\varphi - \lambda\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\varphi)\ \mathrm{d}x\ \mathrm{d}t;$$

- *The energy inequality:*

$$\int_{\Omega_\varepsilon} \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + H(\varrho) \right)(\tau,\cdot)\ \mathrm{d}x + \lambda \int_0^\tau \int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\ \mathrm{d}x\ \mathrm{d}t \le \int_{\Omega_\varepsilon} \left(\frac{|\varrho \mathbf{u}|^2}{2\varrho} + H(\varrho) \right)(0,\cdot)\ \mathrm{d}x,$$

with:

$$H(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z,$$

holds for a.e. $\tau \in (0,T)$.

The existence of dissipative solutions can be shown by the method of Lions [12], with the necessary modifications introduced in [5].

Remark 2.3. In the Navier-Stokes limit, we will impose an extra boundary condition:

$$\mathbf{u}(x_h, z) = 0 \qquad \forall (x_h, z) \in \overline{\Omega}_\varepsilon, \quad z \in \{0,1\}. \tag{2.8}$$

Accordingly, the class of admissible test functions in the momentum balance (2.7) is restricted to:

$$\varphi \in C^\infty([0,T] \times \overline{\Omega}_\varepsilon; \mathbb{R}^3), \qquad \varphi \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \qquad \varphi \text{ compactly supported in } z \in (0,1).$$

2.2 Main results

Our goal is to identify the asymptotic limit for solutions of system (1.5–1.7), (1.9)/(2.8) if the diameter ε of the cylinder Ω_ε tends to zero. To measure the distance to the solutions of the limit system, we use the relative energy functional:

$$\mathcal{E}_\varepsilon \left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U} \right) = \int_{\Omega_\varepsilon} \left(\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H(\varrho) - H'(r)(\varrho - r) - H(r) \right)\ \mathrm{d}x. \tag{2.9}$$

Since $H''(\varrho) = p'(\varrho)/\varrho$ and the pressure p is a strictly increasing differentiable function of the density, the pressure potential H is strictly convex and it is easy to check that for $r > 0$:

$$\mathcal{E}_\varepsilon \left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U} \right) = 0 \ \Leftrightarrow\ \varrho = r,\ \mathbf{u} = \mathbf{U}.$$

Moreover, it follows from (2.1) that:

$$\begin{aligned} C_1(K)\left(|\mathbf{u} - \mathbf{U}|^2 + |\varrho - r|^2\right) &\le \frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H(\varrho) - H'(r)(\varrho - r) - H(r) \\ &\le C_2(K)\left(|\mathbf{u} - \mathbf{U}|^2 + |\varrho - r|^2\right) \qquad \forall \varrho, r \in K \subset (0,\infty),\ K \text{ compact} \\ \text{and} \quad \frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + H(\varrho) - H'(r)(\varrho - r) - H(r) &\ge C(K,\tilde{K})\left(1 + \varrho|\mathbf{u} - \mathbf{U}|^2 + \varrho^\gamma\right) \\ &\forall r \in K \subset \mathrm{int}[\tilde{K}],\ \varrho \in [0,\infty) \setminus \tilde{K},\ \tilde{K} \subset (0,\infty) \text{ compact}. \end{aligned} \tag{2.10}$$

2.2.1 Inviscid limit

The system (1.1), (1.2) can be written as a semilinear perturbation of the standard isentropic Euler system in the following form:

$$\partial_t \varrho_E + \partial_z(\varrho_E u_E) + \frac{\partial_z A}{A} \varrho_E u_E = 0,$$
$$\partial_t(\varrho_E u_E) + \partial_z(\varrho_E u_E^2) + \partial_z p(\varrho_E) + \frac{\partial_z A}{A} \varrho_E u_E^2 = 0.$$

In view of the standard theory of hyperbolic conservation laws, see e.g. Majda [13], one can therefore anticipate the existence of *local* in time smooth solutions to problem (1.1), (1.2) provided the initial data are smooth enough. As shown in the following theorem, these solutions may be seen as suitable limits of those of the Navier-Stokes system (1.5–1.7), (1.9) in Ω_ε in the regime $\varepsilon, \lambda \to 0$.

Theorem 2.4. *Let Ω_ε be given by (1.8), where $\Omega = \Omega_1$ is determined through (2.2), with $\mathbf{V}_h \in C^1(\overline{\Omega}; \mathbb{R}^2)$. Let the pressure p satisfy hypothesis (2.1). Set:*

$$A(z) = |\omega(z)|.$$

Let $[\varrho_E, u_E]$ be a classical solution of the Euler system (1.1), (1.2) on a time interval $[0,T]$ such that:

$$u_E|_{z\in\{0,1\}} = 0. \tag{2.11}$$

Let $[\varrho, \mathbf{u}]$ be a (weak) dissipative solution of the Navier-Stokes system (1.5–1.7), (1.9) in $(0,T)\times\Omega_\varepsilon$.

Then there is a constant C, depending only on time T, on the norm of the solution $[\varrho_E, u_E]$, on the C^1 norm of $\mathbf{V}_h$, but independent of $[\varrho, \mathbf{u}]$ and of the scaling parameters λ and ε, such that:

$$\frac{1}{|\Omega_\varepsilon|}\mathcal{E}_\varepsilon\left(\varrho, \mathbf{u} \;\middle|\; \varrho_E, \mathbf{u}_E\right)(\tau) \leq C\left(\lambda + \varepsilon + \frac{1}{|\Omega_\varepsilon|}\mathcal{E}_\varepsilon\left(\varrho, \mathbf{u} \;\middle|\; \varrho_E, \mathbf{u}_E\right)(0)\right) \tag{2.12}$$

for any $\tau \in (0,T)$, where we have set $\mathbf{u}_E = [0, 0, u_E]$.

Theorem 2.4 will be shown in Section 4.1.

2.2.2 Positive viscosity limit

Similarly to the preceding section, we may rewrite (1.3), (1.4) as:

$$\partial_t \varrho_{NS} + \partial_z(\varrho_{NS} u_{NS}) + \frac{\partial_z A}{A} \varrho_{NS} u_{NS} = 0,$$

$$\partial_t(\varrho_{NS} u_{NS}) + \partial_z(\varrho_{NS} u_{NS}^2) + \partial_z p(\varrho_{NS}) + \frac{\partial_z A}{A} \varrho_{NS} u_{NS}^2 = \left(\frac{4\mu}{3} + \eta\right) \partial_z^2 u_{NS} + \left(\frac{\mu}{3} + \eta\right) \partial_z \left(\frac{\partial_z A}{A} u_{NS}\right).$$

Thus, by analogy to its inviscid counterpart, we may anticipate the existence of at least local-in-time smooth solutions to system (1.3), (1.4), supplemented with the boundary conditions:

$$u_{NS}|_{z \in \{0,1\}} = 0,$$

for sufficiently smooth initial data. Moreover, in view of the theory developed by Kazhikhov [9], we may even expect those solutions to be global in time, however, we were not able to find a relevant reference. We claim the following result proved in Section 4.2.

Theorem 2.5. *Let Ω_ε be given by (1.8), where Ω is determined through (2.2), with the vector field $\mathbf{V}_h \in C^2(\overline{\Omega}; \mathbb{R}^2)$. Let the pressure p satisfy hypothesis (2.1). Set:*

$$A(z) = |\omega(z)|.$$

Let $[\varrho_{NS}, u_{NS}]$ be a classical solution of the Navier-Stokes system with drift (1.3), (1.4) on a time interval $[0, T]$, satisfying:

$$u_{NS}|_z \in \{0, 1\} = 0.$$

Let $[\varrho, \mathbf{u}]$ be a (weak) dissipative solution of the Navier-Stokes system (1.5–1.7), (1.9) in $(0, T) \times \Omega_\varepsilon$ with $\lambda = 1$ and strictly positive bulk viscosity $\eta > 0$, satisfying, in addition, the no-slip boundary condition (2.8) at the horizontal part of the boundary of the cylinder Ω_ε.

Then there is a constant C, depending only on time T, on the norm of the solution $[\varrho_{NS}, u_{NS}]$, on the C^2 norm of the vector field $\mathbf{V}_h$, but independent of $[\varrho, \mathbf{u}]$ and of the scaling parameter ε, such that:

$$\frac{1}{|\Omega_\varepsilon|} \mathcal{E}_\varepsilon \left(\varrho, \mathbf{u} \;\middle|\; \varrho_{NS}, \mathbf{u}_{NS}\right)(\tau) \leq C \left(\varepsilon + \frac{1}{|\Omega_\varepsilon|} \mathcal{E}_\varepsilon \left(\varrho, \mathbf{u} \;\middle|\; \varrho_{NS}, \mathbf{u}_{NS}\right)(0)\right)$$

for any $\tau \in (0, T)$, where we have set $\mathbf{u}_{NS} = [0, 0, u_{NS}]$.

As already pointed out, the proof of Theorem 2.5 is based on a version of Korn-Poincaré inequality on thin domains proved in Section 5.

3 The relative energy inequality

As shown in [4], *any* dissipative solution $[\varrho, \mathbf{u}]$ of the Navier-Stokes system (1.5–1.7) satisfies the *relative energy inequality*:

$$
\begin{aligned}
\mathcal{E}_\varepsilon \left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U} \right)(\tau) + \int_0^\tau \int_{\Omega_\varepsilon} \Big(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) \Big) : \Big(\nabla_x \mathbf{u} - \nabla_x \mathbf{U} \Big) \ \mathrm{d}x \ \mathrm{d}t \\
\leq \ \mathcal{E}_\varepsilon \Big(\varrho(0,\cdot), \mathbf{u}(0,\cdot) \ \Big| \ r(0,\cdot), \mathbf{U}(0,\cdot) \Big) + \int_0^\tau \mathcal{R}_\varepsilon(\varrho, \mathbf{u}, r, \mathbf{U}) \ \mathrm{d}t,
\end{aligned}
\tag{3.1}
$$

with the remainder:

$$
\begin{aligned}
\mathcal{R}_\varepsilon \left(\varrho, \mathbf{u}, r, \mathbf{U} \right) = & \int_{\Omega_\varepsilon} \varrho \Big(\partial_t \mathbf{U} + \mathbf{u} \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x + \lambda \int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{U}) : \nabla_x (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x \\
& + \int_{\Omega_\varepsilon} \left((r - \varrho) \partial_t H'(r) + \nabla_x H'(r) \cdot (r\mathbf{U} - \varrho \mathbf{u}) \right) \ \mathrm{d}x \\
& - \int_{\Omega_\varepsilon} \mathrm{div}_x \mathbf{U} \Big(p(\varrho) - p(r) \Big) \ \mathrm{d}x.
\end{aligned}
$$

Here $[r, \mathbf{U}]$ represent arbitrary test functions that are sufficiently smooth and satisfy a kind of compatibility conditions:

$$
r > 0, \qquad \mathbf{U} \cdot \mathbf{n}|_{\partial \Omega_\varepsilon} = 0, \tag{3.2}
$$

and:

$$
\mathbf{U}(x_h, z) = 0 \qquad \forall (x_h, z) \in \overline{\Omega}_\varepsilon, \ z \in \{0, 1\} \tag{3.3}
$$

provided the extra no-slip condition (2.8) is imposed.

