Hysteresis, convexity and dissipation in hyperbolic equations *

Pavel Krejčí

Mathematical Institute
Academy of Sciences of the Czech Republic
Žitná 25
11567 Prague, Czech Republic

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Preface

It is not necessary to make a long introduction in order to justify that the mathematical theory of hysteresis gives a useful tool for solving concrete engineering problems in various branches of applied research. A sufficient evidence is presented in the monographs that recently appeared or will appear in the near future (Krasnosel'skii and Pokrovskii (1983), Mayergoyz (1991), Visintin (1994), Brokate and Sprekels (to appear)) which cover a broad area of the theory and applications.

The present volume is mainly devoted to mathematical aspects of rate independent plastic hysteresis in continuum dynamics. The results of Chapters II and III can however be interpreted also in the framework of Maxwell's equations in ferromagnetic media of Preisach or Della Torre type. In any case, coupling hysteretic constitutive laws with the equations of motion we are led to quasilinear hyperbolic equations with hysteretic terms. This is a completely new branch of applied mathematics at the early stage where, following Hrych (1991), one can say with not so much exaggeration that "fabrication is the most reliable reference".

The situation is very different here from the theory of parabolic equations with hysteresis developed by Visintin in the 80's (see Visintin (1994)) which is an extension (sometimes very nontrivial) of the ideas and techniques derived from the general theory of quasilinear parabolic equations and applied to specific hysteretic nonlinearities. This is by no means the case of hyperbolic equations with hysteresis and the conclusion is surprising: although the (quasilinear) equation of motion with a hysteretic constitutive law preserves its hyperbolicity characterized by the finite speed of propagation, it can be solved considerably more easily than quasilinear hyperbolic equations without hysteresis by the methods of *semilinear* equations.

There is no simple and satisfactory explanation of this fact. We nevertheless make here a comparison of the behavior of solutions to one-dimensional quasilinear wave equations with and without hysteresis. While the latter develop discontinuities (shocks) in a finite time and weak solutions are not uniquely determined, so that additional physically motivated conditions have to be prescribed, hysteresis constitutive operators with convex loops in the former case exhibit a higher order energy dissipation which enables us to derive strong a priori estimates and pass to the limit in a suitable approximation scheme. From the geometrical point of view, if we represent the solutions of the Riemann problem for the equation without hysteresis by their trajectories in the strain - stress diagram, then shocks correspond to straight segments connecting two points on the constitutive graph. We observe that shocks are always organized in such a way that the corresponding trajectory is convex if the solution increases and concave if it decreases. The maximal dissipation principle then selects the solution with the minimal convex/maximal concave trajectory. We can say that some kind of spontaneous hysteresis occurs even if no hysteresis is assumed in the constitutive law itself. If now the constitutive law is given by a hysteresis operator with convex loops, it is natural to expect that the solution will follow smoothly their convex/concave branches and shocks have no reason to occur.

There are other interesting coincidences which would merit deeper understanding. This is for instance the question of the role of the two maximal dissipation principles in the rigid - plastic constitutive law (Sect. I.1) and in the Riemann problem (Sect. IV.3) which are in some sense responsible for the generation of hysteresis. We also do not comment on the fact that the Preisach operator itself is governed by a hyperbolic equation, where the memory variable plays the role of time (Sect. II.3).

This book is intended to give a consistent and self-contained presentation of the theory and its connection to other disciplines. In Chapter I we interpret hysteresis within the classical approach to continuum mechanics and derive analytical properties of hysteresis operators arising from rheological models. The efficiency of the hysteretic description depends on the complexity of the memory structure. In Chapter II we study the memory induced by scalar hysteresis models of Prandtl - Ishlinskii, Preisach, Della Torre and two models for fatigue and damage.

The main and rather nontrivial feature of hysteresis operators consists in the fact that they dissipate energy of two orders which relate to the area of closed hysteresis loops and to the curvature of their branches, respectively. We derive corresponding energy inequalities which enable us subsequently in Chapter III to construct solutions to hyperbolic equations with hysteretic constitutive laws. Chapter IV gives a detailed study of the Riemann problem with a not necessarily monotone nonlinearity without hysteresis and shows how hysteresis appears in the physically relevant solutions. Chapter V is an appendix, where we try to incorporate specific auxiliary functional-analytic results into a larger theory in order to make them more accessible to the reader.

Statements and formulae in the text are numbered consecutively in each section. References to results from other chapters are preceded by the roman number of the chapter. Thus, for example, Proposition I.3.9 refers to Proposition 3.9 of Chapter I, equation (3.26) means the corresponding formula in the chapter where the reference is made etc.

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P.K.

I. Hysteresis operators in mechanics

The equation of motion of a deformable body $\Omega \subset \mathbb{R}^N$ for some $N \in \mathbb{N}$, where $\mathbb{N}$ denotes the set of positive integers and $\mathbb{R}^N$ is the N-dimensional Euclidian space, is in classical continuum mechanics (Landau, Lifschitz (1953)) considered in the form

$$\varrho \frac{\partial^2 u_i}{\partial t^2} = \sum_{j=1}^{N} \frac{\partial \sigma_{ij}}{\partial x_j} + g_i, \quad i = 1, \dots, N, \tag{0.1}$$

where $x \in \Omega$, $t > 0$ are the space and time variables, respectively, $u = (u_i)$ is the displacement vector, ϱ is the density, $\sigma = (\sigma_{ij})$ is the stress tensor and $g = (g_i)$ is the applied force density, $i, j = 1, \dots, N$. The meaningful choice in applications is usually $N = 3$. We shall see in Chapter III that well-posedness of equation (0.1) can be obtained if it is coupled with initial and boundary conditions and with a suitable constitutive law between the stress tensor $\sigma = (\sigma_{ij})$ and strain tensor $\varepsilon = (\varepsilon_{ij})$ defined by the symmetric derivative of u, namely

$$\varepsilon_{ij} = \frac{1}{2}\Big(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\Big), \quad i, j = 1, \dots, N. \tag{0.2}$$

While (0.1) is a general physical law, the constitutive relation characterizes specific properties of a concrete material subject to time-dependent loading.

This chapter will mainly be devoted to the classification and mathematical properties of constitutive operators corresponding to models of elasticity and plasticity with or without hardening and fatigue effects.

We shall not treat in detail models for viscous, viscoelastic and viscoelastoplastic materials. The first reason is that there exists already an extensive literature in this area, for instance the modern monograph by Ionescu and Sofonea (1993), where an interested reader can find a good information about the current state of research. On the other hand, the objective of this book is to develop a theory of *rate independent* constitutive operators which, coupled with the equation of motion, lead to hyperbolic systems. The question of approximating equations of rate independent elastoplasticity by vanishing viscosity models which was studied already for instance by Duvaut, Lions (1972) and is still of high general interest will be briefly considered only in Chapter IV in a special situation of the Riemann problem.

The first section introduces the basic physical concepts used in the plasticity theory and their mathematical interpretation. In Section I.2 we recall some elements of convex analysis. Sections I.3, I.4 then present analytical properties of constitutive operators, in particular their dependence on given data.

I.1 Rheological models

We denote by $\mathbb{T}$ the space of symmetric tensors $\xi = (\xi_{ij})$, $i,j = 1,\ldots,N$, $N \in \mathbb{N}$, endowed with the scalar product $\langle \xi, \eta \rangle = \sum_{i,j=1}^{N} \xi_{ij}\eta_{ij}$. The strain and stress tensors ε and σ, respectively, are in general functions of the space variable $x \in \Omega \subset \mathbb{R}^N$ and time variable $t \geqslant 0$ with values in $\mathbb{T}$. We consider here only *homogeneous media*, where the constitutive law is independent of the spatial variable x which thus plays the role of a parameter.

Definition 1.1. *A system consisting of*

(1.1) (i) a constitutive relation *between* ε *and* σ,

(ii) a potential energy $U \geqslant 0$

is called a rheological element.

A rheological element is said to be thermodynamically consistent*, if the quantity*

$$\dot{q} := \langle \dot{\varepsilon}, \sigma \rangle - \dot{U} \tag{1.2}$$

called dissipation rate*, where dot denotes the time derivative, is nonnegative in the sense of distributions for all* ε, σ, U *satisfying conditions (i), (ii).*

Example 1.2. The elastic element $\mathcal{E}$.
In mechanics, elastic materials are characterized by a *linear stress-strain relation* and by the *complete reversibility* of dynamical processes. In mathematical terminology, it is assumed that there exists a matrix $A = (A_{ijk\ell})$ over $\mathbb{T}$ such that

$$\sigma = A\varepsilon \quad \text{or equivalently} \quad \sigma_{ij} = \sum_{k,\ell=1}^{N} A_{ijk\ell}\varepsilon_{k\ell},\ i,j = 1,\ldots,N. \tag{1.3}$$

Reversibility means that the potential energy U involves no memory and can be chosen in such a way that the dissipation rate $\dot{q}$ vanishes, i.e. the value of $U(t)$ for each $t > 0$ depends only on the instantaneous value of $\varepsilon(t)$ and $\dot{U} = \langle \dot{\varepsilon}, A\varepsilon \rangle$ almost everywhere for every absolutely continuous ε. This necessarily implies that the matrix A is symmetric with respect to the scalar product $\langle \cdot, \cdot \rangle$ and U has the form

$$U = \frac{1}{2}\langle A\varepsilon, \varepsilon \rangle \tag{1.4}$$

up to an additive constant. Indeed, for an arbitrary $\varepsilon \in W^{1,1}(0,T;\mathbb{T})$ and $t \in]0,T[$ put $\tilde{\varepsilon}(\tau) := \varepsilon(0) + \frac{\tau}{t}(\varepsilon(t) - \varepsilon(0))$ for $\tau \in [0,t]$. We can choose the initial value for U arbitrarily, for instance $U(0) := \frac{1}{2}\langle A\varepsilon(0), \varepsilon(0) \rangle$. We have by hypothesis

$$U(t) = U(0) + \int_0^t \langle \dot{\tilde{\varepsilon}}(\tau), A\tilde{\varepsilon}(\tau) \rangle d\tau = \frac{1}{2}\langle A\varepsilon(t), \varepsilon(t) \rangle + \frac{1}{2}\langle \varepsilon(t), (A - A^T)\varepsilon(0) \rangle, \tag{1.5}$$

where $(A^T)_{ijk\ell} = A_{k\ell ij}$, hence

$$\dot{U}(t) = \langle \dot{\varepsilon}(t), A\varepsilon(t) \rangle + \frac{1}{2} \langle \dot{\varepsilon}(t), (A - A^T)(\varepsilon(0) - \varepsilon(t)) \rangle$$

and we easily conclude that the matrix $A = A^T$ is *symmetric* and (1.4) holds.

To guarantee that the stress-strain relation is one-to-one and the material law is deterministic we assume that the matrix A is *positive definite*.

The elastic element is said to be *isotropic*, if the matrix A has the form

$$A = 2\mu I + \lambda J, \tag{1.6}$$

where μ, λ are positive numbers called *Lamé's constants* (see Rabotnov (1988)), I is the identity matrix $I\xi = \xi$ and J is the matrix of the symmetric bilinear form $\langle J\xi, \eta \rangle = \xi_I \eta_I$. We denote by $\xi_I := \sum_{i=1}^{N} \xi_{ii}$ the *first invariant* (trace) of a symmetric tensor $\xi \in \mathbb{T}$ and by $(\xi_{\text{dev}})_{ij} := \xi_{ij} - \frac{1}{N} \xi_I \delta_{ij}$, where δ_{ij} is the Kronecker symbol, the *deviatoric part* of ξ. We also introduce the *deviatoric subspace* $\mathbb{T}_{\text{dev}} := \{\xi \in \mathbb{T};\, \xi_I = 0\}$ of $\mathbb{T}$ and its orthogonal complement $\mathbb{T}_{\text{diag}} := \{\xi \in \mathbb{T};\, \xi_{ij} = \lambda \delta_{ij},\, \lambda \in \mathbb{R}^1,\, i,j = 1, \ldots, N\}$.

Example 1.3. The viscous element $\mathcal{V}$.
Modeling of rate dependent relaxation effects makes often use of the concept of *viscosity* based on the hypothesis that there exist two coefficients $\eta > 0, \zeta > 0$ of proportionality between the deviators and first invariants of the stress and the strain rate, i.e.

$$\sigma_{\text{dev}} = \eta \dot{\varepsilon}_{\text{dev}}, \; \sigma_I = \zeta \dot{\varepsilon}_I. \tag{1.7}$$

The assumption that no reversible energy can be stored by the viscous element $(U = 0)$ ensures its thermodynamical consistency.

Example 1.4. The rigid-plastic element $\mathcal{R}$.
The basic concept in plasticity is the *yield surface* in the stress space which can be described as the boundary ∂Z of a convex closed set $Z \subset \mathbb{T}$.

The rigid-plastic behavior consists of two different phases characterized by the instantaneous value σ of the stress tensor. The material remains rigid as long as $\sigma \in \operatorname{Int} Z$ (the *interior* of Z). In this case no deformation occurs and $\dot{\varepsilon} = 0$. The material becomes plastic if σ reaches the boundary ∂Z of Z. Plasticity is governed by three physical principles:

(1.8) $\sigma \in Z$ (the stress values do not exceed the threshold ∂Z),

(1.9) $U = 0$ (no reversible energy is stored),

(1.10) $\langle \dot{\varepsilon}, \sigma - \tilde{\sigma} \rangle \geqslant 0, \quad \forall \tilde{\sigma} \in Z$ (principle of maximal dissipation rate with respect to all admissible stress values).

Geometrically, $\dot{\varepsilon}$ has the direction of the outward normal cone, and condition (1.10) is also called *von Mises normality rule*. We see that the variational inequality (1.10) includes the rigid behavior (for $\sigma \in \operatorname{Int} Z$ it entails $\dot{\varepsilon} = 0$). In order to ensure the thermodynamical consistency we assume $0 \in Z$. In fact, it is natural to assume that no deformation occurs for $\sigma = 0$. This is equivalent to the hypothesis $0 \in \operatorname{Int} Z$.

It has been observed that volume changes are negligible during plastic deformation (Rabotnov (1988)). Combining constitutive relation (1.8) - (1.10) with the *volume invariance condition*

$$\dot{\varepsilon}_I = 0, \tag{1.11}$$

we conclude from Proposition 2.13 and Remark 3.10 below that Z has the form of a cylinder

$$Z = Z_0 + \mathbb{T}_{\text{diag}}, \tag{1.12}$$

where $Z_0 \subset \mathbb{T}_{\text{dev}}$ is a bounded convex closed set. The classical models of Tresca and von Mises are special cases of (1.8)-(1.12) with Z_0 a ball (von Mises) or $Z_0 := \{\xi \in \mathbb{T}_{\text{dev}};\ \sum_{k=1}^{N} |\xi_k| \leqslant r\}$ for some $r > 0$ (Tresca), where $\{\xi_k\}$ are the eigenvalues of the symmetric matrix $\xi = (\xi_{ij})$. Note that we have $\sum_{k=1}^{N} \xi_k = 0$ for $\xi \in \mathbb{T}_{\text{dev}}$. The Tresca set Z_0 is usually represented for $N = 3$ by a hexagon in the plane $\xi_1 + \xi_2 + \xi_3 = 0$.

Example 1.5. The rigid-plastic element with isotropic hardening $\mathcal{J}$. Following Nečas and Hlaváček (1981) we introduce a scalar hardening parameter α of physical dimension of stress into the constitutive relations. We assume analogously as in Example 1.4 that a bounded convex closed set $Z_0 \subset \mathbb{T}_{\text{dev}}$ is given such that $0 \in \operatorname{Int} Z_0$, and we denote by $M_0 : \mathbb{T}_{\text{dev}} \to [0, \infty[$ the Minkowski functional associated to Z_0 by formula (2.9) below. Let further a concave nondecreasing function $\varphi : [1, \infty[\to [1, \infty[$ be given, $\varphi(1) = 1$.

We denote by $\mathbb{T}_1$ the space $\mathbb{T} \times \mathbb{R}^1$ endowed with the scalar product $\left[\binom{\xi}{\beta}, \binom{\eta}{\gamma}\right] := \langle \xi, \eta \rangle + \beta\gamma$ and by Z_1 the convex closed subset of $\mathbb{T}_1$ (see Fig. 1)

$$Z_1 := \left\{ \begin{pmatrix} \xi \\ \alpha \end{pmatrix} \in \mathbb{T}_1;\ \alpha \geqslant 0,\ M_0(\xi_{\text{dev}}) \leqslant \varphi(1 + \alpha) \right\}. \tag{1.13}$$

The constitutive relations are analogous to (1.8)-(1.10), namely

$$\begin{pmatrix} \sigma \\ \alpha \end{pmatrix} \in Z_1, \tag{1.14}$$

$$U = 0,\ \alpha(0) = 0, \tag{1.15}$$

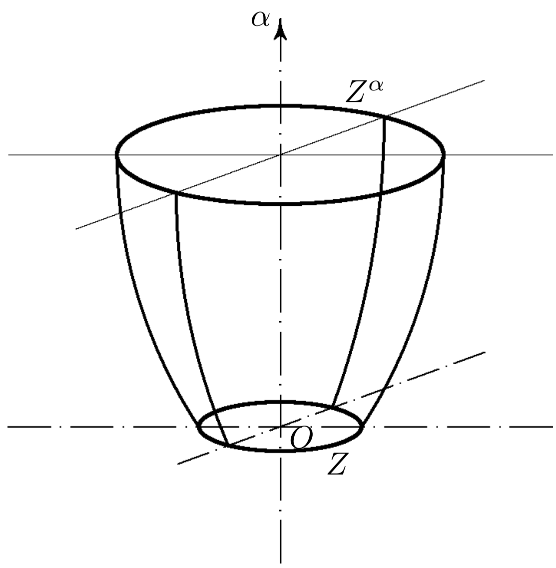
$$\left[\begin{pmatrix} \dot{\varepsilon} \\ -\frac{1}{c}\dot{\alpha} \end{pmatrix}, \begin{pmatrix} \sigma \\ \alpha \end{pmatrix} - \begin{pmatrix} \tilde{\sigma} \\ \tilde{\alpha} \end{pmatrix} \right] \geqslant 0 \quad \forall \begin{pmatrix} \tilde{\sigma} \\ \tilde{\alpha} \end{pmatrix} \in Z_1, \tag{1.16}$$

where $c > 0$ is a given physical constant.

We immediately observe that choosing $\tilde{\sigma} = \sigma$ in (1.16) we obtain $\dot{\alpha}(\alpha - \tilde{\alpha}) \leqslant 0 \quad \forall \tilde{\alpha} \geqslant \alpha$, hence $\dot{\alpha} \geqslant 0$.

Let $Z^{\alpha} := \left\{ \xi \in \mathbb{T}; \binom{\xi}{\alpha} \in Z_1 \right\}$ be the domain of admissible stresses for an instantaneous value α of the hardening parameter. We see that Z^{α} increases without changing its shape with increasing α (see Fig. 1)

Fig. 1



Example 1.6. The brittle element $\mathcal{B}$.
An application of the notion of brittleness to modeling of fatigue and damage will be shown in Sect. II.5. To introduce the concept, we assume again the existence of a convex open domain of rigidity $\operatorname{Int} Z \subset \mathbb{T}$ in the stress space; as soon as the value σ of the stress reaches the *fragility surface* ∂Z, the material breaks, the stress drops to 0 and we lose any control on the strain.

Under the same assumptions on Z as in Example 1.4 we denote by M_Z the associated Minkowski functional (see (2.9) below), by H the Heaviside function

$$H(r) = \begin{cases} 1 & \text{for } r > 0, \\ 0 & \text{for } r \leqslant 0 \end{cases} \tag{1.17}$$

and we introduce the *damage function* d by the formula

$$d(t) := 1 - H(1 - \|M_Z(\sigma)\|_{[0,t]}), \tag{1.18}$$

where we put $\|f\|_{[0,t]} := \sup\{|f(s)|; s \in [0,t]\}$ for each function $f : [0,T] \to \mathbb{R}^1$ and $t \in [0,T]$. We see that $d = 0$ characterizes the rigid (unperturbed) state, $d = 1$

corresponds to the irreversible damaged state. We define the constitutive relations in the form, see Visintin (1994)

$$\begin{cases} (1 - d(t))\varepsilon(t) = 0, \\ d(t)\sigma(t) = 0. \end{cases} \tag{1.19}$$

It is natural to admit that no reversible energy can be stored by the brittle element, so we put $U = 0$. The thermodynamical consistency will be discussed in Example 1.7 below.

Composition of rheological elements

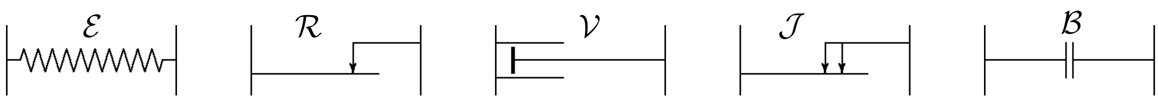
A large variety of models for the behavior of materials can be obtained by composing rheological elements from Examples 1.2 - 1.6 in series or in parallel.

Let G_1, G_2 be two rheological elements and let $\varepsilon_i, \sigma_i, U_i$ be the strain, stress and potential energy, respectively, corresponding to the element $G_i, i = 1, 2$.

The total strain ε, stress σ and potential energy U for the combination in parallel $G_1|G_2$ and in series $G_1 - G_2$ are defined by the following natural relations

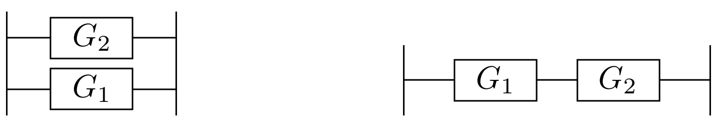
$G_1\|G_2$	$G_1 - G_2$
$\varepsilon = \varepsilon_1 = \varepsilon_2$	$\varepsilon = \varepsilon_1 + \varepsilon_2$
$\sigma = \sigma_1 + \sigma_2$	$\sigma = \sigma_1 = \sigma_2$
$U = U_1 + U_2$	$U = U_1 + U_2$

Fig. 2: *Rheological elements*



Composition of rheological elements

in parallel $G_1|G_2$ *in series* $G_1 - G_2$



in analogy with the theory of electrical circuits. It is easy to see that every combination of thermodynamically consistent elements is thermodynamically consistent.

Example 1.7. The parallel elasto-brittle element $\mathcal{E}|\mathcal{B}$.
According to general rheological principles, the constitutive law has the form

$$\begin{aligned} \varepsilon &= \varepsilon^e = \varepsilon^b, \\ \sigma &= \sigma^e + \sigma^b, \\ U &= \frac{1}{2}\langle \varepsilon, \sigma^e \rangle, \end{aligned}$$

where ε^e, σ^e and ε^b, σ^b are strain and stress tensors corresponding to the elastic and brittle element, respectively. We therefore have

$$\begin{cases} \sigma^e = A\varepsilon, \\ \varepsilon(t) H(1 - \|M_Z(\sigma^b)\|_{[0,t]}) = 0, \\ \sigma^b(t)(1 - H(1 - \|M_Z(\sigma^b)\|_{[0,t]})) = 0 \end{cases} \tag{1.20}$$

with the same notation as in Examples 1.2, 1.6. One immediately notices that these identities are contradictory if $M_Z(\sigma(0)) \geqslant 1$. Indeed, we have either $M_Z(\sigma^b(0)) < 1$, hence $\varepsilon(0) = \sigma^e(0) = 0$ and $\sigma^b(0) = \sigma(0)$, which is a contradiction, or $M_Z(\sigma^b(0)) \geqslant 1$, hence $\sigma^b(0) = 0$, which is a contradiction, too. A similar contradiction is obtained for any $t \in]0, T[$ whenever $\sup\{|M_Z(\sigma(s))|; s \in [0, t[\} < 1$ and $M_Z(\sigma(t+)) \geqslant 1$. To preserve the consistency, we assume

$$M_Z(\sigma(0)) < 1, \ \sigma : [0, T] \to \mathbb{T} \quad \text{is continuous.} \tag{1.21}$$

Assuming (1.21) we obtain from (1.20) $\sigma^e(t) H(1 - \|M_Z(\sigma^b)\|_{[0,t]}) = 0$, hence $\sigma^b(t) = \sigma(t) H(1 - \|M_Z(\sigma^b)\|_{[0,t]}) = \sigma(t) H(1 - \|M_Z(\sigma)\|_{[0,t]})$.

The constitutive law can therefore be written in operator form

$$\begin{cases} \varepsilon(t) = A^{-1}\sigma(t)[1 - H(1 - \|M_Z(\sigma)\|_{[0,t]})], \\ U(t) = \frac{1}{2}\langle A^{-1}\sigma(t), \sigma(t) \rangle [1 - H(1 - \|M_Z(\sigma)\|_{[0,t]})] \end{cases} \tag{1.22}$$

with input $\sigma \in C([0, T]; \mathbb{T})$ and output $\varepsilon \in L^\infty(0, T; \mathbb{T})$. Formulas (1.22) are now meaningful without any restriction on $\sigma(0)$.

Let us verify that constitutive equations (1.22) define a thermodynamically consistent element. We choose an absolutely continuous input σ and an arbitrary time interval $]t_1, t_2[\subset [0, T]$. The total dissipation is given by the formula

$$D := q(t_2-) - q(t_1+) = \langle \varepsilon, \sigma \rangle_{t \to t_2-} - \langle \varepsilon, \sigma \rangle_{t \to t_1+} - \int_{t_1}^{t_2} \langle \varepsilon, \dot{\sigma} \rangle dt - U(t_2-) + U(t_1+).$$

We distinguish 3 cases.

a) $\|M_Z(\sigma)\|_{[0,t]} < 1$ for all $t \in]t_1, t_2[$. Then $\varepsilon(t) = U(t) = 0$ in $[0, t_2[$, hence $D = 0$.

b) $\|M_Z(\sigma)\|_{[0,t_1]} \geqslant 1$. Then $\varepsilon(t) = A^{-1}\sigma(t), U(t) = \frac{1}{2}\langle A^{-1}\sigma(t), \sigma(t)\rangle$ in $]t_1, t_2[$ with the same conclusion.

c) $\exists t_0 \in]t_1, t_2[: \|M_Z(\sigma)\|_{[0,t_0]} = 1, \sigma(t) \in \operatorname{Int} Z$ for $t \in [0, t_0[$. Then we have $\varepsilon(t) = U(t) = 0$ for $t \in [0, t_0[$ and $D = \frac{1}{2}\langle A^{-1}\sigma(t_0), \sigma(t_0)\rangle > 0$.

