Differentiability of the metric projection onto a convex set with singular boundary points

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Abstract. The differentiability of the metric projection P onto a closed convex set K in $\mathbf{R}^n$ is examined. The boundary ∂K can have singular points of orders $k = -1, 0, 1 \ldots, n-1$. Here $k = -1$ corresponds to the interior points of K, $k = 0$ to regular points of the boundary (i.e., faces), $k = 1, \ldots, n-2$ to edges and $k = n-1$ to vertices. It is assumed that for every k the set of all singular points forms an $n-k-1$ dimensional manifold T_{k+1} (possibly empty) of class $p \geq 2$. Under a mild continuity assumption it is shown that then P is of class $p-1$ on an open set W whose complement has null Lebesgue measure. The set W is the union of the interiors of inverse images of T_{k+1} under P. Moreover, a formula for the Fréchet derivative $\mathrm{D}P$ on each of these regions is given that relates $\mathrm{D}P$ to the second fundamental form of the manifold T_{k+1}. The results are illustrated (a) on the metric projection P from the space Sym of symmetric matrices onto the convex cone Sym^+ of positive semidefinite symmetric matrices and (b) on the metric projection from Sym onto the unit ball under the operator norm. We prove the indefinite differentiability of these projections on explicitly determined open sets with complements of measure 0 and give explicit formulas for the derivatives. In (a) the method of proof, based on the above general result, is different from the previous treatment in [17] and applies to situations [21] where the special methods of [17] cannot be used. The case (b) is new.

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Contents

Chapter 1

Introduction

Let K be a nonempty closed convex set in $\mathbf{R}^n$. Denote for every $x \in \mathbf{R}^n$ by $P(x)$ the unique element of K such that

$$\|x - P(x)\| = \inf\{\|x - w\| : w \in K\}$$

where $\|\cdot\|$ denotes the euclidean norm on $\mathbf{R}^n$. The map P is called the metric projection of $\mathbf{R}^n$ onto K. The purpose of this paper is to examine the differentiability of P on $U := \mathbf{R}^n \sim K$. It is well–known that P is nonexpansive, i.e.,

$$\|P(x) - P(y)\| \le \|x - y\|$$

for every $x, y \in \mathbf{R}^n$ and thus Rademacher's theorem [7; Theorem 3.1.6] implies that P has a Fréchet derivative at almost every point of $\mathbf{R}^n$ with respect to the Lebesgue measure. Alternatively, since P is the derivative of the convex function

$$f(x) = \tfrac{1}{2}\|x\|^2 - \tfrac{1}{2}\|x - P(x)\|,$$

$x \in \mathbf{R}^n$ [27, 12], the differentiability of P almost everywhere follows from the Alexandrov theorem on the second differentiability of convex functions [3, 2], or from the differentiability almost everywhere of maximal monotone maps [18, 2].

Holmes [12], considering projections onto convex sets in a possibly infinite dimensional Hilbert space H proved that if K has non–empty interior and if the boundary ∂K of K is of locally class p, where $p \ge 2$, then P is locally of class $p - 1$. He also gives a formula for $\mathrm{D}\,P(x)$. We refer to Corollary 2.3.5, below, for a more detailed discussion of Holmes' result in the case of a finite dimensional space. Fitzpatrick & Phelps [8] proved that under a natural invertibility condition on the derivative the smoothness of class p of the boundary of K is also necessary for P to be of class $p - 1$ on $H \sim K$. The differentiability in [12] and [8] is the classical Gateaux or Fréchet differentiability. The use of the generalized derivatives was suggested by Hiriart-Urruty [11], explored by Noll [20], and applied to the convex cone of positive semidefinite matrices by Malick & Sendov [17].

We here restrict to finite dimensional euclidean spaces, adhere to Fréchet derivatives, and note that many useful convex sets arising in mathematical analysis have boundaries with singularities forming edges and corners:

1.1 Examples. The following convex sets have boundaries with singular points:
(i) the n dimensional simplexes in $\mathbf{R}^n$,
(ii) the orthant of points with nonnegative coordinates in $\mathbf{R}^n$,
(iii) the set of all positive semidefinite $m \times m$ real symmetric matrices,
(iv) the unit ball $\mathbf{B}$ in the space Sym of $m \times m$ real symmetric matrices under the operator norm

$$v(a) = \max\left\{ \|a\xi\| : \xi \in \mathbf{R}^m,\ \|\xi\| \le 1 \right\} \tag{1.1}$$

$a \in \mathrm{Sym}$, where $\|\cdot\|$ is the euclidean norm on $\mathbf{R}^m$.

The singular points in (i) and (ii) are obvious; Examples 1.1(iii) and (iv) are treated below in Chapters 4 and 5, respectively, with a detailed description of singularities.

The boundaries with singularities are not covered by the aforementioned works, and the purpose of the present paper is to examine the differentiability of P in these situations. Our main motivation is the differentiability of the stress function of no–tension masonry materials of continuum mechanics [6, 9, 16, 21], which is closely related to Example 1.1(iii); similarly a Hencky plastic material [24, 23] with the Tresca yield criterion is related to Example 1.1(iv).

Throughout the note, let n be a positive integer and K a closed convex subset of $\mathbf{R}^n$.

For every $y \in K$ define the normal cone $\mathrm{Nor}^+(K, y) \subset \mathbf{R}^n$ to K at y by

$$\mathrm{Nor}^+(K, y) = \{ b \in \mathbf{R}^n : (y - z) \cdot b \ge 0 \ \text{ for every } \ z \in K \}.$$

For every integer r with $0 \le r \le n$ define the set

$$T_r = \{ y \in K : \dim \mathrm{Nor}^+(K, y) = r \} \tag{1.2}$$

where the dimension of $\mathrm{Nor}^+(K, r)$ is defined to be the dimension of the span of $\mathrm{Nor}^+(K, x)$. We say that $y \in K$ is a singular point of order $k = -1, 0, 1, \dots, n-1$ if it belongs to T_{k+1}. Here $k = -1$ corresponds to the interior points of K, $k = 0$ to regular points of the boundary, $k = 1, \dots, n-2$ to edges and $k = n-1$ to vertices. For convenience we stipulate the equality $\dim \mathrm{Nor}^+(K, y) = r$ in (1.2) and note that a more standard definition of a singular point of order k requires $\dim \mathrm{Nor}^+(K, y) \ge r$, i.e., $\dim \mathrm{Nor}^+(K, y) \ge k + 1$.

It is well known that the set of all singular points of K (i.e., those with $k \ge 2$) is small; see Anderson & Klee, Jr. [5], Zajíček [26] and Alberti [1] and the references therein. Alberti [1] proves the following result:

1.2 Theorem. *Let r be an integer such that $0 \le r \le n$. Then T_r is $(\mathscr{H}^{n-r}, n-r)$ rectifiable subset of $\mathbf{R}^n$ of class 2.*

Here for any integer s such that $0 \le s \le n$, the symbol $\mathscr{H}^s$ denotes the s–dimensional Hausdorff measure in $\mathbf{R}^n$ and a Borel subset A of $\mathbf{R}^n$ is said to be $(\mathscr{H}^s, s)$ rectifiable subset of $\mathbf{R}^n$ of class 2 if there exist s dimensional submanifolds $M_i \subset \mathbf{R}^n$, $i = 1, \dots,$ of class 2 such that

$$\mathscr{H}^s(A \sim \bigcup_{i=1}^{\infty} M_i) = 0.$$

Alberti [1] also shows that the regularity of T_r, described in Theorem 1.2 cannot be improved. We note that the regularity of T_r is 2, not 1 as in many cases in the geometric measure theory [7, 4].

We shall not employ Theorem 1.2 in this paper. However, we use it to motivate our main assumption, in which r is an integer with $0 \le r \le n$.

1.3 Assumption $\mathbf{A}_r$. The set T_r is (a possibly empty) $n - r$ dimensional manifold of class $p \ge 2$.

This will be combined with the following technical continuity assumption. Denote by $\operatorname{ri} \operatorname{Nor}^+(K, y)$ is the relative interior of $\operatorname{Nor}^+(K, y)$ in $\operatorname{span} \operatorname{Nor}^+(K, y)$ and by $\operatorname{Nor}(T_r, y)$ the normal space to T_r, i.e., the orthogonal complement in $\mathbf{R}^n$ of the tangent space $\operatorname{Tan}(T_r, y)$ to T_r at y.

1.4 Assumption $\mathbf{B}_r$. If $y \in T_r$ and $z \in \operatorname{ri} \operatorname{Nor}^+(K, y)$ then there exists an $\varepsilon > 0$ such that $\{z' \in \operatorname{Nor}(T_r, y') : \|z' - z\| < \varepsilon\} \subset \operatorname{Nor}^+(K, y')$ for all $y' \in T_r$ sufficiently close to y.

Both these assumptions are satisfied by Examples 1.1(i)–(iv) with $p = \infty$; in Examples 1.1(iii), (iv) only certain dimensions are effective, i.e., $T_r \neq \emptyset$ only for certain values of r (see below).

The author does not know if Assumption $\mathbf{A}_r$ and the convexity of K does not imply $\mathbf{B}_r$. For $r = 1$, the situation is very simple.

1.5 Remark. *If $\mathbf{A}_1$ holds then also $\mathbf{B}_1$ holds.*

Proof For each $y \in T_1$ we have

$$\operatorname{Nor}^+(K, y) = \{tm(y) : t \ge 0\} \tag{1.3}$$

where $m : T_1 \to \mathbf{R}^n$ is the unit normal of class 1. Then if $z \in \operatorname{ri} \operatorname{Nor}^+(K, y) = \{tm(y) : t > 0\}$ and $\varepsilon = \|z\|/2$, the assertion holds by the continuity of m. □

Under $\mathbf{A}_r$ and $\mathbf{B}_r$, the small sets T_r are the images, under the projection P, of large sets, i.e., sets with nonempty interior in $\mathbf{R}^n$. Namely, we put

$$V_r = \bigcup \{y + \operatorname{Nor}^+(K, y) : x \in T_r\} \tag{1.4}$$

and define W_r as the interior of V_r. The following result will be proved:

1.6 Theorem. *Let $\mathbf{A}_r$ and $\mathbf{B}_r$ hold for all r with $0 \le r \le n$. Then P is of class $p - 1$ on the open set $\bigcup_{r=0}^{n} W_r$ whose complement E has null n dimensional Lebesgue measure.*

Moreover, a formula is given for the derivative of P on each W_r (see (2.3.6), below) which relates $\mathrm{D}P$ to the second fundamental form of riemannian geometry of the manifold T_r (see, e.g., [15; Section VII.3] and the definition in Section 2.2, below). For $r = 1$ (regular points of the boundary) the general formula for $\mathrm{D}P$ reduces to that derived in [12].

The general results are illustrated in two matrix cases. The first case is the differentiability of the metric projection P from the space Sym of symmetric $m \times m$ matrices onto the convex cone Sym^+ of positive semidefinite symmetric matrices and on a formula for the derivative. These problems have been definitively treated in the paper by Malick and Sendov [17], where it was proved that P is of class ∞ on the set InvSym of invertible symmetric matrices and a formula was given for the derivative at points $x \in \mathrm{InvSym}$. The proof in [17] is based on the formula for the second derivative of a function of the eigenvalues of the matrix argument. The proof in the present note is based on the decomposition of the boundary of Sym^+ into sets Sym_ρ^+, $\rho = 0, \dots, m$, of matrices of rank ρ. Each of these sets forms a class ∞ manifold of dimension $\rho(2m - \rho + 1)/2$ (see Proposition 4.2.1, below), which gives singularities of order $k = [m(m+1) - \rho(2m - \rho + 1)]/2 - 1$.