3.1 Extending the velocity field

The proofs of Theorems 2.4, 2.5 are based on the idea to use the solutions of the target systems to construct test functions for the relative energy inequality (3.1). This cannot be done directly as the velocity fields u_E, u_{NS} or, more specifically, their extensions $\mathbf{u}_E = [0, 0, u_E]$, $\mathbf{u}_{NS} = [0, 0, u_{NS}]$ do not comply with the boundary conditions (3.2), (3.3), respectively. Instead, we consider a tilted extension of a velocity field of the form:

$$
\mathbf{U}_\varepsilon = [\mathbf{V}_{h,\varepsilon}, 1] \, v, \quad v = u_E, u_{NS}, \tag{3.4}
$$

where:
$$\mathbf{V}_{h,\varepsilon}(\mathbf{x}_h, z) := \varepsilon \mathbf{V}_h \left(\frac{\mathbf{x}_h}{\varepsilon}, z \right) \qquad \forall (\mathbf{x}_h, z) \in \overline{\Omega}_\varepsilon \tag{3.5}$$

and $\mathbf{V}_h$ is the vector field introduced in (2.2). As the vector field $[\mathbf{V}_{h,\varepsilon}, 1]$ is tangent to $\partial\Omega_\varepsilon$ at any point of the lateral boundary $\partial\Omega_\varepsilon \cap \{0 < z < 1\}$, $\mathbf{U}_\varepsilon$ is an admissible test function in (3.1) as soon as u_E, u_{NS} vanish at $z \in \{0, 1\}$. The following result shows that the extension defined through (3.4) satisfies also the equation of continuity.

Lemma 3.1. *Let $\mathbf{U}_\varepsilon$ be the velocity field defined by (3.4) and suppose that the functions $r = r(z)$, $v = v(z)$ satisfy:*

$$\partial_t (rA) + \operatorname{div}_x (rvA) = \partial_t (rA) + \partial_z (rvA) = 0 \qquad \forall z \in (0, 1),$$

where $A(z) = |\omega_h(z)|$. Then:
$$\partial_t r + \operatorname{div}_x (r\mathbf{U}_\varepsilon) = 0 \qquad \text{in } \Omega_\varepsilon.$$

Proof. On one hand, we have:

$$\partial_t(rA) + \operatorname{div}_x(r\mathbf{U}_\varepsilon A) = A \left(\partial_t r + \operatorname{div}_x(r\mathbf{U}_\varepsilon) \right) + rv\partial_z A.$$

On the other hand, in accordance with (2.3), we get:

$$\partial_t(rA) + \operatorname{div}_x(r\mathbf{U}_\varepsilon A) = \partial_t(rA) + \partial_z(rvA) + \operatorname{div}_h(r\mathbf{V}_{h,\varepsilon} vA) = rvA\operatorname{div}_h \mathbf{V}_{h,\varepsilon} = rvA\operatorname{div}_h \mathbf{V}_h = rv\partial_z A,$$

and the desired conclusion follows. □

3.2 Relative energy inequality and the asymptotic limits

We start by rewriting $\mathcal{R}_\varepsilon$ as:

$$\begin{aligned} \mathcal{R}_\varepsilon (\varrho, \mathbf{u}, r, \mathbf{U}) = & \int_{\Omega_\varepsilon} \varrho \Big(\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} \Big) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x - \int_{\Omega_\varepsilon} \varrho (\mathbf{u} - \mathbf{U}) \cdot \nabla_x \mathbf{U} \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x \\ & + \lambda \int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{U}) : \nabla_x (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x \\ & + \int_{\Omega_\varepsilon} \left((r - \varrho) \partial_t H'(r) + \nabla_x H'(r) \cdot (r\mathbf{U} - \varrho \mathbf{u}) \right) \, \mathrm{d}x \\ & - \int_{\Omega_\varepsilon} \operatorname{div}_x \mathbf{U} \Big(p(\varrho) - p(r) \Big) \, \mathrm{d}x. \end{aligned} \tag{3.6}$$

3.2.1 Relative energy inequality in the inviscid limit

Take $r = \varrho_E$, $\mathbf{U} = \mathbf{U}_\varepsilon = [\mathbf{V}_{h,\varepsilon}, 1]u_E$ as test functions in the relative energy inequality (3.1), where $\varrho_E = \varrho_E(z)$, $u_E = u_E(z)$ is a (smooth) solution of the 1D-Euler system (1.1), (1.2) satisfying the boundary conditions (2.11). Going back to (3.6) we compute:

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \varrho\Big(\partial_t \mathbf{U}_\varepsilon + \mathbf{U}_\varepsilon \cdot \nabla_x \mathbf{U}_\varepsilon\Big) \cdot (\mathbf{U}_\varepsilon - \mathbf{u}) \ \mathrm{d}x \\
&\qquad = \int_{\Omega_\varepsilon} \varrho\Big(\partial_t \mathbf{U}_\varepsilon - \partial_t \mathbf{u}_E + \mathbf{U}_\varepsilon \cdot \nabla_x \mathbf{U}_\varepsilon - \mathbf{u}_E \cdot \nabla_x \mathbf{u}_E\Big) \cdot (\mathbf{U}_\varepsilon - \mathbf{u}) \ \mathrm{d}x \\
&\qquad\quad + \int_{\Omega_\varepsilon} \varrho\Big(\partial_t u_E + u_E \cdot \partial_z u_E\Big)(u_E - u_3) \ \mathrm{d}x \\
&\qquad = E_1(\varrho, \mathbf{U}_\varepsilon, u_E, \mathbf{u}) - \int_{\Omega_\varepsilon} \frac{\varrho}{\varrho_E} \partial_z p(\varrho_E)(u_E - u_3) \ \mathrm{d}x,
\end{aligned}
\tag{3.7}
$$

where the last equality follows from $\varrho_E(\partial_t u_E + u_E \cdot \partial_z u_E + \partial_z p(\varrho_E)) = 0$, which is a consequence of (1.1) and (1.2), and the error term has the form:

$$
E_1(\varrho, \mathbf{U}_\varepsilon, u_E, \mathbf{u}) = \int_{\Omega_\varepsilon} \varrho\Big(\partial_t \mathbf{U}_\varepsilon - \partial_t \mathbf{u}_E + \mathbf{U}_\varepsilon \cdot \nabla_x \mathbf{U}_\varepsilon - \mathbf{u}_E \cdot \nabla_x \mathbf{u}_E\Big) \cdot (\mathbf{U}_\varepsilon - \mathbf{u}) \ \mathrm{d}x.
$$

Next, the terms containing the pressure, coming from the last line in (3.6), can be written as:

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \left((\varrho_E - \varrho) \frac{1}{\varrho_E} p'(\varrho_E) \partial_t \varrho_E + \frac{1}{\varrho_E} p'(\varrho_E) \partial_z \varrho_E \left(\varrho_E u_E - \varrho u_3 \right) \right) \ \mathrm{d}x \\
&= \int_{\Omega_\varepsilon} \partial_t p(\varrho_E) + \partial_z p(\varrho_E) u_E \ \mathrm{d}x - \int_{\Omega_\varepsilon} \frac{\varrho}{\varrho_E} p'(\varrho_E) \left(\partial_t \varrho_E + \partial_z \varrho_E u_3 \right) \ \mathrm{d}x \\
&= \int_{\Omega_\varepsilon} \partial_t p(\varrho_E) + \partial_z p(\varrho_E) u_E \ \mathrm{d}x - \int_{\Omega_\varepsilon} \frac{\varrho}{\varrho_E} p'(\varrho_E) \left(\partial_t \varrho_E + \partial_z \varrho_E u_E \right) \ \mathrm{d}x + \int_{\Omega_\varepsilon} \frac{\varrho}{\varrho_E} \partial_z p(\varrho_E)(u_E - u_3) \ \mathrm{d}x.
\end{aligned}
$$

Finally, we use the fact established in Lemma 3.1, namely that $[\varrho_E, \mathbf{U}_\varepsilon]$ solve the equation of continuity, to conclude:

$$
\begin{aligned}
&\int_{\Omega_\varepsilon} \left((\varrho_E - \varrho) \frac{1}{\varrho_E} p'(\varrho_E) \partial_t \varrho_E + \frac{1}{\varrho_E} p'(\varrho_E) \partial_z \varrho_E \left(\varrho_E u_E - \varrho u_3 \right) \right) \ \mathrm{d}x \\
&\qquad = \int_{\Omega_\varepsilon} p'(\varrho_E) \Big(\varrho - \varrho_E \Big) \mathrm{div}_x \mathbf{U}_\varepsilon \ \mathrm{d}x + \int_{\Omega_\varepsilon} \frac{\varrho}{\varrho_E} \partial_z p(\varrho_E)(u_E - u_3) \ \mathrm{d}x.
\end{aligned}
\tag{3.8}
$$

Thus, summing up (3.7), (3.8) and comparing the resulting expression with (3.6), we may infer that:

$$
\begin{aligned}
\mathcal{R}_\varepsilon\left(\varrho, \mathbf{u}, \varrho_E, \mathbf{U}_\varepsilon\right) = & \int_{\Omega_\varepsilon} \operatorname{div}_x \mathbf{U}_\varepsilon \Big(p(\varrho) - p'(\varrho_E)(\varrho - \varrho_E) - p(\varrho_E) \Big) \, \mathrm{d}x \\
& - \int_{\Omega_\varepsilon} \varrho (\mathbf{u} - \mathbf{U}_\varepsilon) \cdot \nabla_x \mathbf{U}_\varepsilon \cdot (\mathbf{U}_\varepsilon - \mathbf{u}) \, \mathrm{d}x \\
& + \lambda \int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{U}_\varepsilon) : \nabla_x (\mathbf{U}_\varepsilon - \mathbf{u}) \, \mathrm{d}x + E_1(\varrho, \mathbf{U}_\varepsilon, u_E, \mathbf{u}).
\end{aligned} \tag{3.9}
$$