We therefore have $\dot{q} \geqslant 0$ in the sense of distributions, hence the element $\mathcal{E}/\mathcal{B}$ is thermodynamically consistent.

Example 1.8. Elastoplastic models $\mathcal{E} - \mathcal{R}, \mathcal{E}/\mathcal{R}$.
There are good reasons for rewriting constitutive variational inequalities in plasticity in operator form. This enables us to distinguish clearly between input and output quantities: while the input can be controlled, the output can be determined by solving the constitutive equation.

Let us compare the constitutive relations for two elastoplastic models $\mathcal{E} - \mathcal{R}, \mathcal{E}|\mathcal{R}$. We denote by ε^e, σ^e and ε^p, σ^p the strain and stress on the elastic and rigid-plastic element, respectively.

$\underline{\mathcal{E}/\mathcal{R}}$		$\underline{\mathcal{E} - \mathcal{R}}$
$\varepsilon = \varepsilon^e = \varepsilon^p$		$\varepsilon = \varepsilon^e + \varepsilon^p$
$\sigma = \sigma^e + \sigma^p$		$\sigma = \sigma^e = \sigma^p$
$\sigma^e = A\varepsilon$		$\sigma = A\varepsilon^e$
$\sigma^p \in Z$		$\sigma \in Z$
$\langle \dot{\varepsilon}, \sigma^p - \tilde{\sigma}\rangle \geqslant 0$	$\forall \tilde{\sigma} \in Z$	$\langle \dot{\varepsilon}^p, \sigma - \tilde{\sigma}\rangle \geqslant 0$
$U = \frac{1}{2}\langle \varepsilon, \sigma^e\rangle$		$U = \frac{1}{2}\langle \varepsilon^e, \sigma\rangle$

Recall that $Z \subset \mathbb{T}$ is a given convex closed set, $0 \in \operatorname{Int} Z$. We see that both models are governed by a variational inequality of the same type, namely

$$\left.\begin{aligned} \mathcal{E}/\mathcal{R}: \ &\langle A^{-1}(\dot{\sigma} - \dot{\sigma}^p), \sigma^p - \tilde{\sigma}\rangle \geqslant 0 \\ \mathcal{E} - \mathcal{R}: \ &\langle A^{-1}(A\dot{\varepsilon} - \dot{\sigma}), \sigma - \tilde{\sigma}\rangle \geqslant 0 \end{aligned}\right\} \forall \tilde{\sigma} \in Z. \tag{1.23}$$

The solvability of such equations is ensured by the following Theorem whose detailed proof (in a more general setting) will be given in Sect. I.3 below. Definition and general information about the space $W^{1,1}(0, T; X)$ of absolutely continuous Hilbert space valued functions is given in Chapter V.

Theorem 1.9. *Let X be a real separable Hilbert space endowed with a scalar product $\langle .,. \rangle_X$. Let $Z \subset X$ be a convex closed set, $0 \in Z$ and let $x^0 \in Z$ be a given element. Then for every function $u \in W^{1,1}(0,T;X)$ there exists a unique $x \in W^{1,1}(0,T;Z)$ satisfying the variational inequality*

$$\langle \dot{u}(t) - \dot{x}(t), x(t) - \tilde{x} \rangle_X \geqslant 0 \quad \text{a.e.} \quad \forall \tilde{x} \in Z \tag{1.24}$$

and the initial condition

$$x(0) = x^0. \tag{1.25}$$

We now define the solution operators $\mathcal{S}, \mathcal{P} : Z \times W^{1,1}(0,T;X) \to W^{1,1}(0,T;X)$ of the problem (1.25), (1.24) by the formula

$$\mathcal{S}(x^0, u) := x, \quad \mathcal{P}(x^0, u) := u - \mathcal{S}(x^0, u). \tag{1.26}$$

According to Krasnosel'skii and Pokrovskii (1983), the operators $\mathcal{S}, \mathcal{P}$ are called *stop* and *play*, respectively (see Fig. 3). The set Z is called the *characteristic* of $\mathcal{S}$ and $\mathcal{P}$.

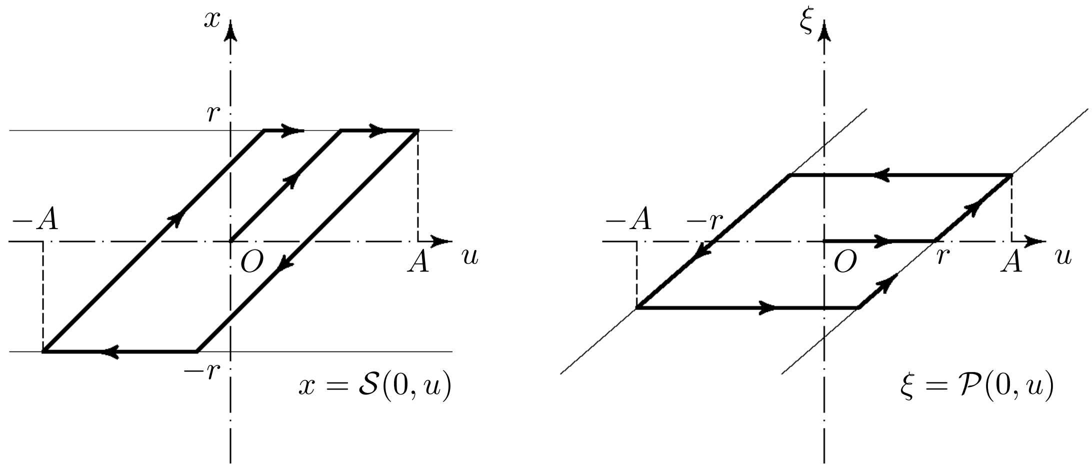


Fig. 3: *Input-output diagram for the stop and play in the case* $\dim X = 1$, $Z = [-r, r]$, $u(t) = A \sin \omega t$ *for* $A > r > 0$.

Exercise 1.10. Prove that the constitutive relations for the elastoplastic models above can be written in the form

$$\mathcal{E}/\mathcal{R}: \quad \varepsilon = A^{-1}\mathcal{P}(\sigma_0^p, \sigma),\, U = \frac{1}{2}\big\langle A^{-1}\,\mathcal{P}(\sigma_0^p, \sigma), \mathcal{P}(\sigma_0^p, \sigma)\big\rangle,$$

$$\mathcal{E} - \mathcal{R}: \quad \sigma = \mathcal{S}(\sigma^0, A\varepsilon),\, U = \frac{1}{2}\big\langle A^{-1}\,\mathcal{S}(\sigma^0, A\varepsilon), \mathcal{S}(\sigma^0, A\varepsilon)\big\rangle,$$

where $\mathcal{S}, \mathcal{P}$ are the stop and play in $X = \mathbb{T}$ endowed with the scalar product $\big\langle \xi, \eta \big\rangle_X := \big\langle A^{-1}\xi, \eta \big\rangle$, and σ_0^p, σ^0 are given initial output values.

It is clear that the roles of input and output in the models $\mathcal{E}/\mathcal{R}$ and $\mathcal{E} - \mathcal{R}$ cannot be reversed.

The definition immediately suggests that the stop has the

(1.27) *Semigroup property*: For $u \in W^{1,1}(0,T;X), s \in]0,T[$ and $t \in [0, T-s]$ put $u_s(t) := u(s+t)$. Then for every $x^0 \in Z$ we have

$$\mathcal{S}(x^0, u)(t+s) = \mathcal{S}(\mathcal{S}(x^0, u)(s), u_s)(t).$$

An operator F acting in some function space $R(0,T;X)$ of functions $[0,T] \to X$ is called

(1.28) *Rate independent*, if for every $u \in R(0,T;X)$ and every nondecreasing mapping α of $[0,T]$ onto $[0,T]$ such that $u_\alpha(t) := u(\alpha(t))$ belongs to $R(0,T;X)$ we have

$$F(u_\alpha)(t) = F(u)(\alpha(t)) \qquad \text{for all } t \in [0,T],$$

(1.29) *Causal*, if $F(u)(t) = F(v)(t)$ for all $t \in [0, t_0] \subset [0,T]$ whenever $u(t) = v(t)$ for all $t \in [0, t_0]$.

Rate independence and causality characterize *hysteresis operators* according to the classification of Visintin (1994). By definition, the stop and play are hysteresis operators in $W^{1,1}(0,T;X)$ (we shall see later in Sect. I.3 that they can be extended to the space of continuous functions $C([0,T];X)$). We notice on Fig. 3 that the input-output diagram for the stop and play forms simple hysteresis loops. More complicated loop structures including internal loops can be observed in scalar multiyield models with a more complex memory structure (Prandtl-Ishlinskii, Preisach, Della Torre, cf. Sect. II.3); we however do not pursue the question of hierarchy of loops here. Instead, we describe in Sect. II.2 the scalar hysteresis memory by means of the *memory sequence* associated to the input which is independent of the concrete choice of the hysteresis operator.

We now show how stop and play can be used for modeling the phenomena of kinematic and isotropic hardening in elastoplastic materials, cf. Lemaitre, Chaboche (1985).

KINEMATIC HARDENING

Let us consider the model $\mathcal{E} - (\mathcal{E}/\mathcal{R})$ (see Fig. 4). The general rheological rules yield

$$
\begin{aligned}
&\sigma = \sigma^e + \sigma^p \\
&\varepsilon = \varepsilon^e + \varepsilon^p \\
&\sigma^e = A\varepsilon^p \\
&\sigma = B\varepsilon^e \\
&\sigma^p \in Z \\
&\langle \dot\varepsilon^p, \sigma^p - \tilde\sigma \rangle \geqslant 0 \quad \forall \tilde\sigma \in Z \\
&U = \tfrac{1}{2}\big(\langle \varepsilon^e, \sigma\rangle + \langle \varepsilon^p, \sigma^e\rangle\big),
\end{aligned}
$$

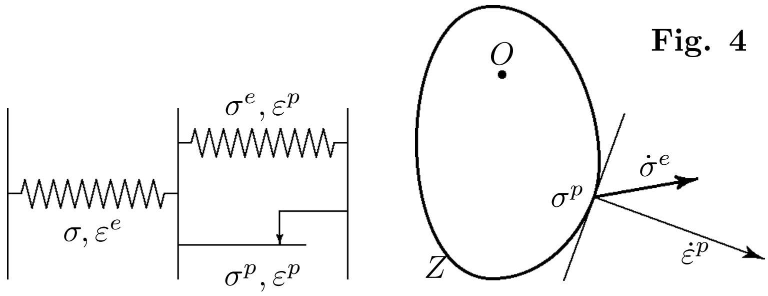


Fig. 4

where A, B are given constant symmetric positive definite matrices and $Z \subset \mathbb{T}$ is a convex closed set, $0 \in \operatorname{Int} Z$. For $t \in [0,T]$ put

$$Z(t) := Z + \sigma^e(t). \tag{1.30}$$

Then $\sigma(t) \in Z(t)$ for all $t \in [0,T]$. We can imagine that relation (1.30) defines a translation of Z in the stress space $\mathbb{T}$ driven by the elastic component σ^e of the stress without changing shape and size. This phenomenon is called *kinematic hardening* and is typical for metals, see Lemaitre and Chaboche (1985).

The evolution of σ^e is governed by the variational inequality

$$\langle A^{-1}\dot\sigma^e, \sigma^p - \tilde\sigma\rangle \geqslant 0, \quad \forall \tilde\sigma \in Z. \tag{1.31}$$

Inequality (1.31) can be interpreted as a normality condition for the hardening rate $\dot\sigma^e$ with respect to the scalar product $\langle .,. \rangle_A := \langle A^{-1}.,.\rangle$; both the hardening rate $\dot\sigma^e$ and the plastic strain rate $\dot\varepsilon^p$ have the outward normal direction to ∂Z at the point σ, but *with respect to different scalar products* (see Fig.4).

With the intention to deal with several scalar products in $\mathbb{T}$ we introduce the subscript A for the play $\mathcal{P}_A$ and stop $\mathcal{S}_A$ corresponding to the scalar product $\langle .,.\rangle_A$.

Using Exercise 1.10 we can express the constitutive law for the model $\mathcal{E} - (\mathcal{E}\,/\,\mathcal{R})$ in the form

$$\varepsilon = B^{-1}\sigma + A^{-1}\underset{A}{\mathcal{P}}(\sigma^p_0, \sigma) \tag{1.32}$$

with input σ and output ε. We now prove that the constitutive operator $B^{-1} + A^{-1}\mathcal{P}_A$ is invertible. Identity (1.34) below gives an equivalent expression for (1.32) with input ε and output σ.

Lemma 1.11. *Let* $\sigma_0^p \in Z$ *be given and let* A, C *be given constant matrices such that* A, CA *are symmetric and positive definite. Put* $\hat{A} := A + CA$*. Then for all* $\sigma \in W^{1,1}(0,T;\mathbb{T})$ *we have*

$$\underset{\hat{A}}{\mathcal{S}}(\sigma_0^p, \sigma + C \underset{A}{\mathcal{P}}(\sigma_0^p, \sigma)) = \underset{A}{\mathcal{S}}(\sigma_0^p, \sigma). \tag{1.33}$$

P r o o f. Put $x := \mathcal{S}_A(\sigma_0^p, \sigma)$, $y := \mathcal{S}_{\hat{A}}(\sigma_0^p, \sigma + C\,\mathcal{P}_A(\sigma_0^p, \sigma))$. Then $y = \mathcal{S}_{\hat{A}}(\sigma_0^p, (I+C)\sigma - Cx)$, where I is the identity matrix. Putting $\tilde{\sigma} := \frac{1}{2}(x+y)$ in the variational inequalities

$$\big\langle A^{-1}(\dot{\sigma} - \dot{x}), x - \tilde{\sigma}\big\rangle \geqslant 0$$

$$\big\langle \hat{A}^{-1}((I+C)\dot{\sigma} - C\dot{x} - \dot{y}), y - \tilde{\sigma}\big\rangle \geqslant 0$$

and using the identity $\hat{A}^{-1} + \hat{A}^{-1}C = A^{-1}$ we conclude $\big\langle \dot{x} - \dot{y}, x - y\big\rangle_{\hat{A}} \leqslant 0$, hence $x = y$. □

We now apply Lemma 1.11 with $C = BA^{-1}$ to the constitutive equation (1.32). We obtain

$$\underset{\hat{A}}{\mathcal{S}}(\sigma_0^p, B\varepsilon) = \underset{A}{\mathcal{S}}(\sigma_0^p, \sigma) \quad \text{for} \quad \hat{A} = A + B,$$

hence $(I + BA^{-1})\sigma = B\varepsilon + BA^{-1}\,\mathcal{S}_{\hat{A}}(\sigma_0^p, B\varepsilon)$, or equivalently

$$\sigma = (A^{-1} + B^{-1})^{-1}\varepsilon + B\hat{A}^{-1}\underset{\hat{A}}{\mathcal{S}}(\sigma_0^p, B\varepsilon) = B\varepsilon - B\hat{A}^{-1}\underset{\hat{A}}{\mathcal{P}}(\sigma_0^p, B\varepsilon), \tag{1.34}$$

where ε is the input and σ is the output.

In the particular case $B = I, A = \frac{1}{\gamma}I$ for some $\gamma > 0$ we obtain $\mathcal{P}_A = \mathcal{P}_{\hat{A}} = \mathcal{P}_I$ and the inversion formula

$$(\mathcal{I} + \gamma\underset{I}{\mathcal{P}}(x^0, .))^{-1} = \mathcal{I} - \frac{\gamma}{1+\gamma}\underset{I}{\mathcal{P}}(x^0, .) \tag{1.35}$$

holds for all $x^0 \in Z$, where $\mathcal{I}$ is the identity mapping in $W^{1,1}(0,T;X)$.

Exercise 1.12. Assume that the matrices A, B commute, i.e. $AB = BA$. Prove that (1.34) is the constitutive equation of the model $\mathcal{E}_1 / (\mathcal{E}_2 - \mathcal{R})$ with

$$\underset{1}{\mathcal{E}} \,:\, \sigma = \tilde{A}\varepsilon, \quad \tilde{A} = (A^{-1} + B^{-1})^{-1},$$

$$\underset{2}{\mathcal{E}} \,:\, \sigma = \tilde{B}\varepsilon, \quad \tilde{B} = B^2(A+B)^{-1},$$

$$\mathcal{R} \,:\, \tilde{Z} = B(A+B)^{-1}(Z),\ \sigma \in \tilde{Z},\ \big\langle \varepsilon, \sigma - \tilde{\sigma}\big\rangle \geqslant 0 \quad \forall \tilde{\sigma} \in \tilde{Z}.$$

H i n t. Use the identity $C\mathcal{S}_A(x^0,u) = \tilde{\mathcal{S}}_{CAC}(Cx^0,Cu)$ for each positive definite symmetric matrix C, where $\tilde{\mathcal{S}}$ is the stop with characteristic $\tilde{Z} = C(Z)$.

The commutativity hypothesis $AB = BA$ is satisfied for instance if both elastic elements are isotropic. In this case the models $\mathcal{E} - (\mathcal{E} \,/\, \mathcal{R})$ and $\mathcal{E} \,/ (\mathcal{E} - \mathcal{R})$ are equivalent.

Isotropic and kinematic hardening

Let us consider now the model $\mathcal{E} - (\mathcal{E} \,/\, \mathcal{J})$. With the notation taken from Example 1.5, the constitutive relations are analogous to the model $\mathcal{E} - (\mathcal{E} \,/\, \mathcal{R})$, namely

$$\begin{pmatrix}\sigma\\0\end{pmatrix} = \begin{pmatrix}\sigma^e\\-\alpha\end{pmatrix} + \begin{pmatrix}\sigma^p\\\alpha\end{pmatrix}, \quad \begin{pmatrix}\sigma^p\\\alpha\end{pmatrix} \in Z_1$$

$$\begin{pmatrix}\varepsilon\\-\frac{1}{c}\alpha\end{pmatrix} = \begin{pmatrix}\varepsilon^e\\0\end{pmatrix} + \begin{pmatrix}\varepsilon^p\\-\frac{1}{c}\alpha\end{pmatrix}, \quad \sigma = B\varepsilon^e, \sigma^e = A\varepsilon^p,$$

$$\left[\begin{pmatrix}\dot{\varepsilon}^p\\-\frac{1}{c}\dot{\alpha}\end{pmatrix}, \begin{pmatrix}\sigma^p\\\alpha\end{pmatrix} - \begin{pmatrix}\tilde{\sigma}\\\tilde{\alpha}\end{pmatrix}\right] \geqslant 0 \quad \forall \begin{pmatrix}\tilde{\sigma}\\\tilde{\alpha}\end{pmatrix} \in Z_1, \tag{1.36}$$

where A, B are symmetric positive definite matrices.

Let $A_1, B_1 : \mathbb{T}_1 \to \mathbb{T}_1$ be the linear mappings defined by the identities $A_1\binom{\xi}{\alpha} := \binom{A\xi}{c\alpha}$, $B_1\binom{\xi}{\alpha} := \binom{B\xi}{c\alpha}$. We have $\left[A_1^{-1}\left(\binom{\dot\sigma}{0} - \binom{\dot\sigma^p}{\dot\alpha}\right), \binom{\sigma^p}{\alpha} - \binom{\tilde\sigma}{\tilde\alpha}\right] \geqslant 0 \quad \forall \binom{\tilde\sigma}{\tilde\alpha} \in Z_1$, hence $\binom{\sigma^p}{\alpha} = \mathcal{S}_1\left(\binom{\sigma^p_0}{0}, \binom{\sigma}{0}\right)$, $\binom{\sigma^e}{-\alpha} = \mathcal{P}_1\left(\binom{\sigma^p_0}{0}, \binom{\sigma}{0}\right)$, where $\mathcal{S}_1, \mathcal{P}_1$ are the stop and play in $\mathbb{T}_1$ endowed with scalar product $\left[A_1^{-1}\cdot, \cdot\right]$ with characteristics Z_1, with a given initial condition $\binom{\sigma^p_0}{0} \in Z_1$. The constitutive equation has the form

$$\begin{pmatrix}\varepsilon\\-\frac{1}{c}\alpha\end{pmatrix} = B_1^{-1}\begin{pmatrix}\sigma\\0\end{pmatrix} + A_1^{-1}\mathcal{P}_1\left(\begin{pmatrix}\sigma^p_0\\0\end{pmatrix}, \begin{pmatrix}\sigma\\0\end{pmatrix}\right). \tag{1.37}$$

We derive now some consequences of the constitutive equation.

Lemma 1.13. *Let $\sigma \in W^{1,1}(0,T;\mathbb{T})$ be given and assume $\sigma(0) = \sigma^p_0 = 0$. Let ε, α be given by the equation (1.37). Then we have*

$$\varphi(1+\alpha(t)) = \max\{1, \|M_0(\sigma^p_{\mathrm{dev}})\|_{[0,t]}\}, \tag{1.38}$$

where φ, M_0 are as in (1.13) and σ^p_{dev} is the deviatoric part of the plastic stress σ^p.

P r o o f. We have $\binom{\sigma^p(t)}{\alpha(t)} \in Z_1$ for all $t \in [0,T]$, hence $M_0\big(\sigma^p_{\mathrm{dev}}(t)\big) \leqslant \varphi(1+\alpha(t))$ by definition. The fact that α is nondecreasing (cf. (1.16)) entails $\|M_0(\sigma^p_{\mathrm{dev}})\|_{[0,t]} \leqslant \varphi(1+\alpha(t))$. In the case $\|M_0(\sigma^p_{\mathrm{dev}})\|_{[0,t]} < 1$ we obviously have $\alpha(t) = 0$ and (1.38) holds. Let us assume now $1 \leqslant \|M_0(\sigma^p_{\mathrm{dev}})\|_{[0,t]} < \varphi(1+\alpha(t))$ for some $t \in]0,T[$. Then there exists $\tau \in]0,t[$ such that $\dot\alpha(\tau) > 0$ and $\|M_0(\sigma^p_{\mathrm{dev}})\|_{[0,\tau]} < \varphi(1+\alpha(\tau))$, hence $\binom{\sigma^p(\tau)}{\alpha(\tau)} \in \mathrm{Int}\, Z_1$. From (1.36) we conclude $\dot\alpha(\tau) = 0$, which is a contradiction. $\square$

We associate to the model $\mathcal{E} - (\mathcal{E} / \mathcal{J})$ the potential energy $U = \frac{1}{2}\left(\langle \varepsilon^e, \sigma \rangle + \langle \varepsilon^p, \sigma^e \rangle\right)$. The dissipated energy $q(t)$ is then equal to the plastic work $\int_0^t \langle \dot\varepsilon^p(\tau), \sigma(\tau) \rangle d\tau$ and is related to $\alpha(t)$ by the following identity.

Proposition 1.14. *Let the assumptions of Lemma 1.13 hold. Put* $r := \inf\{\beta > 0; \varphi'(1+\beta) = 0\} \in [0, \infty]$. *Let* $\Phi : [0, r] \to [0, \infty[$ *be the function* $\Phi(p) := \int_0^p \frac{\varphi(1+\beta)}{c\varphi'(1+\beta)} d\beta$. *Then we have* $\alpha(t) \in [0, r]$ *for all* $t \in [0, T]$ *and*

$$q(t) = \Phi(\alpha(t)) \quad \textit{provided} \quad \alpha(t) \in [0, r[. \tag{1.39}$$

P r o o f. Assume $\alpha(t) > r$ for some $t \in]0, T[$. Then there exists $\tau < t$ such that $\dot\alpha(\tau) > 0$ and $\alpha(\tau) > r$. Putting $\tilde\sigma := \sigma^p(\tau), \tilde\alpha = r$ we have $\varphi(1 + \tilde\alpha) = \varphi(1 + \alpha(\tau))$, hence $\binom{\tilde\sigma}{\tilde\alpha} \in Z_1$ and (1.36) yields $\dot\alpha(\tau) \leqslant 0$, which is a contradiction.

Identity (1.39) can be equivalently written in the form

$$\dot q(t) = \dot\alpha(t) \frac{\varphi(1 + \alpha(t))}{c\varphi'(1 + \alpha(t))} \quad \text{a.e. provided } \alpha(t) < r. \tag{1.40}$$

To prove (1.40) we distinguish two cases.

a) $\dot\alpha(t) = 0$.

Put $\tilde\sigma := (1 + a)\sigma^p(t)$ and $\tilde\alpha := \alpha(t) + b$ for $a > 0$ sufficiently small and $b > 0$ sufficiently large such that $M_0(\tilde\sigma_{\text{dev}}) - \varphi(1 + \tilde\alpha) = (1 + a)\big(M_0(\sigma^p_{\text{dev}}(t)) - \varphi(1 + \alpha(t))\big) + (1 + a)\varphi(1 + \alpha(t)) - \varphi(1 + \alpha(t) + b) \leqslant 0$, hence $\binom{\tilde\sigma}{\tilde\alpha} \in Z_1$. From inequality (1.36) we infer $a\langle \dot\varepsilon^p, \sigma^p \rangle \leqslant 0$, hence $\dot q(t) = 0$.

b) $\dot\alpha(t) > 0$.

The play depends continuously on the characteristic with respect to the Hausdorff distance (see Sect. I.3 below). It therefore suffices to assume that φ and M_0 are smooth functions. We have $\binom{\sigma^p(t)}{\alpha(t)} \in \partial Z_1$ and according to (1.36), the vector $\binom{\dot\varepsilon^p(t)}{-\frac{1}{c}\dot\alpha(t)}$ has the direction of the outward normal vector $n := \binom{\operatorname{grad} M_0(\sigma^p_{\text{dev}}(t))}{-\varphi'(1+\alpha(t))}$, i.e. $\binom{\dot\varepsilon^p(t)}{-\frac{1}{c}\dot\alpha(t)} = \frac{\dot\alpha(t)}{c\varphi'(1+\alpha(t))} n$. This yields $\dot q(t) = \langle \dot\varepsilon^p(t), \sigma^p(t) \rangle = \langle \dot\varepsilon^p(t), \sigma^p_{\text{dev}}(t) \rangle = \frac{\dot\alpha(t)}{c\varphi'(1+\alpha(t))} \langle \operatorname{grad} M_0(\sigma^p_{\text{dev}}(t)), \sigma^p_{\text{dev}}(t) \rangle$. We have $\langle \operatorname{grad} M_0(\sigma^p_{\text{dev}}), \sigma^p_{\text{dev}} \rangle = M_0(\sigma^p_{\text{dev}})$ by Exercise 2.10(iii) and $M_0(\sigma^p_{\text{dev}}(t)) = \varphi(1 + \alpha(t))$ by hypothesis, hence identity (1.40) holds. □

As a consequence of Proposition 1.14, we see that the isotropic hardening can be equivalently characterized by the plastic work (or dissipation) q. For this reason it is sometimes referred to as *work hardening*, see Nečas and Hlaváček (1981), Lemaitre and Chaboche (1985).