The second example is the metric projection onto the unit ball Sym^1 under the operator norm. The structure of singularities of the boundary of Sym^1 is somewhat more complicated but explicitly tractable. The manifolds forming the faces, edges, vertices, etc., are parameterized by two parameters σ and τ giving the multiplicity of the occurrence of the eigenvalues 1 and -1 of the boundary matrix and the formula for the derivative reflects this. Each of these sets forms a manifold of dimension $[m(m+1) - \sigma(\sigma+1) - \tau(\tau+1)]/2$.

The main step in the proofs of these examples is the evaluation of the second fundamental forms of the manifolds on the boundary of the specific convex sets. The application of the general result then follows easily.

Chapter 2

The general theory

This chapter is devoted to the differentiability of the metric projection onto a closed convex set if a finite dimensional euclidean space.

2.1 Properties of metric projections

We here summarize the main properties of metric projections. For every set M let $\mathbf{1}_M$ denote the identity transformation on M.

2.1.1 Proposition. *The following assertions hold:*

(i) *for every* $y \in K$, $\operatorname{Nor}^+(K,y)$ *is a closed convex cone in* $\mathbf{R}^n$ *with vertex at the origin* [22; Proposition 6.5];

(ii) *putting* $\operatorname{Nor}^+(K,x) = \emptyset$ *if* $x \in \mathbf{R}^n \sim K$ *then* $\operatorname{Nor}^+(K,\cdot)$, *interpreted as a multivalued map form* $\mathbf{R}^n$ *to* $\mathbf{R}^n$ [22; Chapter 5], *is maximal monotone, i.e.,*

$$(a-b)\cdot(x-y) \geq 0 \tag{2.1.1}$$

for every $x, y \in \mathbf{R}^n$ *and every* $a \in \operatorname{Nor}^+(K,x)$, $b \in \operatorname{Nor}^+(K,y)$ *and* $\operatorname{Nor}^+(K,\cdot)$ *cannot be genuinely extended to a multivalued map satisfying* (2.1.1) [22; Chapter 12];

(iii) *the inverses of the indicated multivalued maps satisfy*

$$\operatorname{Nor}^+(K,\cdot) = P^{-1} - \mathbf{1}_{\mathbf{R}^n}, \quad P = (\mathbf{1}_{\mathbf{R}^n} + \operatorname{Nor}^+(K,\cdot))^{-1}, \tag{2.1.2}$$

where $\mathbf{1}_{\mathbf{R}^n}$ *is the identity map on* $\mathbf{R}^n$, [22; Proposition 6.17].

It follows from (2.1.2) that the collection of closed convex sets

$$\{y + \operatorname{Nor}^+(K,y) : y \in K\}$$

is pairwise disjoint and the union is the whole of $\mathbf{R}^n$; hence the sets V_r from Section 1 satisfy

$$\bigcup_{r=0}^{n} V_r = \mathbf{R}^n \tag{2.1.3}$$

with the union disjoint.

We say that a map $F : \mathbf{R}^n \to Z$, where Z is a finite dimensional normed space, is differentiable at $x \in \mathbf{R}^n$ if there exists a linear transformation L from $\mathbf{R}^n$ into Z such that

$$\lim_{z \to x} \|F(z) - F(x) - L(z - x)\| / \|z - x\| = 0.$$

We call L the derivative of F at x and write $\mathrm{D}F(x)[h] = Lh$ for each $h \in \mathbf{R}^n$.

2.1.2 Proposition. *The following assertions hold:*

(i) *the map P is nonexpansive and monotone, i.e.*

$$\|P(x) - P(y)\| \le \|x - y\|, \quad (P(x) - P(y)) \cdot (x - y) \ge 0$$

for any $x, y \in \mathbf{R}^n$ [22; Corollary 12.20];

(ii) *if the derivative $\mathrm{D}P(x)$ of P at $x \in \mathbf{R}^n$ exists, then it is positive semidefinite, i.e., $b \cdot \mathrm{D}P(x)[b] \ge 0$ for any $b \in \mathbf{R}^n$* [22; Proposition 12.3] *and symmetric, i.e., $b \cdot \mathrm{D}P(x)[a] = a \cdot \mathrm{D}P(x)[b]$ for any $a, b \in \mathbf{R}^n$* [8; Proposition 2.2].

2.1.3 Proposition. *Let $y \in K$ and $x \in \operatorname{ri}\operatorname{Nor}^+(K, y)$. If P is differentiable at x and if R and Q denote the orthogonal projections onto the span of $\operatorname{Nor}^+(K, y)$ and onto the orthogonal complement of $\operatorname{Nor}^+(K, y)$ then*

$$\mathrm{D}P(x)Q = Q\mathrm{D}P(x) = \mathrm{D}P(x) \tag{2.1.4}$$

and

$$\mathrm{D}P(x)R = 0. \tag{2.1.5}$$

Proof Equation (2.1.5): If $h \in \operatorname{span}\operatorname{Nor}^+(K, y)$ then $x + th \in \operatorname{Nor}^+(K, y)$ for all $t \in \mathbf{R}$ with $|t|$ sufficiently small. For all such t then $P(x + th) = P(x)$ and thus $\mathrm{D}P(x)[h] = 0$, which proves (2.1.5). Combining that relation with $Q = \mathbf{1}_{\mathbf{R}^n} - P$ we obtain $\mathrm{D}P(x)Q = \mathrm{D}P(x)$ and taking the transpose using the symmetry of $\mathrm{D}P(x)$ we obtain $Q\mathrm{D}P(x) = \mathrm{D}P(x)$. Thus we have (2.1.4). □

2.2 Second fundamental form

The second fundamental form of a manifold M imbedded in an ambient riemannian manifold M' measures the discrepancy between the covariant derivative relative to M' and M, respectively. In the present case, the ambient manifold is an euclidean space.

2.2.1 Definitions.

(i) For any manifold M of class 1 in $\mathbf{R}^v$ [7; Subsection 3.1.19] (where v is a positive integer) and for any $y \in M$, we define the tangent space $\operatorname{Tan}(M, y)$ as the set of all $b \in \mathbf{R}^v$ such that there exists a class 1 map a satisfying

$$a : (-\varepsilon, \varepsilon) \to M, \quad a(0) = y, \quad \dot{a}(0) = b. \tag{2.2.1}$$

(ii) We define the normal space $\mathrm{Nor}(M, y)$ to M at z as the orthogonal complement of $\mathrm{Tan}(M, y)$ in $\mathbf{R}^{\nu}$.

(iii) If $F : M \to V$ is a map from M to a finite dimensional vectorspace V and $y \in M$ and M is of class σ, we say that F is of class $s \le \sigma$ on M if $f \circ \xi$ is of class s on an open subset of $\mathbf{R}^{\dim M}$ for any class σ local parameterization ξ of M.

(iv) If F is of class 1 on M then for every $y \in M$ there exists a linear transformation $\mathrm{D} F(y)[\cdot]$ from $\mathrm{Tan}(M, y)$ into V satisfying

$$\mathrm{D} F(y)[b] = \frac{d}{dt} F(a(t))\Big|_{t=0}$$

for any class 1 map a as in (2.2.1). We call $\mathrm{D} F(y)$ the derivative of F at y. We do not indicate the fact that $\mathrm{D} F(y)[\cdot]$ is a surface derivative relative to M as this is uniquely given by the domain of F.

(v) We denote by $Q : M \to \mathbb{M}^{\nu \times \nu}$ the map which associates with each $y \in M$ the orthogonal projection $Q(y)$ from $\mathbf{R}^{\nu}$ onto $\mathrm{Tan}(M, y)$. Here $\mathbb{M}^{\nu \times \nu}$ is the set of all linear transformations from $\mathbf{R}^{\nu}$ into itself.

(vi) A vectorfield $\beta : M \to \mathbf{R}^{n}$ is said to be tangential (to M) if $Q\beta = \beta$.

2.2.2 Proposition (See [15; Section VII.3]). *Let $M \subset \mathbf{R}^{\nu}$ be a class $p \ge 2$ manifold. There exists a class $p - 2$ map B on M, which associates with any $y \in M$ a symmetric bilinear form $B(y) : \mathrm{Tan}(M, y) \times \mathrm{Tan}(M, y) \to \mathrm{Nor}(M, y)$ such that for every two class 1 tangential vectorfields β, $\gamma : M \to \mathbf{R}^{\nu}$ we have*

$$B(y)(\beta(y), \gamma(y)) = \mathrm{D}\, \gamma(y)[\beta(y)] - Q(y)\, \mathrm{D}\, \gamma(y)[\beta(y)].$$

One has

$$B(y)(b, c) = \mathrm{D}\, Q(y)[b]\, c$$

for any $y \in M$ and b, $c \in \mathrm{Tan}(M, y)$.

2.2.3 Definition. The form B from the above proposition is called the second fundamental form of M (more precisely of the imbedding of M in $\mathbf{R}^{\nu}$).

2.2.4 Proposition. *Let M be a class $p \ge 2$ manifold with $M \subset \partial K$ where $K \subset \mathbf{R}^{n}$ is a closed convex set. Then for any $y \in M$, any $z \in \mathrm{Nor}^{+}(K, y)$ and $b \in \mathrm{Tan}(M, y)$ we have $z \cdot (B(y)(b, b)) \le 0$.*

Proof Let a be a class 2 map as in (2.2.1). We have

$$z \cdot (a(t) - a(0)) \le 0$$

for all t with the equality at $t = 0$ and hence

$$z \cdot \ddot{a}(0) \le 0. \tag{2.2.2}$$

By differentiating $\dot{a}(t) = Q(a(t))\dot{a}(t)$ we obtain

$$\ddot{a}(0) = B(y)(b, b) + Q(y)\ddot{a}(0);$$

inserting into (2.2.2) we obtain the result. □

2.3 The derivative of P

In this section we prove that under Assumption $\mathbf{A}_r$ the map P is of class $p-1$ on the interior W_r of the set V_r in (1.4) and establish a formula for $\mathrm{D}P(x)$ for $x \in W_r$. The situation $W_r = \emptyset$ is not excluded and only in the following section we invoke Assumption $\mathbf{B}_r$ to show that $W_r \neq \emptyset$ if $T_r \neq \emptyset$ and that the union $\bigcup_{r=0}^{n} W_r$ has the complement of null Lebesgue measure.

2.3.1 Definitions. Let r be an integer, $0 \le r \le n$.

(i) We denote by $\mathscr{M}_r$ the set

$$\mathscr{M}_r = \{(y,z) \in \mathbf{R}^n \times \mathbf{R}^n : y \in T_r,\ z \in \mathrm{Nor}(T_r,y)\}$$

and by $\mathscr{N}_r$ its subset

$$\mathscr{N}_r = \{(y,z) \in \mathbf{R}^n \times \mathbf{R}^n : y \in T_r,\ z \in \mathrm{Nor}^+(K,y)\}.$$

(ii) Define a map $\Phi_r : \mathscr{M}_r \to \mathbf{R}^n$ by

$$\Phi_r(y,z) = y + z$$

for every $(y,z) \in \mathscr{M}_r$.

2.3.2 Proposition. *Φ_r maps $\mathscr{N}_r$ homeomorphically onto V_r with the inverse*

$$(\Phi_r|\mathscr{N}_r)^{-1}(x) = (P(x), x - P(x)) \tag{2.3.1}$$

for every $x \in V_r$ when $\mathscr{N}_r$ and V_r are endowed with the relative topologies.