3.2.2 Relative entropy inequality in the viscous limit

The viscous (Navier-Stokes) limit can be handled in a similar way. An analogue of (3.7), derived using (1.3) and (1.4), reads:

$$
\begin{aligned}
& \int_{\Omega_\varepsilon} \varrho \Big(\partial_t \mathbf{U}_\varepsilon + \mathbf{U}_\varepsilon \cdot \nabla_x \mathbf{U}_\varepsilon \Big) \cdot (\mathbf{U}_\varepsilon - \mathbf{u}) \, \mathrm{d}x \\
& \quad = E_1(\varrho, \mathbf{U}_\varepsilon, u_{NS}, \mathbf{u}) - \int_{\Omega_\varepsilon} \frac{\varrho}{\varrho_{NS}} \partial_z p(\varrho_{NS}) (u_{NS} - u_3) \, \mathrm{d}x \\
& \qquad + \int_{\Omega_\varepsilon} \frac{\varrho}{\varrho_{NS}} \left(\nu \partial_z^2 u_{NS} + (\mu/3 + \eta) \partial_z (\partial_z (\ln A) u_{NS}) \right) (u_{NS} - u_3) \, \mathrm{d}x,
\end{aligned}
$$

which, after a similar treatment as in Section 3.2.1 gives rise to the remainder:

$$
\begin{aligned}
\mathcal{R}_\varepsilon(\varrho, \mathbf{u}, \varrho_{NS}, \mathbf{U}_\varepsilon) = & \int_{\Omega_\varepsilon} \operatorname{div}_x \mathbf{U}_\varepsilon \Big(p(\varrho) - p'(\varrho_{NS})(\varrho - \varrho_{NS}) - p(\varrho_{NS}) \Big) \, \mathrm{d}x \\
& - \int_{\Omega_\varepsilon} \varrho (\mathbf{u} - \mathbf{U}_\varepsilon) \cdot \nabla_x \mathbf{U}_\varepsilon \cdot (\mathbf{U}_\varepsilon - \mathbf{u}) \, \mathrm{d}x \\
& + \int_{\Omega_\varepsilon} \frac{\varrho}{\varrho_{NS}} \left(\nu \partial_z^2 u_{NS} + (\mu/3 + \eta) \partial_z (\partial_z (\ln A) u_{NS}) \right) (u_{NS} - u_3) \, \mathrm{d}x \\
& - \int_{\Omega_\varepsilon} \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{U}_\varepsilon) \cdot (\mathbf{U}_\varepsilon - \mathbf{u}) \, \mathrm{d}x + E_1(\varrho, \mathbf{U}_\varepsilon, u_{NS}, \mathbf{u}) \\
= & \int_{\Omega_\varepsilon} \operatorname{div}_x \mathbf{U}_\varepsilon \Big(p(\varrho) - p'(\varrho_{NS})(\varrho - \varrho_{NS}) - p(\varrho_{NS}) \Big) \, \mathrm{d}x \\
& - \int_{\Omega_\varepsilon} \varrho (\mathbf{u} - \mathbf{U}_\varepsilon) \cdot \nabla_x \mathbf{U}_\varepsilon \cdot (\mathbf{U}_\varepsilon - \mathbf{u}) \, \mathrm{d}x \\
& + \int_{\Omega_\varepsilon} \frac{1}{\varrho_{NS}} \Big(\varrho - \varrho_{NS} \Big) \left(\nu \partial_z^2 u_{NS} + (\mu/3 + \eta) \partial_z (\partial_z (\ln A) u_{NS}) \right) \Big(u_{NS} - u_3 \Big) \, \mathrm{d}x \\
& + E_1(\varrho, \mathbf{U}_\varepsilon, u_{NS}, \mathbf{u}) + E_2(\mathbf{U}_\varepsilon, u_{NS}, \mathbf{u}),
\end{aligned} \tag{3.10}
$$

where we have set:

$$E_2(\mathbf{U}_\varepsilon, u_{NS}, \mathbf{u}) = \int_{\Omega_\varepsilon} \left(\nu\partial_z^2 u_{NS} + (\mu/3+\eta)\partial_z(\partial_z(\ln A)u_{NS})\right)\Big(u_{NS} - u_3\Big)\ \mathrm{d}x$$
$$- \int_{\Omega_\varepsilon} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}_\varepsilon) \cdot (\mathbf{U}_\varepsilon - \mathbf{u})\ \mathrm{d}x.$$

3.3 Estimates of the error terms

Our goal is to show that the error terms E_1, E_2 vanish in the asymptotic limit $\varepsilon \to 0$. As for E_1, we first observe that:

$$\sup_{x\in\Omega_\varepsilon} |\mathbf{U}_\varepsilon - \mathbf{u}_E| = \sup_{x\in\Omega_\varepsilon} |u_E \mathbf{V}_{h,\varepsilon}| \leq C\varepsilon.$$

Moreover, seeing that:

$$\partial_t \mathbf{U}_\varepsilon - \partial_t \mathbf{u}_E = \partial_t u_E \mathbf{V}_{h,\varepsilon},$$

we deduce:

$$\|\partial_t \mathbf{U}_\varepsilon - \partial_t \mathbf{u}_E\|_{C([0,T]\times\overline{\Omega}_\varepsilon)} + \|\mathbf{U}_\varepsilon \cdot \nabla_x \mathbf{U}_\varepsilon - u_E \partial_z \mathbf{U}_\varepsilon\|_{C([0,T]\times\overline{\Omega}_\varepsilon)} \leq C\varepsilon.$$

Finally, we estimate:

$$\|u_E \partial_z \mathbf{U}_\varepsilon - \mathbf{u}_E \cdot \nabla_x \mathbf{u}_E\|_{C([0,T]\times\overline{\Omega}_\varepsilon)} = \|u_E \partial_z \left(u_E \mathbf{V}_{h,\varepsilon}\right)\|_{C([0,T]\times\overline{\Omega}_\varepsilon)} \leq C\varepsilon,$$

obtaining:

$$|E_1(\varrho, \mathbf{U}_\varepsilon, u_E, \mathbf{u})| \leq C\varepsilon \int_{\Omega_\varepsilon} \varrho|\mathbf{U}_\varepsilon - \mathbf{u}|\ \mathrm{d}x, \tag{3.11}$$

provided that u_E is continuously differentiable in $[0,T]\times[0,1]$. Similarly, we can show that:

$$|E_1(\varrho, \mathbf{U}_\varepsilon, u_{NS}, \mathbf{u})| \leq C\varepsilon \int_{\Omega_\varepsilon} \varrho|\mathbf{U}_\varepsilon - \mathbf{u}|\ \mathrm{d}x \tag{3.12}$$

provided u_{NS} is continuously differentiable in $[0,T]\times[0,1]$.

To control E_2, we use:

$$\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}_\varepsilon) = \mu\Delta \mathbf{U}_\varepsilon + \left(\frac{\mu}{3} + \eta\right)\nabla_x \mathrm{div}_x \mathbf{U}_\varepsilon \quad \text{and} \quad \mathbf{U}_\varepsilon(\mathbf{x}_h, z) = [\mathbf{V}_{h,\varepsilon}(\mathbf{x}_h, z), 1]u_{NS}(z)$$

and we write:

$$
\begin{aligned}
\int_{\Omega_\varepsilon} \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{U}_\varepsilon) \cdot (\mathbf{U}_\varepsilon - \mathbf{u}) \, \mathrm{d}x &= \int_{\Omega_\varepsilon} \mu \Delta \mathbf{U}_\varepsilon \cdot (\mathbf{U}_\varepsilon - \mathbf{u}) + \left(\frac{\mu}{3} + \eta\right) \nabla_x \operatorname{div}_x \mathbf{U}_\varepsilon \cdot (\mathbf{U}_\varepsilon - \mathbf{u}) \, \mathrm{d}x \\
&= \int_{\Omega_\varepsilon} \mu [\Delta_h (\mathbf{V}_{h,\varepsilon}) u_{NS} + \partial_z^2 (\mathbf{V}_{h,\varepsilon} u_{NS})] \cdot (\mathbf{U}_\varepsilon - \mathbf{u})_h \, \mathrm{d}x \\
&\quad + \int_{\Omega_\varepsilon} \mu \partial_z^2 u_{NS} (u_{NS} - u_3) \, \mathrm{d}x \\
&\quad + \int_{\Omega_\varepsilon} \left(\frac{\mu}{3} + \eta\right) [\nabla_h \operatorname{div}_h (\mathbf{V}_{h,\varepsilon}) u_{NS} + \nabla_h \partial_z u_{NS}] \cdot (\mathbf{U}_\varepsilon - \mathbf{u})_h \, \mathrm{d}x \\
&\quad + \int_{\Omega_\varepsilon} \left(\frac{\mu}{3} + \eta\right) [\partial_z (\operatorname{div}_h (\mathbf{V}_{h,\varepsilon}) u_{NS}) + \partial_z^2 u_{NS}] (u_{NS} - u_3) \, \mathrm{d}x.
\end{aligned}
$$

Since $\mathbf{V}_{h,\varepsilon}(\mathbf{x}_h, z)$ is given by (3.5) with $\mathbf{V}_h$ satisfying (2.2), by assumptions of Theorem 2.5 the first and second derivative of $\mathbf{V}_{h,\varepsilon}$ in the z-variable are bounded by $C\varepsilon$. Moreover: $\Delta_h \mathbf{V}_{h,\varepsilon} = 0$, $\nabla_h \operatorname{div}_h (\mathbf{V}_{h,\varepsilon}) = 0$, and $|\partial_z^2 (\mathbf{V}_{h,\varepsilon} u_{NS})| \le C\varepsilon$ provided that $\partial_z^2 u_{NS}$ is bounded in $[0,T] \times [0,1]$. Since u_{NS} is a function of z only, we also see that $\nabla_h \partial_z u_{NS} = 0$. Using $\operatorname{div}_h \mathbf{V}_{h,\varepsilon} = \operatorname{div}_h \mathbf{V}_h = \partial_z(\ln(A))$ in view of (2.3), the above implies:

$$
\begin{aligned}
\Bigg| \int_{\Omega_\varepsilon} \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{U}_\varepsilon) \cdot (\mathbf{U}_\varepsilon - \mathbf{u}) \, \mathrm{d}x - \left(\frac{4}{3}\mu + \eta\right) \int_{\Omega_\varepsilon} \partial_z^2 u_{NS} (u_{NS} - u_3) \, \mathrm{d}x \\
- \left(\frac{\mu}{3} + \eta\right) \int_{\Omega_\varepsilon} \partial_z (\partial_z (\ln(A) u_{NS}) (u_{NS} - u_3) \, \mathrm{d}x \Bigg| \le C\varepsilon \int_{\Omega_\varepsilon} |\mathbf{u} - \mathbf{U}_\varepsilon| \, \mathrm{d}x.
\end{aligned} \tag{3.13}
$$

Consequently, we get:

$$
|E_2(\mathbf{U}_\varepsilon, u_{NS}, \mathbf{u})| \le C\varepsilon \int_{\Omega_\varepsilon} |\mathbf{u} - \mathbf{U}_\varepsilon| \, \mathrm{d}x \tag{3.14}
$$

provided that $\partial_z^2 u_{NS}$ is bounded in $[0,T] \times [0,1]$.

4 Convergence

Having collected the necessary material, we are now ready to complete the proofs of Theorems 2.4, 2.5. As the solutions of the limit systems are regular, we may assume:

$$
0 < \underline{\varrho} \le \varrho_E \le \overline{\varrho}, \qquad 0 < \underline{\varrho} \le \varrho_{NS} \le \overline{\varrho}
$$

for certain positive constants $\underline{\varrho}$, $\overline{\varrho}$. Next, it is convenient to introduce the *essential* and *residual* component of an integrable function h as:

$$
h_{\text{ess}} = \chi(\varrho) h, \qquad h_{\text{res}} = (1 - \chi(\varrho)) h,
$$

where:

$$\chi \in C_c^\infty(0,\infty), \qquad 0 \leq \chi \leq 1, \qquad \chi(z) = 1 \qquad \forall z \in [\underline{\varrho}/2, 2\overline{\varrho}].$$