Multiyield models

Models of plasticity involving a single yield surface cannot provide a satisfactory description of the real material behavior. In concrete experiments, the transition between the elastic and the plastic regime is smooth. If we neglect relaxation effects and assume that the process is rate independent, the most natural way to proceed is to combine a continuum of plastic elements which are not all active (i.e. in the plastic regime) at the same time. We briefly describe three standard models in this category, namely the Prandtl-Ishlinskii model of stop type, Prandtl-Ishlinskii model of play type and the Mróz model. We shall see in Sect. II.3 that all these models are equivalent in the one-dimensional case.

Example 1.15. Prandtl-Ishlinskii model of stop type.
Following Visintin (1994), we call *Prandtl-Ishlinskii model of stop type* the rheological element defined by the formula $\mathcal{E}_0 \mid \prod_{r\in D} (\mathcal{E}_r - \mathcal{R}_r)$, where $\prod_{r\in D}$ denotes the (possibly uncountable) combination in parallel parametrized by elements of an index set D with measure μ. The sum in the rheological equation is formally replaced by the integration with respect to the measure μ. Combining the rheological equations

$$\sigma = \sigma^e + \sigma^p,\ \sigma^p = \int_D \sigma_r \, d\mu(r),\ \varepsilon = \varepsilon_r^e + \varepsilon_r^p, \tag{1.41}$$

the elastic constitutive laws

$$\sigma^e = A\varepsilon,\ \sigma_r = A_r \varepsilon_r^e, \quad \forall r \in D \tag{1.42}$$

with symmetric positive definite matrices $A, \{A_r; r \in D\}$, and the rigid-plastic variational inequalities

$$\sigma_r \in Z_r,\ \langle \dot{\varepsilon}_r^p, \sigma_r - \tilde{\sigma}_r \rangle \geqslant 0, \quad \forall \tilde{\sigma}_r \in Z_r,\ \forall r \in D, \tag{1.43}$$

where $\{Z_r;\, r \in D\}$ is a given system of convex closed sets in $\mathbb{T}$ with $0 \in \operatorname{Int} Z_r$, we can use the results of Exercise 1.10 to derive formally the constitutive law in operator form

$$\sigma = A\varepsilon + \int_D \underset{r}{\mathcal{S}}(\sigma_r^0, A_r\varepsilon) d\mu(r), \tag{1.44}$$

$$U = \frac{1}{2}\langle A\varepsilon, \varepsilon \rangle + \frac{1}{2}\int_D \langle A_r^{-1} \underset{r}{\mathcal{S}}(\sigma_r^0, A_r\varepsilon), \underset{r}{\mathcal{S}}(\sigma_r^0, A_r\varepsilon) \rangle d\mu(r), \tag{1.45}$$

where $\{\sigma_r^0;\, r \in D\}$ is the given initial distribution of stresses at each individual element $\mathcal{E}_r - \mathcal{R}_r$ and $\mathcal{S}_r$ is the stop with characteristic Z_r.

Example 1.16. Prandtl-Ishlinskii model of play type.
The dual model to the one considered in the previous Example 1.15 is characterized by the formula $\mathcal{E} - \sum_{r\in D} \mathcal{E}_r \,|\, \mathcal{R}_r$, where $\sum_{r\in D}$ denotes the combination in series. Similarly as above, we combine the rheological equations

$$(1.46) \qquad \varepsilon = \varepsilon^e + \varepsilon^p,\ \varepsilon^p = \int_D \varepsilon_r \, d\mu(r),\ \sigma = \sigma_r^e + \sigma_r^p \quad \forall r \in D,$$

the elastic constitutive laws

$$(1.47) \qquad \varepsilon^e = A^{-1}\sigma, \quad \varepsilon_r = A_r^{-1}\sigma_r^e \quad \forall r \in D$$

and the rigid-plastic variational inequalities

$$(1.48) \qquad \sigma_r^p \in Z_r,\ \langle \dot{\varepsilon}_r, \sigma_r^p - \tilde{\sigma}_r \rangle \geqslant 0 \quad \forall \tilde{\sigma}_r \in Z_r \quad \forall r \in D$$

to derive the constitutive equation in operator form

$$(1.49) \qquad \varepsilon = A^{-1}\sigma + \int_D A_r^{-1} \underset{r}{\mathcal{P}}(\sigma_{or}^p, \sigma) d\mu(r),$$

$$(1.50) \qquad U = \frac{1}{2}\langle A^{-1}\sigma, \sigma\rangle + \frac{1}{2}\int_D \langle A_r^{-1} \underset{r}{\mathcal{P}}(\sigma_{or}^p, \sigma), \underset{r}{\mathcal{P}}(\sigma_{or}^p, \sigma)\rangle d\mu(r),$$

where $\{\sigma_{or}^p; r \in D\}$ is the initial distribution of plastic stresses and $\mathcal{P}_r$ is the play with characteristic Z_r.

Constitutive laws (1.44), (1.49) involve a memory which is completely described by the system $\{\mathcal{S}_r;\ r \in D\}$ or $\{\mathcal{P}_r;\ r \in D\}$ of stops and plays, respectively. More precisely, the value of the output for $t \geqslant t_0$ is uniquely determined by the value of the input for $t \geqslant t_0$ and by the distribution of stops and plays at $t = t_0$. This follows from the fact that stop and play are solution operators of variational inequalities. It is therefore justified to introduce the concept of *memory state functions*.

$$(1.51) \qquad \psi_{\mathcal{S}}(r,t) := \underset{r}{\mathcal{S}}(\sigma_r^0, A_r\varepsilon)(t),$$

$$(1.52) \qquad \psi_{\mathcal{P}}(r,t) := \underset{r}{\mathcal{P}}(\sigma_{or}^p, \sigma)(t)$$

which characterize the instantaneous state of the system.

The efficiency of the hysteresis approach in plasticity is related to the possibility to find a simple *memory structure* (i.e. to describe the evolution of $\psi_{\mathcal{S}}, \psi_{\mathcal{P}}$ without solving infinite systems of variations inequalities). This has been done in the uniaxial case (see Chapter II); in the multiaxial case no particular memory structure has been discovered yet except for trivial cases where Z_r are parallelepipeds. This restricts considerably the possibilities of practical application of the multidimensional Prandtl-Ishlinskii operator. One way how to overcome this difficulty is shown in the next example.

Example 1.17. The model of Mróz.
The idea of Mróz (1967) was to consider the family $\{Z_r\}$ of finitely many characteristics of von Mises type

$$Z_r := (B_r(0) \cap \mathbb{T}_{\text{dev}}) + \mathbb{T}_{\text{diag}} \tag{1.53}$$

where $B_r(0)$ is the ball in $\mathbb{T}$ centered at 0 with radius $r > 0$. The concept was then extended to the whole system $\{Z_r;\, r > 0\}$ by Chu (1984).

The rheological structure of the model is essentially analogous to the Prandtl-Ishlinskii model of play type. Equations (1.46), (1.47) are assumed to hold, provided μ in chosen to be the Lebesgue measure in $]0, \infty[$ and the matrices A_r have the form $A_r = h(r)A$, where A is an isotropic matrix of the form (1.6) and $h \in L^1_{\text{loc}}(0, \infty)$ is a nonnegative function.

The Prandtl-Ishlinskii variational inequality (1.48) can be decomposed into two implications:

$$\sigma_r^p \in \operatorname{Int} Z_r \Rightarrow \dot{\sigma}_r^e = 0, \tag{1.54}$$

$$\sigma_r^p \in \partial Z_r \Rightarrow \dot{\sigma}_r^e \in N_{Z_r}(\sigma_r^p), \tag{1.55}$$

where $N_{Z_r}(\sigma_r^p)$ is the normal cone to Z_r at σ_r^p.

The model of Mróz preserves property (1.54), i.e. no plastic deformation occurs in the elastic domain. Denoting by $Z_r(t) := Z_r + \sigma_r^e(t)$ the ball (or cylinder, more precisely) centered at $\sigma_r^e(t)$ with radius r, we obtain similarly as in Example 1.11

$$\sigma(t) \in Z_r(t) \quad \forall r > 0, \quad \forall t \in [0, T]. \tag{1.56}$$

The boundary part (1.55) of the maximal dissipation principle is replaced with the *nonintersection condition*

$$Z_r(t) \subset Z_s(t) \quad \forall r < s, \quad \forall t \in [0, T], \tag{1.57}$$

which represents the main distinctive feature of the Mróz model.

We do not deal with mathematical details here. A more complete information about geometrical and analytical properties of the model can be found in Brokate, Dressler, Krejčí (to appear/a) or Krejčí (1993/c). We just note that conditions (1.46), (1.47), (1.53), (1.54), (1.56), (1.57) coupled with the volume invariance condition

$$\varepsilon_r \in \mathbb{T}_{\text{dev}} \quad \text{or equivalently} \quad \sigma_r^e \in \mathbb{T}_{\text{dev}} \quad \forall r > 0 \tag{1.58}$$

determine a well defined constitutive relation with input σ and output ε in suitable function spaces.

The memory state function

$$\varphi :]0, \infty[\times [0, T] \to \mathbb{T}_{\text{dev}} : (r, t) \mapsto \sigma_r^e(t) \tag{1.59}$$

describes the kinematic hardening and possesses a relatively simple memory structure. This is particularly appreciated in numerical computations. Moreover, putting

$$U(t) := \frac{1}{2} \int_0^\infty \big\langle A^{-1}\varphi(r,t), \varphi(r,t)\big\rangle \, h(r)\, dr \tag{1.60}$$

we obtain a thermodynamically consistent model.

The Mróz model can easily be generalized to the case where $Z \cap \mathbb{T}_{\mathrm{dev}}$ is an ellipsoid of the form $\{\sigma \in \mathbb{T}_{\mathrm{dev}}; \big\langle \tilde{A}^{-1}\sigma, \sigma\big\rangle \leqslant 1\}$ with a non necessarily isotropic positive definite symmetric matrix $\tilde{A}$. To preserve the thermodynamical consistency we must use the same matrix $\tilde{A}$ in (1.47) and (1.60). We see that the shape of the yield surface and the elastic constitutive law *cannot be chosen independently of each other.*

In analogy to the Prandtl-Ishlinskii model, it is possible to define a "Mróz model of stop type" with input ε and output σ by the relations

(1.61)

$$\begin{aligned}
&\text{(i)} && \sigma = \sigma_0 + \int_0^\infty \sigma_r h(r)\, dr,\\
&\text{(ii)} && \varepsilon = \varepsilon_r^e + \varepsilon_r^p,\\
&\text{(iii)} && \sigma_r = A\varepsilon_r^e,\\
&\text{(iv)} && \sigma_r \in rZ,\\
&\text{(v)} && Z_r(t) := \varepsilon_r^p(t) + rA^{-1}Z,\\
&\text{(vi)} && \varepsilon(t) \in \operatorname{Int} Z_r(t) \Rightarrow \dot{\varepsilon}_r^p(t) = 0.
\end{aligned}$$

coupled with the nonintersection condition (1.57). The Mróz models of play and stop type are equivalent in the sense that they generate mutually inverse constitutive operators similarly as scalar Prandtl-Ishlinskii operators in Chapter II. Note that for general multidimensional Prandtl-Ishlinskii operators this is an open problem.

Originally (Mróz (1967), Chu (1984)) the condition $\varepsilon^p = \int_0^\infty \varepsilon_r \, dr$ used to be replaced with a global normality condition analogous to (1.55), namely

$$\sigma_r^p \in \partial Z_r \Rightarrow \dot{\varepsilon}^p \in N_{Z_r}(\sigma_r^p) \tag{1.62}$$

(the nonintersection condition (1.57) guarantees that $N_{Z_r}(\sigma_r^p)$ is independent of r, hence the implication is meaningful). It turns out that this model does not exclude the existence of perpetual motion (see Example 3.2 of Brokate, Dressler, Krejčí (to appear/a) which violates the second law of thermodynamics. The Mróz model coupled with the normality condition (1.62) thus becomes thermodynamically inconsistent. It is therefore natural to conclude that the normality condition cannot be regarded as an independent physical property: it is or is not a mathematical consequence of the constitutive law.

I.2 Geometry of convex sets

The aim of this section is to recall some basic elements of analysis of convex sets in a Hilbert space. Most of the results are well-known. We present them in order to fix the notation and to keep the presentation consistent (for more information we refer the reader to the monographs Rockafellar (1970) and Aubin, Ekeland (1984)). The only concept which is probably new is the *complementary function* of a convex set (Def. 2.4 below) which plays an important role in the study of vector-valued hysteresis operators with unbounded characteristics that occur e.g. in identity (1.37).

Throughout the section, X denotes a real separable Hilbert space endowed with a scalar product $\langle \cdot, \cdot \rangle$ and norm $|x|_X := \langle x, x \rangle^{1/2}$. By Z we denote a convex closed subset of X such that $0 \in Z$. We fix the number

$$m := \operatorname{dist}(0, \partial Z) := \inf\{|z|_X ; z \in \partial Z\} \geqslant 0. \tag{2.1}$$

It is clear that $m > 0$ if and only if $0 \in \operatorname{Int} Z$.

We start with a simple lemma.

Lemma 2.1. *For each $x \in X$ there exists a unique $z \in Z$ such that $|x - z|_X = \operatorname{dist}(x, Z) = \min\{|x - y|_X ; y \in Z\}$.*

P r o o f. Let $x \in X$ be given. Put $p := \inf\{|x - y|_X ; y \in Z\}$ and let $\{y_n\} \subset Z$ be a sequence such that $|x - y_n|_X \to p$. Using the identity

$$|u - v|_X^2 + |u + v|_X^2 = 2(|u|_X^2 + |v|_X^2) \tag{2.2}$$

for $u = x - y_n, v = x - y_k$ we obtain

$$\frac{1}{2}|y_n - y_k|_X^2 = |x - y_n|_X^2 + |x - y_k|_X^2 - 2\left|x - \frac{y_n + y_k}{2}\right|_X^2 \leqslant |x - y_n|_X^2 + |x - y_k|_X^2 - 2p^2,$$

hence $\{y_n\}$ is a convergent sequence and it suffices to put $z := \lim_{n\to\infty} y_n$. Uniqueness follows from identity (2.2). □

Using Lemma 2.1 we can define the projections $Q : X \to Z, P := I - Q$ (I is the identity) associated to Z by the formulae

$$Qx \in Z,\ |Px|_X = \operatorname{dist}(x, Z) \text{ for } x \in X. \tag{2.3}$$

We shall make extensive use of the following properties of P, Q.

Lemma 2.2. *For every* $x, y \in X$ *we have*
(i) $\langle Px, Qx - z\rangle \geqslant 0 \quad \forall z \in Z$,
(ii) $\langle Px - Py, Qx - Qy\rangle \geqslant 0$,
(iii) $\langle Px, x\rangle \geqslant m|Px|_X + |Px|_X^2$ *with* m *given by (2.1),*
(iv) $Q(x + \alpha Px) = Qx \quad \forall \alpha \geqslant -1$.

P r o o f. (i) For $z \in Z, z \neq Qx$ and $\gamma \in]0,1[$ we have $|x - \gamma z - (1-\gamma)Qx|_X^2 > |Px|_X^2$, hence $2\langle Px, Qx - z\rangle + \gamma|Qx - z|_X^2 > 0$ and the assertion follows easily. Statement (ii) is an obvious consequence of (i). We obtain (iii) from (i) by putting $z := \frac{m}{|Px|_X} Px$ if $x \notin Z$, the case $x \in Z$ is trivial. To prove (iv) we notice that for all $z \in Z$ we have $|x + \alpha Px - z|_X^2 = |Qx - z|_X^2 + (1+\alpha)^2|Px|_X^2 + 2(1+\alpha)\langle Px, Qx - z\rangle$, hence the minimum of $|x + \alpha Px - z|_X$ is attained for $z = Qx$. □

It is perhaps not necessary to emphasize that statement (i) of Lemma 2.2 is a Hilbert space version of the Hahn-Banach Convex Separation Theorem. We pass now to a more detailed study of geometrical properties of convex sets.

Recession cone

Definition 2.3. *A nonempty closed convex set* $C \subset X$ *is called* a closed cone, *if the implication* $x \in C \Rightarrow \alpha x \in C$ *holds for all* $x \in X$ *and* $\alpha \geqslant 0$.

Definition 2.4. *Let* $Z \subset X$ *be a convex closed set,* $0 \in \operatorname{Int} Z$. *The set*

$$C_Z := \{x \in Z;\ \alpha x \in Z \quad \forall \alpha \geqslant 0\} \tag{2.4}$$

is called the recession cone *of* Z *and the function* $K_Z : [0,\infty[\to [0,\infty[$ *defined by the formula*

$$K_Z(r) := \sup\{\operatorname{dist}(x, C_Z);\ x \in Z \cap B_r(0)\} \text{ for } r \geqslant 0 \tag{2.5}$$

is called the complementary function of Z, *where* $B_r(x_0) := \{x \in Z; |x - x_0|_X \leqslant r\}$ *denotes the ball centered at* x_0 *with radius* r.

Proposition 2.5. *Let* $Z \subset X$ *be a convex closed set with* $0 \in \operatorname{Int} Z$ *and with the recession cone* C_Z *and complementary function* K_Z. *Then*
(i) $x + y \in Z \quad \forall x \in C_Z, \quad \forall y \in B_m(0)$,
(ii) K_Z *is nondecreasing in* $[0,\infty[$, $K_Z(0) = 0$, $\frac{K_Z(s)}{s} \geqslant \frac{K_Z(r)}{r}$ *for all* $0 < s < r$,
(iii) *if* $\dim X < \infty$, *then*

$$\lim_{r\to\infty} \frac{K_Z(r)}{r} = 0. \tag{2.6}$$

P r o o f. (i) Since Z is closed, it suffices to assume $0 < |y|_X < m$. Put $\gamma := \frac{|y|_X}{m} \in]0,1[$. For $x \in C_Z$ we have $x_\gamma := \frac{1}{1-\gamma}x \in C_Z \subset Z$, $y_\gamma := \frac{1}{\gamma}y \in B_m(0) \subset Z$, hence $x + y = (1-\gamma)x_\gamma + \gamma y_\gamma \in Z$.

(ii) The function K_Z is obviously nondecreasing, $K_Z(0) = 0$. Let us fix positive numbers $r > s$ and $\varepsilon > 0$ and an element $x_\varepsilon \in Z$ such that $|x_\varepsilon|_X \leqslant r, K_Z(r) \leqslant \operatorname{dist}(x_\varepsilon, C_Z) + \varepsilon$. We have either $|x_\varepsilon|_X \leqslant s$ and $K_Z(s) \geqslant K_Z(r) - \varepsilon$ or $|x_\varepsilon|_X > s$ and $\frac{s}{|x_\varepsilon|_X}x_\varepsilon \in Z, K_Z(s) \geqslant \operatorname{dist}\left(\frac{s}{|x_\varepsilon|_X}x_\varepsilon, C_Z\right) = \frac{s}{|x_\varepsilon|_X}\operatorname{dist}(x_\varepsilon, C_Z) \geqslant \frac{s}{|x_\varepsilon|_X}(K_Z(r) - \varepsilon)$. For $\varepsilon \to 0+$ we obtain in both cases $\frac{K_Z(s)}{s} \geqslant \frac{K_Z(r)}{r}$.

(iii) We see that the limit of $\frac{K_Z(r)}{r}$ as $r \to \infty$ exists. Assume that it is positive, say $\lim\limits_{r\to\infty} \frac{K_Z(r)}{r} = \varepsilon > 0$. For each $r > 0$ there exists $z_r \in Z \cap B_r(0)$ such that $a_r := |z_r|_X \geqslant \operatorname{dist}(z_r, C_Z) = K_Z(r) \geqslant \varepsilon r$. We have in particular $a_r \to \infty$ as $r \to \infty$. There exists $y \in X$ and a sequence $r_n \to \infty$ such that $y = \lim\limits_{n\to\infty} \frac{z_{r_n}}{a_{r_n}}, |y|_X = 1$. For an arbitrary $\alpha > 0$ and for n sufficiently large the element $\frac{\alpha}{a_{r_n}} z_{r_n}$ belongs to Z, hence $y \in C_Z$. By hypothesis we have $|z_{r_n} - a_{r_n} y|_X \geqslant \varepsilon r_n$, hence $\left|\frac{z_{r_n}}{a_{r_n}} - y\right|_X \geqslant \varepsilon$, which is a contradiction. $\square$

We immediately see that $C_Z = \{0\}$ if Z is bounded. The converse is true provided $\dim X < \infty$ as a consequence of Proposition 2.5(iii) and is false if $\dim X = \infty$. It suffices to consider the example of the convex "infinite-dimensional cube" $Z := \{x \in X; |\langle x, e_k\rangle| \leqslant 1\ \forall k \in \mathbb{N}\}$, where $\{e_k\}$ is an orthonormal basis in X.

Let us note that for $r > s > 0$ we have by Proposition 2.5 (ii) $K_Z(r) - K_Z(s) \leqslant \frac{K_Z(r)}{r}(r - s)$, hence K_Z is Lipschitz.

Property (iii) in Proposition 2.5 is crucial for the extension of the vector stop and play to the space of continuous functions. We therefore introduce the following terminology.

Definition 2.6. *A convex closed set $Z \subset X$ is called* a recession set *if* $0 \in \operatorname{Int} Z$ *and the complementary function K_Z satisfies (2.6).*

Indeed, every convex closed set $Z \subset X$ with $0 \in \operatorname{Int} Z$ is a recession set if $\dim X < \infty$. This is not true for infinitely dimensional spaces, but the system of recession sets still contains for instance all sets of the form $Z = C + Z_B$, where C is a cone and Z_B is bounded, $0 \in \operatorname{Int} Z_B$.

Definition 2.7. *Let $A, B \subset X$ be two closed subsets of X. The* Hausdorff distance *$\mathcal{H}(A, B)$ of A and B in X is defined by the expression*

$$\mathcal{H}(A,B) := \max\left\{\sup\{\operatorname{dist}(y,A); y \in B\}, \sup\{\operatorname{dist}(x,B); x \in A\}\right\}.$$

The complementary function depends continuously on the convex set Z with respect to the Hausdorff distance.

Lemma 2.8. *Let $Z, \tilde{Z}$ be convex closed subsets of X such that $B_m(0) \subset Z$ and $\mathcal{H}(Z, \tilde{Z}) =: \varepsilon < m$. Then*
(i) $B_{m-\varepsilon}(0) \subset \tilde{Z}$,
(ii) $C_Z = C_{\tilde{Z}}$,
(iii) $|K_Z(r) - K_{\tilde{Z}}(r)| \leqslant \varepsilon \quad \forall r \geqslant 0$.

P r o o f. (i) Assume that for some $x \in B_{m-\varepsilon}(0)$ we have $\operatorname{dist}(x, \tilde{Z}) = |\tilde{P}x|_X > 0$, where $\tilde{P}, \tilde{Q}$ are the projections associated to $\tilde{Z}$ by (2.3). Using Lemma 2.2(iv) for $\alpha = \frac{\varepsilon}{|\tilde{P}x|_X}$ we obtain $\operatorname{dist}(x+\alpha\tilde{P}x, \tilde{Z}) = |\tilde{P}(x+\alpha\tilde{P}x)|_X = (1+\alpha)|\tilde{P}x|_X = \varepsilon+|\tilde{P}x|_X > \varepsilon$ and $x + \alpha\tilde{P}x \in B_m(0) \subset Z$, which is a contradiction.

(ii) It suffices to prove $C_Z \subset \tilde{Z}$. Let $x \in C_Z$ be given. For every $n \subset \mathbb{N}$ there exists $\tilde{z}_n \in \tilde{Z}$ such that $|\tilde{z}_n - nx|_X \leqslant \varepsilon + \frac{1}{n}$. This yields $\frac{1}{n}\tilde{z}_n \to x \in \tilde{Z}$.

(iii) Using (ii) we denote by C the recession cone $C_Z = C_{\tilde{Z}}$. Let P_C, Q_C be the projections associated to C by (2.3).