Proof That Φ_r maps $\mathscr{N}_r$ bijectively onto V_r and Formula (2.3.1) holds is a direct verification; clearly $\Phi_r|\mathscr{N}_r$ is continuous and the continuity of P and (2.3.1) imply that $(\Phi_r|\mathscr{N}_r)^{-1}$ is continuous. □

Let r be a fixed integer with $0 \le r \le n$ and Assume that $\mathbf{A}_r$ holds. For each $y \in T_r$ we denote by $\mathrm{Tan}(T_r,y) \subset \mathbf{R}^n$ and $\mathrm{Nor}(T_r,y) \subset \mathbf{R}^n$ the tangent and normal spaces to T_r at y, and by $Q_r(y)$ and $R_r(y)$ the orthogonal projections onto $\mathrm{Tan}(T_r,y)$ and $\mathrm{Nor}(T_r,y)$, respectively. The maps Q_r and R_r on T_r are of class $p-1$ with values in the space of linear transformations on $\mathbf{R}^n$. Further, for every $y \in T_r$ let $B_r(y) : \mathrm{Tan}(T_r,y) \times \mathrm{Tan}(T_r,y) \to \mathrm{Nor}(T_r,y)$ denote the second fundamental form of T_r. Let furthermore for any $z \in \mathbf{R}^n$ the symbol $C_r(y,z)$ denote a linear transformation from $\mathrm{Tan}(T_r,y)$ into itself defined by

$$c \cdot C_r(y,z)b = z \cdot B_r(y)(b,c) \tag{2.3.2}$$

for every b, $c \in \mathrm{Tan}(T_r,y)$. Note that Proposition 2.2.4 implies that for any $y \in T_r$ and $z \in \mathrm{Nor}^+(K,y)$ the linear transformation $C_r(y,z)$ is symmetric and negative semidefinite on $\mathrm{Tan}(T_r,y)$.

2.3.3 Lemma. *Let r be a fixed integer with $0 \le r \le n$ and Assume that $\mathbf{A}_r$ holds. Then*

(i) *$\mathscr{M}_r \subset \mathbf{R}^n \times \mathbf{R}^n$ is an n dimensional manifold of class $p-1$ and the map Φ_r is of class $p-1$;*

(ii) *for every $(y,z) \in \mathscr{M}_r$ the derivative $\mathrm{D}\,\Phi_r$ is given by*

$$\mathrm{D}\,\Phi_r(y,z)(\xi,\alpha) = \xi - C_r(y,z)\xi + R_r(y)\alpha$$

for every $(\xi, \alpha) \in \operatorname{Tan}(\mathscr{M}_r, (y,z))$;

(iii) *for every* $(y,z) \in \mathscr{N}_r$ *the derivative* $\mathrm{D}\,\Phi_r(y,z)$ *maps* $\operatorname{Tan}(\mathscr{M}_r, (y,z))$ *bijectively onto* $\mathbf{R}^n$ *and we have*

$$\mathrm{D}\,\Phi_r(y,z)^{-1}\lambda = (\xi, \alpha)$$

for any $\lambda \in \mathbf{R}^n$ *where*

$$\xi = \left[\mathbf{1}_{\operatorname{Tan}(T_r, y)} - C_r(y,z)\right]^{-1} Q_r(y)\lambda, \tag{2.3.3}$$

$$\alpha = \lambda - \left[\mathbf{1}_{\operatorname{Tan}(T_r, y)} - C_r(y,z)\right]^{-1} Q_r(y)\lambda; \tag{2.3.4}$$

the existence of the inverse follows from the negative semidefinite character of $C_r(y,z)$.

Proof We note that (i) is immediate.

(ii): Let $(y,z) \in \mathscr{M}_r$ and $(\xi, \alpha) \in \operatorname{Tan}(\mathscr{M}_r, (y,z))$. There exists a class 1 map $\gamma = (a,c) : (-\varepsilon, \varepsilon) \to \mathscr{M}_r$ such that $\gamma(0) = (y,z)$ and $\dot{\gamma}(0) = (\xi, \alpha)$. We have $c(t) = R_r(a(t))c(t)$ for every $t \in (-\varepsilon, \varepsilon)$. Differentiating with respect to t at $t = 0$, using $B_r = \mathrm{D}\,Q_r = -\mathrm{D}\,R_r$ and invoking (2.3.2) we obtain

$$\alpha = -C_r(y,z)\xi + R_r(y)\alpha; \tag{2.3.5}$$

differentiating

$$\Phi_r(a(t), c(t)) = a(t) + c(t)$$

and using (2.3.5), we obtain

$$\mathrm{D}\,\Phi_r(y,z)(\xi, \alpha) = \xi + \alpha = \xi - C_r(y,z)\xi + R_r(y)\alpha.$$

(iii): Let us prove that for every $(y,z) \in \mathscr{N}_r$ the derivative $\mathrm{D}\,\Phi_r(y,z)$ maps $\operatorname{Tan}(\mathscr{M}_r, (y,z))$ bijectively onto $\mathbf{R}^n$. Indeed, let $(y,z) \in \mathscr{N}_r$ and $(\xi, \alpha) \in \operatorname{Tan}(\mathscr{M}_r, (y,z))$ and assume that

$$\mathrm{D}\,\Phi_r(y,z)(\xi, \alpha) = \xi - C_r(y,z)\xi + R_r(y)\alpha = 0.$$

Multiplying by $Q_r(y)$, we obtain

$$\xi - C_r(y,z)\xi = 0,$$

and since $\xi \in \operatorname{Tan}(T_r, y)$ and since $C_r(y,z)$ is negative semidefinite, we see that the last equation gives $\xi = 0$. The proof of (i) provides $\mathrm{D}\,\Phi_r(y,z)(\xi, \alpha) = \xi + \alpha$ and as this must vanish, we have $\alpha = 0$. Thus $\mathrm{D}\,\Phi_r(y,z)$ maps $\operatorname{Tan}(\mathscr{M}_r, (y,z))$ injectively into $\mathbf{R}^n$ and as the dimensions of these two spaces coincide, we see that $\mathrm{D}\,\Phi_r(y,z)$ is a bijection. Finally, solve the equation

$$\mathrm{D}\,\Phi_r(y,z)(\xi, \alpha) = \xi - C_r(y,z)\xi + R_r(y)\alpha = \lambda$$

where $\lambda \in \mathbf{R}^n$. Multiplying by $Q_r(y)$, we obtain

$$\xi - C_r(y,z)\xi = Q_r(y)\lambda,$$

and hence (2.3.3). Equation (2.3.4) then follows from $\xi + \alpha = \lambda$. □

2.3.4 Theorem. *Let r be an integer with $0 \le r \le n$ and assume that $\mathbf{A}_r$ holds. Then we have the following assertions:*

(i) *for every $y \in T_r$ and $x \in y + \mathrm{Nor}^+(T_r, y)$ the transformation $C_r(y, x-y)$ is negative semidefinite in the sense that $b \cdot C_r(y, x-y)b \le 0$ for every $b \in \mathrm{Tan}(T_r, y)$;*

(ii) *the map P is of class $p-1$ on W_r, and we have, for every $x \in W_r$,*

$$\mathrm{D}\,P(x) = \left[\mathbf{1}_{\mathrm{Tan}(T_r, y)} - C_r(y, x-y)\right]^{-1} Q_r(y) \tag{2.3.6}$$

where $y = P(x)$; the existence of the inverse is guaranteed by (i).

Proof (i): Follows from Proposition 2.2.4.

(ii): On $\mathscr{N}_r$ the map Φ_r is injective by Proposition 2.3.2 and of class $p-1$ on the relative interior of $\mathscr{N}_r$ by (2.3.1). By Lemma 2.3.3 the derivative of Φ_r is injective at any point of $\mathscr{N}_r$. The inverse of Φ_r on $\mathscr{N}_r$ is given by (2.3.1); differentiating this relation, we obtain

$$\mathrm{D}(\Phi_r|\mathscr{N}_r)^{-1}(x) = (\mathrm{D}\,P(x), \mathbf{1}_{\mathrm{Tan}(T_r, y)} - \mathrm{D}\,P(x)).$$

Combining with the value of the inverse of $\mathrm{D}\,\Phi_r$ calculated in Lemma 2.3.3(iii), we obtain the formula for $\mathrm{D}\,P(x)$ in (2.3.6). □

2.3.5 Corollary. *Let ∂K be an $n-1$ dimensional surface of class $p \ge 2$. Then P is of class $p-1$ on $\mathbf{R}^n \sim K$ and we have*

$$\mathrm{D}\,P(x) = \left[\mathbf{1}_{\mathrm{Tan}(\partial K, y)} + \|x-y\| L(y)\right]^{-1} Q(y) \tag{2.3.7}$$

for every $x \in \mathbf{R}^n \sim K$ with $y = P(x)$, where $Q(y)$ is the orthogonal projection onto $\mathrm{Tan}(\partial K, y)$, *and $L(y) = \mathrm{D}\,m(y)$ is the surface derivative of the outer normal m to K at y. If $f : \mathbf{R}^n \to \mathbf{R}$ is a class 2 convex function such that $f = 1$ and $\mathrm{D}\,f \ne 0$ on ∂K with $f \le 1$ on K then*

$$\mathrm{D}\,P(x) = \left[\mathbf{1}_{\mathrm{Tan}(\partial K, y)} + \|x-y\| Q(y)\,\mathrm{D}^2 f(y)/\|\,\mathrm{D}\,f(y)\,\|\right]^{-1} Q(y). \tag{2.3.8}$$

We here interpret $\mathrm{D}^2 f(y)$ as a symmetric linear transformation $\mathbf{R}^n \to \mathbf{R}^n$ associated to the equally denoted quadratic form. In particular, if f is the Minkowski functional of K relative to any fixed interior point $x_\circ$ of K, i.e.,

$$f(y) = \inf\{t > 0 : y \in t(K - x_\circ) + x_\circ\}, \quad y \in \mathbf{R}^n,$$

then $f(y) = 1$ on ∂K; hence (2.3.8) is then the formula [12; Equation (3.3) and Lemma 4], here restricted to the finite dimensional case.

Proof Under the hypothesis,

$$\mathrm{Nor}(\partial K, y) = \{tm(y); t \in \mathbf{R}\}$$

for each $y \in \partial K$ where m is the outer normal to ∂K in the sense of differential geometry. Formula (1.3) then follows and thus $\mathbf{A}_1$ holds. From the riemannian geometry we then have

$$B(y)(b, c) = -m(y)(L(y)b \cdot c)$$

for any $b, c \in \mathrm{Tan}(\partial K, y)$. Formula (2.3.6) reduces to (2.3.7). If f is as in the statement of the corollary, then

$$m(y) = \mathrm{sgn}\,\mathrm{D}\,f(y)$$

for each $y \in \partial K$ where $\mathrm{sgn}\,b = b/|b|$ for any nonzero $b \in \mathbf{R}^n$. Consequently

$$\mathrm{D}\,\mathrm{sgn}(\mathrm{D}\,f) = (\|\,\mathrm{D}\,f\,\|^2\,\mathrm{D}^2 f - \mathrm{D}\,f \otimes \mathrm{D}^2 f\,\mathrm{D}\,f)/\|\,\mathrm{D}\,f\,\|^3 \equiv Q_r\,\mathrm{D}^2 f/\|\,\mathrm{D}\,f\,\|$$

and (2.3.8) follows. □

2.4 Exceptional points

In Theorem 2.3.4 we proved the differentiability of P on each open set W_r, $0 \le r \le n$, under $\mathbf{A}_r$. The differentiability is not guaranteed on the closed exceptional set

$$E = \mathbf{R}^n \sim \bigcup_{r=0}^{n} W_r.$$

We shall now invoke Assumption $\mathbf{B}_r$ to prove that E is small in the sense that it is a closed Lebesgue null set (and hence in particular, $\bigcup_{r=0}^{n} W_r$ is an open dense set).