4.1 Convergence to the Euler system - the proof of Theorem 2.4

It follows from the relative energy inequality (3.1), the coercivity (2.10), and the bounds (3.9), (3.11) that:

$$\begin{aligned}\Big[\mathcal{E}_\varepsilon\left(\varrho, \mathbf{u} \;\Big|\; \varrho_E, \mathbf{U}_\varepsilon\right)(t)\Big]_{t=0}^{t=\tau} &+ \lambda \int_0^\tau \int_{\Omega_\varepsilon} \left(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}_\varepsilon)\right) : \left(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}_\varepsilon\right) \,\mathrm{d}x \,\mathrm{d}t \\ &\leq C \int_0^\tau \mathcal{E}_\varepsilon\left(\varrho, \mathbf{u} \;\Big|\; \varrho_E, \mathbf{U}_\varepsilon\right)(t) \,\mathrm{d}t + C\varepsilon \int_0^\tau \int_{\Omega_\varepsilon} \varrho |\mathbf{u} - \mathbf{U}_\varepsilon| \,\mathrm{d}x \\ &\quad + \lambda \int_0^\tau \int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{U}_\varepsilon) : \nabla_x (\mathbf{U}_\varepsilon - \mathbf{u}) \,\mathrm{d}x \,\mathrm{d}t,\end{aligned}$$

where, furthermore:

$$\begin{aligned}\varepsilon \int_0^\tau \int_{\Omega_\varepsilon} \varrho |\mathbf{u} - \mathbf{U}_\varepsilon| \,\mathrm{d}x &\leq \frac{\varepsilon}{2} \int_0^\tau \int_{\Omega_\varepsilon} \varrho |\mathbf{u} - \mathbf{U}_\varepsilon|^2 \,\mathrm{d}x + \frac{\varepsilon}{2} \int_0^\tau \int_{\Omega_\varepsilon} \varrho \,\mathrm{d}x \\ &\leq C \left(\varepsilon |\Omega_\varepsilon| + \int_0^\tau \mathcal{E}\left(\varrho, \mathbf{u} \;\Big|\; \varrho_E, \mathbf{U}_\varepsilon\right)(t) \,\mathrm{d}t\right).\end{aligned}$$

Next, setting $\tilde{\mathbf{u}} := \mathbf{U}_\varepsilon - \mathbf{u}$ for notational convenience, we write:

$$\begin{aligned}\mathbb{S}(\nabla_x \mathbf{U}_\varepsilon) : \nabla_x (\mathbf{U}_\varepsilon - \mathbf{u}) &= \mu(\nabla_x \mathbf{U}_\varepsilon + \nabla_x^t \mathbf{U}_\varepsilon - \frac{2}{3} \mathrm{div}_x \mathbf{U}_\varepsilon \mathbb{I}) : \nabla_x \tilde{u} + \eta \mathrm{div}_x \mathbf{U}_\varepsilon \mathrm{div}_x \tilde{u} \\ &= \frac{\mu}{2}(\nabla_x \mathbf{U}_\varepsilon + \nabla_x^t \mathbf{U}_\varepsilon - \frac{2}{3} \mathrm{div}_x \mathbf{U}_\varepsilon \mathbb{I}) : (\nabla_x \tilde{\mathbf{u}} + \nabla_x^t \tilde{\mathbf{u}} - \frac{2}{3} \mathrm{div}_x \tilde{\mathbf{u}}) + \eta \mathrm{div}_x \mathbf{U}_\varepsilon \mathrm{div}_x \tilde{u},\end{aligned}$$

where we used the fact that $\nabla_x \mathbf{U}_\varepsilon + \nabla_x^t \mathbf{U}_\varepsilon - \frac{2}{3} \mathrm{div}_x \mathbf{U}_\varepsilon \mathbb{I}$ is symmetric and traceless to smuggle in $\nabla_x^t \tilde{\mathbf{u}}$ and $\frac{2}{3} \mathrm{div}_x \tilde{\mathbf{u}}$. In a similar way, we observe that:

$$\left(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}_\varepsilon)\right) : \left(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}_\varepsilon\right) = \mathbb{S}(\nabla_x \tilde{\mathbf{u}}) : \nabla_x \tilde{\mathbf{u}} = \frac{\mu}{2} \left| \nabla_x \tilde{\mathbf{u}} + \nabla_x^t \tilde{\mathbf{u}} - \frac{2}{3} \mathrm{div}_x \tilde{\mathbf{u}} \mathbb{I} \right|^2 + \eta |\mathrm{div}_x \tilde{\mathbf{u}}|^2,$$

and so, using the above, we may estimate:

$$\begin{aligned}&\left| \lambda \int_0^\tau \int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{U}_\varepsilon) : \nabla_x (\mathbf{U}_\varepsilon - \mathbf{u}) \,\mathrm{d}x \,\mathrm{d}t \right| \\ &\qquad \leq C\lambda \int_0^\tau \int_{\Omega_\varepsilon} |\nabla_x \mathbf{U}_\varepsilon|^2 \,\mathrm{d}x + \frac{\lambda}{2} \int_0^\tau \int_{\Omega_\varepsilon} \frac{\mu}{2} \left| \nabla_x \tilde{\mathbf{u}} + \nabla_x^t \tilde{\mathbf{u}} - \frac{2}{3} \mathrm{div}_x \tilde{\mathbf{u}} \mathbb{I} \right|^2 + \eta |\mathrm{div}_x \tilde{\mathbf{u}}|^2 \,\mathrm{d}x \\ &\qquad \leq C\lambda |\Omega_\varepsilon| + \frac{\lambda}{2} \int_0^\tau \int_{\Omega_\varepsilon} \left(\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}_\varepsilon)\right) : \left(\nabla_x \mathbf{u} - \nabla_x \mathbf{U}_\varepsilon\right) \,\mathrm{d}x.\end{aligned}$$

Combining the previous estimates and a Gronwall-type argument, we conclude:

$$\mathcal{E}_\varepsilon\left(\varrho,\mathbf{u}\;\Big|\;\varrho_E,\mathbf{U}_\varepsilon\right)(\tau)\leq C\left[(\varepsilon+\lambda)\,|\Omega_\varepsilon|+\mathcal{E}_\varepsilon\left(\varrho,\mathbf{u}\;\Big|\;\varrho_E,\mathbf{U}_\varepsilon\right)(0)\right]\qquad\forall 0\leq\tau\leq T,$$

from which we easily deduce (2.12). We have proved Theorem 2.4. $\square$

4.2 Convergence to the Navier-Stokes system - the proof of Theorem 2.5

Proving similar estimates for the Navier-Stokes limit is more delicate. We start observing that (3.10), (3.12) together with (3.14) and the coercivity property (2.10), give rise to:

$$\begin{aligned}
&\left[\mathcal{E}_\varepsilon\left(\varrho,\mathbf{u}\;\Big|\;\varrho_{NS},\mathbf{U}_\varepsilon\right)(t)\right]_{t=0}^{t=\tau}+\int_0^\tau\int_{\Omega_\varepsilon}\left(\mathbb{S}(\nabla_x\mathbf{u})-\mathbb{S}(\nabla_x\mathbf{U}_\varepsilon)\right):\left(\nabla_x\mathbf{u}-\nabla_x\mathbf{U}_\varepsilon\right)\,\mathrm{d}x\,\mathrm{d}t\\
&\leq C\int_0^\tau\mathcal{E}_\varepsilon\left(\varrho,\mathbf{u}\;\Big|\;\varrho_{NS},\mathbf{U}_\varepsilon\right)(t)\,\mathrm{d}t+C\varepsilon\int_0^\tau\int_{\Omega_\varepsilon}\varrho|\mathbf{u}-\mathbf{U}_\varepsilon|\,\mathrm{d}x+C\varepsilon\int_0^\tau\int_{\Omega_\varepsilon}|\mathbf{u}-\mathbf{U}_\varepsilon|\,\mathrm{d}x\\
&+\int_0^\tau\int_{\Omega_\varepsilon}\frac{1}{\varrho_{NS}}\Big(\varrho-\varrho_{NS}\Big)\left((4\mu/3+\eta)\,\partial_z^2u_{NS}+(\mu/3+\eta)\partial_z(\partial_z(\ln A)u_{NS})\right)\Big(u_{NS}-u_3\Big)\,\mathrm{d}x\mathrm{d}t,
\end{aligned}\tag{4.1}$$

where the integral in the last line, using the notation:

$$F(z):=\left((4\mu/3+\eta)\partial_z^2u_{NS}+(\mu/3+\eta)\partial_z(\partial_z(\ln A)u_{NS})\right),$$

the fact that $|F|\leq C$ and (2.10), can be estimated by the following:

$$\begin{aligned}
&\left|\int_{\Omega_\varepsilon}\frac{F}{\varrho_{NS}}\Big(\varrho-\varrho_{NS}\Big)\Big(u_{NS}-u_3\Big)\,\mathrm{d}x\right|\\
&\qquad\leq\left|\int_{\Omega_\varepsilon}\frac{F}{\varrho_{NS}}\Big[\varrho-\varrho_{NS}\Big]_{\mathrm{ess}}\Big[u_{NS}-u_3\Big]_{\mathrm{ess}}\,\mathrm{d}x\right|+\left|\int_{\Omega_\varepsilon}\frac{F}{\varrho_{NS}}\Big[\varrho-\varrho_{NS}\Big]_{\mathrm{res}}\Big[u_{NS}-u_3\Big]_{\mathrm{res}}\,\mathrm{d}x\right|\\
&\qquad\leq C\left[\mathcal{E}\left(\varrho,\mathbf{u}\;\Big|\;\varrho_{NS},\mathbf{U}_\varepsilon\right)+C(\delta)\int_{\Omega_\varepsilon}|[\varrho-\varrho_{NS}]_{\mathrm{res}}|\,\mathrm{d}x\right]+\delta\int_{\Omega_\varepsilon}(1+\varrho)\,|[\mathbf{u}-\mathbf{U}_\varepsilon]_{\mathrm{res}}|^2\,\mathrm{d}x\\
&\qquad\leq C(\delta)\mathcal{E}\left(\varrho,\mathbf{u}\;\Big|\;\varrho_{NS},\mathbf{U}_\varepsilon\right)+\delta\int_{\Omega_\varepsilon}|\mathbf{u}-\mathbf{U}_\varepsilon|^2\,\mathrm{d}x,
\end{aligned}$$

for any $\delta>0$. Applying a similar treatment to the remaining integrals in (4.1), we obtain that:

$$\begin{aligned}
&\left[\mathcal{E}_\varepsilon\left(\varrho,\mathbf{u}\;\Big|\;\varrho_{NS},\mathbf{U}_\varepsilon\right)(t)\right]_{t=0}^{t=\tau}+\int_0^\tau\int_{\Omega_\varepsilon}\left(\mathbb{S}(\nabla_x\mathbf{u})-\mathbb{S}(\nabla_x\mathbf{U}_\varepsilon)\right):\left(\nabla_x\mathbf{u}-\nabla_x\mathbf{U}_\varepsilon\right)\,\mathrm{d}x\,\mathrm{d}t\\
&\qquad\leq C(\delta)\left[\int_0^\tau\mathcal{E}\left(\varrho,\mathbf{u}\;\Big|\;\varrho_{NS},\mathbf{U}_\varepsilon\right)(t)\,\mathrm{d}t+\varepsilon|\Omega_\varepsilon|\right]+(\varepsilon+\delta)\int_0^\tau\int_{\Omega_\varepsilon}|\mathbf{u}-\mathbf{U}_\varepsilon|^2\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

for any $\delta > 0$. Consequently, in order to conclude, we use the following variant of *Korn-Poincaré inequality*:

$$\int_{\Omega_\varepsilon} \left|\nabla_x \mathbf{v} + \nabla_x^T \mathbf{v}\right|^2 \, \mathrm{d}x \geq C \int_{\Omega_\varepsilon} |\mathbf{v}|^2 \, \mathrm{d}x, \tag{4.2}$$

with a constant C *independent of* $\varepsilon \to 0$, see Theorem 5.1 in Section 5. This allows to estimate $\int_0^\tau \int_{\Omega_\varepsilon} |\mathbf{u} - \mathbf{U}_\varepsilon|^2 \, \mathrm{d}x \, \mathrm{d}t$ with $\int_0^\tau \int_{\Omega_\varepsilon} (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}_\varepsilon)) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}_\varepsilon) \, \mathrm{d}x \, \mathrm{d}t$. For that to work we had to assume that the bulk viscosity coefficient η is strictly positive, since otherwise (4.2) would need to be replaced with its conformal version (1.11).