For an arbitrary $\delta > 0$ we find $\tilde{x}_\delta \in \tilde{Z} \cap B_r(0)$ such that $\operatorname{dist}(\tilde{x}_\delta, C) = |P_C\tilde{x}_\delta|_X \geqslant K_{\tilde{Z}}(r) - \delta$. Put $x_\delta := Q\tilde{x}_\delta \in Z$. We have by hypothesis $\varepsilon \geqslant \operatorname{dist}(\tilde{x}_\delta, Z) = |P\tilde{x}_\delta|_X = |x_\delta - \tilde{x}_\delta|_X$ and $|x_\delta|_X \leqslant |\tilde{x}_\delta|_X \leqslant r$ as a consequence of Lemma 2.2 (i) for $z = 0$. From Lemma 2.2 (ii) it follows $|P_C x_\delta - P_C\tilde{x}_\delta|_X \leqslant |x_\delta - \tilde{x}_\delta|_X \leqslant \varepsilon$ and we conclude

$$K_Z(r) \geqslant \operatorname{dist}(x_\delta, C) = |P_C x_\delta|_X \geqslant K_{\tilde{Z}}(r) - \delta - \varepsilon. \tag{2.7}$$

We similarly prove the counterpart to (2.7), namely $K_{\tilde{Z}}(r) \geqslant K_Z(r) - \delta - \varepsilon$. Letting δ tend to 0 we obtain the assertion. □

Lemma 2.9. *Let the hypotheses of Lemma 2.8 hold and let $P, Q, \tilde{P}, \tilde{Q}$ be as above. Then for all $x, y \in X$ we have*

$$\max\left\{|Px - \tilde{P}y|_X, |Qx - \tilde{Q}y|_X\right\} \leqslant |x - y|_X + \left[\varepsilon(|x|_X + |y|_X)\right]^{1/2}. \tag{2.8}$$

P r o o f. Let $z \in Z, \tilde{z} \in \tilde{Z}$ be such that $|Qx - \tilde{z}|_X \leqslant \varepsilon, |\tilde{Q}y - z|_X \leqslant \varepsilon$. The inequalities $\langle Px, Qx - z\rangle \geqslant 0, \langle\tilde{P}y, \tilde{Q}y - \tilde{z}\rangle \geqslant 0$ entail $\langle Px - \tilde{P}y, Qx - \tilde{Q}y\rangle \geqslant \langle Px, z - \tilde{Q}y\rangle + \langle\tilde{P}y, \tilde{z} - Qx\rangle$, hence

$$|Px - \tilde{P}y|_X^2 \leqslant \langle Px - \tilde{P}y, x - y\rangle + \varepsilon(|Px|_X + |\tilde{P}y|_X)$$
$$|Qx - \tilde{Q}y|_X^2 \leqslant \langle Qx - \tilde{Q}y, x - y\rangle + \varepsilon(|Px|_X + |\tilde{P}y|_X)$$

and (2.8) follows. □

Another useful concept in applications (see Sect.I.1) is the *Minkowski functional* M_Z (also called *gauge*, cf. Rockafellar (1970)) associated to a convex closed set Z with $0 \in \operatorname{Int} Z$ by the formula

$$M_Z(x) := \inf \Big\{ r > 0; \frac{1}{r} x \in Z \Big\} \text{ for } x \in X. \tag{2.9}$$

The proof of the following properties of the Minkowski functional is left to the reader.

Exercise 2.10. Prove that

(i) $M_Z(rx) = rM_Z(x) \quad \forall r \geqslant 0, \quad \forall x \in X$;

(ii) $M_Z(x+y) \leqslant M_Z(x) + M_Z(y) \quad \forall x, y \in X$;

(iii) Let ∂M_Z denote the subdifferential of M_Z. Then $\langle w, x \rangle = M_Z(x)$ for all $x \in X$ and $w \in \partial M_Z(x)$;

(iv) $C_Z = \{x \in X; M_Z(x) = 0\}$, $Z = \{x \in X; M_Z(x) \leqslant 1\}$;

(v) M_Z is a norm in X if and only if $Z = -Z$ and $C_Z = \{0\}$;

(vi) The space X endowed with the norm M_Z is a Banach space if and only if Z is bounded;

(vii) The space X endowed with the norm M_Z is a Hilbert space if and only if there exists a bounded linear selfadjoint strictly positive operator $A : X \to X$ such that $Z = \{x \in X; \langle Ax, x \rangle \leqslant 1\}$.

Tangent and normal cones

A natural generalization of normal vectors and tangent hyperplanes which in general are not uniquely determined, is the concept of *normal cone* $N_Z(x)$ and *tangent cone* $T_Z(x)$ to a convex closed set $Z \subset X$ at a point $x \in Z$. They are defined by the formula

$$\begin{cases} N_Z(x) := \{y \in X; \langle y, x - z \rangle \geqslant 0 \quad \forall z \in Z\}, \\ T_Z(x) := \{w \in X; \langle w, y \rangle \leqslant 0 \quad \forall y \in N_Z(x)\}. \end{cases} \tag{2.10}$$

It is easy to check for each $x \in Z$ using Lemma 2.2 that every element $u \in X$ can be decomposed in a unique way into the sum $u = v + w$ of the normal component $v \in N_Z(x)$ and the tangential component $w \in T_Z(x)$ such that $\langle v, w \rangle = 0$.

For $x \in \operatorname{Int} Z$ we obviously have $N_Z(x) = \{0\}, T_Z(x) = X$. One might expect that for $x \in \partial Z$ the normal cone should contain nonzero elements. The example $Z := \{x \in X; |\langle x, e_k \rangle| \leqslant \frac{1}{k} \ \forall k \in \mathbb{N}\}$, where $\{e_k\}$ is an orthonormal basis, shows that this conjecture is false, since $0 \in \partial Z$ and $N_Z(0) = \{0\}$. In regular cases this cannot happen.

Proposition 2.11. *Let* $\operatorname{Int} Z \neq \emptyset$. *Then for every* $x \in \partial Z$ *we have* $N_Z(x) \setminus \{0\} \neq \emptyset$.

P r o o f. Let $\{z_n; n \in \mathbb{N}\} \subset X \setminus Z$ be a sequence such that $\lim\limits_{n\to\infty} |z_n - x|_X = 0$. Put $\varepsilon_n := |Pz_n|_X > 0$, $y_n := z_n + \frac{1}{\varepsilon_n} Pz_n$. We have $\varepsilon_n \leqslant |z_n - x|_X$ and Lemma 2.2(iv) yields $Qy_n = Qz_n$, $Py_n = (1 + \frac{1}{\varepsilon_n})Pz_n$. By Lemma 2.2(i) we further have $|Qy_n - x|_X^2 = |Qz_n - x|_X^2 = |z_n - x|_X^2 - |Pz_n|_X^2 - 2\langle Pz_n, Qz_n - x\rangle \leqslant |z_n - x|_X^2$ and

$$\langle Py_n, Qy_n - z\rangle \geqslant 0 \quad \forall z \in Z, \quad \forall n \in \mathbb{N}. \tag{2.11}$$

Passing to subsequences we can assume that $\{Py_n\}$ converges weakly to an element ξ which belongs to $N_Z(x)$ by (2.11). It remains to verify that $\xi \neq 0$. We fix an arbitrary ball $B_\delta(x_0) \subset \operatorname{Int} Z$. Putting $z := x_0 + \frac{\delta}{1+\varepsilon_n} Py_n$ in (2.11) we obtain $\delta \leqslant \langle \xi, x - x_0\rangle$, hence $\xi \neq 0$. □

It seems justified with respect to applications in plasticity (see Example 1.4) to mention the important particular case of *cylinders* in X.

Definition 2.12. *Let $Y \subset X$ be a closed subspace of X, let $Y^\perp$ be its orthogonal complement and let $\tilde{Z} \subset Y$ be a convex closed set. Then the set $Z := \tilde{Z} + Y^\perp$ is called a* cylinder *in X.*

Proposition 2.13. *A convex closed set $Z \subset X$ is a cylinder of the form $Z = \tilde{Z} + Y^\perp$ with $\tilde{Z} \subset Y$ if and only if $N_Z(x) \subset Y$ for all $x \in Z$.*

P r o o f. The "only if" part is trivial. To prove the converse we put $\tilde{Z} := Z \cap Y$ and choose arbitrarily $u \in \tilde{Z}$ and $w \in Y^\perp$. From Lemma 2.2(i) we infer $\langle P(u + w), Q(u + w) - u\rangle \geqslant 0$, hence $|P(u + w)|_X^2 \leqslant \langle P(u + w), w\rangle$. On the other hand, we have $P(u+w) \in N_Z(Q(u+w)) \subset Y$, and we conclude $\langle P(u+w), w\rangle = |P(u+w)|_X^2 = 0$. Consequently, $\tilde{Z} + Y^\perp \subset Z$ and equality follows from the convexity of Z. □

Remark 2.14. Cylinders of the form $Z = \tilde{Z} + Y^\perp$ with $\tilde{Z} \subset Y$ are characterized by the condition $Px \in Y$ for all $x \in X$. Denoting by $\tilde{P}, \tilde{Q}$ the projections associated to $\tilde{Z}$ in Y we obtain for every $x \in X$ of the form $x = u + w$, $u \in Y$, $w \in Y^\perp$ the identities $Px = \tilde{P}u$, $Qx = \tilde{Q}u + w$.

Strict convexity

In general, the boundary ∂Z of a convex closed set $Z \subset X$ can contain straight segments. We recall two criteria for their existence. It is easy to verify that ∂Z contains a segment of length $r > 0$ if one of the following conditions is satisfied.

A. *Internal criterion*: There exist $x, y \in \partial Z, |x - y|_X = r, \frac{1}{2}(x + y) \in \partial Z$.

B. *External criterion*: There exists $z \in \partial Z$ and a sequence $\{w_n; n \in \mathbb{N}\} \subset X \setminus T_Z(z)$ such that $|w_n|_X = 1, \lim\limits_{n\to\infty} w_n = w, z + rw \in \partial Z$.

The terminology is justified by the fact that we always have $\frac{1}{2}(x+y) \in Z$ for $x, y \in Z$ and $z + rw \notin Z$ for $z \in \partial Z, w \in X \setminus T_Z(z)$ and $r > 0$.

According to these criteria we introduce the functions $\alpha, \delta : [0,\infty[\to [0,\infty[$ by the formulae

$$\text{(2.12)} \qquad \begin{cases} \delta(r) := \inf\left\{\operatorname{dist}\left(\frac{1}{2}(x+y), \partial Z\right);\ x, y \in Z,\ |x-y|_X = 2r\right\}, \\ \alpha(r) := \inf\left\{|P(z+rw)|_X;\ z \in \partial Z,\ w \in X \setminus T_Z(z),\ |w|_X = 1\right\}, \end{cases}$$

where P is the projection (2.3). We naturally have $\delta(r) = +\infty$ if $2r > \operatorname{diam} Z := \sup\{|x-y|_X; x, y \in Z\}$ (the *diameter* of Z) and $\delta(r) \leqslant \left|\frac{1}{2}(x+y) - x\right|_X = r$ for $0 \leqslant r < \frac{1}{2}\operatorname{diam} Z$. Choosing an arbitrary $x \in X \setminus Z$ we obtain $\alpha(r) \leqslant \left|P(Qx + \frac{r}{|Px|_X}Px)\right|_X = r$ by Lemma 2.2.

The case $\dim X = 1$ is trivial (then $\delta(r) = r$ for $r < \frac{1}{2}\operatorname{diam} Z$, $\alpha(r) = r$ for all $r \geqslant 0$), as well as the case $\operatorname{Int} Z = \emptyset$ (then $\delta(r) = \alpha(r) = 0$ for $r < \frac{1}{2}\operatorname{diam} Z$).

Proposition 2.15. *Let $Z \subset X$ be a convex closed set, $\operatorname{Int} Z \neq \emptyset$. Then for all $0 \leqslant p < r$ we have*

(i) $\frac{\alpha(p)}{p} \leqslant \frac{\alpha(r)}{r}$,

(ii) $\frac{\delta(p)}{p} \leqslant \frac{\delta(r)}{r}$,

(iii) $\alpha(r) \leqslant \delta(r)$.

P r o o f. (i) Let $0 \leqslant p < r$ and $\varepsilon > 0$ be given. Put $\gamma := \frac{p}{r}$. We fix $z \in \partial Z$ and $w \in X \setminus T_Z(z), |w|_X = 1$ such that $|P(z+rw)|_X < \alpha(r) + \varepsilon$. For $v := (1-\gamma)z + \gamma Q(z+rw) \in Z$ we have

$$\alpha(p) \leqslant |P(z+pw)|_X \leqslant |z + pw - v|_X = \gamma|P(z+rw)|_X < \frac{p}{r}(\alpha(r) + \varepsilon)$$

hence (i) holds.

(ii) It suffices to assume $\delta(r) < \infty$. We find $x, y \in Z$ and $z \in \partial Z$ such that $|x-y|_X = 2r$ and

$$\text{(2.13)} \qquad \left|\frac{x+y}{2} - z\right|_X - \frac{\varepsilon}{2} \leqslant \operatorname{dist}\left(\frac{x+y}{2}, \partial Z\right) \leqslant \delta(r) + \frac{\varepsilon}{2}.$$

Put $\hat{x} := \gamma x + (1-\gamma)z, \hat{y} := \gamma y + (1-\gamma)z$ with γ as above. Then $\hat{x}, \hat{y} \in Z, |\hat{x} - \hat{y}|_X = 2p$ and $\delta(p) \leqslant \left|\frac{\hat{x}+\hat{y}}{2} - z\right|_X = \gamma\left|\frac{x+y}{2} - z\right|_X \leqslant \frac{p}{r}(\delta(r) + \varepsilon)$.

(iii) Let x, y, z, ε be as in (ii). We fix an arbitrary $\psi \in N_Z(z), |\psi|_X = 1$ and assume $\langle\psi, x-y\rangle \geqslant 0$ (otherwise we interchange x and y). Put $v_\varepsilon := \frac{x-y}{2} + \varepsilon\psi \in X \setminus T_Z(z)$. Then $\alpha(r) \leqslant \left|P\left(z + \frac{r}{|v_\varepsilon|_X}v_\varepsilon\right)\right|_X \leqslant \left|z + \frac{r}{|v_\varepsilon|_X}v_\varepsilon - x\right|_X \leqslant \left|z - \frac{x+y}{2}\right|_X + \left|\varepsilon\psi - (1 - \frac{r}{|v_\varepsilon|_X})v_\varepsilon\right|_X \leqslant \left|z - \frac{x+y}{2}\right|_X + \varepsilon$. Letting ε tend to 0 we obtain (iii) from (2.13). $\square$

We see that both α, δ are nondecreasing in their domains. One can derive by elementary means further interesting properties of these functions. Details are left to the reader as an exercise.

Exercise 2.16. Let $Z \subset X$ be a closed convex domain with a nonempty interior. Prove that

(i) $\delta(r) \leqslant \frac{1}{2}\alpha(2r + 2\delta(r))$ for $r \in [0, \frac{1}{2}\operatorname{diam} Z[$,

(ii) $\alpha(r) - \alpha(p) \leqslant r - p$ for $0 \leqslant p < r$,

(iii) if $\dim X \geqslant 2$, then for every $x \in \operatorname{Int} Z$, $c := \operatorname{dist}(x, \partial Z)$ and $r \in [0, c]$ we have $2c\delta(r) \leqslant r^2 + \delta^2(r)$;

(iv) if $\delta(r) > 0$ for some $r \in]0, \frac{1}{2}\operatorname{diam} Z[$, then $\operatorname{diam} Z \leqslant \frac{r}{\delta^2(r)}(r^2 + \delta^2(r))$.

Hint. (i) Assume $\alpha(2r + 2\delta(r)) < 2\delta(r) - \varepsilon$ for some $r > 0, \varepsilon > 0$. Find $x \in \partial Z$, $w \in \partial B_1(0) \cap \big(X \setminus T_Z(x)\big)$ such that $\big|P\big(x + (2r + 2\delta(r))w\big)\big|_X < 2\delta(r)$ and put $z := Q\big(x + (2r + 2\delta(r))w\big)$. Then $z \in Z$, $|x - z|_X > 2r$, $x + (r + \delta(r))w \notin Z$, hence $\big|x + (r + \delta(r))w - \frac{x+z}{2}\big|_X > \delta(r)$ which is a contradiction.

(ii) Use the Lipschitz continuity of P which follows from Lemma 2.2(ii).

(iii) Let $z_\varepsilon \in \partial Z$ be such that $|z_\varepsilon - x|_X \leqslant c + \varepsilon$. Find $w \in B_1(0)$ such that $\big\langle w, z_\varepsilon - x\big\rangle = 0$ and put $u_\pm := x + \sqrt{c^2 - r^2}\frac{z_\varepsilon - x}{|z_\varepsilon - x|_X} \pm rw$. Then $u_\pm \in B_c(x) \subset Z$, $|u_+ - u_-|_X = 2r$ and $\delta(r) \leqslant \big|z_\varepsilon - \frac{u_+ + u_-}{2}\big|_X$.

(iv) Assume $s := \frac{1}{2}|x - y|_X > \frac{r}{2\delta^2(r)}\big(r^2 + \delta^2(r)\big)$ for some $x, y \in Z$. Then $s > r$, hence $\delta(s) \geqslant \frac{s}{r}\delta(r) > \frac{1}{2\delta(r)}\big(r^2 + \delta^2(r)\big) \geqslant r$. By (iii) we have $2\delta(s)\delta(r) \leqslant r^2 + \delta^2(r)$ which is a contradiction.

The upper bound for $\operatorname{diam} Z$ in Exercise 2.16(iv) does not seem to be optimal. If Z is a ball, then we obtain for instance $\operatorname{diam} Z = \frac{1}{\delta(r)}(r^2 + \delta^2(r))$. We can nevertheless conclude that Z *is unbounded if and only if* $\alpha(r) = 0$ *for all* $r \geqslant 0$. The opposite situation is of some interest in applications.

Definition 2.17. *A convex closed set $Z \subset X$ is said to be* strictly convex, *if $\alpha(r) > 0$ for all $r > 0$.*

Proposition 2.18. *Let $Z \subset X$ be a strictly convex set, $\dim X \geqslant 2$, $B_m(x) \subset Z$ for some $x \in \operatorname{Int} Z$. Then $\alpha^{-1} : [0, \infty[\to [0, \infty[$ is locally Lipschitz in $]0, \infty[$, and we have $\lim_{s \to \infty} \frac{\alpha^{-1}(s)}{s} = 1$, $\alpha^{-1}(s) \geqslant \sqrt{ms}$ for all $s \geqslant 0$.*

Proof. Proposition 2.15 (i) entails $\alpha(r) - \alpha(p) \geqslant \frac{\alpha(p)}{p}(r - p)$ for all $r > p > 0$, hence α^{-1} is locally Lipschitz in $]0, \infty[$. We obviously have $r \geqslant \alpha(r) \geqslant r - \operatorname{diam} Z$, hence $\lim_{s \to \infty} \frac{\alpha^{-1}(s)}{s} = 1$. To conclude, notice that Exercise 2.16(iii) and Proposition 2.15(iii) yield $m\alpha(r) \leqslant r^2$ for $r \in [0, m]$ and the trivial inequality $\alpha(r) \leqslant r < \frac{1}{m}r^2$ for $r > m$ completes the proof. $\square$

I.3 The play and stop operators

The elementary hysteresis operators called *stop* and *play* have already been introduced in Sect. I.1. The rigorous construction presented here is slightly different from the approach of Krasnosel'skii and Pokrovskii (1983) and Visintin (1987). We admit the infinitely dimensional case and we start with nonsmooth input functions. More precisely, we define the inputs and outputs in the space $C([0,T];X) \cap BV(0,T;X)$ of continuous functions of bounded variation with values in a Hilbert space X. We further prove that the play and stop operators can be extended to continuous (but not necessarily bounded) operators from $C([0,T];X)$ to $C([0,T];X)$ and that they depend continuously on the characteristic Z. The play operator has an interesting smoothening property in $C([0,T];X)$, namely that the total variation of the output remains bounded. The restriction of these operators to Sobolev-type spaces $W^{1,p}(0,T;X)$ (for details about vector-valued absolutely continuous functions see Chapter V) is shown to be continuous if $1 \leqslant p < \infty$ and discontinuous for $p = +\infty$. We consider also the problem of periodicity if the input is periodic and derive two energy inequalities.

The first step consists in proving the following generalization of Theorem 1.9.

Theorem 3.1. *Let a real separable Hilbert space X, a convex closed set $Z \subset X$ with $0 \in Z$, an element $x_0 \in Z$ and a function $u \in C([0,T];X) \cap BV(0,T;X)$ be given. Then there exists a unique $\xi \in C([0,T];X) \cap BV(0,T;X)$ and $x \in C([0,T];Z)$ such that*

$$\text{(3.1)} \qquad \begin{aligned} &\text{(i)} && x(t) + \xi(t) = u(t) \quad \forall t \in [0,T], \\ &\text{(ii)} && x(0) = x_0, \\ &\text{(iii)} && \int_0^T \langle x(t) - \varphi(t), d\xi(t) \rangle \geqslant 0 \quad \forall \varphi \in C([0,T];Z), \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ is the scalar product in X.

The integral in (3.1)(iii) is the Riemann-Stieltjes integral (see Sect. V.1). Theorem 1.9 will turn out to be the consequence of Theorem 3.1 after we prove the regularity in Proposition 3.9.

Exercise 3.2. Prove that condition (3.1)(iii) is equivalent to

$$\text{(3.2)} \qquad \int_s^t \langle x(\tau) - \psi(\tau), d\xi(\tau) \rangle \geqslant 0 \quad \forall \psi \in C([s,t];Z) \qquad \text{for all } 0 \leqslant s < t \leqslant T.$$

H i n t. For $0 < s < t < T$, $\psi \in C([s,t];Z)$ and $\delta \in]0, \min\{s, T-t\}[$ put

$$\varphi_\delta(\tau) := \begin{cases} x(\tau) & \text{for } \tau \in [0, s-\delta] \cup [t+\delta, T], \\ \psi(\tau) & \text{for } \tau \in [s,t], \\ x(s-\delta) + \frac{\tau - s + \delta}{\delta}\big(\psi(s) - x(s-\delta)\big), & \tau \in]s-\delta, s[, \\ x(t+\delta) + \frac{t+\delta-\tau}{\delta}\big(\psi(t) - x(t+\delta)\big), & \tau \in]t, t+\delta[. \end{cases}$$

Use formulae V(1.22), V(1.23) to prove that

$$\int_s^t \big\langle x(\tau) - \psi(\tau), d\xi(\tau)\big\rangle \geqslant \int_t^{t+\delta} \big\langle x(t+\delta) - x(\tau), d\xi(\tau)\big\rangle - \int_{s-\delta}^{s} \big\langle x(\tau) - x(s-\delta), d\xi(\tau)\big\rangle$$
$$+ \frac{1}{\delta}\int_{s-\delta}^{s} \big\langle \psi(s) - x(s-\delta), \xi(s) - \xi(\tau)\big\rangle d\tau + \frac{1}{\delta}\int_t^{t+\delta} \big\langle \psi(t) - x(t+\delta), \xi(\tau) - \xi(t)\big\rangle d\tau$$

and pass to the limit as $\delta \to 0+$.

P r o o f of Theorem 3.1. The uniqueness is easy. Indeed, let (x, ξ) and (y, η) be two solutions of (3.1). Putting $\varphi = \frac{1}{2}(x+y)$ we obtain for all $t \in [0,T]$

$$\int_0^t \big\langle x(\tau) - y(\tau), d(\xi - \eta)(\tau)\big\rangle = -\int_0^t \big\langle \xi(\tau) - \eta(\tau), d(\xi-\eta)(\tau)\big\rangle \geqslant 0$$

and formula V(1.21) yields $\xi = \eta, x = y$.

The solution will be constructed by a simple time-discretization scheme. For a fixed $n \in \mathbb{N}$ we define

$$u_j := u\left(\frac{jT}{n}\right), \quad j = 0, \ldots, n. \tag{3.3}$$

Let P, Q be the projections defined by formula (2.3). We construct the sequences

$$\begin{cases} x_j := Q(x_{j-1} + u_j - u_{j-1}), & j = 1, \ldots, n, \\ \xi_j := u_j - x_j, & j = 0, \ldots, n. \end{cases} \tag{3.4}$$

We have $\xi_j - \xi_{j-1} = P(x_{j-1} + u_j - u_{j-1})$ and Lemma 2.2(i) yields

$$\big\langle \xi_j - \xi_{j-1}, x_j - z\big\rangle \geqslant 0 \qquad \forall z \in Z, \quad \forall j \in \{1, \ldots, n\}. \tag{3.5}$$

Putting $z := x_{j-1}$ and $M := \operatorname*{Var}_{[0,T]} u$ we immediately obtain from (3.5)

$$\sum_{j=1}^{n} |\xi_j - \xi_{j-1}|_X \leqslant M. \tag{3.6}$$

We now define piecewise linear functions $u^{(n)}, \xi^{(n)}, x^{(n)} \in W^{1,1}([0,T];X)$ by the formula

$$
\begin{cases} u^{(n)}(t) := u_{j-1} + n\big(\frac{t}{T} - \frac{j-1}{n}\big)(u_j - u_{j-1}), \\ \xi^{(n)}(t) := \xi_{j-1} + n\big(\frac{t}{T} - \frac{j-1}{n}\big)(\xi_j - \xi_{j-1}), \\ x^{(n)}(t) := x_{j-1} + n\big(\frac{t}{T} - \frac{j-1}{n}\big)(x_j - x_{j-1}) \end{cases} \tag{3.7}
$$

for $t \in \big[\frac{(j-1)T}{n}, \frac{jT}{n}\big[$ and $j = 1, \dots, n$ continuously extended to $t = T$.

Let μ_u be the modulus of continuity of u defined by V(1.19). For every $\tau \in \big]\frac{(j-1)T}{n}, \frac{jT}{n}\big[$ and $z \in Z$ we have by (3.5) and Lemma (2.2)(i)

$$
\begin{aligned}
\big\langle \dot{\xi}^{(n)}(\tau), x^{(n)}(\tau) - z \big\rangle &\geqslant -\frac{n}{T}\big\langle \xi_j - \xi_{j-1}, x_j - x_{j-1} \big\rangle \geqslant -\frac{n}{T}\big\langle \xi_j - \xi_{j-1}, u_j - u_{j-1} \big\rangle \\
&\geqslant -\frac{n}{T}\mu_u\big(\frac{T}{n}\big)|\xi_j - \xi_{j-1}|_X
\end{aligned}
$$

and estimate (3.6) yields

$$
\int_0^t \big\langle x^{(n)}(\tau) - \varphi(\tau), d\xi^{(n)}(\tau) \big\rangle \geqslant -M\mu_u\big(\frac{T}{n}\big) \tag{3.8}
$$

for all $n \in \mathbb{N}$, $t \in [0,T]$ and $\varphi \in C([0,T];Z)$.

The proof of Theorem 3.1 will be complete if we prove that

$$
\{\xi^{(n)}; n \in \mathbb{N}\} \quad \text{is a uniformly convergent sequence.} \tag{3.9}
$$

Indeed, in this case it suffices to use formula (3.8) and Theorem V.1.26, since the sequence $\{u^{(n)}\}$ is uniformly convergent and $\operatorname*{Var}_{[0,T]} \xi^{(n)} \leqslant M$ by (3.6).