2.4.1 Lemma. *For each integer r with* $0 \le r \le n$ *such that* $\mathbf{B}_r$ *holds we have*

$$W_r = \bigcup \{ y + \operatorname{ri} \operatorname{Nor}^+(K,y) : y \in T_r \} \tag{2.4.1}$$

and

$$V_r \subset \operatorname{cl} W_r.$$

Proof If $x \in V_r$ is a point of W_r then the ball B of center x and sufficiently small radius belongs to W_r which implies that $B \cap (P(x) + \operatorname{Nor}^+(K, P(x)))$ is a relatively open subset of $P(x) + \operatorname{Nor}^+(K, P(x))$ and hence $x \in P(x) + \operatorname{ri} \operatorname{Nor}^+(K, P(x))$. This shows that we have the inclusion "$\subset$" in (2.4.1). Conversely, Assumption $\mathbf{B}_r$ implies that if $z \in \operatorname{ri} \operatorname{Nor}^+(K,y)$ then (y,z) is an interior point of the set $\mathscr{N}_r$. Since Φ_r is a homeomorphism, we see that $y+z$ is an interior point of V_r. This proves "$\supset$" in (2.4.1) and hence (2.4.1) holds. Furthermore, if $b \in V_r \sim \bigcup \{ y + \operatorname{ri} \operatorname{Nor}^+(K,y) : y \in T_r \}$ then $b \in y + \operatorname{Nor}^+(K,y) \sim (y + \operatorname{ri} \operatorname{Nor}^+(K,y))$ for some $y \in T_r$ and hence there exists a sequence c_i, $i = 1, \ldots$, in $\operatorname{ri} \operatorname{Nor}^+(K,y)$ such that $y + c_i \to b$. We have $y + c_i \in \bigcup \{ y + \operatorname{ri} \operatorname{Nor}^+(K,y) : y \in T_r \}$. □

For each convex set C we denote by $\operatorname{rbd} C$ the relative boundary of C, i.e., $\operatorname{rbd} C = \operatorname{cl} C \sim \operatorname{ri} C$. As we have (2.1.3), it follows from Lemma 2.4.1 that if $\mathbf{B}_r$ holds then

$$E = \bigcup_{r=0}^{n} E_r \tag{2.4.2}$$

where

$$E_r = \bigcup \{ y + \operatorname{rbd} \operatorname{Nor}^+(K,y) : y \in T_r \}.$$

We now invoke the coarea and area formulas of the geometric measure theory to show that E_r is Lebesgue negligible.

2.4.2 Lemma. *Let r be an integer with* $0 \le r \le n$ *and assume that* $\mathbf{A}_r$ *holds. Then*

(i) *the set*

$$\tilde{E}_r := (\Phi_r | \mathscr{N}_r)^{-1}(E_r) \subset \mathscr{M}_r \subset \mathbf{R}^n \times \mathbf{R}^n \tag{2.4.3}$$

has the $\mathscr{H}^n$ *measure* 0 (*here* $\mathscr{H}^n$ *is the n dimensional Hausdorff measure in* $\mathbf{R}^{2n} \equiv \mathbf{R}^n \times \mathbf{R}^n$);

(ii) *the set* E_r *has null Lebesgue measure in* $\mathbf{R}^n$.

Proof (i): By $\mathbf{A}_r$, the set T_r is $(\mathscr{H}^{n-r}, n-r)$ rectifiable and if $f : \mathscr{M}_r \to T_r$ is a map defined by $f(x,z) = x$ for every $(x,z) \in \mathscr{M}_r$ then f is class $p - 1 \ge 1$. The general coarea formula (see [19; Theorem 2.4]) gives

$$\mathscr{H}^n(\tilde{E}_r) \equiv \int_{\tilde{E}_r} d\mathscr{H}^n = \int_{T_r} \mathscr{H}^r(\tilde{E}_r \cap f^{-1}\{y\}) \, d\mathscr{H}^{n-r}(y)$$

where we note that the jacobian of f is 1. For each $y \in T_r$ we have

$$\tilde{E}_r \cap f^{-1}\{y\} = \operatorname{rbd} \operatorname{Nor}^+(K,y) \times \{y\}$$

and the convex subset rbd $\operatorname{Nor}^+(K,y)$ of the r dimensional linear space $\operatorname{Nor}(T_r,y)$ has vanishing r dimensional measure $\mathscr{H}^r$. Indeed, on $\operatorname{Nor}(T_r,y)$ the Hausdorff measure $\mathscr{H}^r$ coincides with the r dimensional Lebesgue measure on $\operatorname{Nor}(T_r,y)$ and the relative boundary of any convex set in an r dimensional space has vanishing Lebesgue measure as a general assertion (this follows, e.g., as a very special case of Theorem 1.2).

(ii): Since by $(2.4.3)_1$ the map $(\Phi_r|\mathscr{N}_r)^{-1}$ maps E_r bijectively onto $\tilde{E}_r$, and its derivative is injective everywhere, by the area formula [7; Theorem 3.2.3(1)] we have

$$\int_{E_r} J \, d\mathscr{L}^n = \mathscr{H}^n(\tilde{E}_r) = 0$$

by (i), where J is the everywhere positive jacobian of the diffeomorphism $(\Phi_r|\mathscr{N}_r)^{-1}$. Thus $\mathscr{L}^n(E_r) = 0$. □

2.4.3 Proof of Theorem 1.6. In view of Theorem 2.3.4, only $\mathscr{L}^n(E) = 0$ remains to be proved. But this follows from Lemma 2.4.2 and (2.4.2). □

Chapter 3

The space of symmetric matrices and the generalized inverse

In Chapters 4 and 5, below, the results of Chapter 2 will be used to determine the derivative of the projection from the space Sym of symmetric $m \times m$ matrices onto the convex cone Sym^+ of positive semidefinite matrices and of the projection from Sym onto the unit ball Sym^1 under the operator norm. Here m is a positive integer. The preceding theory applies with $n = m(m+1)/2$.

We denote by Lin the space of linear maps from $\mathbf{R}^m$ to itself endowed with the euclidean scalar product $a \cdot b = \mathrm{tr}(ab^{\mathrm{T}})$, $a, b \in \mathrm{Lin}$; we denote by $\| \cdot \|$ the associated euclidean norm. As mentioned above, Sym^+ is the convex cone of positive semidefinite matrices; we further denote by Sym^- the convex cone of negative semidefinite matrices. For each $y \in \mathrm{Sym}$ we denote by $q(y)$ and $r(y)$ the (linear) orthogonal projectors onto $\mathrm{ran}\, y$ and $\ker y$, respectively, $q(y) + r(y) = \mathbf{1}_{\mathbf{R}^m}$.

Let $\mathbf{N}_0$ denote the set of all nonnegative integers.

We shall use the following notation:

3.1 Proposition. *For each* $x \in \mathrm{Sym}$ *there exists a unique* $x^{-1} \in \mathrm{Sym}$ *such that*

$$x^{-1}x = xx^{-1} = q(x).$$

We call x^{-1} the generalized inverse of x in the present paper. Note that if $x = \mathrm{diag}(x_1, \ldots, x_m)$ then $x^{-1} = \mathrm{diag}(\xi_1, \ldots, \xi_m)$ where

$$\xi_i = \begin{cases} 1/x_i & \text{if} \quad x_i \neq 0, \\ 0 & \text{if} \quad x_i = 0. \end{cases} \tag{3.4}$$

Proof This follows from (3.4) and the spectral theorem for symmetric matrices. □

Chapter 4

Projection onto the set of positive semidefinite matrices

As mentioned in the introduction, the metric projection P onto the closed convex cone Sym^+ is closely related to the no–tension masonry materials of continuum mechanics. We here calculate the derivative of P and detail the singularities of the boundary of Sym^+.

4.1 The formula for P; orthogonal invariance

Throughout the chapter, let ρ be an integer with $0 \le \rho \le m$. We denote by Sym_ρ and Sym_ρ^+ the set of all elements y of Sym and Sym^+, respectively, with $\operatorname{rank} y = \rho$. We put

$$\hat{r}(\rho) = [m(m+1) - \rho(2m - \rho + 1)]/2.$$

Let $P : \mathrm{Sym} \to \mathrm{Sym}^+$ be the metric projection onto the closed convex cone Sym^+ relative to the metric $\|\cdot\|$.

4.1.1 Remarks.

(i) *We have*

$$P(uxu^{\mathrm{T}}) = uP(x)u^{\mathrm{T}}$$

for each $x \in \mathrm{Sym}$ *and each orthogonal transformation* $u \in \mathrm{Lin}$.

(ii) *If* $x = \operatorname{diag}(x_1, \dots, x_m)$ *where* $x_i > 0$ *for* $i = 1, \dots, \rho$ *and* $x_\alpha \le 0$ *for* $\alpha = \rho+1, \dots, m$ *then*

$$P(x) = \operatorname{diag}(x_1, \dots, x_\rho, 0, \dots, 0).$$

(iii) *If P is differentiable at* $x \in \mathrm{Sym}$ *then it is also differentiable at* uxu^{T} *for any orthogonal transformation u and the derivatives satisfy*

$$\mathrm{D}P(uxu^{\mathrm{T}})[uhu^{\mathrm{T}}] = u\,\mathrm{D}P(x)[h]u^{\mathrm{T}}$$

for each $h \in \mathrm{Sym}$. *Because of that, it suffices to calculate the derivative of P at diagonal matrices only.*

Proof All this follows from the relation $u\mathrm{Sym}^+ u^{\mathrm{T}} = \mathrm{Sym}^+$ for each orthogonal transformation $u \in \mathrm{Lin}$. The details are left to the reader. □

4.2 Second fundamental form of Sym^+_ρ

In this section we view Sym^+_ρ as a riemannian manifold imbedded in the euclidean space Sym. The riemannian structure of Sym^+_ρ has been recently explored in [25] but the second fundamental form of the imbedding, the main goal of this section, is not treated there.

We denote by q_ρ and r_ρ the restrictions of q and r to Sym^+_ρ.

4.2.1 Proposition.

(i) *The set* Sym^+_ρ *is a connected manifold of dimension* $\rho(2m-\rho+1)/2$ *of class* ∞;

(ii) *if* $y \in \mathrm{Sym}^+_\rho$ *then the tangent and normal spaces to* Sym^+_ρ *at* y *are given, respectively, by*

$$\mathrm{Tan}(\mathrm{Sym}^+_\rho, y) = \{b \in \mathrm{Sym} : r_\rho(y) b r_\rho(y) = 0\}, \tag{4.2.1}$$

$$\mathrm{Nor}(\mathrm{Sym}^+_\rho, y) = \{z \in \mathrm{Sym} : r_\rho(y) z r_\rho(y) = z\}, \tag{4.2.2}$$

with

$$\dim \mathrm{Nor}(\mathrm{Sym}^+_\rho, y) = \hat{r}(\rho);$$

(iii) *the orthogonal projections* $Q_{\hat{r}(\rho)}(y)$ *and* $R_{\hat{r}(\rho)}(y)$, *onto* $\mathrm{Tan}(\mathrm{Sym}^+_\rho, y)$ *and* $\mathrm{Nor}(\mathrm{Sym}^+_\rho, y)$ *respectively, are given by*

$$R_{\hat{r}(\rho)}(y)c = r_\rho(y) c r_\rho(y), \quad Q_{\hat{r}(\rho)}(y)c = c - r_\rho(y) c r_\rho(y) \tag{4.2.3}$$

for any $y \in \mathrm{Sym}^+_\rho$ *and* $c \in \mathrm{Sym}$.