Finally, as a consequence of a Gronwall-type argument, we obtain:

$$\mathcal{E}_\varepsilon \left(\varrho, \mathbf{u} \;\middle|\; \varrho_{NS}, \mathbf{U}_\varepsilon\right)(\tau) \leq C \left[\varepsilon|\Omega_\varepsilon| + \mathcal{E}_\varepsilon \left(\varrho, \mathbf{u} \;\middle|\; \varrho_{NS}, \mathbf{U}_\varepsilon\right)(0)\right] \qquad \forall 0 \leq \tau \leq T,$$

completing the proof of Theorem 2.5. $\square$

5 A Korn inequality in thin channels

In this section we discuss various variants of Korn and Korn-Poincaré inequalities that may be of independent interest. In particular, we show the Korn-Poincaré inequality (4.2). We assume that:

$$\Omega_\varepsilon = \left\{ x = (\varepsilon \mathbf{x}_h, z) \;\middle|\; z \in (0,1), \; \mathbf{x}_h \in \omega_h(z) \right\} \subset \mathbb{R}^n, \tag{5.1}$$

where $\{\omega_h(z)\}_{z \in [0,1]}$ is a uniformly Lipschitz family of simply connected bounded domains $\omega(z) \subset \mathbb{R}^{n-1}$, such that the boundary of Ω_1 is Lipschitz. We use the following notation: $\mathrm{sym}\,\mathbb{M} = \frac{1}{2}(\mathbb{M}+\mathbb{M}^t)$ and $\mathrm{skew}\,\mathbb{M} = \frac{1}{2}(\mathbb{M} - \mathbb{M}^t)$ for the symmetric and the skew-symmetric parts of a given matrix $\mathbb{M} \in \mathbb{R}^{n\times n}$, and $so(n)$ for the space of all skew-symmetric matrices $\mathbb{M} = \mathrm{skew}\,\mathbb{M} \in \mathbb{R}^{n\times n}$.

Theorem 5.1. *Let $\mathbf{v} \in W^{1,2}(\Omega_\varepsilon; \mathbb{R}^n)$ satisfy:*

$$\mathbf{v} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \quad \mathbf{v}(\mathbf{x}_h, z) = 0 \qquad \forall z \in \{0,1\}, \; \mathbf{x}_h \in \varepsilon\omega(z). \tag{5.2}$$

Then, we have the following bounds with a constant C independent of ε and $\mathbf{v}$:

$$\int_{\Omega_\varepsilon} |\nabla_x \mathbf{v}|^2 \, \mathrm{d}x \leq \frac{C}{\varepsilon^2} \int_{\Omega_\varepsilon} |\mathrm{sym}\nabla_x \mathbf{v}|^2 \, \mathrm{d}x \tag{5.3}$$

$$\int_{\Omega_\varepsilon} |\mathbf{v}|^2 \, \mathrm{d}x \leq C \int_{\Omega_\varepsilon} |\mathrm{sym}\nabla_x \mathbf{v}|^2 \, \mathrm{d}x. \tag{5.4}$$

5.1 An approximation theorem

Towards the proof of Theorem 5.1, we first recall the classical Korn's inequality:

Theorem 5.2. *Let $\Omega \subset \mathbb{R}^n$ be an open bounded connected and Lipschitz domain. For every $\mathbf{v} \in W^{1,2}(\Omega;\mathbb{R}^n)$ there exists a matrix $\mathbb{A} \in so(n)$ such that:*

$$\int_\Omega |\nabla_x \mathbf{v} - \mathbb{A}|^2 \, \mathrm{d}x \leq C \int_\Omega |\mathrm{sym}\nabla_x \mathbf{v}|^2 \, \mathrm{d}x. \tag{5.5}$$

The constant C above depends only on the domain Ω, but not on $\mathbf{v}$. The constant is invariant under dilations of Ω and it is uniform for the class of domains that are bilipschitz equivalent with controlled Lipschitz constants.

It is easy to check that the optimal $\mathbb{A}$ in the left hand side of (5.5) equals $\mathbb{A} = \mathrm{skew} \fint_\Omega \nabla_x \mathbf{v} \, \mathrm{d}x$. Armed with this observation, we derive a fine approximation of $\nabla_x \mathbf{v}$ that is suitable for the thin limit problem in Theorem 5.1. This approach is motivated by a similar construction in [7].

Theorem 5.3. *Let $\mathbf{v} \in W^{1,2}(\Omega_\varepsilon;\mathbb{R}^n)$ satisfy the boundary conditions (5.2). Then, there exists a smooth mapping $\mathbb{A} : [0,1] \to so(n)$ such that:*

$$\int_{\Omega_\varepsilon} |\nabla_x \mathbf{v}(\mathbf{x}_h, z) - \mathbb{A}(z)|^2 \, \mathrm{d}x \leq C \int_{\Omega_\varepsilon} |\mathrm{sym}\nabla_x \mathbf{v}|^2 \, \mathrm{d}x, \tag{5.6}$$

$$\int_0^1 |\mathbb{A}|^2 \, \mathrm{d}z + \int_0^1 |\partial_z \mathbb{A}|^2 \, \mathrm{d}z \leq \frac{C}{\varepsilon^2} \fint_{\Omega_\varepsilon} |\mathrm{sym}\nabla_x \mathbf{v}|^2 \, \mathrm{d}x, \tag{5.7}$$

where the constant C above is independent of ε and $\mathbf{v}$.

Proof. **1.** We identify $\mathbf{v}$ with its extension on an infinite curvilinear cylinder as in (5.1) with $z \in \mathbb{R}$, where we put $\omega_h(z) = \omega_h(0)$ for $z < 0$, $\omega_h(z) = \omega_h(1)$ for $z > 1$ and $\mathbf{v}(\mathbf{x}_h, z) = 0$ for $z < 0$ and $z > 1$, and $\mathbf{x}_h \in \varepsilon\omega(z)$. For each $z_0 \in \mathbb{R}$, we define the sets:

$$B_{z_0,\varepsilon} = \Big\{ x = (\mathbf{x}_h, z) \;\Big|\; z \in (z_0 - \varepsilon, z_0 + \varepsilon), \; \mathbf{x}_h \in \varepsilon\omega_h(z) \Big\},$$

and the approximation fields:

$$\tilde{\mathbb{A}}(z_0) = \fint_{\varepsilon\omega_h(z)} \mathrm{skew}\nabla_x \mathbf{v}(\mathbf{x}_h, z_0) \, \mathrm{d}\mathbf{x}_h \quad \text{ and } \quad \mathbb{A} = \kappa_\varepsilon * \tilde{\mathbb{A}}$$

by means of a convolution with a regularization kernel $\kappa_\varepsilon = \kappa_\varepsilon(z)$. We set $\kappa_\varepsilon(z) = \frac{1}{\varepsilon}\kappa(\frac{z}{\varepsilon})$ for some smooth nonnegative $\kappa \in C_c^\infty$ supported in $(-\frac{1}{2}, \frac{1}{2})$ and with integral 1. Note that $\mathbb{A} \in C_c^\infty(\mathbb{R}; so(n))$ and in particular:

$$\mathbb{A}(-1) = \mathbb{A}(2) = 0. \tag{5.8}$$

Application of Korn's inequality (5.5) on sets $B_{z_0,\varepsilon}$ gives:

$$\fint_{B_{z_0,\varepsilon}} |\nabla_x \mathbf{v} - \mathbb{A}_{z_0,\varepsilon}|^2 \,\mathrm{d}x \leq C \fint_{B_{z_0,\varepsilon}} |\mathrm{sym}\nabla_x \mathbf{v}|^2 \,\mathrm{d}x, \tag{5.9}$$

with a uniform constant C (independent of z_0, ε and $\mathbf{v}$) and some appropriate $\mathbb{A}_{z_0,\varepsilon} \in so(n)$. Note that for every $z' \in \mathbb{R}$ we have:

$$\tilde{\mathbb{A}}(z') - \mathbb{A}_{z_0,\varepsilon} = \fint_{\varepsilon\omega_h(z')} \mathrm{skew}\nabla_x \mathbf{v} - \mathbb{A}_{z_0,\varepsilon} \,\mathrm{d}\mathbf{x}_h = \fint_{\varepsilon\omega_h(z')} \nabla_x \mathbf{v} - \mathbb{A}_{z_0,\varepsilon} - \mathrm{sym}\nabla_x \mathbf{v} \,\mathrm{d}\mathbf{x}_h.$$

Using the above for $z' \in (z_0 - \varepsilon, z_0 + \varepsilon)$ we obtain, in view of (5.9):

$$\begin{aligned}
|\mathbb{A}(z'') - \mathbb{A}_{z_0,\varepsilon}|^2 &= \Big|(\kappa_\varepsilon * (\tilde{\mathbb{A}} - \mathbb{A}_{z_0,\varepsilon}))(z'')\Big|^2 \leq C \fint_{z''-\varepsilon/2}^{z''+\varepsilon/2} |\tilde{\mathbb{A}}(z') - \mathbb{A}_{z_0,\varepsilon}|^2 \,\mathrm{d}z' \\
&\leq C \fint_{z''-\varepsilon/2}^{z''+\varepsilon/2} \fint_{\varepsilon\omega_h(z')} |\nabla_x \mathbf{v} - \mathbb{A}_{z_0,\varepsilon} - \mathrm{sym}\nabla_x \mathbf{v}|^2 \,\mathrm{d}\mathbf{x}_h \,\mathrm{d}z' \\
&\leq C \fint_{B_{z_0,\varepsilon}} |\nabla_x \mathbf{v} - \mathbb{A}_{z_0,\varepsilon}|^2 + |\mathrm{sym}\nabla_x \mathbf{v}|^2 \,\mathrm{d}x \\
&\leq C \fint_{B_{z_0,\varepsilon}} |\mathrm{sym}\nabla_x \mathbf{v}|^2 \,\mathrm{d}x \qquad\qquad \forall z'' \in (z_0 - \frac{\varepsilon}{2}, z_0 + \frac{\varepsilon}{2}).
\end{aligned} \tag{5.10}$$

Similarly, we deal with the derivative $\partial_z \mathbb{A}$:

$$\begin{aligned}
|\partial_z \mathbb{A}(z'')|^2 &= \Big|\partial_z(\kappa_\varepsilon * (\tilde{\mathbb{A}} - \mathbb{A}_{z_0,\varepsilon}))(z'')\Big|^2 = \Big|((\partial_z \kappa_\varepsilon) * (\tilde{\mathbb{A}} - \mathbb{A}_{z_0,\varepsilon}))(z'')\Big|^2 \\
&\leq \frac{C}{\varepsilon^2} \fint_{B_{z_0,\varepsilon}} |\mathrm{sym}\nabla_x \mathbf{v}|^2 \,\mathrm{d}x \qquad\qquad \forall z'' \in (z_0 - \frac{\varepsilon}{2}, z_0 + \frac{\varepsilon}{2}).
\end{aligned} \tag{5.11}$$