To prove (3.9) we choose arbitrarily $n, \ell \in \mathbb{N}$ and put $\varphi(\tau) := \frac{1}{2}\big(x^{(n)}(\tau) + x^{(\ell)}(\tau)\big)$. From (3.8) we infer

$$
\int_0^t \big\langle \dot{\xi}^{(n)}(\tau) - \dot{\xi}^{(\ell)}(\tau), x^{(n)}(\tau) - x^{(\ell)}(\tau) \big\rangle d\tau \geqslant -M\left(\mu_u\big(\frac{T}{n}\big) + \mu_u\big(\frac{T}{\ell}\big)\right), \tag{3.10}
$$

hence by inequality V(1.20)

$$
\frac{1}{2}\big|\xi^{(n)}(t) - \xi^{(\ell)}(t)\big|_X^2 \leqslant \big|u^{(n)} - u^{(\ell)}\big|_\infty \Big(\operatorname*{Var}_{[0,T]} \xi^{(n)} + \operatorname*{Var}_{[0,T]} \xi^{(\ell)}\Big) + M\Big(\mu_u\big(\frac{T}{n}\big) + \mu_u\big(\frac{T}{\ell}\big)\Big).
$$

The sequence $\{\xi^{(n)}\}$ is therefore fundamental in $C([0,T],X)$, hence (3.9) holds and Theorem 3.1 is proved. □

Definition 3.3. *Let* $u \in C([0,T];X) \cap BV(0,T;X)$ *be a given function and let* $Z \subset X$ *be a convex closed set,* $0 \in Z$. *Let* (x,ξ) *be the solution of (3.1). We define the values* $\mathcal{P}(x_0,u), \mathcal{S}(x_0,u)$ *of the* play *and* stop *operators* $\mathcal{P}, \mathcal{S} : Z \times C([0,T];X) \cap BV(0,T;X) \to C([0,T];X) \cap BV(0,T;X)$, *respectively, by the formula*

$$\mathcal{P}(x_0,u) := \xi, \qquad \mathcal{S}(x_0,u) := x. \tag{3.11}$$

Remark 3.4. The initially unperturbed state ("virginal state" in the terminology of Visintin (1984)) is characterized by the choice $x_0 = Qu(0)$ of the initial condition (3.1) (ii). In this case we use the simplified notation

$$\mathcal{P}(u) := \mathcal{P}(Qu(0),u), \quad \mathcal{S}(u) := \mathcal{S}(Qu(0),u). \tag{3.12}$$

We next study the dependence of $\mathcal{P}, \mathcal{S}$ on Z in terms of the Hausdorff distance of sets in X (cf. Def. 2.7).

Proposition 3.5. *Let* $u,v \in C([0,T];X) \cap BV(0,T;X)$ *be given functions, let* $Z, \tilde{Z}$ *be given convex closed sets such that their Hausdorff distance* $\varepsilon := \mathcal{H}(Z,\tilde{Z})$ *is finite,* $0 \in Z \cap \tilde{Z}$ *and let* $x_0 \in Z$, $\tilde{x}_0 \in \tilde{Z}$ *be given. Let* $\mathcal{P}, \tilde{\mathcal{P}}$ *be the play operators corresponding to* $Z, \tilde{Z}$, *respectively.*

Put $\xi := \mathcal{P}(x_0,u)$, $\eta := \tilde{\mathcal{P}}(\tilde{x}_0,v)$, $x := u - \xi$, $y := v - \eta$. *Then for* $0 \leqslant s < t \leqslant T$ *we have*

$$|\xi(t) - \eta(t)|_X^2 \leqslant |\xi(s) - \eta(s)|_X^2 + 2(\varepsilon + |u-v|_\infty)\big(\operatorname*{Var}_{[s,t]} \xi + \operatorname*{Var}_{[s,t]} \eta\big). \tag{3.13}$$

P r o o f. Let $Q, P, \tilde{Q}, \tilde{P}$ be the projections (2.3) corresponding to $Z, \tilde{Z}$, respectively. For $\tau \in [s,t]$ put $\psi(\tau) := \tilde{Q}(x(\tau))$, $\varphi(\tau) := Q(y(\tau))$. We have $|\psi(\tau) - x(\tau)|_X \leqslant \varepsilon$, $|\varphi(\tau) - y(\tau)|_X \leqslant \varepsilon$ and the inequalities

$$\int_s^t \big\langle x(\tau) - \varphi(\tau), d\xi(\tau)\big\rangle \geqslant 0, \quad \int_s^t \big\langle y(\tau) - \psi(\tau), d\eta(\tau)\big\rangle \geqslant 0$$

entail

$$\int_s^t \big\langle x(\tau) - y(\tau), d(\xi - \eta)(\tau)\big\rangle \geqslant -\varepsilon\big(\operatorname*{Var}_{[s,t]} \xi + \operatorname*{Var}_{[s,t]} \eta\big)$$

and the rest follows from Exercise V.1.24. □

Continuous inputs

Theorem 3.7 below enables us to extend the stop and play to the space $C([0,T];X)$. The idea of the proof is due to A.A. Vladimirov (see Krasnosel'skii, Pokrovskii (1983) for $\dim X < \infty$ and Z bounded) and relies on the following Lemma.

Lemma 3.6. *Let $\mathcal{B} \subset C([0,T];X)$ be a compact set. Let $\tilde{Z} \subset X$ be a recession set (see Def. 2.6) with $B_m(0) \subset \tilde{Z}$ and let $r > 0$ be given. Then there exists $M > 0$ such that for every $u \in \mathcal{B} \cap BV(0,T;X)$, every convex closed set $Z \subset X$ such that $\mathcal{H}(Z,\tilde{Z}) < \frac{m}{2}$ and every $x_0 \in Z \cap B_r(0)$ we have*

$$\operatorname*{Var}_{[0,T]} \mathcal{P}(x_0,u) \leqslant M, \tag{3.14}$$

where $\mathcal{P}$ is the play operator corresponding to Z.

Proof. Let Ξ be the system of sets Z satisfying the hypotheses of Lemma 3.6. Every $Z \in \Xi$ is a recession set and $B_{\frac{m}{2}}(0) \subset Z$ by Lemma 2.8.

We find $u_1,\ldots,u_N \in \mathcal{B}$ such that $\mathcal{B} \subset \bigcup_{k=1}^{N} \{u \in C([0,T];X);\ |u-u_k|_\infty < \gamma\}$ for $\gamma := \frac{m}{12}$ and fix $\delta > 0$ such that $\max\{\mu_{u_k}(\delta); k = 1,\ldots,N\} < \gamma$.

We first prove that for every $u \in \mathcal{B} \cap BV(0,T;X)$, $Z \in \Xi$, $x_0 \in Z$ and $0 \leqslant s < t \leqslant T$ such that $|t-s| < \delta$ we have

$$\operatorname*{Var}_{[s,t]} \mathcal{P}(x_0,u) \leqslant \frac{2}{m} K_Z^2(|\,\mathcal{S}(x_0,u)(s)|_X). \tag{3.15}$$

Put $\xi := \mathcal{P}(x_0,u)$, $x := \mathcal{S}(x_0,u)$. We find $\hat{x} \in C_Z$ such that $|x(s)-\hat{x}|_X \leqslant K_Z(|x(s)|_X)$ and put for $\tau \in [s,t]$

$$\psi(\tau) := \hat{x} + u(\tau) - u(s) + \frac{m}{4}\varphi(\tau)$$

for some $\varphi \in C([s,t];X), |\varphi|_\infty \leqslant 1$.

We have $|\psi(\tau)-\hat{x}|_X \leqslant \frac{m}{2}$ for all $\tau \in [s,t]$, hence $\psi \in C([s,t];Z)$. Inequality (3.2) and Exercise V.1.24 then entail

$$\frac{m}{4}\int_s^t \langle \varphi(\tau), d\xi(\tau)\rangle \leqslant \int_s^t \langle u(s)-\hat{x}-\xi(\tau), d\xi(\tau)\rangle = \frac{1}{2}|x(s)-\hat{x}|_X^2 - \frac{1}{2}|u(s)-\hat{x}-\xi(t)|_X^2,$$

and inequality (3.15) follows from Theorem V.1.30.

Putting $R := \frac{T}{\delta} + 1$ we obtain from (3.15)

$$\operatorname*{Var}_{[0,T]} \xi \leqslant \frac{2R}{m} K_Z^2(|x|_\infty). \tag{3.16}$$

Inequality (3.13) for $v = \eta = \varepsilon = 0$ yields $|\xi|_\infty^2 \leqslant |u(0)-x_0|_X^2 + 2|u|_\infty \operatorname*{Var}_{[0,T]} \xi$, and from Lemma 2.8(iii) we infer

$$|x|_\infty^2 \leqslant 4\big(|u|_\infty + r\big)^2 + \frac{4R}{m} 2|u|_\infty \big(K_{\tilde{Z}}(|x|_\infty) + \frac{m}{2}\big)^2$$

The set $\mathcal{B}$ is bounded, hence the last inequality provides an upper bound for $|x|_\infty$ independent of $u \in \mathcal{B}$ and $Z \in \Xi$. Inequality (3.14) then follows from (3.16) and property (2.6) of recession sets. $\square$

Lemma 3.6 immediately implies that for each $u \in C([0,T];X)$ and $x_0 \in Z$ the value of $\mathcal{P}(x_0,u) \in C([0,T];X) \cap BV(0,T,X)$ and $\mathcal{S}(x_0,u) \in C([0,T];Z)$ can be defined in a unique way provided Z is a recession set (this is no restriction if $\dim X < \infty$). Indeed, for any sequence $\{u_n; n \in \mathbb{N}\} \subset C([0,T];X) \cap BV(0,T;X)$ such that $\lim\limits_{n\to\infty} |u - u_n|_\infty = 0$ we conclude from (3.14)

$$\operatorname*{Var}_{[0,T]} \mathcal{P}(x_0,u_n) \leqslant M \tag{3.17}$$

and (3.13) yields

$$|\,\mathcal{P}(x_0,u_n) - \mathcal{P}(x_0,u_k)|_\infty^2 \leqslant |u_n(0) - u_k(0)|_X^2 + 4M|u_n - u_k|_\infty \tag{3.18}$$

for all $k,n \in \mathbb{N}$. The sequence $\{\mathcal{P}(x_0,u_n)\}$ is therefore fundamental in $C([0,T];X)$ and its limit is independent of the concrete choice of the sequence $\{u_n\}$. We therefore can define

$$\mathcal{P}(x_0,u) := \lim_{n\to\infty} \mathcal{P}(x_0,u_n). \tag{3.19}$$

By Proposition V.1.18(ii) we have

$$\operatorname*{Var}_{[0,T]} \mathcal{P}(x_0,u) \leqslant M, \tag{3.20}$$

hence $\mathcal{P}$ maps $Z \times C([0,T];X)$ into $C([0,T];X) \cap BV(0,T;X)$.

The following two continuity results are straightforward consequences of the above considerations and the density of $BV(0,T;X) \cap C([0,T];X)$ in $C([0,T];X)$.

Theorem 3.7. *Let the hypotheses of Lemma 3.6 be satisfied. Let Ξ_ε be a system of recession sets $Z \subset X$ such that $\mathcal{H}(Z,\tilde{Z}) \leqslant \frac{\varepsilon}{2}$ for some $\varepsilon < \frac{m}{2}$. Then there exists a constant $M > 0$ such that for all $u,v \in \mathcal{B}$, $Z_1, Z_2 \in \Xi_\varepsilon$ and $x_i^0 \in Z_i \cap B_r(0), i = 1,2$ we have*

$$|\,\underset{1}{\mathcal{P}}(x_1^0,u) - \underset{2}{\mathcal{P}}(x_2^0,v)|_\infty \leqslant M(\varepsilon + |u-v|_\infty)^{1/2} + |u-v|_\infty + |x_1^0 - x_2^0|_X, \tag{3.21}$$

where $\mathcal{P}_1, \mathcal{P}_2$ are the plays corresponding to Z_1, Z_2, respectively.

Corollary 3.8. *Let $\{u_n; n \in \mathbb{N} \cup \{0\}\} \subset C([0,T];X)$ be a given sequence of functions, let $\{Z_n; n \in \mathbb{N} \cup \{0\}\}$ be a given sequence of recession sets such that $\lim\limits_{n\to\infty} |u_n - u_o|_\infty = 0$, $\lim\limits_{n\to\infty} \mathcal{H}(Z_n, Z_o) = 0$ and let $x_n^0 \in Z_n$ be given initial values, $x_o^0 = \lim\limits_{n\to\infty} x_n^0$. Put $\xi_n := \mathcal{P}_n(x_n^0, u_n)$ for $n \in \mathbb{N} \cup \{0\}$, where $\mathcal{P}_n$ is the play corresponding to Z_n. Then $\lim\limits_{n\to\infty} |\xi_n - \xi_o|_\infty = 0$.*

Notice that the unperturbed initial values defined in Remark 3.4 satisfy the convergence condition by Lemma 2.9.

Regularity

One can expect that play and stop operators act in Sobolev spaces $W^{1,p}(0,T;X)$. Before passing to the continuity result we formulate rigorously the normality rule mentioned in Sect. I.1.

Proposition 3.9. *Let $Z \subset X$ be a convex closed set with $0 \in Z$, let $x_0 \in Z$ be a given initial value and let $u \in W^{1,p}(0,T;X)$ be given for some $p \in [1,+\infty]$. Then $\xi := \mathcal{P}(x_0,u), x := \mathcal{S}(x_0,u)$ belong to $W^{1,p}(0,T;X)$ and satisfy*

$$\text{(3.22)} \qquad \begin{aligned} &\text{(i)} \quad \langle \dot{\xi}(t), x(t) - z \rangle \geqslant 0 \quad \text{a.e.} \quad \forall z \in Z, \\ &\text{(ii)} \quad \langle \dot{\xi}(t), \dot{x}(t) \rangle = 0 \quad \text{a.e.} \end{aligned}$$

P r o o f. For arbitrary $0 \leqslant s < t \leqslant T$ and $\tau \in [s,t]$ put $\psi(\tau) = x(s)$ in (3.2). Then Lemma V.1.25 and formulas V(1.22), V(1.25) yield

$$\begin{aligned} \frac{1}{2}|\xi(t) - \xi(s)|_X^2 \leqslant \int_s^t \langle u(\tau) - u(s), d\xi(\tau) \rangle &= \int_s^t \langle \xi(t) - \xi(\tau), \dot{u}(\tau) \rangle \, d\tau \\ &\leqslant \max_{s \leqslant \tau \leqslant t} |\xi(t) - \xi(\tau)|_X \int_s^t |\dot{u}(\tau)|_X \, d\tau \,, \end{aligned}$$

hence $|\xi(t) - \xi(s)|_X \leqslant 2\int_s^t |\dot{u}(\tau)|_X \, d\tau$ for all $0 \leqslant s < t \leqslant T$. This implies $\xi \in W^{1,p}(0,T;X)$ and by Lemma V.1.25 we have $\int_s^t \langle \dot{\xi}(\tau), x(\tau) - \psi(\tau) \rangle \, d\tau \geqslant 0$ for all $\psi \in C([s,t];Z)$ and $0 \leqslant s < t \leqslant T$, which is equivalent to (3.22)(i). To prove (3.22)(ii) it suffices to put $z := x(t \pm h)$ in (3.22)(i) and let h tend to 0. □

Remarks 3.10.

(i) Formulae (3.22) admit a simple geometrical interpretation in terms of the normal and tangential cones N_Z, T_Z introduced in (2.10). If u is absolutely continuous, then $\dot{\xi}(t) \in N_Z(x(t))$ and $\dot{x}(t) \in T_Z(x(t))$ a.e., so $\dot{u} = \dot{\xi} + \dot{x}$ is the (unique) orthogonal decomposition of $\dot{u}$ into the normal and tangential component.

(ii) Putting $x := \mathcal{S}(x_0,u)$, $y := \mathcal{S}(y_0,v)$ for given $x_0, y_0 \in Z$, $u, v \in W^{1,1}(0,T;X)$ we immediately obtain from (3.22)(i)

$$\text{(3.23)} \qquad \frac{1}{2}\frac{d}{dt}|x(t) - y(t)|_X^2 \leqslant \langle x(t) - y(t), \dot{u}(t) - \dot{v}(t) \rangle \quad \text{a.e.,}$$

consequently

$$\text{(3.24)} \qquad |x(t) - y(t)|_X \leqslant |x_0 - y_0|_X + 2\int_0^t |\dot{u}(\tau) - \dot{v}(\tau)|_X d\tau \qquad \forall t \in [0,T].$$

This immediately implies that the mapping $\mathcal{S} : Z \times W^{1,1}(0,T;X) \to C([0,T];X)$ is Lipschitz.

Theorem 3.12 below concerns the continuity of the play and stop as mappings $Z \times W^{1,p}(0,T;X) \to W^{1,p}(0,T;X)$ for $1 \leqslant p < +\infty$ and their continuous dependence on Z. We first prove that the piecewise linear approximations (3.7) converge strongly in $W^{1,1}(0,T;X)$.

Proposition 3.11. *Let the hypotheses of Proposition 3.9 be fulfilled for $p = 1$ and let $u^{(n)}, \xi^{(n)}, x^{(n)}$ be defined by (3.7). Then $\lim_{n\to\infty} |\xi^{(n)} - \xi|_{1,1} = 0$, where $\xi = \mathcal{P}(x_0, u)$ and $|\cdot|_{1,1}$ is the norm in $W^{1,1}(0,T;X)$.*

P r o o f. Inequality (3.5) entails

$$\big\langle \dot{\xi}^{(n)}(t), \dot{x}^{(n)}(t) \big\rangle \geqslant 0 \quad \text{a.e.} \tag{3.25}$$

Put $y^{(n)} := \xi^{(n)} - x^{(n)}$, $y := \xi - x$. We have by (3.22)(ii) and (3.25)

$$|\dot{y}^{(n)}(t)|_X \leqslant |\dot{u}^{(n)}(t)|_X,\ |\dot{y}(t)|_X = |\dot{u}(t)|_X \quad \text{a.e.}$$

The hypotheses of Theorem V.1.15 are now satisfied for $v_o := \dot{y}$, $v_n := \dot{y}^{(n)}$, $g_o := |\dot{u}|_X$, $g_n := |\dot{u}^{(n)}|_X$ (in particular, hypothesis (i) follows from (3.9)) and we conclude

$$\lim_{n\to\infty} |y^{(n)} - y|_{1,1} = 0, \quad \text{hence} \quad \lim_{n\to\infty} |\xi^{(n)} - \xi|_{1,1} = 0.$$

□

Theorem 3.12. *Let $\{Z_n;\, n \in \mathbb{N} \cup \{0\}\}$ be a sequence of convex closed sets in X such that $0 \in \bigcap_{n=0}^{\infty} Z_n$, $\lim_{n\to\infty} \mathcal{H}(Z_o, Z_n) = 0$ and let $\{x_n^0\}$ be a sequence of initial values such that $x_o^0 = \lim_{n\to\infty} x_n^0$. Let $\{u_n;\, n \in \mathbb{N} \cup \{0\}\} \subset W^{1,p}(0,T;X)$ be a sequence such that $\lim_{n\to\infty} |u_n - u_o|_{1,p} = 0$ for some $p \in [1, +\infty[$. Put $\xi_n := \mathcal{P}_n(x_n^0, u_n)$ for $n \in \mathbb{N} \cup \{0\}$, where $\mathcal{P}_n$ is the play corresponding to Z_n. Then $\lim_{n\to\infty} |\xi_n - \xi_o|_{1,p} = 0$.*

P r o o f. Put $\delta_n := \mathcal{H}(Z_o, Z_n)$, $x_n := u_n - \xi_n$. Inequality (3.22)(i) and the argument of the proof of Proposition 3.5 yield $\big\langle \dot{x}_n - \dot{x}_o, x_n - x_o \big\rangle \leqslant \big\langle x_n - x_o, \dot{u}_n - \dot{u}_o \big\rangle + \delta_n(|\dot{u}_n|_X + |\dot{u}_o|_X)$ a.e., hence $|x_n - x_o|_\infty \to 0$, $|\xi_n - \xi_o|_\infty \to 0$ as $n \to \infty$. For $y_n := \xi_n - x_n$ we further obtain from (3.22)(ii)

$$|\dot{y}_n(t)|_X = |\dot{u}_n(t)|_X \quad \text{a.e. for all} \quad n \in \mathbb{N} \cup \{0\}.$$

Similarly as in the proof of Proposition 3.11 we use Theorem V.1.15 to prove that $\lim_{n\to\infty} |\xi_n - \xi|_{1,1} = 0$. The assertion of Proposition V.1.13 for $g_n := |\dot{u}_n|_X$, $v_n := \dot{y}_n$ then completes the proof for $p > 1$. □

A counterpart to Theorem 3.12 does not hold for $p = +\infty$ even if $\dim X = 1$. It suffices to consider $Z = [-1, 1]$, $T = 1$ and the sequence $u_n(t) := \big(1+\frac{1}{n}\big)t$ for $t \in [0,1]$, $n \in \mathbb{N}$ with $u_0(t) := t$, $x_n^0 := 0$. We then have

$$\xi_0(t) \equiv 0, \quad \xi_n(t) := \begin{cases} 0 \quad \text{for} \quad t \in \big[0, \frac{n}{n+1}\big], \\ \big(1 + \frac{1}{n}\big)t - 1 \quad \text{for} \quad t \in \big]\frac{n}{n+1}, 1\big] \end{cases} \qquad \text{for } n \in \mathbb{N},$$

hence $|u_n - u_0|_{1,\infty} \to 0$, $|\xi_n - \xi_0|_{1,\infty} \geqslant 1$.

We shall see in Chapter II that for $\dim X = 1$ the operators $\mathcal{P}, \mathcal{S}$ are Lipschitz continuous in $C([0,T];X)$ and in $W^{1,1}(0,T;X)$. Better continuity results will also be obtained in Section I.4 in the vector case provided Z has a special shape.

In general, the situation is more complicated. The following example similar to those which occur in models of plasticity with isotropic hardening of the type (1.37) shows that the play operator (3.12) does not necessarily map bounded sets in $C([0,T];X)$ into bounded sets. The operator $F := I + \mathcal{P}$ thus provides an elegant example of general interest in functional analysis such that both F and $F^{-1} = I - \frac{1}{2}\mathcal{P}$ (cf. identity (1.35)) are continuous and unbounded in $C([0,T];X)$.

Example 3.13. Put $X := \mathbb{R}^2$, $Z := \{(a,b) \in X;\, 1 \geqslant b \geqslant g(a)\}$, where $g(a) := e^{-a-1} - 1$, $\xi_n := \mathcal{P}(u_n)$, $x_n := u_n - \xi_n$, $u_n(t) := \binom{0}{\cos n\pi t}$ for $t \in [0,2]$ and $n \in \mathbb{N}$. Let $\Gamma_U := \{(a,b) \in Z; b = 1\}$, $\Gamma_L := \{(a,b) \in Z; b = g(a)\}$ denote the upper and lower boundary of Z, respectively.

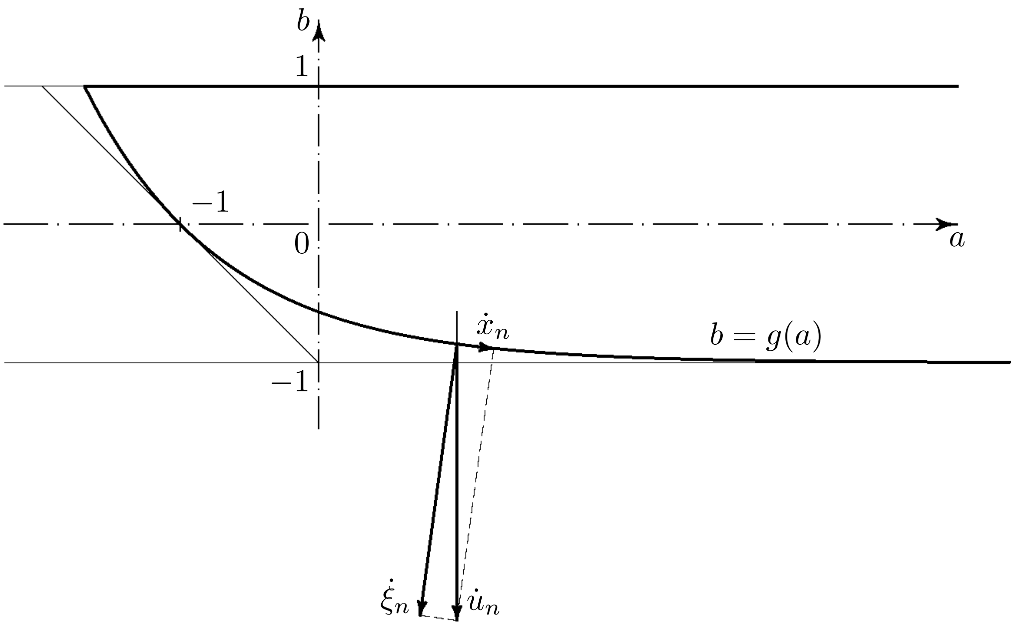


Fig. 5

In component form we can write $x_n(t) = \binom{\alpha_n(t)}{\beta_n(t)}$. For $x_n(t) \in \operatorname{Int} Z$ we have $\dot{x}_n(t) = \dot{u}_n(t)$ and for $x_n(t) \in \Gamma_U$ we have $\dot{x}_n(t) = 0$ by Remark 3.10, so α_n is nonconstant only if $x_n(t) \in \Gamma_L$. In this case $\dot{x}_n(t)$ is the tangential component of $\dot{u}_n(t)$, hence,

$$\dot{\alpha}_n(t)\,\cosh(\alpha_n(t)+1) = -\frac{1}{2}\frac{d}{dt}(\cos n\pi t). \tag{3.26}$$

Put $a_k := \alpha_n\big(\frac{2k-1}{n}\big)$. There exists $\tau_k \in \big]\frac{2k}{n}, \frac{2k+1}{n}\big[$ such that $\alpha_n(t) = a_k$ for $t \in \big[\frac{2k-1}{n}, \tau_k\big]$, $\cos n\pi\tau_k = g(a_k)$ and in $\big]\tau_k, \frac{2k+1}{n}\big[$ equation (3.26) holds. After integration we obtain

$$\sinh(a_{k+1}+1) - \sinh(a_k+1) = \frac{1}{2}\big(1+g(a_k)\big) = \frac{1}{2}e^{-a_k-1},$$

hence $a_{k+1} > a_k$ for all k and $\lim\limits_{k\to\infty} a_k = +\infty$. We therefore have $\lim\limits_{n\to\infty} \alpha_n\big(2-\frac{1}{n}\big) = \lim\limits_{n\to\infty} a_n = +\infty$ and the sequence $\{x_n;\, n \in \mathbb{N}\}$ is unbounded in $C([0,2];X)$.