If for $y \in \mathrm{Sym}^+_\rho$ we denote by $H \subset \mathbf{R}^m$ the range of y with $\dim H = \rho$ so that corresponding to the decomposition $\mathbf{R}^m = H \oplus H^\perp$ the matrix y has the block form

$$y = \begin{bmatrix} y_0 & 0 \\ 0 & 0 \end{bmatrix},$$

where y_0 is a symmetric $\rho \times \rho$ matrix. Then each $b \in \mathrm{Tan}(\mathrm{Sym}^+_\rho, y)$ has the form

$$b = \begin{bmatrix} b_0 & b_1 \\ b_1^{\mathrm{T}} & 0 \end{bmatrix},$$

while each $z \in \mathrm{Nor}(\mathrm{Sym}^+, y)$ has the form

$$z = \begin{bmatrix} 0 & 0 \\ 0 & z_0 \end{bmatrix}.$$

Here b_0 is a symmetric $\rho \times \rho$ matrix, b_1 is a $\rho \times (m-\rho)$ matrix and z_0 is a symmetric $(m-\rho) \times (m-\rho)$ matrix.

We set the indexes of the projections and of the second fundamental form (below) equal to $\hat{r}(\rho)$ to comply with the notation of the general theory in Chapter 2, since we shall see in the next section that $T_{\hat{r}(\rho)} = \mathrm{Sym}_\rho^+$; more precisely we have (4.3.3) (below).

Proof (i): This follows from [10; Proposition 1.1, Section 5.1]. (ii): The same proposition asserts that

$$\mathrm{Tan}(\mathrm{Sym}_\rho^+, y) = \{cy + yc^{\mathrm{T}} : c \in \mathrm{Lin}\}.$$

Thus if $z \in \mathrm{Nor}(\mathrm{Sym}_\rho^+, y)$, we have, for any $c \in \mathrm{Lin}$,

$$cy \cdot z + yc^{\mathrm{T}} \cdot z = 2c \cdot zy = 0$$

which implies $zy = 0$ and as y is positive semidefinite, this further implies $zq_\rho(y) = 0$; hence $zr_\rho(y) = z$ and consequently $r_\rho(y)zr_\rho(y) = z$. This proves (4.2.2). To prove (4.2.1), we note that if $c \in \mathrm{Sym}$ then $r_\rho(y)cr_\rho(y) \in \mathrm{Nor}(\mathrm{Sym}_\rho^+, y)$ by (4.2.2) and hence if $b \in \mathrm{Tan}(\mathrm{Sym}_\rho^+, y)$, we have $b \cdot r_\rho(y)cr_\rho(y) = r_\rho(y)br_\rho(y) \cdot c = 0$. Hence $r_\rho(y)br_\rho(y) = 0$, which proves (4.2.1). (iii): immediate. □

4.2.2 Proposition. *The second fundamental form $B_{\hat{r}(\rho)}$ of Sym_ρ^+ is given by*

$$B_{\hat{r}(\rho)}(y)(b,c) = r_\rho(y)(by^{-1}c + cy^{-1}b)r_\rho(y)$$

for any $y \in \mathrm{Sym}_\rho^+$ and b, $c \in \mathrm{Tan}(\mathrm{Sym}_\rho^+, y)$.

Proof Note first that the map q_ρ is of class ∞ on Sym_ρ^+. Differentiating the relation $q_\rho(y)y = y$ for each $y \in \mathrm{Sym}_\rho^+$ in the direction $b \in \mathrm{Tan}(\mathrm{Sym}_\rho^+, y)$ we obtain

$$\mathrm{D}\, q_\rho(y)[b]y + q_\rho(y)b = b;$$

multiplying by y^{-1} we then obtain

$$\mathrm{D}\, q_\rho(y)[b]q_\rho(y) = r_\rho(y)by^{-1} \tag{4.2.4}$$

for any $y \in \mathrm{Sym}_\rho^+$.

Differentiating $(4.2.3)_1$ we obtain

$$\begin{aligned} \mathrm{D}\, R_{\hat{r}(\rho)}(y)[b]c &= \mathrm{D}\, r_\rho(y)[b]cr_\rho(y) + r_\rho(y)c\,\mathrm{D}\, r_\rho(y)[b] \\ &= -\mathrm{D}\, q_\rho(y)[b]cr_\rho(y) - r_\rho(y)c\,\mathrm{D}\, q_\rho(y)[b]. \end{aligned}$$

In particular if $c \in \mathrm{Tan}(\mathrm{Sym}_\rho^+, y)$ then from (4.2.1) we find $cr_\rho(y) = q_\rho(y)cr_\rho(y)$ and thus the last line and (4.2.4) provide

$$\begin{aligned} \mathrm{D}\, R_{\hat{r}(\rho)}(y)[b]c &= -\mathrm{D}\, q_\rho(y)[b]q_\rho(y)cr_\rho(y) - r_\rho(y)cq_\rho(y)\,\mathrm{D}\, q_\rho(y)[b] \\ &= -r_\rho(y)(by^{-1}c + cy^{-1}b)r_\rho(y). \end{aligned}$$

Consequently

$$\mathrm{D}\, Q_{\hat{r}(\rho)}(y)[b]c = r_\rho(y)(by^{-1}c + cy^{-1}b)r_\rho(y).$$

The definition of $B_{\hat{r}(\rho)}$ then gives the result. □

4.3 The normal cone to Sym^+

We here determine the sets $\mathrm{Nor}^+(\mathrm{Sym}^+, y)$, $y \in \mathrm{Sym}^+$ and verify that the convex cone $K = \mathrm{Sym}^+$ satisfies Assumptions $\mathbf{A}_r$ and $\mathbf{B}_r$ for all $r = 0, \ldots, m(m+1)/2$.

4.3.1 Proposition. *If* $y \in \mathrm{Sym}^+_\rho$ *then*

$$\mathrm{Nor}^+(\mathrm{Sym}^+, y) = \{z \in \mathrm{Sym}^- : r_\rho(y) z r_\rho(y) = z\}, \tag{4.3.1}$$

$$\mathrm{ri}\,\mathrm{Nor}^+(\mathrm{Sym}^+, y) = \{z \in \mathrm{Sym}^- : r_\rho(y) z r_\rho(y) = z, \mathrm{rank}\, z = m - \rho\}. \tag{4.3.2}$$

Proof Equation (4.3.1): It follows from the fact that Sym^+ is a convex cone that $\mathrm{Nor}^+(\mathrm{Sym}^+, y)$ is the set of all elements of the dual cone that are perpendicular to y (see [22; Example 11.4(b)]). The dual cone is Sym^- and thus

$$\mathrm{Nor}^+(\mathrm{Sym}^+, y) = \{z \in \mathrm{Sym}^- : z \cdot y = 0\};$$

however, since $z \in \mathrm{Sym}^-$ and $y \in \mathrm{Sym}^+$, the relation $z \cdot y = 0$ implies $zy = 0$; this in turn implies that $z q_\rho(y) = 0$. We finally conclude that $r_\rho(y) z r_\rho(y) = z$.

Equation (4.3.2): It follows from (4.3.1) that all elements z of $\mathrm{Nor}^+(\mathrm{Sym}^+, y)$ have all nonpositive eigenvalues and satisfy $\mathrm{rank}\, z \le m - \rho$. Since the ordered m tuple of eigenvalues is a lipschitzian function of the matrix, one sees that the set on the right hand side of (4.3.2) is open in $\mathrm{Nor}(\mathrm{Sym}^+_\rho, y) \equiv \mathrm{span}\,\mathrm{Nor}^+(\mathrm{Sym}^+, y)$. On the other hand, if $\mathrm{rank}\, z < m - \rho$ then each neighborhood of z contains a matrix $\bar{z} \in \mathrm{Nor}(\mathrm{Sym}^+_\rho, y)$ which is not negative semidefinite. Thus each such a z is on the boundary of $\mathrm{Nor}^+(\mathrm{Sym}^+, y)$. □

4.3.2 Corollary.
(i) *For each* $r = 0, \dots, m(m+1)/2$ *the set* T_r *from* (1.2) *is*

$$T_r = \begin{cases} \mathrm{Sym}^+_\rho & \text{if} \quad r = \hat{r}(\rho) \quad \text{where} \quad \rho = 0, \dots, m, \\ \emptyset & \text{else.} \end{cases} \tag{4.3.3}$$

(ii) *Assumption* $\mathbf{A}_r$ *is satisfied for all* $r = 0, \dots, m(m+1)/2$ *with* $p = \infty$.

Proof (i): Comparing (4.3.1) with (4.2.2), we see that $\dim \mathrm{Nor}^+(\mathrm{Sym}^+, y) = \dim \mathrm{Nor}(\mathrm{Sym}^+, y) = \hat{r}(\rho)$ for each $y \in \mathrm{Sym}^+_\rho$, which gives the result.

(ii): Follows from (i) and Proposition 4.2.1(i). □

We denote by InvSym the set of all injective transformations from Sym.

4.3.3 Corollary.
(i) *For each* $r = 0, \dots, m(m+1)/2$ *the set* V_r *from* (1.4) *is*

$$V_r = \begin{cases} \{x \in \mathrm{Sym} : \mathrm{rank}\, P(x) = \rho\} & \text{if } r = \hat{r}(\rho) \\ & \text{where } \rho = 0, \dots, m, \\ \emptyset & \text{else,} \end{cases} \tag{4.3.4}$$

and its interior W_r *is*

$$W_r = \begin{cases} \{x \in \mathrm{InvSym} : \mathrm{rank}\, P(x) = \rho\} & \text{if } r = \hat{r}(\rho) \\ & \text{where } \rho = 0, \dots, m, \\ \emptyset & \text{else.} \end{cases} \tag{4.3.5}$$

(ii) *Assumption* $\mathbf{B}_r$ *is satisfied for all* $r = 0, \dots, m(m+1)/2$.

Proof (i): Equation (4.3.4) follows directly from the definition and from Corollary 4.3.2(i). Equation (4.3.5): The set on the right hand side of (4.3.5) is open since if x belongs to this set then ρ eigenvalues of x are positive and $m-\rho$ eigenvalues negative. Since the ordered m tuple of eigenvalues is a lipschitzian function of x, the assertion about positive and negative eigenvalues is stable under the perturbation of x. This proves that we have "$\supset$" sign in (4.3.5). Conversely, if x is such that $\operatorname{rank} P(x)=\rho$, $\operatorname{rank}(x-P(x))<m-\rho$ then at least one eigenvalue of x vanishes and thus any neighborhood of x contains an element $\bar{x}$ with $\operatorname{rank}\bar{x}=\rho+1$. This element does not belong to $V_{\hat{r}(\rho)}$ which proves that x is a boundary point of $V_{\hat{r}(\rho)}$. Thus we have "$\subset$" in (4.3.5).