2. We now estimate, by (5.9) and (5.10):

$$\begin{aligned}
\fint_{B_{z_0,\varepsilon/2}} |\nabla_x \mathbf{v}(\mathbf{x}_h, z) - \mathbb{A}(z)|^2 \,\mathrm{d}x &\leq C\Big(\fint_{B_{z_0,\varepsilon/2}} |\nabla_x \mathbf{v} - \mathbb{A}_{z_0,\varepsilon}|^2 \,\mathrm{d}x + \fint_{z_0-\varepsilon/2}^{z_0+\varepsilon/2} |\mathbb{A}(z'') - \mathbb{A}_{z_0,\varepsilon}|^2 \,\mathrm{d}z''\Big) \\
&\leq C\Big(\fint_{B_{z_0,\varepsilon}} |\nabla_x \mathbf{v} - \mathbb{A}_{z_0,\varepsilon}|^2 \,\mathrm{d}x + \fint_{z_0-\varepsilon/2}^{z_0+\varepsilon/2} \fint_{B_{z_0,\varepsilon}} |\mathrm{sym}\nabla_x \mathbf{v}|^2 \,\mathrm{d}x \,\mathrm{d}z''\Big), \\
&\leq C \fint_{B_{z_0,\varepsilon}} |\mathrm{sym}\nabla_x \mathbf{v}|^2 \,\mathrm{d}x,
\end{aligned}$$

which implies (5.6) through an easy covering argument. Likewise, (5.11) yields:

$$\fint_{z_0-\varepsilon/2}^{z_0+\varepsilon/2} |\partial_z \mathbb{A}(z'')|^2 \, \mathrm{d}z'' \leq \frac{C}{\varepsilon^2} \fint_{z_0-\varepsilon/2}^{z_0+\varepsilon/2} \fint_{B_{z_0,\varepsilon}} |\mathrm{sym}\nabla_x \mathbf{v}|^2 \, \mathrm{d}x \, \mathrm{d}z'' = \frac{C}{\varepsilon^2} \fint_{B_{z_0,\varepsilon}} |\mathrm{sym}\nabla_x \mathbf{v}|^2 \, \mathrm{d}x,$$

and a further covering argument results in:

$$\int_{-1}^{2} |\partial_z \mathbb{A}|^2 \, \mathrm{d}z \leq \frac{C}{\varepsilon^2} \fint_{\Omega_\varepsilon} |\mathrm{sym}\nabla_x \mathbf{v}|^2 \, \mathrm{d}x.$$

Using Poincaré's inequality to the function $\mathbb{A}$ and noting (5.8), we finally obtain (5.7). $\square$

5.2 A uniform Poincaré inequality for vector fields

In the proof of Theorem 5.1 we need yet another result, which is a Poincaré inequality for vector fields that are tangent on the boundary of $\omega_h(z)$ (see (5.1)), and with constant independent of $z \in [0, 1]$. Let us point out that there are many results [2, 3, 15] regarding the dependence of C on an open bounded connected and Lipschitz set $\Omega \subset \mathbb{R}^n$ in:

$$\int_\Omega |v - \fint_\Omega v|^2 \, \mathrm{d}x \leq C \int_\Omega |\nabla_x v|^2 \, \mathrm{d}x \qquad \forall v \in W^{1,2}(\Omega). \tag{5.12}$$

These results are linked to the fact that the smallest C in (5.12) is the inverse of the first nonzero eigenvalue λ_2 of the Neumann problem for $-\Delta$ on Ω. It is then known [3], that $\lambda_2 \geq C_n \frac{r^n}{\bar{r}^{n+2}}$ where r and $\bar{r}$ are the inner and outer radii of the star-shaped Ω.

Further, in [2] it has been proved that (5.12) is valid with C that is uniform for all Ω which are uniformly Lipschitz with uniformly bounded diameter. More precisely, C depends only on constants n, $\bar{r}$, γ and M below, for any open and connected Ω satisfying the following two conditions:

(i) Ω is a subset of the ball $B(0, \bar{r}) \subset \mathbb{R}^n$.

(ii) At each point $x \in \partial\Omega$ there exists a local orthonormal coordinate system such that writing, in this system, $x = (\hat{x}, x^n)$ we have the following. There exists a Lipschitz function $\phi : \hat{\mathcal{O}} \to \mathbb{R}$ with Lipschitz constant M and we have:

$$z = (\hat{z}, z^n) \in \mathcal{O} \cap \Omega \qquad \text{if and only if} \qquad z \in \mathcal{O} \;\text{ and }\; z^n > \phi(\hat{z}),$$

where we denoted:

$$\hat{\mathcal{O}} = \big\{\hat{z} \in \mathbb{R}^{n-1} \;\big|\; |(\hat{z} - \hat{x}) \cdot e_i| < \gamma \;\text{ for all } i = 1, \ldots, n-1\big\}$$
$$\mathcal{O} = \big\{z = (\hat{z}, z^n) \in \mathbb{R}^n \;\big|\; \hat{z} \in \hat{\mathcal{O}} \;\text{ and }\; |z^n - x^n| < M\gamma\sqrt{n-1}\big\}.$$

Note that boundary of each Ω as above is uniformly Lipschitz continuous. This results had been recently extended in [15] to more general classes of domains, that are uniformly bounded in: the diameter, the interior cone condition, and an appropriate measure of connectedness.

We now deduce the needed vectorial Poincaré inequality:

Theorem 5.4. *Let $\Omega \subset \mathbb{R}^n$ be an open connected domain satisfying conditions (i) and (ii) above. Let $\mathbf{v} \in W^{1,2}(\Omega;\mathbb{R}^n)$ satisfy $\mathbf{v}\cdot\mathbf{n}|_{\partial\Omega} = 0$. Then:*

$$\int_\Omega |\mathbf{v}|^2 \,\mathrm{d}x \leq C \int_\Omega |\nabla_x \mathbf{v}|^2 \,\mathrm{d}x, \tag{5.13}$$

with C independent of $\mathbf{v}$ and depending on Ω only through n, $\bar{r}$, γ and M.

Proof. It is easy to note that conditions (i) and (ii) ensure the following uniform bound:

$$|\mathbf{a}|^2 \leq C \int_{\partial\Omega} |\mathbf{a}\cdot\mathbf{n}|^2 \qquad \forall \mathbf{a} \in \mathbb{R}^n. \tag{5.14}$$

Indeed, sliding the plane perpendicular to $\mathbf{a}$ along the direction $\mathbf{a}$, at the first point $x \in \partial\Omega$ where this plane touches the boundary, vector $\mathbf{a}$ has scalar product bounded away from zero, with every element of Clarke's subdifferential of ϕ at $\hat{x}$. Consequently, C in (5.14) depends only on n, γ and M.

Applying (5.14) to the vector $\mathbf{a} = \fint_\Omega \mathbf{v}\,\mathrm{d}x$, we get the following chain of uniform inequalities:

$$\Big|\fint_\Omega \mathbf{v}\Big|^2 \leq C\int_{\partial\Omega} \Big(\Big(\fint_\Omega \mathbf{v}\Big)\cdot\mathbf{n}\Big)^2 = C\int_{\partial\Omega}\Big(\Big(\mathbf{v} - \fint_\Omega \mathbf{v}\Big)\cdot\mathbf{n}\Big)^2 \leq C\int_{\partial\Omega}\Big|\mathbf{v} - \fint_\Omega \mathbf{v}\Big|^2 \leq C\Big\|\mathbf{v} - \fint_\Omega \mathbf{v}\Big\|^2_{W^{1,2}(\Omega;\mathbb{R}^n)},$$

where the last bound follows from the trace theorem [1]. The quoted above result in [2] now implies:

$$\Big|\fint_\Omega \mathbf{v}\,\mathrm{d}x\Big|^2 \leq C\Big(\Big\|\mathbf{v} - \fint_\Omega \mathbf{v}\Big\|^2_{L^2(\Omega;\mathbb{R}^n)} + \|\nabla_x \mathbf{v}\|^2_{L^2(\Omega;\mathbb{R}^n)}\Big) \leq C\int_\Omega |\nabla_x \mathbf{v}|^2\,\mathrm{d}x,$$

resulting in:

$$\int_\Omega |\mathbf{v}|^2\,\mathrm{d}x \leq 2\int_\Omega \Big|\mathbf{v} - \fint_\Omega \mathbf{v}\Big|^2\,\mathrm{d}x + 2|\Omega|\Big|\fint_\Omega \mathbf{v}\,\mathrm{d}x\Big|^2 \leq C\int_\Omega |\nabla_x \mathbf{v}|^2\,\mathrm{d}x$$

and establishing the proof. ∎

Remark 5.5. Note that the uniformity assumptions of Theorem 5.4 clearly hold for the family of cross sections $\{\omega_h(z)\}_{z\in[0,1]}$ because Ω_1 is Lipschitz. In this case, one can alternatively deduce (5.13) by an argument by contradiction that we now sketch.

Assume that there was a sequence $z_k \in [0,1]$, converging to some z_0 and such that $\int_{\omega_h(z_k)} |\mathbf{v}_k|^2 = 1$ but $\int_{\omega_h(z_k)} |\nabla_x \mathbf{v}_k|^2 \leq 1/k$ for some vector fields $\mathbf{v}_k \in W^{1,2}(\omega_h(z_k); \mathbb{R}^{n-1})$ each tangential on the boundary $\partial\omega_h(z_k)$ of its own domain. The uniform Lipschitz continuity of Ω_1 ensures that extending $\mathbf{v}_k$ on the large ball $B = B(0,\bar{r})$ that contains all sets $\omega_h(z_k)$, still obeys the uniform bound $\|\mathbf{v}_k\|_{W^{1,2}(B;\mathbb{R}^{n-1})} \leq C$. Thus without loss of generality $\mathbf{v}_k$ converges to some $\mathbf{v}_0$, weakly in $W^{1,2}(B;\mathbb{R}^{n-1})$. Existence of the Lipschitz continuous homotopy between sets $\omega_h(z_n)$ allows now to deduce that this implies $\int_{\omega_h(z_0)} |\nabla_x \mathbf{v}_0|^2 = 0$ and $\mathbf{v}_0 \cdot \mathbf{n} = 0$ on $\partial\omega_h(z_0)$. Consequently, $\mathbf{v}_0 = 0$ in $\omega_h(z_0)$, contradicting the assumption $\int_{\omega_h(z_k)} |\mathbf{v}_k|^2 = 1$.