Periodic inputs

An interesting particular case arises if the input function u is periodic. We denote by $W^{1,1}_\omega$ the space of absolutely continuous ω-periodic functions $u : \mathbb{R}^1 \to X$, i.e. such that $u(t+\omega) = u(t)$ for all $t \in \mathbb{R}^1$, endowed with the norm of $W^{1,1}(0,\omega;X)$. Example 3.13 above shows that the outputs $\mathcal{P}(u), \mathcal{S}(u)$ are not necessarily periodic. We nevertheless have the following asymptotic periodicity result.

Theorem 3.14. *Let $u \in W^{1,1}_\omega$ be given, let $Z \subset X$ be a convex closed set with $0 \in \operatorname{Int} Z$ and let $x_0 \in Z$ be given. Assume that the trajectory $\{x(t);\, t \in [0,+\infty[\} \subset Z$ of $x = \mathcal{S}(x_0,u)$ is precompact. Then there exists $x^* \in W^{1,1}_\omega$ such that*

$$\lim_{t\to\infty} |x(t) - x^*(t)|_X = 0.$$

P r o o f. We denote as usual $\xi := \mathcal{P}(x_0,u)$. We have $\big\langle \dot{\xi}(t), x(t) - x(t+\omega)\big\rangle \geqslant 0$, $\big\langle \dot{\xi}(t+\omega), x(t+\omega) - x(t)\big\rangle \geqslant 0$ a.e., hence $\frac{d}{dt}|x(t+\omega) - x(t)|_X^2 \leqslant 0$.

Put $r := \lim\limits_{t\to\infty} |x(t+\omega) - x(t)|_X$. By hypothesis, there exists $\bar{x} \in Z$ and a sequence $\{n_j;\, j \in \mathbb{N}\} \subset \mathbb{N}$ such that $\lim\limits_{j\to\infty} |x(n_j\omega) - \bar{x}|_X = 0$. Put

$$\begin{aligned} &\text{(i)} \quad x_j(t) := x(t+n_j\omega) \quad \text{for} \quad t \geqslant 0,\\ &\text{(ii)} \quad x^* \quad := \mathcal{S}(\bar{x},u). \end{aligned} \tag{3.27}$$

Then $\big\langle \dot{x}_j(t) - \dot{x}^*(t), x_j(t) - x^*(t)\big\rangle \leqslant 0$ a.e., hence

$$|x_j(t) - x^*(t)|_X \leqslant |x(n_j\omega) - \bar{x}|_X. \tag{3.28}$$

We therefore have for all $t \geqslant 0$

$$|x^*(t+\omega) - x^*(t)|_X = \lim_{j\to\infty} |x(t+n_j\omega+\omega) - x(t+n_j\omega)|_X = r.$$

For $\xi^*(t) := u(t) - x^*(t)$ this yields $\big\langle \dot\xi^*(t+\omega) - \dot\xi^*(t), x^*(t+\omega) - x^*(t)\big\rangle = 0$, a.e., consequently $\big\langle \dot\xi^*(t+\omega), x^*(t+\omega) - x^*(t)\big\rangle = \big\langle \dot\xi^*(t), x^*(t) - x^*(t+\omega)\big\rangle = 0$ a.e. We now obtain from (3.27)(ii), (3.22)(i) $\big\langle \dot\xi^*(t), x^*(t+\omega) - z\big\rangle \geqslant 0$, $\big\langle \dot\xi^*(t+\omega), x^*(t) - z\big\rangle \geqslant 0$ for all $z \in Z$. As in the proof of Proposition 3.9 we conclude $\big\langle \dot\xi^*(t), \dot x^*(t+\omega)\big\rangle = \big\langle \dot\xi^*(t+\omega), \dot x^*(t)\big\rangle = 0$, hence by (3.22)(ii)

$$|\dot x^*(t) - \dot x^*(t+\omega)|_X^2 = -\big\langle \dot\xi^*(t) - \dot\xi^*(t+\omega), \dot x^*(t) - \dot x^*(t+\omega)\big\rangle = 0 \quad \text{a.e.}$$

For all $t \geqslant 0$ the last identity entails $x^*(t+\omega) - x^*(t) = x^*(\omega) - \bar x$, and in particular $|x^*(n\omega) - \bar x|_X = nr$. The trajectory of x^* is bounded due to the inequality (3.28), hence $r = 0$. We therefore have $x^* \in W^{1,1}_\omega$. Inequality (3.28) then yields

$$|x(t) - x^*(t)|_X \leqslant |x(n_j\omega) - \bar x|_X \quad \text{for} \quad t > n_j\omega,$$

hence the assertion of Theorem 3.14 holds. □

Energy inequalities

The natural definition in Exercise 1.10 of the potential energy associated to the play and stop has the form $U_{\mathcal P}(t) := \frac12 |\mathcal P(x_0,u)(t)|_X^2$, $U_{\mathcal S}(t) := \frac12 |\mathcal S(x_0,u)(t)|_X^2$, respectively. We then have for all $u \in W^{1,1}(0,T;X)$ and almost all $t \in]0,T[$

$$\begin{cases} \big\langle \frac{d}{dt}\mathcal P(x_0,u)(t), u(t)\big\rangle - \dot U_{\mathcal P}(t) \geqslant 0, \\ \big\langle \dot u(t), \mathcal S(x_0,u)(t)\big\rangle - \dot U_{\mathcal S}(t) \geqslant 0. \end{cases} \tag{3.29}$$

The left-hand sides of these inequalities express the rate of dissipation. It can be in some cases computed explicitly.

Example 3.15. (von Mises yield condition, see Example 1.4). Let $Y \subset X$ be a closed subspace of X and let Z be the infinite cylinder $Z := (B_r(0) \cap Y) + Y^\perp$ of radius $r > 0$. In both cases in (3.29), the dissipation rate $\dot q(t)$ is given by the formula

$$\dot q(t) = \Big\langle \frac{d}{dt}\mathcal P(x_0,u)(t), \mathcal S(x_0,u)(t)\Big\rangle = r\Big|\frac{d}{dt}\mathcal P(x_0,u)(t)\Big|_X. \tag{3.30}$$

as a special case of Lemma 4.12 below.

Hysteresis operators arising from variational inequalities have an interesting feature with important consequences, namely that they admit higher order energy inequalities. This property is obvious for linear constitutive operators and very nontrivial for nonlinear operators, like the stop or play. We shall put much emphasis on this fact in the following chapters in connection with hyperbolic equations on one hand and with the geometry of constitutive laws on the other hand. The first general result in this direction is the following.

Theorem 3.16. *Let* $u \in W^{1,\infty}(0,T;X)$ *be given such that* $\dot{u} \in W^{1,1}(0,T;X)$*. For a convex closed* $Z \subset X$ *with* $0 \in Z$ *and for some* $x_0 \in Z$ *put* $x(t) := \mathcal{S}(x_0,u)(t)$, $U_2(t) := \frac{1}{2}|\dot{x}(t)|_X^2$ *for a.e.* $t \in]0,T[$*. Then we have*

$$U_2(t) - U_2(s) \leqslant \int_s^t \big\langle \dot{x}(\tau), \ddot{u}(\tau) \big\rangle d\tau \tag{3.31}$$

for almost all $0 < s < t < T$.

P r o o f. By (3.22)(i) we have for almost all $\tau \in]0,T[$ and $\delta \in]0,T-\tau[$

$$\big\langle \dot{x}(\tau) - \dot{u}(\tau), x(\tau) - x(\tau+\delta) \big\rangle \leqslant 0,\ \big\langle \dot{x}(\tau+\delta) - \dot{u}(\tau+\delta), x(\tau+\delta) - x(\tau) \big\rangle \leqslant 0,$$

hence

$$\frac{1}{2}\frac{d}{dt}|x(\tau+\delta) - x(\tau)|_X^2 \leqslant \big\langle x(\tau+\delta) - x(\tau), \dot{u}(\tau+\delta) - \dot{u}(\tau) \big\rangle.$$

We fix $0 < s < t < T$ such that by Proposition V.1.22 we have

$$\dot{x}(t) = \lim_{\delta \to 0} \frac{x(t+\delta) - x(t)}{\delta},\ \dot{x}(s) = \lim_{\delta \to 0} \frac{x(s+\delta) - x(s)}{\delta}.$$

Then

$$U_2(t) - U_2(s) \leqslant \limsup_{\delta \to 0+} \int_s^t \langle \frac{x(\tau+\delta) - x(\tau)}{\delta}, \frac{\dot{u}(\tau+\delta) - \dot{u}(\tau)}{\delta} \rangle d\tau. \tag{3.32}$$

To prove that inequalities (3.31), (3.32) are equivalent, we use Proposition V.1.13 for an arbitrary sequence $\delta_n \downarrow 0+$ and $p = 1$. We put

$$\begin{aligned}
v_n(\tau) &:= \frac{1}{\delta_n^2} \big\langle x(\tau+\delta_n) - x(\tau), \dot{u}(\tau+\delta_n) - \dot{u}(\tau) \big\rangle \\
v_o(\tau) &:= \big\langle \dot{x}(\tau), \ddot{u}(\tau) \big\rangle, \\
g_n(\tau) &:= |\dot{x}|_\infty \frac{1}{\delta_n} \int_\tau^{\tau+\delta_n} |\ddot{u}(\sigma)|_X d\sigma, \\
g_o(\tau) &:= |\dot{x}|_\infty |\ddot{u}(\tau)|_X.
\end{aligned}$$

We have indeed $|g_n(\tau) - g_o(\tau)| \leqslant \frac{1}{\delta_n} \int_\tau^{\tau+\delta_n} |g_o(\sigma) - g_o(\tau)| d\sigma$, hence the hypotheses of Proposition V.1.13 are satisfied as a consequence of Proposition V.1.14. Note that by (3.22)(ii) we have $|\dot{x}(\tau)|_X \leqslant |\dot{u}(\tau)|_X \leqslant$ const. for a.e. $\tau \in]0,T[$. □

I.4 Special characteristics

According to Theorem 3.7 in the previous section, the play and stop are $\frac{1}{2}$-Hölder continuous on compact sets in $C([0,T];X)$. Krasnosel'skii and Pokrovskii (1983) already pointed out that these operators have better continuity properties for special characteristics Z. In particular, they are uniformly continuous if Z is a strictly convex cylinder and Lipschitz if Z is a polyhedron. We present here elementary proofs of these results and prove that the play transforms uniformly convergent sequences in $C([0,T];X)$ into strictly convergent sequences in $C([0,T];X) \cap BV(0,T;X)$ provided ∂Z is smooth (Proposition 4.11 below). The latter statement is a generalization of a result of Visintin (1994) for $\dim X = 1$.

For the sake of simplicity, we assume throughout this section that the initial conditions (3.1)(ii) for the stop and play are chosen as in Remark 3.4.

Cylinders

In classical models of plasticity, the yield surfaces represented by the boundary ∂Z of the convex characteristic Z have a cylindrical shape in the sense of Definition 2.12. This enables us to reduce the dimension of the problem.

Proposition 4.1. *Let $Y \subset X$ be a closed subspace of X and let $Y^\perp$ be its orthogonal complement. Let $\tilde{Z} \subset Y$ be a recession set, $Z = \tilde{Z} + Y^\perp$. Let $u \in C([0,T];X), v \in C([0,T];Y)$ be given such that $u(t) - v(t) \in Y^\perp$ for all $t \in [0,T]$. Let $\mathcal{P} : C([0,T];X) \to C([0,T];X)$, $\tilde{\mathcal{P}} : C([0,T];Y) \to C([0,T];Y)$ be the play operators corresponding to $Z, \tilde{Z}$, respectively, with initial conditions from Remark 3.4. Then $\mathcal{P}(u) = \tilde{\mathcal{P}}(v)$.*

P r o o f. The assertion follows immediately from the time-discrete construction and from Remark 2.14. □

Strictly convex cylinders

The uniform continuity of the play on a strictly convex cylinder is expressed by Theorem 4.2 below (cf. also Proposition 2.18). By a *strictly convex cylinder* we mean a set $Z \subset X$ which admits a representation of the form

(4.1) $$Z = \tilde{Z} + Y^\perp, \quad \text{where} \quad \tilde{Z} \subset Y \quad \text{is strictly convex,}$$

$Y, Y^\perp$ being complementary orthogonal closed subspaces of X.

Theorem 4.2. *Let* $Z \subset X$ *be a strictly convex cylinder of the form (4.1) and let* α *be the function associated to* $\tilde{Z}$ *by formula (2.12). Then for all* $u, v \in C([0,T];X)$ *we have*

$$|\,\mathcal{P}(u) - \mathcal{P}(v)|_\infty \leqslant \alpha^{-1}\big(|u-v|_\infty\big). \tag{4.2}$$

P r o o f. It suffices to assume $Y = X$ (by Proposition 4.1) and $u, v \in W^{1,1}(0,T;X)$ (by density). Put $\xi := \mathcal{P}(u)$, $\eta := \mathcal{P}(v)$, $x := u - \xi$, $y := v - \eta$, $V(t) := \max\big\{|\xi(t) - \eta(t)|_X; \alpha^{-1}(|u-v|_\infty)\big\}$ for $t \in [0,T]$. Then V is absolutely continuous. Assume that for some $t \in]0,T[$ we have $\dot{V}(t) > 0$. Then

$$|\xi(t) - \eta(t)|_X > \alpha^{-1}(|u-v|_\infty) \tag{4.3}$$

and $\frac{d}{dt}|\xi(t)-\eta(t)|_X^2 = 2\big\langle \dot{\xi}(t) - \dot{\eta}(t), \xi(t) - \eta(t)\big\rangle > 0$.

At least one of the expressions $\big\langle \dot{\xi}(t), \xi(t) - \eta(t)\big\rangle$, $\big\langle \dot{\eta}(t), \eta(t) - \xi(t)\big\rangle$ must therefore be positive. Let us choose for instance $\big\langle \dot{\xi}(t), \xi(t) - \eta(t)\big\rangle > 0$. This implies $\dot{\xi}(t) \neq 0$, hence by Remark 3.10 we have $x(t) := u(t) - \xi(t) \in \partial Z$ and $\xi(t) - \eta(t) \in X \setminus T_Z\big(x(t)\big)$. From the definition of the function α it follows

$$\alpha(|\xi(t) - \eta(t)|_X) \leqslant |P\big(x(t) + \xi(t) - \eta(t)\big)|_X \leqslant |x(t) + \xi(t) - \eta(t) - y(t)|_X = |u(t) - v(t)|_X$$

which contradicts (4.3). We conclude $\dot{V}(t) \leqslant 0$ a.e., consequently

$$|\xi(t) - \eta(t)|_X \leqslant \max\big\{|\xi(0) - \eta(0)|_X, \alpha^{-1}(|u-v|_\infty)\big\} \tag{4.4}$$

for all $t \in [0,T]$ and the assertion follows from the choice of initial conditions. □

Example 4.3. If Z (or more precisely $\tilde{Z}$) is the ball $B_R(0)$ with radius $R > 0$ (the model of von Mises, cf. Examples 1.4 and 3.15), then $\mathcal{P}$ is globally $\frac{1}{2}$-Hölder continuous by Theorem 4.2, since the function α has the form

$$\alpha(r) = \sqrt{(R^2 + r^2)} - R \quad \text{for} \quad r \geqslant 0. \tag{4.5}$$

To verify that the exponent $\frac{1}{2}$ is optimal it suffices to consider the case $X = \mathbb{R}^2$, $u(t) := (R+h)\binom{\cos t}{\sin t}$, $v(t) := R\binom{\cos t}{\sin t}$ for some fixed $R > 0, h > 0$ and for all $t \geqslant 0$ (see Fig. 6). We obviously have $y = v$, $\eta = 0$, $|u - v|_\infty = h$, $x(0) = \binom{R}{0}$ and $|x(t)|_X \leqslant R$ for all $t \geqslant 0$. Put $A := \{t \geqslant 0; |x(t)|_X < R\}$ and assume $A \neq \emptyset$. Let $]a,b[\subset A$ be an arbitrary component of A. For $t \in A$ we have by (3.22) $\dot{x}(t) = \dot{u}(t)$, hence

$$\frac{1}{2}\frac{d^2}{dt^2}|x(t)|_X^2 = \frac{d}{dt}\big\langle \dot{u}(t), x(t)\big\rangle = |\dot{u}(t)|_X^2 + \big\langle x(t), \ddot{u}(t)\big\rangle \geqslant h(R+h) > 0.$$

It follows from this last inequality and from the hypothesis $|x(a)|_X = R$ that $\big\langle \dot u(a), x(a)\big\rangle < 0$, hence $a > 0$ and there exists $\varepsilon > 0$ such that $\big\langle \dot u(t), x(t)\big\rangle < 0$ for a.e. $t \in]a-\varepsilon, a[$. Inequality (3.22)(i) then yields

$$\frac{1}{2}\left(R^2 - |x(a-\varepsilon)|_X^2\right) = \int_{a-\varepsilon}^{a} \big\langle \dot x(t), x(t)\big\rangle dt \leqslant \int_{a-\varepsilon}^{a} \big\langle \dot u(t), x(t)\big\rangle dt < 0,$$

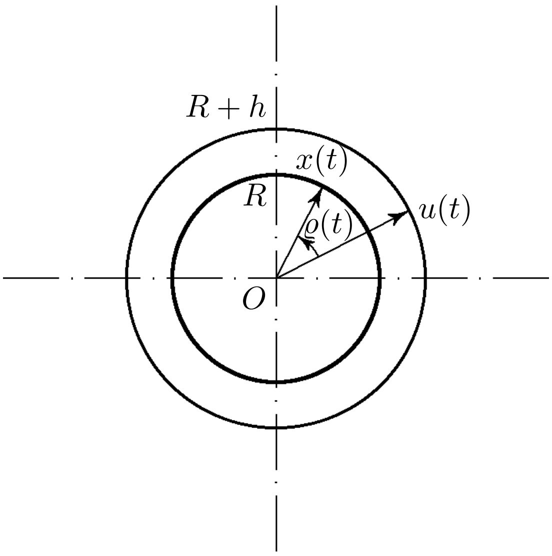
which is a contradiction. We therefore have $A = \emptyset$ and $\dot x(t)$ is the tangential component of $\dot u(t)$ at the point $x(t)$ for all $t > 0$, i.e.

$$\dot x(t) = \dot u(t) - \frac{1}{R^2}\big\langle \dot u(t), x(t)\big\rangle x(t) \quad \text{for all} \quad t > 0.$$

We easily compute x in the form $x(t) = R\binom{\cos(t+\varrho(t))}{\sin(t+\varrho(t))}$, where ϱ is the solution of the differential equation

$$\dot\varrho(t) = \frac{R+h}{R}\cos\varrho(t) - 1,\ \varrho(0) = 0.$$

Fig. 6



An explicit formula for ϱ has the form $\varrho(t) = 2\arctan\left(\sqrt{\frac{h}{2R+h}}\tanh\left(\frac{\sqrt{2Rh+h^2}}{2R}t\right)\right)$, hence $\varrho_0 := \lim_{t\to\infty}\varrho(t) = 2\arctan\sqrt{\frac{h}{2R+h}}$ and $|\xi(t)-\eta(t)|_X = |x(t)-u(t)|_X = \big(R^2 + (R+h)^2 - 2R(R+h)\cos\varrho(t)\big)^{1/2}$. The optimal estimate is obtained for $t\to\infty$ and equals $\big((R+h)^2 - R^2\big)^{1/2}$; it is therefore identical to (4.2), (4.5).

The above example provides also an illustration to Theorem 3.14. We obtain in this case $x^*(t) = R\binom{\cos(t+\varrho_0)}{\sin(t+\varrho_0)}$.

POLYHEDRONS

Definition 4.4. *Let* $n_1, \dots, n_p$ *be given unit vectors and* $\beta_1, \dots, \beta_p$ *given positive numbers,* $p \in \mathbb{N}$*. Then the convex closed set*

$$Z := \big\{ x \in X;\ \big\langle x, n_i \big\rangle \leqslant \beta_i \quad \forall i \in \{1, \dots, p\} \big\}$$

is called a polyhedron.

The linear hull of the system $\{n_1, \dots, n_p\}$ denoted by $Y := \mathrm{Lin}\{n_1, \dots, n_p\}$ is a closed subspace of X with $N := \dim Y \leqslant p$.

We introduce the quantities

$$(4.6)\qquad \begin{aligned} &\text{(i)} \quad \varepsilon := \max \Big\{ \big\langle w, n_{i_N} \big\rangle; w \in \mathrm{Lin}\{n_{i_1}, \dots, n_{i_{N-1}}\},\ |w|_X = 1, \\ &\qquad\qquad \mathrm{Lin}\{n_{i_1}, \dots, n_{i_N}\} = Y \Big\}, \\ &\text{(ii)} \quad \Psi(s) := \frac{1}{1-\varepsilon^2}(1 + s^2 + 2\varepsilon s) \quad \text{for} \quad s \geqslant 0, \\ &\text{(iii)} \quad L_1 := 1, L_{k+1} := \sqrt{\Psi(L_k)} \quad \text{for} \quad k \geqslant 1. \end{aligned}$$

This subsection will be devoted to the proof of the following Lipschitz estimate.

Theorem 4.5. *For every* $u, v \in C([0,T]; X)$ *we have*

$$|\,\mathcal{P}(u) - \mathcal{P}(v)|_\infty \leqslant L_N |u - v|_\infty .$$

It is not known whether the constant L_N is optimal in general except for the trivial case $N = 1$. The optimality for $N = 2$ follows from the next example.

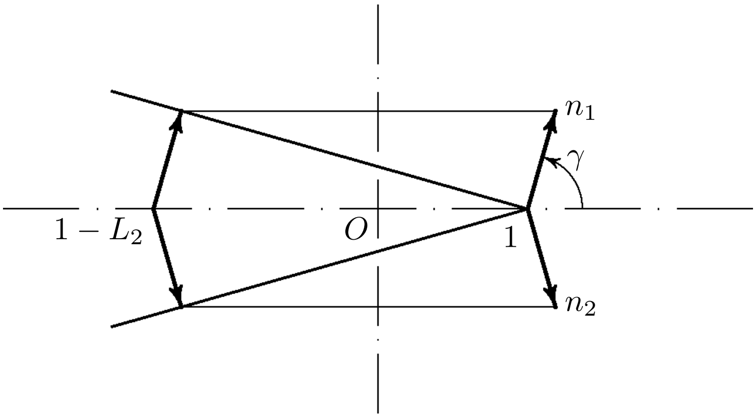
Example 4.6 (see Fig. 7). In $X = \mathbb{R}^2$ we choose $n_1 := \binom{\cos\gamma}{\sin\gamma}, n_2 := \binom{\cos\gamma}{-\sin\gamma}$, $\beta_1 = \beta_2 := \cos\gamma$ for some $\gamma \in]\frac{\pi}{4}, \frac{\pi}{2}[$. For all $t \geqslant 0$ we define $v(t) := \binom{1}{0}$ and

$$u(t) := \begin{cases} v(t) + n_1 \sin t & \text{for} \quad t \in [2k\pi, (2k+1)\pi[, \\ v(t) + n_2 \sin t & \text{for} \quad t \in [(2k+1)\pi, (2k+2)\pi[, \end{cases} \quad k = 0, 1, \dots,$$

hence $|u - v|_\infty = 1$.

With the notation of (4.6) we obtain $\varepsilon = -\cos 2\gamma$, $L_2 = \frac{1}{cos\gamma}$. In our concrete situation we have $\mathcal{P}(v)(t) = 0$ and putting $\hat{n}_1 := \binom{-\sin\gamma}{\cos\gamma}$, $\hat{n}_2 := \binom{-\sin\gamma}{-\cos\gamma}$, $\xi := \mathcal{P}(u)$, $x := u - \xi$, $\hat{Z} := \{z \in X; \big\langle z, \hat{n}_i \big\rangle \leqslant (L_2 - 1)\sin\gamma,\ i = 1, 2\}$ we prove by induction (details

Fig. 7



are left to the reader) that $x(t)$ remains in $Z \cap \hat{Z}$ for all $t \geqslant 0$. From Theorem 3.14 it follows that x is asymptotically periodic with $x^*(t) = u(t) - \binom{L_2}{0}$. The minimal upper bound for $|\mathcal{P}(u) - \mathcal{P}(v)|_\infty$ is therefore equal to $|x^* - u|_\infty = L_2$.

Before proving Theorem 4.5 we start with three auxiliary Lemmas which are due to V. Lovicar, see Picek (1991). Note that by Theorem 4.1 it suffices to assume $X = Y$.

Lemma 4.7. *Let Z be a polyhedron from Definition 4.4. For $z \in Z$ put $\Gamma(z) := \{k \in \{1, \dots, p\}; \langle z, n_k \rangle = \beta_k\}$, $C(z) := \{w \in X;\, w = \sum_{k \in \Gamma(z)} a_k n_k,\, a_k \geqslant 0\}$. Then $C(z) = N_Z(z)$, where $N_Z(z)$ is the normal cone (2.10).*

Proof. We obviously have $C(z) \subset N_Z(z)$. The set $C(z)$ is a convex closed cone and we can associate to it the projections Q_z, P_z according to formula (2.3). Let $w \in N_Z(z)$ be arbitrary. We have by definition

$$\langle P_z w, Q_z w - \varphi \rangle \geqslant 0 \qquad \forall \varphi \in C(z), \tag{4.7}$$

$$\langle w, z - \psi \rangle \geqslant 0 \qquad \forall \psi \in Z. \tag{4.8}$$

For $k \in \Gamma(z)$ we have $Q_z w + n_k \in C(z)$, and (4.7) yields $\langle P_z w, n_k \rangle \leqslant 0$. For $k \in \{1, \dots, p\} \setminus \Gamma(z)$ we have $\langle z, n_k \rangle < \beta_k$. In both cases we obtain $z + \delta P_z w \in Z$ for some sufficiently small $\delta > 0$. Putting $\psi := z + \delta P_z w$ we infer from (4.8) and Lemma 2.2(iii) $|P_z w|_X^2 \leqslant \langle P_z w, w \rangle \leqslant 0$, hence $w \in C(z)$. □

Lemma 4.8. *Let Z be as above and let $u, v \in W^{1,1}(0, T; X)$ be given. For $t \in [0, T]$ put $\xi(t) := \mathcal{P}(u)(t)$, $\eta(t) := \mathcal{P}(v)(t)$, $x(t) := u(t) - \xi(t)$, $y(t) := v(t) - \eta(t)$, $g(t) := \xi(t) - \eta(t)$, $G(t) := |g(t)|_X$. Then for every $j \in \Gamma(x(t))$ we have $\langle n_j, g(t) \rangle \leqslant |u(t) - v(t)|_X$ and for every $i \in \Gamma(y(t))$ we have $\langle n_i, g(t) \rangle \geqslant -|u(t) - v(t)|_X$.*

Proof. For $j \in \Gamma(x(t))$ we have $n_j \in N_Z(x(t))$, hence $\langle n_j, g(t) \rangle \leqslant \langle n_j, u(t) - v(t) \rangle \leqslant |u(t) - v(t)|_X$ and similarly for $i \in \Gamma(y(t))$. □

Lemma 4.9. *Assume that under the hypotheses of Lemma 4.8 the derivatives* $\dot{\xi}(t), \dot{\eta}(t)$ *exist for some* $t \in]0, T[$ *and that* $G(t) > 0$, $\dot{G}(t) > 0$. *Then there exists either* $j \in \Gamma(x(t))$ *such that* $\langle n_j, g(t)\rangle > 0$ *or* $i \in \Gamma(y(t))$ *such that* $\langle n_i, g(t)\rangle < 0$.