(ii): Follows from (4.3.2) and the fact each perturbation $\bar{z}\in\operatorname{Nor}(\operatorname{Sym}_\rho^-,\bar{y})$ of a matrix $z\in\operatorname{Sym}^-$ with $\operatorname{rank} z=n-\rho$ is a matrix with $\operatorname{rank}\bar{z}=n-\rho$. □

4.4 The derivative of P

We can finally determine $\mathrm{D}P$.

4.4.1 Remark. *For any $y\in\operatorname{Sym}_\rho^+$ and $z\in\operatorname{Nor}^+(\operatorname{Sym}^+,y)$, the map $C_r(y,z)$ [see (2.3.2)] is defined only if $r=\hat{r}(\rho)$ for some $\rho=0,\dots,m$ and then*

$$C_{\hat{r}(\rho)}(y,z)b=y^{-1}bz+zby^{-1} \tag{4.4.1}$$

for every $b\in\operatorname{Tan}(\operatorname{Sym}_\rho^+,y)$.

Proof This follows from Proposition 4.2.2. □

4.4.2 Theorem. *The map P is infinitely differentiable on* InvSym; *if $x\in$* InvSym *and $c\in$* Sym *then $\mathrm{D}P(x)[c]=b$ where $b\in$* Sym *is the unique solution of the equation*

$$b-y^{-1}b(x-y)-(x-y)by^{-1}=c-r(y)cr(y) \tag{4.4.2}$$

where $y=P(x)$. Equation (4.4.2) *splits into*

$$\left.\begin{aligned} q(y)bq(y)&=q(y)cq(y),\\ r(y)bq(y)-(x-y)by^{-1}&=r(y)cq(y),\\ r(y)br(y)&=0.\end{aligned}\right\} \tag{4.4.3}$$

If $x=\operatorname{diag}(x_1,\dots,x_m)$ where $x_i>0$ for $i=1,\dots,\rho$ and $x_\alpha<0$ for $\alpha=\rho+1,\dots,m$ then

$$P(x)[c]=\begin{bmatrix}\alpha & \beta\\ \beta^{\mathrm{T}} & 0\end{bmatrix}$$

where α and β are $\rho\times\rho$ and $\rho\times(m-\rho)$ matrices given by

$$\alpha_{ij}=c_{ij},\quad 0\le i,j\le\rho, \tag{4.4.4}$$

$$\beta_{i\alpha}=(1-x_\alpha/x_i)^{-1}c_{i\alpha},\quad 1\le i\le\rho,\ \rho+1\le\alpha\le m. \tag{4.4.5}$$

Formulas (4.4.4) and (4.4.5) show the coincidence with the result [17; Theorem 2.7].

Proof Denoting $\mathrm{D}P(x)[c]=b$ we have from (2.3.6)

$$\left[\mathbf{1}_{\operatorname{Tan}(\operatorname{Sym}_\rho^+,y)}-C_{\hat{r}(\rho)}(y,x-y)\right]b=Q_{\hat{r}(\rho)}(y)c$$

which by (4.4.1) and $Q_{\hat{r}(\rho)}(y)c = c - r_{\hat{r}(\rho)}(y)cr_{\hat{r}(\rho)}(y)$ reads as (4.4.2). Multiplying (4.4.2) by $q(y)$ from the left and right we obtain $(4.4.3)_1$; multiplying (4.4.2) by $r(y)$ from the left and $q(y)$ from the right we obtain $(4.4.3)_2$; multiplying (4.4.2) by $r(y)$ from the left and from the right we obtain $(4.4.3)_3$. Finally, if $x = \operatorname{diag}(x_1, \dots, x_m)$ where $x_i > 0$ for $i = 1, \dots, \rho$ and $x_\alpha < 0$ for $\alpha = \rho + 1, \dots, m$ then $y = P(x) = \operatorname{diag}(x_1, \dots, x_\rho, 0, \dots, 0)$ and $x - y = \operatorname{diag}(0, \dots, 0, x_{\rho+1}, \dots, x_m)$. Then with the notation

$$b = \begin{bmatrix} \alpha & \beta \\ \beta^{\mathrm{T}} & \gamma \end{bmatrix}$$

we see that $(4.4.3)_1$ reduces to (4.4.4), the transpose of $(4.4.3)_2$ reads in the component form

$$(1 - x_\alpha / x_i)\beta_{i\alpha} = c_{i\alpha}, \quad 1 \le i \le \rho, \ \rho + 1 \le \alpha \le n,$$

which gives (4.4.5), and $(4.4.3)_3$ provides $\gamma = 0$. This also shows the uniqueness of the solution of (4.4.2). □

Chapter 5

Projection onto the unit ball under the operator norm

In comparison with Sym^+, the structure of the boundary of the unit ball Sym^1 under the operator norm is more complicated in that the manifolds forming the boundary of Sym^1 must be parameterized by two parameters σ and τ determining the multiplicity of the occurrence of the numbers 1 and −1 in the spectrum of the boundary point. (In particular, it will be clear that the unit matrix $\mathbf{1}_{\mathbf{R}^m}$ is at the corner of Sym^1.) The formula for the derivative of the projection onto Sym^1 is accordingly more complicated also.

5.1 The formula for P; orthogonal invariance

For each $a \in \mathrm{Sym}$ we denote by $m_+(a)$ the orthogonal projection onto the span of all eigenvectors corresponding to eigenvalues bigger than or equal to 1 and by $m_-(a)$ the orthogonal projection onto the span of all eigenvectors corresponding to eigenvalues lower than or equal to −1. We also write

$$m_\circ(a) = \mathbf{1}_{\mathbf{R}^m} - m_+(a) - m_-(a).$$

Let $v : \mathrm{Sym} \to [0, \infty)$ denote the operator norm on Sym, defined in (1.1) and let

$$\mathrm{Sym}^1 = \{y \in \mathrm{Sym} : v(y) \le 1\}$$

be the unit ball under v. Throughout the chapter, let P be the metric projection onto Sym^1.

For each $\sigma, \tau \in \mathbf{N}_0$ such that $\sigma + \tau \le m$ and $\sigma \le \tau$ let

$$\hat{r}(\sigma, \tau) = [\sigma(\sigma + 1) + \tau(\tau + 1)]/2.$$

If $\hat{r}(\sigma, \tau)$ is mentioned in the subsequent treatment, it is always assumed that σ, $\tau \in \mathbf{N}_0$, $\sigma + \tau \le m$ and $\sigma \le \tau$.

5.1.1 Remark. The projection P satisfies Items (i) and (iii) of Remarks 4.1.1.

5.1.2 Proposition. *For each* $x \in \mathrm{Sym}$,

$$P(x) = m_+(x) - m_-(x) + m_\circ(x) x m_\circ(x). \tag{5.1.1}$$

If $x = \mathrm{diag}(x_1, \ldots, x_m)$ *with*

$$x_1 \geq x_2 \geq \ldots \geq x_\sigma \geq 1 > x_{\sigma+1} \geq \ldots \geq x_{m-\tau} > 1 \geq x_{m-\tau+1} \geq \ldots \geq x_m. \tag{5.1.2}$$

then

$$P(x) = \mathrm{diag}(1, \ldots, 1, x_{\sigma+1}, \ldots, x_{m-\tau}, -1, \ldots, -1) \tag{5.1.3}$$

with the number 1 *occurring* σ *times and the number* -1 *occurring* τ *times.*

Proof In view of the spectral theorem and Remarks 5.1.1, it suffices to prove the diagonal case. Thus let $x = \mathrm{diag}(x_1, \ldots, x_m)$ satisfy (5.1.2) and let y be given by the right hand side of (5.1.3). Then

$$\|x - y\|^2 = (x_1 - 1)^2 + \ldots + (x_\sigma - 1)^2 + (x_{m-\tau+1} + 1)^2 + \ldots + (x_m + 1)^2.$$

Let now $w \in \mathrm{Sym}^1$ have the eigenvalues $1 \geq w_1 \geq \ldots \geq w_m \geq -1$. By [13; exercise, p. 370] we have

$$\sum_{i=1}^{m} (x_i - w_i)^2 \leq \|x - w\|^2$$

and since

$$(x_1 - 1)^2 + \ldots + (x_\sigma - 1)^2 + (x_{m-\tau+1} + 1)^2 + \ldots + (x_m + 1)^2 \leq \sum_{i=1}^{m} (x_i - w_i)^2,$$

we have $\|x - y\| \leq \|x - w\|$. This proves (5.1.3). Formula (5.1.1) then easily follows in the diagonal case and the general case is established via Remark 5.1.1. □

5.2 Second fundamental form of $\mathbf{Sym}^{(\sigma,\tau)}$

For any $\sigma, \tau \in \mathbf{N}_0$ such that $\sigma + \tau \leq m$ and $\sigma \leq \tau$ let $\mathrm{Sym}^{(\sigma,\tau)}$ be the set of all $y \in \mathrm{Sym}$ whose ordered m-tuple of eigenvalues satisfies

$$1 = \lambda_1 = \ldots = \lambda_\sigma > \lambda_{\sigma+1} \geq \ldots \geq \lambda_{m-\tau} > \lambda_{m-\tau+1} = \ldots = \lambda_m = -1 \tag{5.2.1}$$

or

$$1 = \lambda_1 = \ldots = \lambda_\tau > \lambda_{\tau+1} \geq \ldots \geq \lambda_{m-\sigma} > \lambda_{m-\sigma+1} = \ldots = \lambda_m = -1. \tag{5.2.2}$$

5.2.1 Proposition. *For any* $\sigma, \tau \in \mathbf{N}_0$ *such that* $\sigma + \tau \leq m$ *and* $\sigma \leq \tau$ *we have the following:*

(i) *the set* $\mathrm{Sym}^{(\sigma,\tau)}$ *is a class* ∞ *manifold and*

$$\dim \mathrm{Sym}^{(\sigma,\tau)} = m(m+1)/2 - \hat{r}(\sigma,\tau);$$

(ii) *for any* $y \in \mathrm{Sym}^{(\sigma,\tau)}$,

$$\mathrm{Tan}(\mathrm{Sym}^{(\sigma,\tau)}, y) = \{b \in \mathrm{Sym} : m_+(y) b m_+(y) = m_-(y) b m_-(y) = 0\}, \tag{5.2.3}$$

$$\mathrm{Nor}(\mathrm{Sym}^{(\sigma,\tau)}, y) = \{z \in \mathrm{Sym} : m_+(y) z m_+(y) + m_-(y) z m_-(y) = z\} \tag{5.2.4}$$

for any $y \in \mathrm{Sym}^1$;

(iii) *the orthogonal projectors onto the tangent and normal spaces to* $\mathrm{Sym}^{(\sigma,\tau)}$ *at* $y \in \mathrm{Sym}^{(\sigma,\tau)}$ *are*

$$Q_{\hat{r}(\sigma,\tau)}(y)c = c - m_+(y)cm_+(y) - m_-(y)cm_-(y), \tag{5.2.5}$$

$$R_{\hat{r}(\sigma,\tau)}(y)c = m_+(y)cm_+(y) + m_-(y)cm_-(y) \tag{5.2.6}$$

for any $c \in \mathrm{Sym}$.