5.3 The proof of Theorem 5.1

Let $\mathbb{A} : [0,1] \to so(n)$ be the approximation function in Theorem 5.3. Using (5.6) and (5.7) we get:

$$\fint_{\Omega_\varepsilon} |\nabla_x \mathbf{v}|^2 \,\mathrm{d}x \leq C\Big(\fint_{\Omega_\varepsilon} |\nabla_x \mathbf{v}(\mathbf{x}_h, z) - \mathbb{A}(z)|^2 \,\mathrm{d}x + \int_0^1 |\mathbb{A}(z)|^2 \,\mathrm{d}z\Big) \leq \frac{C}{\varepsilon^2} \fint_{\Omega_\varepsilon} |\mathrm{sym}\nabla_x \mathbf{v}|^2 \,\mathrm{d}x,$$

which establishes (5.3). Towards proving (5.4), define for a smooth curve $X : [0,1] \to \mathbb{R}^{n-1}$ such that $(X(z), z) \in \Omega_1$ for all $z \in (0,1)$, the following set:

$$S_{r,\varepsilon} = \Big\{ x = (\mathbf{x}_h, z) \;\Big|\; z \in (0,1),\ \mathbf{x}_h \in \varepsilon B(X(z), r) \Big\}.$$

Clearly, $S_{r,\varepsilon} \subset \Omega_\varepsilon$ for a sufficiently small $r > 0$. We have the following Poincaré inequality:

$$\fint_{\Omega_1} |v|^2 \,\mathrm{d}x \leq C\Big(\fint_{\Omega_1} |\nabla_x v|^2 \,\mathrm{d}x + \fint_{S_{r,1}} |v|^2 \,\mathrm{d}x\Big) \qquad \forall v \in W^{1,2}(\Omega_1),$$

which by an easy scaling argument translates to:

$$\fint_{\Omega_\varepsilon} |v|^2 \,\mathrm{d}x \leq C\Big(\varepsilon^2 \fint_{\Omega_\varepsilon} |\nabla_x v|^2 \,\mathrm{d}x + \fint_{S_{r,\varepsilon}} |v|^2 \,\mathrm{d}x\Big) \qquad \forall v \in W^{1,2}(\Omega_\varepsilon). \tag{5.15}$$

If additionally the scalar function v obeys: $v(\cdot,0) = v(\cdot,1) = 0$, then the change of variables and the Poincaré inequality on $[0,1]$ yield:

$$\begin{aligned}\fint_{S_{r,\varepsilon}} |v|^2 \,\mathrm{d}x &= \int_0^1 \fint_{\varepsilon B(X(z),r)} |v|^2 \,\mathrm{d}\mathbf{x}_h \,\mathrm{d}z = \fint_{\varepsilon B(0,r)} \int_0^1 |v(\mathbf{x}_h + \varepsilon X(z), z)|^2 \,\mathrm{d}z \,\mathrm{d}\mathbf{x}_h \\ &\leq C \fint_{\varepsilon B(0,r)} \int_0^1 |\partial_z v + \varepsilon(\nabla_{\mathbf{x}_h} v)\partial_z X|^2 \,\mathrm{d}z \,\mathrm{d}\mathbf{x}_h \leq C \fint_{S_{r,\varepsilon}} |\partial_z v|^2 + \varepsilon^2 |\nabla_x v|^2 \,\mathrm{d}x,\end{aligned} \tag{5.16}$$

where $\nabla_{\mathbf{x}_h} v$ denotes the derivative of v in the horizontal directions in $\mathbf{x}_h$. Applying (5.15) and (5.16) to $v = \mathbf{v} \cdot e_n$ results now in the following bound, in view of the already proven (5.3):

$$\fint_{\Omega_\varepsilon} (\mathbf{v} \cdot e_n)^2 \,\mathrm{d}x \leq C\Big(\fint_{\Omega_\varepsilon} |\partial_z(\mathbf{v} \cdot e_n)|^2 \,\mathrm{d}x + \varepsilon^2 \fint_{\Omega_\varepsilon} |\nabla_x v|^2 \,\mathrm{d}x\Big) \leq C \fint_{\Omega_\varepsilon} |\mathrm{sym}\nabla_x \mathbf{v}|^2 \,\mathrm{d}x. \tag{5.17}$$

Further, we note that for almost every $z \in [0, 1]$, the vector field $\mathbf{v}-(\mathbf{v}\cdot e_n)e_n \in W^{1,2}(\varepsilon\omega_h(z); \mathbb{R}^{n-1})$ is tangential on the boundary $\partial(\varepsilon\omega_h(z))$ and thus we may apply the uniform Poincaré inequality in Theorem 5.4 whose constant on the domain $\varepsilon\omega(z)$ scales like ε^2 with respect to the constant on the domain $\omega(z)$. Consequently:

$$\begin{aligned}\fint_{\Omega_\varepsilon} |\mathbf{v} - (\mathbf{v} \cdot e_n)e_n|^2 \,\mathrm{d}x &\leq C \int_0^1 \fint_{\varepsilon\omega(z)} |\mathbf{v} - (\mathbf{v} \cdot e_n)e_n|^2 \,\mathrm{d}\mathbf{x}_h \,\mathrm{d}z \\ &\leq C\varepsilon^2 \int_0^1 \fint_{\varepsilon\omega(z)} |\nabla_{\mathbf{x}_h} \mathbf{v}|^2 \,\mathrm{d}\mathbf{x}_h \,\mathrm{d}z \leq C \fint_{\Omega_\varepsilon} |\mathrm{sym}\nabla_x \mathbf{v}|^2 \,\mathrm{d}x,\end{aligned} \tag{5.18}$$

where we used (5.3) in the last inequality above. Now, (5.18) and (5.17) imply (5.4) as claimed.

5.4 An optimal Korn inequality for channels with circular cross sections

Let us point out that the Korn constant in (5.3) blows up, in general, at the rate $\frac{C}{\varepsilon^2}$ which is due to a positive measure set $\mathcal{C} \subset [0, 1]$ where each cross section $\omega_h(z)$ with $z \in \mathcal{C}$ has a rotational symmetry.

Example 5.6. Given two Lipschitz functions: $X : [0, 1] \to \mathbb{R}^{n-1}$ and a positive $r : [0, 1] \to (0, \infty)$, let each set $\omega(z)$ be a ball given by:

$$\omega(z) = B(X(z), r(z)) \subset \mathbb{R}^{n-1}. \tag{5.19}$$

For some nonzero function $Q \in W^{1,2}((0, 1), so(n - 1))$ satisfying $Q(0) = Q(1) = 0$, consider the following vector fields:

$$\mathbf{v}^\varepsilon(\mathbf{x}_h, z) = \Big(Q(z)\big(\mathbf{x}_h - \varepsilon X(z)\big), 0\Big). \tag{5.20}$$

Note that $\mathbf{v}^\varepsilon \in W^{1,2}(\Omega_\varepsilon; \mathbb{R}^n)$ and it automatically satisfies the boundary conditions (5.2). The non-zero entries of the matrix $\nabla_x \mathbf{v}^\varepsilon$ are grouped in its principal minor of dimension $(n - 1)$, and its n-th column, that are given by:

$$[\nabla_x \mathbf{v}^\varepsilon]_{(n-1)\times(n-1)} = Q(z), \qquad \partial_z \mathbf{v}^\varepsilon = \Big((\partial_z Q)\big(\mathbf{x}_h - \varepsilon X(z)\big) - \varepsilon Q \partial_z X, 0\Big)$$

Consequently:

$$\fint_{\Omega_\varepsilon} |\nabla_x \mathbf{v}^\varepsilon|^2 \,\mathrm{d}x \geq \fint_{\Omega_\varepsilon} |Q(z)|^2 \,\mathrm{d}x \geq c \quad \text{and} \quad \fint_{\Omega_\varepsilon} |\mathrm{sym}\nabla_x \mathbf{v}^\varepsilon|^2 \,\mathrm{d}x \leq C\varepsilon^2.$$

We will now show that under assumption (5.19) the blow-up of Korn's constant is precisely due to the presence of vector fields $\mathbf{v}^\varepsilon$ in Example 5.6. The result below, although not needed for the fluid dynamics discussion of the present paper, is of independent interest and should be compared with paper [11] where an optimal Korn's inequality was derived for thin n-dimensional shells around a compact boundaryless $(n-1)$-dimensional mid-surface.

Theorem 5.7. *Let Ω_ε be as in (5.1) with $\omega_h(z)$ given in (5.19) by Lipschitz functions: $X : [0,1] \to \mathbb{R}^{n-1}$ and $r : [0,1] \to (0,\infty)$. Define $\mathbf{v}^\varepsilon_Q$ by (5.20), for every $Q \in \mathcal{I}$ where:*

$$\mathcal{I} = \Big\{ Q \in W^{1,2}([0,1]; so(n-1)) \;\Big|\; Q(0) = Q(1) = 0 \Big\}.$$

Let $\alpha \in [0,1)$. Then for every $\mathbf{v} \in W^{1,2}(\Omega_\varepsilon; \mathbb{R}^n)$ satisfying the boundary conditions (5.2) and:

$$\left| \int_{\Omega_\varepsilon} \nabla_x \mathbf{v} : \nabla_x \mathbf{v}^\varepsilon_Q \;\mathrm{d}x \right| \leq \alpha \|\nabla_x \mathbf{v}\|_{L^2(\Omega_\varepsilon)} \|\nabla_x \mathbf{v}^\varepsilon_Q\|_{L^2(\Omega_\varepsilon)} \qquad \forall Q \in \mathcal{I}, \tag{5.21}$$

there holds:

$$\int_{\Omega_\varepsilon} |\nabla_x \mathbf{v}|^2 \;\mathrm{d}x \leq \frac{C}{1-\alpha^2} \int_{\Omega_\varepsilon} |\mathrm{sym} \nabla_x \mathbf{v}|^2 \;\mathrm{d}x, \tag{5.22}$$

with a constant C independent of $\mathbf{v}$, ε and α.

Proof. **1.** The angle condition (5.21) implies that:

$$\int_{\Omega_\varepsilon} |\nabla_x \mathbf{v}|^2 \;\mathrm{d}x \leq \frac{1}{1-\alpha^2} \int_{\Omega_\varepsilon} |\nabla_x \mathbf{v} - \nabla_x \mathbf{v}^\varepsilon_Q|^2 \;\mathrm{d}x \qquad \text{for all } Q \in \mathcal{I}.$$

Let $\mathbb{A} : [0,1] \to so(n)$ be as in Theorem 5.3. Note that, by construction: $\mathbb{A}(z) = 0$ for $z < -\varepsilon$ and $z > 1 + \varepsilon$. Thus, we can modify $\mathbb{A}$ on the intervals $[0,\varepsilon]$ and $[1-\varepsilon, 1]$, so that $\mathbb{A}(0) = \mathbb{A}(1) = 0$ and (5.6), (5.7) still hold. Define $Q_0(z) = \mathbb{A}_{(n-1)\times(n-1)}(z) \in so(n-1)$ as the principal minor of $\mathbb{A}(z)$ of dimension $(n-1)$. Then $Q_0 \in \mathcal{I}$ and using the above we have:

$$\begin{aligned}
(1-\alpha^2) \fint_{\Omega_\varepsilon} |\nabla_x \mathbf{v}|^2 \;\mathrm{d}x &\leq \fint_{\Omega_\varepsilon} |\nabla_x \mathbf{v} - \nabla_x \mathbf{v}^\varepsilon_{Q_0}|^2 \;\mathrm{d}x \\
&\leq C \Big(\fint_{\Omega_\varepsilon} |\nabla_x \mathbf{v}(\mathbf{x}_h, z) - \mathbb{A}(z)|^2 \;\mathrm{d}x + \fint_{\Omega_\varepsilon} |\varepsilon \partial_z Q_0|^2 + |\varepsilon Q_0|^2 \;\mathrm{d}x \\
&\qquad\qquad + \fint_{\Omega_\varepsilon} |\partial_z \mathbf{v}|^2 + |\nabla_x (\mathbf{v} \cdot e_n)|^2 \;\mathrm{d}x \Big) \qquad (5.23) \\
&\leq C \Big(\fint_{\Omega_\varepsilon} |\mathrm{sym} \nabla_x \mathbf{v}|^2 \;\mathrm{d}x + \varepsilon^2 \int_0^1 |\partial_z \mathbb{A}|^2 + |\mathbb{A}|^2 \;\mathrm{d}z + \fint_{\Omega_\varepsilon} |\partial_z \mathbf{v}|^2 \;\mathrm{d}x \Big) \\
&\leq C \Big(\fint_{\Omega_\varepsilon} |\mathrm{sym} \nabla_x \mathbf{v}|^2 \;\mathrm{d}x + \fint_{\Omega_\varepsilon} |\partial_z \mathbf{v}|^2 \;\mathrm{d}x \Big),
\end{aligned}$$

where we applied Theorem 5.3. We now observe that the last term above satisfies:

$$\fint_{\Omega_\varepsilon} |\partial_z \mathbf{v}|^2 \, \mathrm{d}x \leq C\Big(\fint_{\Omega_\varepsilon} |\partial_z \mathbf{v} - \mathbb{A}e_n|^2 \, \mathrm{d}x + \fint_{\Omega_\varepsilon} |\mathbb{A}e_n|^2 \, \mathrm{d}x \Big),$$

and thus (5.23) yields:

$$\fint_{\Omega_\varepsilon} |\nabla_x \mathbf{v}|^2 \, \mathrm{d}x \leq C\Big(\fint_{\Omega_\varepsilon} |\mathrm{sym}\nabla_x \mathbf{v}|^2 \, \mathrm{d}x + \int_0^1 |\mathbb{A}e_n|^2 \, \mathrm{d}z \Big). \tag{5.24}$$