Proof. By hypothesis we have $\frac{d}{dt}G^2(t) = 2\langle\dot{\xi}(t) - \dot{\eta}(t), \xi(t) - \eta(t)\rangle > 0$, hence either $\langle\dot{\xi}(t), g(t)\rangle > 0$ or $\langle\dot{\eta}(t), g(t)\rangle < 0$. We have $\dot{\xi}(t) \in N_Z(x(t)), \dot{\eta}(t) \in N_Z(y(t))$ and it suffices to use Lemma 4.7. □

We now pass to the proof of Theorem 4.5.

Proof of Theorem 4.5. We may assume that $u, v \in W^{1,1}(0, T; X)$, $X = Y$. Let $r > |u - v|_\infty$ be arbitrarily chosen and let $P_{i_1,\ldots,i_k} : X \to \mathrm{Lin}\{n_{i_1}, \ldots, n_{i_k}\}$ denote the orthogonal projection of X onto $\mathrm{Lin}\{n_{i_1}, \ldots, n_{i_k}\}$. We introduce a Lyapunov function $V : X \to \mathbb{R}^1$ by the formula

$$V(x) := \max\{L_N^2 r^2,\ L_k^2 r^2 - |P_{i_1,\ldots,i_k} x|_X^2 + |x|_X^2,\ |x|_X^2\} \tag{4.9}$$

where the maximum is taken over all $k = 1, \ldots, N-1$ and over all linearly independent systems $\{n_{i_1}, \ldots, n_{i_k}\} \subset \{n_1, \ldots, n_p\}$. Let us note that each of the functions $x \mapsto L_k^2 r^2 - |P_{i_1,\ldots,i_k} x|_X^2 + |x|_X^2 = L_k^2 r^2 + |(I - P_{i_1,\ldots,i_k})x|_X^2$, where I is the identity, is convex, hence V is convex. In particular $V(g(t))$ is absolutely continuous.

It suffices to prove

$$\frac{d}{dt} V(g(t)) \leqslant 0 \qquad \text{almost everywhere.} \tag{4.10}$$

Indeed, assuming (4.10) we infer $|g(t)|_X^2 \leqslant V(g(t)) \leqslant V(g(0)) \leqslant L_N^2 r^2$ using the fact that $|g(0)|_X < r$.

It remains to verify inequality (4.10). Assume that for some $t \in]0, T[$ the derivatives $\dot{\xi}(t), \dot{\eta}(t)$ exist and $\frac{d}{dt}V(g(t)) > 0$. We necessarily have $V(g(t)) > L_N^2 r^2$.

Assume first $V(g(t)) = |g(t)|_X^2 = G^2(t)$. Lemmas 4.8, 4.9 then entail that there exists $\ell \in \Gamma(x(t)) \cup \Gamma(y(t))$ such that $|\langle n_\ell, g(t)\rangle| < r$, and the inequality $V(g(t)) \geqslant L_1 r^2 - \langle n_\ell, g(t)\rangle^2 + |g(t)|_X^2 > |g(t)|_X^2$ contradicts the hypothesis.

There exists therefore $k \in \{1, \ldots, N-1\}$ and a linearly independent system $\{n_{i_1}, \ldots, n_{i_k}\} \subset \{n_1, \ldots, n_p\}$ such that

$$V(g(t)) = L_k^2 r^2 - |P_{i_1,\ldots,i_k} g(t)|_X^2 + |g(t)|_X^2. \tag{4.11}$$

The assumption $\frac{d}{dt}V(g(t)) > 0$ yields $\langle\dot{g}(t), (I - P_{i_1,\ldots,i_k})g(t)\rangle > 0$. We can assume $\langle\dot{\xi}(t), (I - P_{i_1,\ldots,i_k})g(t)\rangle > 0$ (otherwise we interchange the roles of u and v). Lemmas 4.7, 4.8 ensure the existence of some $i_{k+1} \in \Gamma(x(t))$ such that

$$r > \langle n_{i_{k+1}}, g(t)\rangle > \langle n_{i_{k+1}}, P_{i_1,\ldots,i_k} g(t)\rangle. \tag{4.12}$$

This implies in particular that $n_{i_{k+1}} \notin \mathrm{Lin}\{n_{i_1}, \dots, n_{i_k}\}$. We find $v \in \mathrm{Lin}\{n_{i_1}, \dots, n_{i_k}\}$, $|v| = 1$ and real numbers a, b such that

$$P_{i_1,\dots,i_{k+1}} g(t) = a\, n_{i_{k+1}} + b\, v. \tag{4.13}$$

Put $\delta := \langle n_{i_{k+1}}, v \rangle \in [-\varepsilon, \varepsilon]$. By definition of the projection we have

$$|P_{i_1,\dots,i_k} g(t)|_X \geqslant |\langle g(t), v\rangle| = |a\delta + b|. \tag{4.14}$$

On the other hand, inequality (4.12) yields

$$r > a + b\delta > a|P_{i_1,\dots,i_k} n_{i_{k+1}}|_X^2 + b\delta, \tag{4.15}$$

hence $a > 0$. From (4.14), (4.15) it follows $a(1-\delta^2) < r - b\delta - a\delta^2 \leqslant r + |\delta||P_{i_1,\dots,i_k} g(t)|_X$ and

$$\begin{aligned} |P_{i_1,\dots,i_{k+1}} g(t)|_X^2 &= a^2 + b^2 + 2ab\delta = (a\delta + b)^2 + a^2(1-\delta^2) \\ &< |P_{i_1,\dots,i_k} g(t)|_X^2 + \frac{1}{1-\delta^2}\left(r + |\delta||P_{i_1,\dots,i_k} g(t)|_X\right)^2 \\ &\leqslant r^2 \Psi\Big(\frac{1}{r}|P_{i_1,\dots,i_k} g(t)|_X\Big). \end{aligned} \tag{4.16}$$

By hypothesis (4.11) we have $|P_{i_1,\dots,i_k} g(t)|_X < L_k r$, and (4.16) entails

$$\begin{aligned} L_{k+1}^2 r^2 - |P_{i_1,\dots,i_{k+1}} g(t)|_X^2 &> r^2\Big(\Psi(L_k) - \Psi\big(\frac{1}{r}|P_{i_1,\dots,i_k} g(t)|_X\big)\Big) \\ &> L_k^2 r^2 - |P_{i_1,\dots,i_k} g(t)|_X^2, \end{aligned}$$

which contradicts assumption (4.11). Consequently, (4.10) holds and Theorem 4.5 is proved. □

Smooth characteristics

We already know that the play $\mathcal{P}$ maps in general $C([0,T];X)$ into $C([0,T];X) \cap BV(0,T;X)$. This mapping is discontinuous with respect to the strong topologies of $C([0,T];X)$ and $BV(0,T;X)$ even in the simplest case $\dim X = 1$. This can easily be verified by the following construction.

Example 4.10. Put $X := \mathbb{R}^1$, $Z = [-1,1]$, $u_0(t) := 1 + t$, $u_n(t) := 1 + t + \frac{1}{n}\sin nt$ for $n \in \mathbb{N}$ and $t \in [0, 2\pi]$, $\xi_n := \mathcal{P}(u_n)$, $x_n := u_n - \xi_n$ for $n \in \mathbb{N} \cup \{0\}$. The functions u_n are nondecreasing, $x_n(0) = 1$. Proposition 3.9 yields $x_n(t) = 1$ for all $n \in \mathbb{N} \cup \{0\}$

and $t \in [0, 2\pi]$, hence $\xi_0(t) = t$, $\xi_n(t) = t + \frac{1}{n}\sin nt$ for $n \in \mathbb{N}$, and we easily check that $\lim_{n\to\infty} |u_n - u_0|_\infty = 0$, $\operatorname*{Var}_{[0,2\pi]}(\xi_n - \xi_0) = 4$.

In some cases it is possible to prove the continuity of the play with respect to the topology of $C([0,T]; X) \cap BV(0,T;X)$ induced by the *strict metric* $d_s(\xi, \eta) := |\xi - \eta|_\infty + |\operatorname*{Var}_{[0,T]} \xi - \operatorname*{Var}_{[0,T]} \eta|$, see Section III.2 of Visintin (1994) for the case $\dim X = 1$. We prove this result without restriction on X only for bounded characteristics with a smooth boundary. The problem whether Proposition 4.11 below holds for an arbitrary convex closed characteristic Z seems to be open.

Proposition 4.11. *Let $Z \subset X$ be a bounded convex closed set such that $0 \in \mathrm{Int}Z$, for every $x \in \partial Z$ there exists a unique outward normal $n(x)$ and the mapping $n : \partial Z \to \partial B_1(0)$ is continuous. Then for every sequence $\{u_i;\, i \in \mathbb{N} \cup \{0\}\} \subset C([0,T]; X)$ such that $\lim_{i\to\infty} |u_i - u_0|_\infty = 0$ we have $\lim_{i\to\infty} \operatorname*{Var}_{[0,T]} \mathcal{P}(u_i) = \operatorname*{Var}_{[0,T]} \mathcal{P}(u_0)$.*

The reader can check in a straightforward way that the mapping $n : \partial Z \to B_1(0)$ is automatically continuous provided $\dim X < \infty$. For $\dim X = \infty$ this need not be true. In the example $Z := \left\{x = \sum_{k=1}^\infty x_k e_k \in X;\, x_1 \geqslant \left[\sum_{k=2}^\infty x_k^2 + \frac{2}{k^2} - \frac{2}{k}\sqrt{x_k^2 + \frac{1}{k^2}}\right]^{1/2}\right\}$, where $\{e_k\}$ is an orthogonal basis in X, the points $x_k^\gamma := \frac{1}{k}\left[(\sqrt{1+\gamma^2} - 1)e_1 + \gamma e_k\right]$ belong to ∂Z for every $\gamma \neq 0$ and $k \in \mathbb{N}$, $x_k^\gamma \to 0$ as $k \to \infty$. The normal cone $N_Z(0)$ obviously contains $-e_1$, since for every $x \in Z$ we have $\langle -e_1, 0 - x\rangle = x_1 \geqslant 0$. On the other hand, each vector $\nu \in N_Z(0)$ must satisfy $\langle \nu, -x_k^\gamma \rangle \geqslant 0\ \forall k \in \mathbb{N}, \forall \gamma \neq 0$, hence

$$\left|\langle \nu, e_k \rangle\right| \leqslant \frac{\sqrt{1+\gamma^2} - 1}{|\gamma|}\langle \nu, -e_1 \rangle$$

and for $\gamma \to 0$ we obtain $n(0) = -e_1$. Since $n(x_k^\gamma) = \frac{1}{\sqrt{1+2\gamma^2}}(-\sqrt{1+\gamma^2}e_1 + \gamma e_k)$, letting $k \to \infty$ for a fixed $\gamma \neq 0$ we conclude that n is discontinuous at $x = 0$.

Proposition 4.11 is an easy consequence of the following Lemma.

Lemma 4.12. *Let the assumptions of Proposition 4.11 be satisfied. Let $\nu : Z \to B_1(0)$ be defined by the formula $\nu(0) := 0$, $\nu(x) := M_Z(x)\, n\Big(\frac{x}{M_Z(x)}\Big)$ for $x \in Z \setminus \{0\}$, where M_Z is the Minkowski functional associated to Z by formula (2.9). Then for every $u \in C([0,T]; X)$ we have*

$$\operatorname*{Var}_{[0,T]} \xi = \int_0^T \langle \nu(x(t)), d\xi(t) \rangle, \tag{4.17}$$

where $\xi = \mathcal{P}(u)$, $x = u - \xi$.

P r o o f of Lemma 4.12. Let us first assume $u \in W^{1,1}(0,T;X)$. Then $\dot{\xi}(t) = 0$ if $x(t) \in \operatorname{Int} Z$, $\dot{\xi}(t) = |\dot{\xi}(t)|_X \, n(x(t))$ if $x(t) \in \partial Z$, hence $|\dot{\xi}(t)|_X = \big\langle \nu(x(t)), \dot{\xi}(t) \big\rangle$ a.e. and (4.17) holds.

Let now $u \in C([0,T];X)$ be arbitrary and let $\{u_i;\, i \in \mathbb{N}\} \subset W^{1,1}(0,T;X)$ be a sequence such that $\lim\limits_{i\to\infty} |u_i - u|_\infty = 0$, and put $\xi_i := \mathcal{P}(u_i), x_i := u_i - \xi_i$. Let $0 = t_0 < t_1 < \ldots < t_N = T$ be an arbitrary partition of $[0,T]$. The mapping ν is continuous. By Lemma 3.6, Corollary 3.8 and Theorem V.1.26 we therefore have $\operatorname*{Var}\limits_{[0,T]} \xi_i \leqslant \text{const.}$, $\lim\limits_{i\to\infty} \operatorname*{Var}\limits_{[0,T]} \xi_i = \int_0^T \big\langle \nu(x(t)), d\xi(t) \big\rangle$ and

$$\sum_{j=1}^{N} |\xi(t_j) - \xi(t_{j-1})|_X = \lim_{i\to\infty} \sum_{j=1}^{N} |\xi_i(t_j) - \xi_i(t_{j-1})|_X \leqslant \int_0^T \big\langle \nu(x(t)), d\xi(t) \big\rangle \leqslant \operatorname*{Var}_{[0,T]} \xi,$$

hence (4.17) holds. □

P r o o f of Proposition 4.11. It suffices to apply formula (4.17), Lemma 3.6, Corollary 3.8 and Theorem V.1.26. □

Remark 4.13. Formula (4.17) generalizes the energy identity (3.30) in Example 3.15, where we have $\nu(x) = \frac{1}{r} x$.

II. Scalar models for hysteresis

In applications to plasticity, scalar hysteresis effects cannot be described simply by putting $N = 1$ in the definition of the space $\mathbb{T}$ of symmetric $N \times N$ tensors in Chapter I. The hypothesis I(1.11) of volume invariance would exclude any plasticity effects, since for $N = 1$ we have $\mathbb{T} = \mathbb{T}_{\text{diag}} = \mathbb{R}^1$. One has to proceed in the following way.

Let us consider for instance the elastoplastic models $\mathcal{E} - \mathcal{R}$, $\mathcal{E} \,|\, \mathcal{R}$ as in Example I.1.8, with characteristics Z of von Mises type, namely

$$Z = \big(B_r(0) \cap \mathbb{T}_{\text{dev}}\big) + \mathbb{T}_{\text{diag}}, \tag{0.1}$$

where $B_r(0)$ is the ball in $\mathbb{T}$ centered at 0 with radius $r > 0$. We further assume that the elastic element is isotropic with a matrix A given by formula I(1.6). Then the inverse matrix A^{-1} has the form

$$A^{-1} = \frac{1}{2\mu} I - \frac{\lambda}{2\mu(3\lambda + 2\mu)} J. \tag{0.2}$$

The constitutive law is governed by the variational inequalities I(1.23). Let us assume that the input is uniaxial. For the model $\mathcal{E} \,|\, \mathcal{R}$ this means

$$\sigma(t) := a(t)\sigma_0, \tag{0.3}$$

where $a : [0, T] \to \mathbb{R}^1$ is a scalar-valued function and $\sigma_0 \in \partial B_1(0) \cap \mathbb{T}_{\text{dev}}$ is a fixed unit vector. Putting $\tilde{\sigma} := \big\langle \sigma^p(t), \sigma_0 \big\rangle \sigma_0$ in the first inequality of I(1.23) we obtain

$$\Big\langle \frac{1}{2\mu} \dot{a}\sigma_0 - A^{-1}\dot{\sigma}^p, \sigma^p - \big\langle \sigma^p, \sigma_0 \big\rangle \sigma_0 \Big\rangle \geqslant 0 \qquad \text{a.e.,} \tag{0.4}$$

hence the projection $\bar{\sigma}^p(t) := \sigma^p(t) - \big\langle \sigma^p(t), \sigma_0 \big\rangle \sigma_0$ of $\sigma^p(t)$ satisfies

$$\frac{1}{2} \frac{d}{dt} \big\langle A^{-1} \bar{\sigma}^p(t), \bar{\sigma}^p(t) \big\rangle \leqslant 0 \qquad \text{a.e.} \tag{0.5}$$

A further hypothesis that the initial output value $\sigma^p(0)$ is proportional to σ_0 then entails $\bar{\sigma}^p(t) \equiv 0$. The output σ^p is therefore uniaxial of the form $\sigma^p(t) = b(t)\sigma_0$, where $b(t) \in [-r, r]$ is the solution of the scalar variational inequality

$$\big(\dot{b}(t) - \dot{a}(t)\big)\big(b(t) - \varphi\big) \leqslant 0 \qquad \text{a.e.} \qquad \forall \varphi \in [-r, r]. \tag{0.6}$$

A similar conclusion is obtained for the model $\mathcal{E} - \mathcal{R}$ when assuming that the input ε has a form analogous to (0.3).

This chapter is devoted to the study of mathematical properties of scalar models of hysteresis which are related to the variational inequality (0.6). The main feature of these models is a particular structure of memory which has important mathematical consequences and, last but not least, represents in itself an important tool in engineering computations related to fatigue and damage in elastoplastic materials.

II.1 Scalar play and stop

The scalar play and stop are the main building blocks for a large class of hysteresis models, such as Prandtl-Ishlinskii and Preisach models, Della Torre's "moving" model and various models for fatigue and damage that will be introduced later in this chapter. This section is devoted to the investigation of analytical properties specific for the scalar case. We obtain additional regularity results related to the total variation of both the output and its derivative and we prove a conjecture of V. Tchernorutskii saying that the play minimizes the total variation among all uniform approximations of a given continuous function.

The most interesting feature of hysteresis models based on the play and stop is their memory structure which will be described in Sect. II.2.

Lipschitz continuity

We restrict ourselves to plays and stops with symmetric characteristics $Z_r = [-r,r]$, $r > 0$.

In Sect. I.3 we already proved that the system

$$
\text{(1.1)} \qquad
\begin{aligned}
&\text{(i)} && |x_r(t)| \leqslant r \quad \forall t \in [0,T],\\
&\text{(ii)} && \big(\dot u(t) - \dot x_r(t)\big)\big(x_r(t) - \varphi\big) \geqslant 0 \quad \text{a.e.} \quad \forall \varphi \in [-r,r],\\
&\text{(iii)} && x_r(0) = x_r^0
\end{aligned}
$$

for a given input function $u \in W^{1,1}(0,T)$ and a given initial condition $x_r^0 \in [-r,r]$ has a unique solution $x_r \in W^{1,1}(0,T)$. The stop and play operators $\mathcal{S}_r, \mathcal{P}_r : [-r,r] \times W^{1,1}(0,T) \to W^{1,1}(0,T)$ are then defined as solution operators of problem (1.1) by the formula

$$
\text{(1.2)} \qquad \underset{r}{\mathcal{S}}(x_r^0, u) := x_r, \quad \underset{r}{\mathcal{P}}(x_r^0, u) := u - x_r.
$$

Theorems I.3.12 and I.4.5 entail that the operators $\mathcal{S}_r(x_r^0,\cdot), \mathcal{P}_r(x_r^0,\cdot)$ are continuous in $W^{1,p}(0,T)$ for $p \in [1,\infty[$ and admit a Lipschitz continuous extension to $C([0,T])$. In fact, we can prove more, namely

Proposition 1.1. *For $x_r^0, y_r^0 \in [-r,r]$ and $u, v \in W^{1,1}(0,T)$ put $x_r := \mathcal{S}_r(x_r^0,u)$, $y_r := \mathcal{S}_r(y_r^0, v)$, $\xi_r := u - x_r$, $\eta_r := v - u_r$. Then we have*

$$
\begin{aligned}
&\text{(i)} \quad \int_0^T |\dot\xi_r(t) - \dot\eta_r(t)|dt \leqslant |x_r^0 - y_r^0| + \int_0^T |\dot u(t) - \dot v(t)|dt,\\
&\text{(ii)} \qquad |\xi_r - \eta_r|_\infty \ \leqslant\ \max\{|\xi_r(0) - \eta_r(0)|, |u - v|_\infty\}.
\end{aligned}
$$

Before proving Proposition 1.1 we mention an auxiliary identity due to Brokate (1989).

Lemma 1.2. *Under the hypotheses of Proposition 1.1 we have for almost all* $t \in]0,T[$

$$(1.3) \qquad |\dot\xi_r(t) - \dot\eta_r(t)| + \frac{d}{dt}|x_r(t) - y_r(t)| = \big(\dot u(t) - \dot v(t)\big) \operatorname{sign}\big(\dot\xi_r(t) - \dot\eta_r(t)\big).$$

P r o o f. Put $A_+ := \{t \in]0,T[;\ x_r(t) > y_r(t)\}$, $A_- := \{t \in]0,T[;\ x_r(t) < y_r(t)\}$, $A_0 := \{t \in]0,T[;\ x_r(t) = y_r(t)\}$. The inequalities $\dot\xi_r(t)\big(x_r(t) - \varphi\big) \geqslant 0$, $\dot\eta_r(t)\big(y_r(t) - \varphi\big) \geqslant 0$ a.e. for all $\varphi \in [-r,r]$ entail

$$(1.4) \qquad \big(\dot\xi_r(t) - \dot\eta_r(t)\big)\big(x_r(t) - y_r(t)\big) \geqslant 0 \quad \text{a.e.}$$

For a.e. $t \in A_+$ we therefore have $\dot\xi_r(t) - \dot\eta_r(t) \geqslant 0$ and (1.3) follows. The same argument works in A_-. For a.e. $t \in A_0$ we have $\dot x_r(t) = \dot y_r(t)$, hence $\dot\xi_r(t) - \dot\eta_r(t) = \dot u(t) - \dot v(t)$ and we conclude that (1.3) holds. □

P r o o f of Proposition 1.1. Inequality (i) follows immediately from Lemma 1.2. The proof of (ii) is an elementary one-dimensional version of the proof of Theorems I.4.2 or I.4.5 with a Lyapunov function $V(t) := \max\{|\xi_r(t) - \eta_r(t)|^2, |u - v|_\infty^2\}$. We leave the details to the reader. □

Remark 1.3. For every $0 \leqslant s < t \leqslant T$ we have with the same notation as above

$$(1.5) \qquad |\xi_r(t) - \xi_r(s)| \leqslant \max\{|u(\tau) - u(s)|;\ \tau \in [s,t]\}.$$

This follows from Proposition 1.1(ii), where we put

$$v(\tau) := \begin{cases} u(\tau) & \text{for} \quad \tau \in [0,s] \\ u(s) & \text{for} \quad \tau \in [s,t] \end{cases}, \quad T := t.$$

It is particularly simple to solve Problem (1.1) if the input is monotone in an interval $[t_1, t_2] \subset [0,T]$. We then have

$$(1.6) \quad x_r(t) = \begin{cases} \min\{r,\ x_r(t_1) + u(t) - u(t_1)\} & \text{for } t \in]t_1,t_2] \text{ if } u \text{ is nondecreasing,} \\ \max\{-r,\ x_r(t_1) + u(t) - u(t_1)\} & \text{for } t \in]t_1,t_2] \text{ if } u \text{ is nonincreasing.} \end{cases}$$

Identity (1.6) is obvious if u is absolutely continuous in $[t_1,t_2]$; the general case follows from the density of $W^{1,1}(t_1,t_2)$ in $C([t_1,t_2])$.

Note that formula (1.6) is sometimes used as an alternative definition of the stop (Krasnosel'skii, Pokrovskii (1983)) for piecewise monotone functions.

Output variation

In Sections I.3 and I.4 we already pointed out regularization properties of the play operator. The assumptions of Proposition I.4.11 are trivially fulfilled here, hence the play operator maps $C([0,T])$ continuously into $BV(0,T)$ endowed with the strict metric. Furthermore, formula I(4.17) is of independent interest here and reads for $u \in C([0,T])$ (cf. also Example I.3.15)

$$\operatorname*{Var}_{[0,T]} \xi_r = \frac{1}{r}\int_0^T x_r(t)d\xi_r(t) \tag{1.7}$$

or, in pointwise form for $u \in W^{1,1}(0,T)$

$$|\dot{\xi}_r(t)| = \frac{1}{r}x_r(t)\dot{\xi}_r(t) \quad \text{a.e.,} \tag{1.8}$$

where $\xi_r = \mathcal{P}_r(x_r^0, u)$, $x_r = \mathcal{S}_r(x_r^0, u)$.

This result can be improved in the following way.