Corresponding to the decomposition

$$\mathbf{R}^m = m_+(y)\mathbf{R}^m \oplus m_\circ(y)\mathbf{R}^m \oplus m_-(y)\mathbf{R}^m$$

the elements $b \in \mathrm{Tan}(\mathrm{Sym}^{(\sigma,\tau)}, y)$ have the block form

$$b = \begin{bmatrix} 0 & b^{(+,\circ)} & b^{(+,-)} \\ (b^{(+,\circ)})^{\mathrm{T}} & b^{(\circ,\circ)} & (b^{(-,\circ)})^{\mathrm{T}} \\ (b^{(+,-)})^{\mathrm{T}} & b^{(-,\circ)} & 0 \end{bmatrix}$$

while the elements $z \in \mathrm{Nor}(\mathrm{Sym}^{(\sigma,\tau)}, y)$ the block form

$$z = \begin{bmatrix} z^{(+,+)} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & z^{(-,-)} \end{bmatrix}.$$

Proof Consider the part of $\mathrm{Sym}^{(\sigma,\tau)}$ defined by (5.2.1); the case (5.2.2) is entirely analogous. To prove that the indicated part of $\mathrm{Sym}^{(\sigma,\tau)}$ is a manifold of the indicated dimension, we take $y \in \mathrm{Sym}^{(\sigma,\tau)}$, and write

$$y = \mu_+ + w - \mu_- \tag{5.2.7}$$

where $\mu_\pm = m_\pm(y)$. Let $\{f_1, \dots, f_m\}$ be an orthonormal basis in $\mathbf{R}^m$ such that

$$\left.\begin{aligned} &\mu_+ \text{ is the projection on the span of } \{f_1, \dots, f_\sigma\}, \\ &\mu_- \text{ is the projection on the span of } \{f_{\sigma+1}, \dots, f_{m-\tau}\} \end{aligned}\right\} \tag{5.2.8}$$

and let $\zeta = [\zeta_{ij}]_{i,j=1}^{m-\tau-\sigma}$ be the matrix of w in the basis $\{f_{\sigma+1}, \dots, f_{m-\tau}\}$, more precisely let

$$w = \sum_{i,i=1}^{m-\sigma-\tau} \zeta_{ij} f_{\sigma+i} \otimes f_{\sigma+j}. \tag{5.2.9}$$

The array

$$\alpha = (\{f_1, \dots, f_m\}, \zeta) \tag{5.2.10}$$

completely determines the transformation y uniquely by the requirements (5.2.7), (5.2.8), and (5.2.9). To obtain a one–to–one relationship, we must introduce an equivalence relation identifying arrays α that are related by orthogonal matrices. To this end, for any positive integer ω, let $O(\omega)$ be the Lie group of orthogonal $\omega \times \omega$ matrices and for any $q \in O(\omega)$ and any orthonormal system $\{o_1, \dots, o_\omega\}$ in $\mathbf{R}^m$ let $q\{o_1, \dots, o_\omega\}$ be the orthogonal system resulting from $\{o_1, \dots, o_\omega\}$ by the action of the tranformation q. Let $\mathscr{O}$ be the set of all orthonormal bases in $\mathbf{R}^m$ and $\mathscr{Z}$ the system of all $(m-\sigma-\tau)\times(m-\sigma-\tau)$ symmetric matrices, and consider the set $\tilde{\mathscr{M}}=\mathscr{O}\times\mathscr{Z}$ of all arrays α as in (5.2.10). Then the arrays α and

$$\bar{\alpha}=(\{\bar{f}_1,\dots,\bar{f}_m\},\bar{\zeta})$$

from $\tilde{\mathscr{M}}$ lead via (5.2.7), (5.2.8), and (5.2.9) to the same matrix if and only if there exist $q\in O(\sigma)$, $r\in O(m-\sigma-\tau)$ and $s\in O(t)$ such that

$$\left.\begin{aligned}\{\bar{f}_1,\dots,\bar{f}_\sigma\}&=q\{f_1,\dots,f_\sigma\},\\ \{\bar{f}_{\sigma+1},\dots,\bar{f}_{m-\tau}\}&=r\{f_{\sigma+1},\dots,f_{m-\tau}\},\\ \{\bar{f}_{m-\tau+1},\dots,\bar{f}_\tau\}&=s\{f_{m-\tau+1},\dots,f_\tau\},\\ \bar{\zeta}&=r\zeta r^{\mathrm{T}}.\end{aligned}\right\}\qquad(5.2.11)$$

Relations (5.2.11) introduce an equivalence $\approx$ on $\tilde{\mathscr{M}}$ and we denote by $\mathscr{M}'$ the quotient space $\mathscr{M}'=\tilde{\mathscr{M}}/\approx$ modulo this equivalence. For a given equivalence class $\alpha'\in\mathscr{M}'$ the value y determined by (5.2.7), (5.2.8), and (5.2.9) is independent of the choice of $\alpha\in\alpha'$ and these relations establish a one–to–one correspondepnce between $\mathrm{Sym}^{(\sigma,\tau)}$ and $\mathscr{M}'$. To complete the proof of (i), it now suffices to prove that $\mathscr{M}'$ is a class ∞ manifold with

$$\dim\mathscr{M}'=m(m+1)/2-\hat{r}(\sigma,\tau)\qquad(5.2.12)$$

and that the above correspondence is of class ∞. To prove that $\mathscr{M}'$ is a class ∞ manifold, one notes that $\tilde{\mathscr{M}}=\mathscr{O}\times\mathscr{Z}$ is a class ∞ manifold since $\mathscr{O}$ is isomorphic with $O(m)$ since each $\{f_1,\dots,f_m\}\in\mathscr{O}$ can be written uniquely as

$$\{f_1,\dots,f_m\}=u\{e_1,\dots,e_m\}$$

with $u\in O(m)$ and $\{e_1,\dots,e_m\}$ is the canonical basis in $\mathbf{R}^m$. Hence

$$\dim\mathscr{O}=m(m-1)/2.$$

Similarly $\mathscr{Z}$ is a class ∞ manifold and

$$\dim\mathscr{Z}=(m-\sigma-\tau)(m-\sigma-\tau+1)/2;$$

consequently

$$\dim\tilde{\mathscr{M}}=[m(m-1)+(m-\sigma-\tau)(m-\sigma-\tau+1)]/2.$$

That the quotient space $\mathscr{M}'$ is a class ∞ manifold follows from [14; Proposition 4.3, p. 44] by noting that the action of the product group $O(\sigma)\times Q_{\hat{r}(\sigma,\tau)}(m-\sigma-\tau)\times O(t)$ defined by (5.2.11) is properly discontinuous. The correspondence is of class ∞ since so are the involved relations. Finally, the dimension is calculated by noting that

$$\begin{aligned}\dim[O(\sigma)\times Q_{\hat{r}(\sigma,\tau)}(m-\sigma-\tau)\times O(t)]&=[\sigma(\sigma-1)\\&\quad+(m-\sigma-\tau)(m-\sigma-\tau+1)+\tau(\tau-1)]/2\end{aligned}$$

and thus

$$\dim\mathscr{M}'=\dim\tilde{\mathscr{M}}-\dim[O(\sigma)\times Q_{\hat{r}(\sigma,\tau)}(m-\sigma-\tau)\times O(t)]$$

which gives (5.2.12).

(ii): Let us prove the formula for the tangent space. Let a be a class 1 curve in $\mathrm{Sym}^{(\sigma,\tau)}$ satisfying (2.2.1). Write

$$\mu_\pm(t) = m_\pm(a(t)). \tag{5.2.13}$$

From $\mu_\pm(t) = \pm a(t)\mu_\pm(t)$ we obtain the following relations for the values of the derivatives at $\tau = 0$:

$$\dot\mu_\pm = \pm b\mu_\pm \pm y\dot\mu_\pm \tag{5.2.14}$$

Multiplying from the left by $\mu_\pm$ we obtain

$$\mu_\pm\dot\mu_\pm = \pm\mu_\pm b\mu_\pm \pm \mu_\pm y\dot\mu_\pm = \pm\mu_\pm b\mu_\pm + \mu_\pm\dot\mu_\pm$$

and hence

$$\mu_\pm b\mu_\pm = 0.$$

This prove that we have the inclusion sign "⊂" in (5.2.3). However, since the dimension of the linear space on the right hand side of (5.2.3) is $m(m+1)/2 - \hat r(\sigma,\tau)]$ and the dimension of the manifold $\mathrm{Sym}^{(\sigma,\tau)}$ is the same, we have actually the equality sign in (5.2.3).

Equation (5.2.4) is a direct consequence of (5.2.3).

(iii): Equations (5.2.5) and (5.2.6) define the symmetric idempotent transformations with the required ranges. □

5.2.2 Proposition. *For any $\sigma, \tau \in \mathbf{N}_0$ such that $\sigma + \tau \le m$ and $\sigma \le \tau$, the second fundamental form of $\mathrm{Sym}^{(\sigma,\tau)}$ at $y \in \mathrm{Sym}^{(\sigma,\tau)}$ is given by*

$$\begin{aligned} B_{\hat r(\sigma,\tau)}(y)(b,c) = & -m_+(y)\big(b(\mathbf{1}_{\mathbf{R}^m} - y)^{-1}c + c(\mathbf{1}_{\mathbf{R}^m} - y)^{-1}b\big)m_+(y) \\ & + m_-(y)\big(b(\mathbf{1}_{\mathbf{R}^m} + y)^{-1}c + c(\mathbf{1}_{\mathbf{R}^m} + y)^{-1}b\big)m_-(y) \end{aligned} \tag{5.2.15}$$

for any $b, c \in \mathrm{Tan}(\mathrm{Sym}^{(\sigma,\tau)}, y)$.

Proof Let y, b and c be as in the statement and let a be a class 1 function satisfying (2.2.1) and define $\mu_\pm$ by (5.2.13). Equation (5.2.14) can be rearranged as

$$(1 \mp y)\dot\mu_\pm = \pm b\mu_\pm.$$

Noting that the projector on the range of $1 \mp y$ is $1 - \mu_\pm$ we obtain

$$(1 - \mu_\pm)\dot\mu_\pm = \pm(1 \mp y)^{-1}b\mu_\pm. \tag{5.2.16}$$

Differentiating $t \mapsto Q_{\hat r(\sigma,\tau)}(a(t))c$ at $\tau = 0$ by using (5.2.5) we obtain

$$Q_{\hat r(\sigma,\tau)}(y)[b]c = -\dot\mu_+ c\mu_+ - \mu_+ c\dot\mu_+ - \dot\mu_- c\mu_- - \mu_- c\dot\mu_-.$$

Since $c \in \mathrm{Tan}(\mathrm{Sym}^{(\sigma,\tau)}, y)$, we have $\mu_\pm c = 0$ and thus

$$\begin{aligned} Q_{\hat r(\sigma,\tau)}(y)[b]c = & -\dot\mu_+(1-\mu_+)c\mu_+ - \mu_+ c(1-\mu_+)\dot\mu_+ \\ & -\dot\mu_-(1-\mu_-)c\mu_- - \mu_- c(1-\mu_-)\dot\mu_- \end{aligned}$$

and combining with (5.2.16) we obtain (5.2.15). □

5.3 The normal cone to Sym^1

We here determine the normal cone at the points of the ball and verify Assumptions $\mathbf{A}_r$ and $\mathbf{B}_r$.