2. For each integral term of the form $\int_0^1 (\mathbb{A}e_n \cdot e_i)^2 \, \mathrm{d}z$, $i = 1 \ldots (n-1)$, we recall the Hilbert space identity $\|a\|^2 + \|b\|^2 = \|a-b\|^2 + 2a\cdot b$ to estimate:

$$\begin{aligned} \int_0^1 (\mathbb{A}e_n \cdot e_i)^2 \, \mathrm{d}z \leq \frac{C}{\varepsilon^{2(n-1)}} \Bigg(& \int_0^1 \Big| \partial_z \Big(\int_{\varepsilon\omega(z)} (\mathbf{v}\cdot e_i) \, \mathrm{d}\mathbf{x}_h \Big) - \int_{\varepsilon\omega(z)} (\mathbb{A}e_n \cdot e_i) \, \mathrm{d}\mathbf{x}_h \Big|^2 \, \mathrm{d}z \\ & + \int_0^1 \Big(\int_{\varepsilon\omega(z)} (\mathbb{A}e_n \cdot e_i) \, \mathrm{d}\mathbf{x}_h \Big) \partial_z \Big(\int_{\varepsilon\omega(z)} (\mathbf{v}\cdot e_i) \, \mathrm{d}\mathbf{x}_h \Big) \, \mathrm{d}z \Bigg). \end{aligned} \tag{5.25}$$

Using Reynolds transport theorem, we find the derivative:

$$\partial_z \Big(\int_{\varepsilon\omega_h(z)} (\mathbf{v}\cdot e_i) \, \mathrm{d}\mathbf{x}_h \Big) = \int_{\varepsilon\omega_h(z)} (\partial_z \mathbf{v} \cdot e_i) + \mathrm{div}\Big((\mathbf{v}\cdot e_i) \partial_z \phi^\varepsilon(\mathbf{x}_h, z) \Big) \, \mathrm{d}\mathbf{x}_h$$

in terms of the derivative $\partial_z \phi^\varepsilon$ of the flow of diffeomorphisms $\phi^\varepsilon(\cdot, z) : \varepsilon\omega_h(0) \to \mathbb{R}^{n-1}$ such that $\phi^\varepsilon(\varepsilon\omega(0), z) = \varepsilon\omega_h(z)$. In fact, we can take $\phi^\varepsilon(\mathbf{x}_h, z) = \varepsilon\phi^1(\frac{1}{\varepsilon}\mathbf{x}_h, z)$, whereas the simple form of the cross sections in (5.19) ensures that:

$$\phi^\varepsilon(\mathbf{x}_h, z) = \varepsilon\partial_z X(z) + \frac{\partial_z r(z)}{r(0)} \mathbf{x}_h \qquad \forall \mathbf{x}_h \in B(\varepsilon X(z), \varepsilon r(0)), \quad z \in [0,1],$$

so that $\partial_z \phi^\varepsilon(\mathbf{x}_h, z) = \varepsilon\partial_z^2 X + \frac{\partial_z^2 r(z)}{r(0)} \mathbf{x}_h$.

Thus, we may bound the first term in the right hand side of (5.25) by:

$$\begin{aligned} & \frac{1}{\varepsilon^{2(n-1)}} \int_0^1 \Big| \partial_z \Big(\int_{\varepsilon\omega_h(z)} (\mathbf{v}\cdot e_i) \, \mathrm{d}\mathbf{x}_h \Big) - \int_{\varepsilon\omega_h(z)} (\mathbb{A}e_n \cdot e_i) \, \mathrm{d}\mathbf{x}_h \Big|^2 \, \mathrm{d}z \\ & \qquad \leq C \int_0^1 \Big| \fint_{\varepsilon\omega_h(z)} (\partial_z \mathbf{v} - \mathbb{A}e_n)\cdot e_i + \mathrm{div}\Big((\mathbf{v}\cdot e_i)\partial_z \phi^\varepsilon(\mathbf{x}_h, z) \Big) \, \mathrm{d}\mathbf{x}_h \Big|^2 \, \mathrm{d}z \\ & \qquad \leq C \Big(\fint_{\Omega_\varepsilon} |\partial_z \mathbf{v} - \mathbb{A}e_n|^2 \, \mathrm{d}x + \varepsilon^2 \fint_{\Omega_\varepsilon} |\nabla_x (\mathbf{v}\cdot e_i)|^2 \, \mathrm{d}x + \fint_{\Omega_\varepsilon} (\mathbf{v}\cdot e_i)^2 \, \mathrm{d}x \Big) \\ & \qquad \leq C \fint_{\Omega_\varepsilon} |\mathrm{sym}\nabla_x \mathbf{v}|^2 \, \mathrm{d}x, \end{aligned} \tag{5.26}$$

where the last inequality above follows from Theorem 5.3 and Theorem 5.1. For the second term in (5.25), we integrate by parts to get:

$$
\begin{aligned}
\frac{1}{\varepsilon^{n-1}}\Big|\int_0^1 (\mathbb{A}e_n\cdot e_i)\partial_z\big(\int_{\varepsilon\omega(z)}(\mathbf{v}\cdot e_i)\ \mathrm{d}\mathbf{x}_h\big)\ \mathrm{d}z\Big| &\leq C\left|\int_0^1 (\partial_z(\mathbb{A}e_n\cdot e_i))\big(\fint_{\varepsilon\omega_h(z)}(\mathbf{v}\cdot e_i)\ \mathrm{d}\mathbf{x}_h\big)\ \mathrm{d}z\right| \\
&\leq \left(\int_0^1 |\partial_z\mathbb{A}|^2\ \mathrm{d}z\right)^{1/2}\left(\fint_{\Omega_\varepsilon}(\mathbf{v}\cdot e_i)\ \mathrm{d}x\right)^{1/2} \\
&\leq C\left(\fint_{\Omega_\varepsilon}|\mathrm{sym}\nabla_x\mathbf{v}|^2\ \mathrm{d}x\right)^{1/2}\left(\fint_{\Omega_\varepsilon}|\nabla_x\mathbf{v}|^2\ \mathrm{d}x\right)^{1/2},
\end{aligned}
\tag{5.27}
$$

using Theorem 5.3 and (5.18).

3. Finally, (5.24), (5.25), (5.26) and (5.27) imply:

$$
\fint_{\Omega_\varepsilon}|\nabla_x\mathbf{v}|^2\ \mathrm{d}x \leq \frac{C}{1-\alpha^2}\fint_{\Omega_\varepsilon}|\mathrm{sym}\nabla_x\mathbf{v}|^2\ \mathrm{d}x + \frac{C}{1-\alpha^2}\left(\fint_{\Omega_\varepsilon}|\mathrm{sym}\nabla_x\mathbf{v}|^2\ \mathrm{d}x\right)^{1/2}\left(\fint_{\Omega_\varepsilon}|\nabla_x\mathbf{v}|^2\ \mathrm{d}x\right)^{1/2},
$$

which yields (5.22) and achieves the proof. □

Remark 5.8. It would be interesting to prove an optimal Korn's inequality in the spirit of Theorem 5.7, for the general case of thin channels as in (5.1). A natural candidate for the functional kernel $\mathcal{I}$ in (5.21) is then the following space:

$$
\begin{aligned}
\mathcal{I} = \Big\{(Q,X)\in W^{1,2}([0,1];so(n-1)\times\mathbb{R}^{n-1})\ \Big|\ &(Q,X)(0)=(Q,X)(1)=0\quad\text{and} \\
&\mathbf{x}_h\mapsto Q(z)\mathbf{x}_h + X(z) \text{ is tangent on } \partial\omega(z),\ \text{for a.a. } z\in[0,1]\Big\}.
\end{aligned}
$$

Each element $(Q,X)\in\mathcal{I}$ generates a vector field $\mathbf{v}^\varepsilon_{Q,X}\in W^{1,2}(\Omega_\varepsilon;\mathbb{R}^n)$ satisfying (5.2), where we set:

$$
\mathbf{v}^\varepsilon_{Q,X}(\mathbf{x}_h,z) = \Big(Q(z)\mathbf{x}_h + \varepsilon X(z), 0\Big).
$$

Note that if $\omega(z)$ has no rotational symmetry, then automatically $(Q,X)(z)=0$. Further, observe that every closed set $\mathcal{C}\subset[0,1]$ is the locus of rotationally symmetric sections in some smooth channel Ω_1. Namely, let $r:[0,1]\to R$ be a smooth nonnegative function such that $r^{-1}(0)=\mathcal{C}$. Let ω_0 be a smooth domain with no rotational symmetry, satisfying $B(0,1)\subset\omega_0\subset B(0,2)\subset\mathbb{R}^{n-1}$. Define:

$$
\omega(z) = B(0,1)\cup\Big\{\Big(1+r(z)(|\mathbf{x}_h|-1)\Big)\mathbf{x}_h\ \Big|\ \mathbf{x}_h\in\omega_0,\ |\mathbf{x}_h|\geq 1\Big\}.
$$

Then $\omega(z)$ equals $B(0,1)$ for all $z \in \mathcal{C}$, and otherwise $\omega(z)$ has no rotational symmetry, so that:

$$\int_{\omega(z)} |\nabla_x \mathbf{u}|^2 \, \mathrm{d}x \leq C \int_{\omega(z)} |\mathrm{sym}\nabla_x \mathbf{u}|^2 \, \mathrm{d}x, \tag{5.28}$$

valid for all $\mathbf{u} \in W^{1,2}(\omega(z); \mathbb{R}^{n-1})$ tangent on $\partial\omega(z)$ and all $z \notin \mathcal{C}$, with a uniform C.

Observe now that taking the set $\mathcal{C}$ nowhere dense implies that $\mathcal{I} = \{0\}$, indicating that (5.22) holds for all $\mathbf{v}$ satisfying (5.2) (here $\alpha = 0$). On the other hand, if $\mathcal{C}$ has positive measure (as valid for the "fattened" Cantor set), then Korn's inequality (5.28) still fails at all $z \in \mathcal{C}$.

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