5.3.1 Proposition. *If $\sigma, \tau \in \mathbf{N}_0$ satisfy $\sigma + \tau \le m$ and $\sigma \le \tau$ and $y \in \mathrm{Sym}^{(\sigma,\tau)}$ then $\mathrm{Nor}^+(\mathrm{Sym}^1, y)$ is the set of all $z \in \mathrm{Sym}$ such that*

$$m_+(y)zm_+(y) + m_-(y)zm_-(y) = z, \tag{5.3.1}$$

$$m_\pm(y)zm_\pm(y) \in \mathrm{Sym}^\pm \tag{5.3.2}$$

with

$$\dim \mathrm{Nor}^+(\mathrm{Sym}^1, y) = \hat{r}(\sigma, \tau) \tag{5.3.3}$$

and $\mathrm{ri}\,\mathrm{Nor}^+(\mathrm{Sym}^1, y)$ *is the set of all z from* $\mathrm{Nor}^+(\mathrm{Sym}^1, y)$ *such that exactly $\sigma + \tau$ eigenvalues of z are different from* 0.

Proof Equation (5.3.1) follows from the characterization of the normal space in Proposition 5.2.1(ii). To prove (5.3.2), we note that any $z \in \mathrm{Nor}(\mathrm{Sym}^{(\sigma,\tau)}, y)$ is of the form

$$z = z_+ + z_-$$

where

$$m_\pm(y)z_\pm m_\pm(y) = z_\pm$$

Then for any $b \in \mathrm{Sym}^1$ we have

$$0 \ge z_+ \cdot (b - y) = z_+ \cdot (m_+(y)bm_+(y) - m_+(y))$$

This is satisfied by all $b \in \mathrm{Sym}^1$ if and only if $z_+ \in \mathrm{Sym}^+$. Similarly, the inequality

$$0 \ge z_- \cdot (b - y) = z_- \cdot (m_-(y)bm_-(y) + m_-(y))$$

leads to $z_- \in \mathrm{Sym}^-$. The assertion about the relative interior is a consequence. □

5.3.2 Corollary.

(i) *For each integer r with $0 \le r \le m(m+1)/2$ we have*

$$T_r = \begin{cases} \mathrm{Sym}^{(\sigma,\tau)} & \text{if} \quad r = \hat{r}(\sigma,\tau) \quad \text{for some} \quad \sigma, \tau, \\ \emptyset & \text{else.} \end{cases}$$

(ii) *Assumption $\mathbf{A}_r$ is satisfied for all $r = 0, \dots, m(m+1)/2$.*

Proof (i) follows from (5.3.3) and the fact that the value $r = \hat{r}(\sigma, \tau)$ determines σ and τ uniquely up to a permutation. (ii) follows from (i) and Proposition 5.2.1(i). □

5.3.3 Corollary.

(i) *For each integer r with $0 \le r \le m(m+1)/2$ we have $V_r \ne \emptyset$ if an only if $r = \hat{r}(\sigma,\tau)$ for some σ, τ. If this is the case, V_r is the set of all $x \in \mathrm{Sym}$ such that exactly σ eigenvalues are ≥ 1 and exactly τ eigenvalues are ≤ -1 or exactly τ eigenvalues are ≥ 1 and exactly σ eigenvalues are ≤ -1.*

(ii) *The interior W_r of V_r is the set of all $x \in V_r$ such that exactly $\sigma + \tau$ eigenvalues of x have absolute value > 1.*

(iii) *Assumption $\mathbf{B}_r$ is satisfied for all $r = 0, \dots, m(m+1)/2$.*

Proof (i): This follows from the definition of V_r and (5.3.2) of Proposition 5.3.1. (ii): This follows from (i) and the lipschitzian continuity of the eigenvalues. (iii) follows from (i) and (ii). □

5.4 The derivative of P

We can finally determine $\mathrm{D}\,P$.

5.4.1 Remark. *For any $y \in \mathrm{Sym}^1$, $z \in \mathrm{Nor}^+(\mathrm{Sym}^{(\sigma,\tau)}, y)$, the map $C_r(y,z)$ is defined only if $r = \hat{r}(\sigma,\tau)$ for some σ, τ and then*

$$\begin{aligned} C_r(y,z)b = &-(\mathbf{1}_{\mathbf{R}^m} - y)^{-1} b z_+ - z_+ b(\mathbf{1}_{\mathbf{R}^m} - y)^{-1} \\ &+ (\mathbf{1}_{\mathbf{R}^m} + y)^{-1} b z_- + z_- b(\mathbf{1}_{\mathbf{R}^m} + y)^{-1} \end{aligned} \tag{5.4.1}$$

$b \in \mathrm{Tan}(\mathrm{Sym}^{(\sigma,\tau)}, y)$ where

$$z_\pm = m_\pm(y) z m_\pm(y). \tag{5.4.2}$$

Proof This follows from Proposition 5.2.2. □

5.4.2 Theorem. *The map P is infinitely differentiable on the set $\mathrm{Sym}^\#$ of all $x \in \mathrm{Sym}$ such that $v(x) \neq 1$. If $x \in \mathrm{Sym}^\#$ then $x \in W_r$ where $r = \hat{r}(\sigma,\tau)$ for some σ, τ. For any $d \in \mathrm{Sym}$ one has $\mathrm{D}\,P(x)[d] = b$ where $b \in \mathrm{Tan}(\mathrm{Sym}^{(\sigma,\tau)}, y)$ is the unique solution of the equation*

$$[\mathbf{1}_{\mathrm{Tan}(\mathrm{Sym}^{(\sigma,\tau)},\,y)} - C_r(y, x-y)]b = c \tag{5.4.3}$$

where $y = P(x)$, $c = Q_{\hat{r}(\sigma,\tau)}(y)d$. If we write b and c in the block form

$$b = \begin{bmatrix} 0 & b^{(+,\circ)} & b^{(+,-)} \\ (b^{(+,\circ)})^{\mathrm{T}} & b^{(\circ,\circ)} & (b^{(-,\circ)})^{\mathrm{T}} \\ (b^{(+,-)})^{\mathrm{T}} & b^{(-,\circ)} & 0 \end{bmatrix}, \quad c = \begin{bmatrix} 0 & c^{(+,\circ)} & c^{(+,-)} \\ (c^{(+,\circ)})^{\mathrm{T}} & c^{(\circ,\circ)} & (c^{(-,\circ)})^{\mathrm{T}} \\ (c^{(+,-)})^{\mathrm{T}} & c^{(-,\circ)} & 0 \end{bmatrix}$$

corresponding to the decomposition

$$\mathbf{R}^m = m_+(y)\mathbf{R}^m \oplus m_\circ(y)\mathbf{R}^m \oplus m_-(y)\mathbf{R}^m$$

then the block components satisfy the following system of decoupled equations

$$\left.\begin{aligned} b^{(+,\circ)} + z_+ b^{(+,\circ)}(\mathbf{1}_{\mathbf{R}^m} - y)^{-1} &= c^{(+,\circ)}, \\ b^{(-,\circ)} - z_- b^{(-,\circ)}(\mathbf{1}_{\mathbf{R}^m} + y)^{-1} &= c^{(-,\circ)}, \\ b^{(+,-)} + \tfrac{1}{2} z_+ b^{(+,-)} - \tfrac{1}{2} b^{(+,-)} z_- &= c^{(+,-)}, \\ b^{(\circ,\circ)} &= c^{(\circ,\circ)}, \end{aligned}\right\} \tag{5.4.4}$$

where we use the notation (5.4.2). *If $x = \mathrm{diag}(x_1, \dots, x_m)$ where the diagonal elements satisfy* (5.1.2) *then*

$$\left.\begin{aligned} b^{(+,\circ)}_{\mathfrak{a}i} &= \tfrac{1 - x_i}{x_\mathfrak{a} - x_i} c^{(+,\circ)}_{\mathfrak{a}i} && \text{for} \quad 1 \le \mathfrak{a} \le \sigma,\ \sigma + 1 \le i \le m - \tau, \\ b^{(-,\circ)}_{\mathfrak{b}i} &= \tfrac{1 + x_i}{x_i - x_\mathfrak{b}} c^{(-,\circ)}_{\mathfrak{b}i} && \text{for} \quad \sigma + 1 \le i \le m - \tau,\ m - \tau + 1 \le \mathfrak{b} \le m, \\ b^{(+,-)}_{\mathfrak{a}\mathfrak{b}} &= \tfrac{2}{x_\mathfrak{a} - x_\mathfrak{b}} c^{(+,-)}_{\mathfrak{a}\mathfrak{b}} && \text{for} \quad 1 \le \mathfrak{a} \le \sigma,\ m - \tau + 1 \le \mathfrak{b} \le m, \\ b^{(\circ,\circ)}_{ij} &= c^{(\circ,\circ)}_{ij} && \text{for} \quad \sigma + 1 \le i,j \le m - \tau. \end{aligned}\right\} \tag{5.4.5}$$

Proof Theorem 2.3.4 asserts the infinite differentiability of P on the union of all sets W_r, $0 \le r \le m(m+1)/2$. The characterization of W_r in Corollary 5.3.3 shows that this union is exactly $\mathrm{Sym}^{\#}$. This proves the assertion about the infinite differentiability of P. The assertion that $x \in \mathrm{Sym}^{\#}$ must belong to some W_r for $r = \hat{r}(\sigma, \tau)$ for some σ, τ then follows that for all other values of r the set W_r is empty. The formula (2.3.6) gives (5.4.3), including its unique solvability.

Let us now prove the system (5.4.4). We combine (5.4.3) with the value of C_r calculated in (5.4.1). Multiplying (5.4.3) by $m_+(y)$ from the left and using $m_+(y)(\mathbf{1}_{\mathbf{R}^m} + y)^{-1} = \frac{1}{2} m_+(y)$, we obtain

$$m_+(y)b + z_+ b(\mathbf{1}_{\mathbf{R}^m} - y)^{-1} - \tfrac{1}{2} m_+(y) b z_- = m_+(y)c.$$

Multiplying by $m_\circ(y)$ and $m_-(y)$ from the right, noting that

$$b^{(+,\circ)} = m_+(y) b m_\circ(y), \quad b^{(+,-)} = m_+(y) b m_-(y),$$

$$c^{(+,\circ)} = m_+(y) c m_\circ(y), \quad c^{(+,-)} = m_+(y) c m_-(y),$$

and using the mutual commutativity of all encountered matrices except for b, we obtain $(5.4.4)_{1,3}$. Similarly, multiplying (5.4.3) by $m_-(y)$ from the left and using $m_-(y)(\mathbf{1}_{\mathbf{R}^m} - y)^{-1} = \frac{1}{2} m_-(y)$ provides

$$m_-(y)b + \tfrac{1}{2} m_-(y) b z_+ - z_- b(\mathbf{1}_{\mathbf{R}^m} + y)^{-1} = m_-(y)c$$

and a multiplication by $m_\circ(y)$ from the right yields $(5.4.4)_2$. Finally, $(5.4.4)_4$ is obtained by noting that $m_\circ(y) C_r(y, x-y) m_\circ(y) = 0$. The proof of (5.4.4) is complete.

The system (5.4.5) is obtained from the system (5.4.4) by a direct substitution using that $y = \mathrm{diag}(y_1, \dots, y_m)$ where

$$y_{\mathfrak{a}} = 1 \text{ for } 1 \le \mathfrak{a} \le \sigma, \quad y_i = x_i \text{ for } \sigma+1 \le i \le m-\tau, \quad y_{\mathfrak{b}} = -1 \text{ for } m-\tau+1 \le \mathfrak{b} \le m,$$

and

$$z_+ = \mathrm{diag}(x_1 - 1, \dots, x_\sigma - 1, 0, \dots, 0), \quad z_- = \mathrm{diag}(0, \dots, 0, x_{m-\tau+1} + 1, \dots, x_m + 1).$$

□

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Chapter 6

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