Proposition 1.4. *Let $u \in C([0,T])$, $r > 0$ and $x_r^0 \in [-r,r]$ be given. Let μ_u be the continuity modulus of u defined by formula V(1.19). Put $\xi_r := \mathcal{P}_r(x_r^0, u)$, $\delta_r := \inf\{\delta > 0; \mu_u(\delta) \geqslant 2r\}$. Then there exists an integer $N \leqslant \frac{T}{\delta_r} + 1$ and a partition $0 = t_N < t_{N-1} < \ldots < t_0 \leqslant T$ such that ξ_r is monotone in $[t_i, t_{i-1}]$ for $i = 1, \ldots, N$ and constant in $[t_0, T]$, $|x_r(t_i)| = r$ for $i = 0, \ldots, N-1$ and*

$$\operatorname*{Var}_{[0,T]} \xi_r = \sum_{i=1}^{N} |u(t_{i-1}) - u(t_i)| - 2Nr + a_0, \quad \textit{where} \tag{1.9}$$

$$a_0 := \begin{cases} r + x_r^0 & \textit{if} \quad x_r(t_{N-1}) = r, \\ r - x_r^0 & \textit{if} \quad x_r(t_{N-1}) = -r. \end{cases}$$

P r o o f. Put $A_\pm := \{t \in [0,T]; x_r(t) = \pm r\}$ and $t_0 := \max\{0, \sup A_\pm\}$ with the convention $\sup \emptyset = -\infty$. If $t_0 = 0$, then we put $N := 0$. For $t_0 > 0$ assume for instance $t_0 \in A_-$ and put recursively $t_{2k-1} := \max\{0, \sup(A_+ \cap [0, t_{2k-2}])\}$, $t_{2k} := \max\{0, \sup(A_- \cap [0, t_{2k-1}])\}$ for $k = 1, 2, \ldots$ until $t_N = 0$.

We first prove that ξ_r is monotone in each interval $[t_i, t_{i-1}]$ and constant in $[t_0, T]$. Choosing i odd for instance, say $i = 2k-1$, we obtain $]t_i, t_{i-1}[\cap A_+ = \emptyset$, hence $x_r(\tau) \in [-r, r[$ for all $\tau \in]t_i, t_{i-1}[$.

Let $[s,t] \subset]t_i, t_{i-1}[$ be an arbitrary subinterval and put $\varrho := \min\{r - x_r(\tau); \tau \in [s,t]\} > 0$. From Exercise I.3.2 we obtain

$$\int_s^t (x_r(\tau) - \psi(\tau))d\xi_r(\tau) \geqslant 0 \quad \forall \psi \in C([s,t]),\ |\psi|_\infty \leqslant r.$$

For $\psi(\tau) := x_r(\tau)+\varrho$ the last inequality yields $\xi_r(t) \leqslant \xi_r(s)$, hence ξ_r is nonincreasing. We similarly prove that ξ_r is nondecreasing in $[t_{2k}, t_{2k-1}]$ and constant in $[t_0, T]$, $0 \geqslant \xi_r(t_{2k-2}) - \xi_r(t_{2k-1}) = u(t_{2k-2}) - u(t_{2k-1}) + 2r$, $0 \leqslant \xi_r(t_{2k-1}) - \xi_r(t_{2k}) = u(t_{2k-1}) - u(t_{2k}) - 2r$. By definition of δ_r we have $t_{i-1} - t_i \geqslant \delta_r$ for $i = 1, \ldots, N-1$. This yields $T \geqslant t_0 - t_{N-1} \geqslant (N-1)\delta_r$, hence $N \leqslant \frac{T}{\delta_r} + 1$.

We therefore have

$$\operatorname*{Var}_{[t_{N-1},t_0]} \xi_r = \sum_{i=1}^{N-1} |u(t_{i-1}) - u(t_i)| - 2(N-1)r$$

and formula (1.9) follows easily. □

By definition of the play we always have

$$|\xi_r - u|_\infty \leqslant r \tag{1.10}$$

for $u \in C([0,T])$ and $\xi_r := \mathcal{P}_r(x_r^0, u)$. Proposition 1.3 says that the play operators define a uniform approximation of u by piecewise monotone functions as $r \to 0+$. Tchernorutskii (1993) pointed out during the Trento Hysteresis Meeting that this approximation minimizes the total variation in the following sense.

Corollary 1.5. *Let $u \in C([0,T])$, $r > 0$ and $x_r^0 \in [-r, r]$ be given and let $\eta \in BV(0,T)$ be a function such that $\eta(0) = u(0) - x_r^0$, $|\eta - u|_\infty \leqslant r$. Then*

$$\operatorname*{Var}_{[0,T]} \eta \geqslant \operatorname*{Var}_{[0,T]} \mathcal{P}_r(x_r^0, u).$$

P r o o f. Let $0 = t_N < t_{N-1}, < \ldots < t_0 \leqslant T$ be the partition defined in Proposition 1.4. Then

$$\operatorname*{Var}_{[0,T]} \eta \geqslant \sum_{i=1}^{N} |\eta(t_{i-1}) - \eta(t_i)| \geqslant \sum_{i=1}^{N-1} |u(t_{i-1}) - u(t_i)| - 2(N-1)r + |\eta(t_{N-1}) - \eta(0)|.$$

We now have either $x_r(t_{N-1}) = r$ and $\eta(t_{N-1}) - \eta(0) \geqslant u(t_{N-1}) - r - u(0) + x_r^0 = |\xi_r(t_{N-1}) - \xi_r(t_N)|$, or $x_r(t_{N-1}) = -r$ and $\eta(0) - \eta(t_{N-1}) \geqslant u(0) - x_r^0 - u(t_{N-1}) - r = |\xi_r(t_{N-1}) - \xi_r(t_N)|$ and identity (1.9) completes the proof. □

In Sect.I.3 we proved that the general Hilbert-space-valued play maps $W^{1,p}$ into $W^{1,p}$ for $1 \leqslant p \leqslant +\infty$ and is continuous in $W^{1,p}$ only if $1 \leqslant p < +\infty$ (Theorem I.3.12). In the scalar case the play preserves more regularity: if the derivative of the input has bounded variation, then the same holds for the output.

Proposition 1.6. *Let* $u \in W^{1,\infty}(0,T)$ *be given such that there exists* $v \in BV(0,T)$, $\dot{u} = v$ *a.e. For* $r > 0$ *and* $x_r^0 \in [-r,r]$ *put* $\xi_r := \mathcal{P}_r(x_r^0, u)$. *Then there exists* $w \in BV(0,T)$ *such that* $\dot{\xi}_r = w$ *a.e. and*

$$\operatorname*{Var}_{[0,T]} w \leqslant |v(0+)| + \operatorname*{Var}_{[0,T]} v. \tag{1.11}$$

For the proof we need two auxiliary lemmas.

Lemma 1.7. *Let* $v : [0,T] \to \mathbb{R}^1$ *be a given function. Assume that there exists a closed set* $C \subset [0,T]$ *such that* $v(t) = 0$ *for* $t \in C$. *Let* Ω *be the open set* $\Omega :=]0,T[\backslash C = \bigcup_{k=1}^{\infty}]a_k, b_k[$ *with* $]a_k, b_k[$ *pairwise disjoint. Then the following two conditions are equivalent.*

(i) $v \in BV(0,T)$,

(ii) $v\big|_{[a_k,b_k]} \in BV(a_k,b_k) \quad \forall k \in \mathbb{N}$ *and* $\displaystyle\sum_{k=1}^{\infty} \operatorname*{Var}_{[a_k,b_k]} v < \infty$.

If moreover one of the conditions (i), (ii) is satisfied, then

$$\operatorname*{Var}_{[0,T]} v = \sum_{k=1}^{\infty} \operatorname*{Var}_{[a_k,b_k]} v \tag{1.12}$$

P r o o f. The implication (i) $\Rightarrow$ (ii) is obvious. Indeed, for each $m \in \mathbb{N}$ we have $\sum_{k=1}^{m} \operatorname*{Var}_{[a_k,b_k]} v \leqslant \operatorname*{Var}_{[0,T]} v$, hence also

$$\sum_{k=1}^{\infty} \operatorname*{Var}_{[a_k,b_k]} v \leqslant \operatorname*{Var}_{[0,T]} v. \tag{1.13}$$

Let us assume now that (ii) holds and let $0 = t_0 < t_1 < \ldots < t_N = T$ be an arbitrary partition of $[0,T]$. Put $M := \big\{j \in \{0,\ldots,N\};\, t_j \notin C\big\}$. For every $j \in M \setminus \{0,N\}$ there exists an interval $]a_{k_j}, b_{k_j}[\ni t_j$; in the case $0 \in M$ put $a_{k_0} := 0$, $b_{k_0} := \min C$ and similarly $a_{k_N} := \max C$, $b_{k_N} := T$ if $N \in M$. To ensure that each interval $]a_{k_j}, b_{k_j}[$ is counted exactly once we choose a set $M' \subset M$ such that $\bigcup_{j \in M}]a_{k_j}, b_{k_j}[= \bigcup_{j \in M'}]a_{k_j}, b_{k_j}[$, $a_{k_j} \neq a_{k_i}$ for $i,j \in M'$, $i \neq j$.

We now construct the partition $0 = s_0 < s_1 < \ldots < s_K = T$ by putting $\{s_0,\ldots,s_K\} := \{t_j;\, j = 0,\ldots,N\} \cup \{a_{k_j}, b_{k_j};\, j \in M'\}$. If for some $0 < i_1 < i_2 < K$

and $j \in M'$ we have $s_{i_1} = b_{k_j}$, $s_{i_2} = a_{k_{j+1}}$, then necessarily $v(s_i) = 0$ for all $i_1 \leqslant i \leqslant i_2$. We thus obtain

$$\sum_{j=1}^{N} |v(t_j) - v(t_{j-1})| \leqslant \sum_{i=1}^{K} |v(s_i) - v(s_{i-1})| \leqslant \sum_{j \in M'} \operatorname*{Var}_{[a_{k_j}, b_{k_j}]} v \tag{1.14}$$

for every partition t_j and the converse of (1.13) follows. Lemma 1.7 is proved. $\square$

Lemma 1.8. *Let $u \in W^{1,\infty}(0,T)$ be given such that there exists $v \in BV(0,T), \dot{u} = v$ a.e. Then the right derivative $\dot{u}_+(t)$ exists and is equal to $v(t+)$ for all $t \in [0,T[$ and the left derivative $\dot{u}_-(t)$ exists and is equal to $v(t-)$ for all $t \in]0,T]$.*

P r o o f. It suffices to pass to the limit as $h \to 0+$ in the formulas

$$\frac{1}{h}\big(u(t+h) - u(t)\big) = \frac{1}{h}\int_0^h \big(v(t+\eta) - v(t+)\big)d\eta + v(t+),$$
$$\frac{1}{h}\big(u(t) - u(t-h)\big) = \frac{1}{h}\int_0^h \big(v(t-\eta) - v(t-)\big)d\eta + v(t-).$$

$\square$

P r o o f of Proposition 1.6. Put $C := \big\{t \in [0,T]\ ;\ 0 \in \operatorname{Conv}\{v(t-), v(t+)\}\big\}$. Then C is closed and choosing a representative of v with minimal total variation we can assume that

$$v(t) = \begin{cases} 0 & \text{for } t \in C, \\ \frac{1}{2}\big(v(t+) + v(t-)\big) & \text{for } t \in]0,T[\setminus C, \\ v(0+) & \text{for } t = 0, \\ v(T-) & \text{for } t = T. \end{cases}$$

In each component $]a_k, b_k[$ of the set $\Omega :=]0,T[\setminus C = \bigcup_{k=1}^{\infty}]a_k, b_k[$ the function v does not change sign. Consequently, u is strictly monotone in each interval $[a_k, b_k]$ and the value of ξ_r can be determined from formula (1.6). For $t \in [a_k, b_k]$ we have

$$\xi_r(t) = \begin{cases} \max\{\xi_r(a_k), u(t) - r\} & \text{if } u \text{ increases,} \\ \min\{\xi_r(a_k), u(t) + r\} & \text{if } u \text{ decreases} \end{cases}$$

For each $k \in \mathbb{N}$ there exists $\tau_k \in [a_k, b_k]$ such that

$$\dot{\xi}_r(t) = \begin{cases} 0 & \text{for a.e. } \ t \in]a_k, \tau_k[, \\ \dot{u}(t) & \text{for a.e. } \ t \in]\tau_k, b_k[. \end{cases}$$

Put $w(t) := v(t)$ for $t \in \bigcup_{k\in\mathbb{N}}]\tau_k, b_k[, w(t) := 0$ otherwise. Notice that for almost all $t \in C$ we have $v(t+) = v(t-) = 0$ and Lemma 1.8 entails $\dot{u}(t) = 0$; inequality (1.5) then yields $\dot{\xi}_r(t) = 0$ for a.e. $t \in C$, hence $\dot{\xi}_r(t) = w(t)$ a.e.

In each interval $[a_k, b_k]$ we have

$$\operatorname*{Var}_{[a_k,b_k]} w = |v(\tau_k)| + \operatorname*{Var}_{[\tau_k,b_k]} v \leqslant |v(a_k)| + \operatorname*{Var}_{[a_k,b_k]} v$$

with $v(a_k) = 0$ whenever $a_k > 0$.

Consequently,

$$\sum_{k=1}^{\infty} \operatorname*{Var}_{[a_k,b_k]} w \leqslant |v(0+)| + \sum_{k=1}^{\infty} \operatorname*{Var}_{[a_k,b_k]} v$$

and it suffices to use Lemma 1.7. □

II.2 Memory of the play-stop system

The concept of memory in connection to hysteresis operators is related to the fact that the instantaneous output value may depend not only on the instantaneous input value and the initial condition, but also on other input values in the history of the process. For the scalar play-stop system it is possible to characterize explicitly the memory in the form of memory sequences associated to each input and each initial configuration. Below, we give a precise meaning to these concepts and we prove a formula (Proposition 2.5) which enables us to compute the output value from the memory sequence without solving variational inequalities. The knowledge of the memory structure of the play-stop system will have important consequences for Preisach-type operators in Sect. II.3.

Already Madelung (1905) formulated axiomatic rules for the behavior of scalar hysteretic systems (we refer the reader to the monograph Brokate, Sprekels (to appear), where the connection between Madelung's rules and Preisach-type hysteresis operators is explained in detail). It has been discovered only recently (Krasnosel'skii, Pokrovskii (1983), Krejčí (1989), (1991/a), Brokate (1990)) that the play - stop system provides a unified approach to Preisach-type models, scalar Prandtl - Ishlinskii models and Madelung's rules.

We start with a superposition formula due to M. Brokate (Brokate, Sprekels (to appear)) which has no counterpart in the vector case.

Lemma 2.1. *Let* $u \in C([0,T])$ *and* $r, s \in]0,\infty[$ *be given. For* $x_r^0 \in [-r,r], y_s^0 \in [-s,s]$ *put* $\xi_r := \mathcal{P}_r(x_r^0, u)$, $\eta_s := \mathcal{P}_s(y_s^0, \xi_r)$, $\eta_{r+s} := \mathcal{P}_{r+s}(x_r^0 + y_s^0, u)$. *Then* $\eta_s = \eta_{r+s}$.

Proof. It suffices to assume $u \in W^{1,1}(0,T)$. By definition we have for almost every $t \in]0,T[$ and every $\varphi \in [-1,1]$

$$(2.1) \qquad \begin{aligned} &\text{(i)} \quad \dot{\eta}_{r+s}(t)\big(u(t) - \eta_{r+s}(t) - (r+s)\varphi\big) \geqslant 0, \\ &\text{(ii)} \quad \dot{\xi}_r(t)\big(u(t) - \xi_r(t) - r\varphi\big) \geqslant 0, \\ &\text{(iii)} \quad \dot{\eta}_s(t)\big(\xi_r(t) - \eta_s(t) - s\varphi\big) \geqslant 0. \end{aligned}$$

The normality rule I(3.22)(ii) here reads $\dot{\eta}_s(\dot{\xi}_r - \dot{\eta}_s) = 0$ a.e., and (2.1)(ii) entails

$$(2.2) \qquad \dot{\eta}_s(t)\big(u(t) - \xi_r(t) - r\varphi\big) \geqslant 0 \quad \text{a.e.} \quad \forall \varphi \in [-1,1].$$

The sum of (2.1)(iii) and (2.2) yields

$$(2.3) \qquad \dot{\eta}_s(t)\big(u(t) - \eta_s(t) - (r+s)\varphi\big) \geqslant 0 \quad \text{a.e.} \quad \forall \varphi \in [-1,1].$$

We obviously have $|u - \eta_s|_\infty \leqslant |u - \xi_r|_\infty + |\xi_r - \eta_s|_\infty \leqslant r + s$ like in (1.10), hence putting $\varphi := \frac{1}{2(r+s)}\big(2u(t) - \eta_s(t) - \eta_{r+s}(t)\big)$ in (2.1)(i) and (2.3) we obtain

$$\big(\dot{\eta}_{r+s}(t) - \dot{\eta}_s(t)\big)\big(\eta_{r+s}(t) - \eta_s(t)\big) \leqslant 0 \quad \text{a.e.}$$

The choice of initial conditions ensures that $\eta_{r+s}(0) = \eta_s(0) = u(0) - x_r^0 - y_s^0$, hence $\eta_s \equiv \eta_{r+s}$. □

As an immediate consequence of Lemma 2.1 we have

Corollary 2.2. *For every* $r, s \in]0,\infty[$, $x_r^0 \in [-r,r]$, $x_{r+s}^0 \in [x_r^0 - s, x_r^0 + s]$ *and* $u \in C([0,T])$ *we have*

$$|\xi_{r+s} - \xi_r|_\infty \leqslant s,$$

where $\xi_r := \mathcal{P}_r(x_r^0, u)$, $\xi_{r+s} := \mathcal{P}_{r+s}(x_{r+s}^0, u)$.

It is clear by definition (more precisely, by existence and uniqueness of solutions of the variational problem (1.1)) that the evolution of the output for $t > t_0$ is uniquely determined by the input values for $t \geqslant t_0$ and the initial output value for $t = t_0$. The curve $\lambda^{t_0} :]0,\infty[\to \mathbb{R}^1 : r \longmapsto \mathcal{P}_r(x_r^0, u)(t_0)$ thus expresses the instantaneous memory created during the interval $[0, t_0]$. Its structure is described below in Proposition 2.5.

Assuming that the initial configuration $\{x_r^0;\ r>0\}$ is chosen in such a way that $|x_{r+s}^0 - x_r^0| \leqslant s$ for all $r>0, s>0$, we infer from Corollary 2.2 that for every $u \in C([0,T])$, $r,s \in]0,\infty[$ and $t \in [0,T]$ we have $|\lambda^t(r+s) - \lambda^t(r)| \leqslant s$.

It is therefore natural to define the *configuration space*

$$\Lambda := \{\lambda \in W^{1,\infty}(0,\infty)\,;\ \left|\frac{d\lambda(r)}{dr}\right| \leqslant 1 \quad \text{a.e.}\} \tag{2.4}$$

of *memory configurations* λ, and its subspaces

$$\Lambda_R := \{\lambda \in \Lambda\,;\ \lambda(r) = 0 \ \text{ for } \ r \geqslant R\}, \quad \Lambda_0 := \bigcup_{R>0} \Lambda_R. \tag{2.5}$$

We now introduce a more convenient notation. For $\lambda \in \Lambda$, $u \in C([0,T])$ and $r>0$ put

$$p_r(\lambda, u) := \underset{r}{\mathcal{P}}(x_r^0, u), \tag{2.6}$$

where x_r^0 is given by the formula

$$x_r^0 := Q_r\big(u(0) - \lambda(r)\big) \tag{2.7}$$

and $Q_r : \mathbb{R}^1 \to [-r,r]$ is the projection

$$Q_r(x) := \operatorname{sign}(x) \min\{r, |x|\}. \tag{2.8}$$

For the sake of consistency put $p_0(\lambda,u) := u$. We immediately see that $p_r(-\lambda,-u) = -p_r(\lambda,u)$ for all r,λ and u. Moreover, the operator $p_r : \Lambda \times C([0,T]) \to C([0,T])$ is Lipschitz in the following sense.

Lemma 2.3. *For every $u,v \in C([0,T])$, $\lambda,\mu \in \Lambda$ and $r>0$ we have*

$$|p_r(\lambda,u) - p_r(\mu,v)|_\infty \leqslant \max\big\{|\lambda(r) - \mu(r)|, |u-v|_\infty\big\}. \tag{2.9}$$

P r o o f. Put $\xi_r := p_r(\lambda,u)$, $\eta_r := p_r(\mu,v)$. Proposition 1.1(ii) yields

$$|\xi_r - \eta_r|_\infty \leqslant \max\big\{|\xi_r(0) - \eta_r(0)|, |u-v|_\infty\big\}, \tag{2.10}$$

where $\xi_r(0) = u(0) - Q_r\big(u(0) - \lambda(r)\big)$, $\eta_r(0) = v(0) - Q_r\big(v(0) - \mu(r)\big)$. Assume for instance $u(0) - \lambda(r) \geqslant v(0) - \mu(r)$. The function Q_r is nondecreasing and $Q_r'(z) \leqslant 1$ for a.e. $z \in \mathbb{R}^1$, hence $\xi_r(0) - \eta_r(0) \leqslant u(0) - v(0)$, $\eta_r(0) - \xi_r(0) \leqslant \mu(r) - \lambda(r)$ and (2.9) follows from (2.10). □

To an arbitrary $\lambda \in \Lambda_0$ we associate a function $m_\lambda : \mathbb{R}^1 \to [0, +\infty[$ by the formula

$$m_\lambda(v) := \min\{r \geqslant 0\,;\, |v - \lambda(r)| = r\}. \tag{2.11}$$

The function $r \mapsto r - |v - \lambda(r)|$ is nondecreasing. This immediately implies that $|v - \lambda(r)| > r$ for $r \in [0, m_\lambda(v)[$, $|v - \lambda(r)| \leqslant r$ for $r \in [m_\lambda(v), +\infty[$; the function m_λ is increasing and left-continuous in $[\lambda(0), +\infty[$ and decreasing and right-continuous in $]-\infty, \lambda(0)]$, $m_\lambda\big(\lambda(0)\big) = 0$. For $\lambda \in \Lambda_0$, $u \in C([0,T])$ and $t \in [0,T]$ we define the quantity

$$M(\lambda, u, t) := \max\big\{m_\lambda\big(u(\tau)\big)\,;\, \tau \in [0,t]\big\}. \tag{2.12}$$

It is clear that the function $t \mapsto M(\lambda, u, t)$ is nondecreasing and left-continuous in $[0,T]$ for fixed λ and u. The following lemma is substantial for the memory description of stops and plays.

Lemma 2.4. *Assume $M(\lambda, u, t) = m_\lambda(u(t))$ for some $\lambda \in \Lambda_0, u \in C([0,T])$ and $t \in [0,T]$. Then*

$$p_r(\lambda, u)(t) = \begin{cases} \lambda(r) & \text{for} \quad r \geqslant M(\lambda, u, t), \\ u(t) + r & \text{for} \quad r < M(\lambda, u, t) \quad \text{if} \quad u(t) < \lambda(0), \\ u(t) - r & \text{for} \quad r < M(\lambda, u, t) \quad \text{if} \quad u(t) > \lambda(0). \end{cases} \tag{2.13}$$

P r o o f. We first prove Lemma 2.4 for $u \in W^{1,1}(0,T)$. Put $\xi_r := p_r(\lambda, u)$ and $\bar{r} := M(\lambda, u, t)$. By (2.6)-(2.8) we have $\xi_r(0) = \lambda(r)$ for $r \geqslant \bar{r}$. Assume that $\xi_r(t) \neq \lambda(r)$ for some $r \geqslant \bar{r}$. Then there exists $\tau \in]0, t[$ such that

$$\begin{aligned} \text{either} \quad \text{(i)} \quad & \dot{\xi}_r(\tau) > 0,\ \xi_r(\tau) > \lambda(r) \\ \text{or} \quad \text{(ii)} \quad & \dot{\xi}_r(\tau) < 0,\ \xi_r(\tau) < \lambda(r). \end{aligned}$$

The variational inequality (1.1) implies $\xi_r(\tau) = u(\tau) - r$ in the case (i) and $\xi_r(\tau) = u(\tau) + r$ in the case (ii), hence $r < |u(\tau) - \lambda(r)|$, which contradicts the definition of $\bar{r}$. We therefore have $\xi_r(t) = \lambda(r)$ for all $r \geqslant \bar{r}$, $\bar{r} = |u(t) - \xi_{\bar{r}}(t)|$.

The case $\bar{r} = 0$ is trivial. For $\bar{r} > 0$ we have $u(t) = \bar{r} + \lambda(\bar{r})$ if $u(t) > \lambda(0)$, $u(t) = -\bar{r} + \lambda(\bar{r})$ if $u(t) < \lambda(0)$, and for $r \in]0, \bar{r}[$ Corollary 2.2 entails

$$\bar{r} - r \geqslant |\xi_{\bar{r}}(t) - \xi_r(t)| \geqslant |\xi_{\bar{r}}(t) - u(t)| - |\xi_r(t) - u(t)| \geqslant \bar{r} - r,$$

hence

$$\xi_r(t) = \frac{r}{\bar{r}}\lambda(\bar{r}) + \frac{\bar{r} - r}{\bar{r}}u(t) = u(t) - \frac{r}{\bar{r}}\big(u(t) - \lambda(\bar{r})\big)$$

and the assertion follows.

Let now $u \in C([0,T])$ be arbitrary and assume $M(\lambda, u, t) = m_\lambda(u(t))$ for some t. We find a sequence $\{u_n\} \subset W^{1,1}(0,T)$ such that $|u_n - u|_\infty \to 0$ as $n \to \infty$, $\min\limits_{[0,t]} u_n = \min\limits_{[0,t]} u$, $\max\limits_{[0,t]} u_n = \max\limits_{[0,t]} u$, $u_n(t) = u(t)$. Then $M(\lambda, u_n, t) = m_\lambda(u_n(t)) = m_\lambda(u(t))$ and it suffices to pass to the limit in formula (2.13) for u_n as $n \to \infty$. $\square$

The general situation $m_\lambda(u(t)) \leqslant M(\lambda, u, t)$ will be treated in the following way. For $u \in C([0,T])$, $\lambda \in \Lambda_0$ and $t \in [0,T]$ put

$$\begin{cases} \bar{r} := M(\lambda, u, t), \\ \bar{t} := \max\{\tau \in [0,t]\,;\, m_\lambda(u(\tau)) = \bar{r}\}, \end{cases} \tag{2.14}$$

$$\begin{cases} t_0 := \bar{t},\ r_0 := \bar{r} & \text{if} \quad u(\bar{t}) = \lambda(\bar{r}) - \bar{r}, \\ t_1 := \bar{t},\ r_1 := \bar{r} & \text{if} \quad u(\bar{t}) = \lambda(\bar{r}) + \bar{r}, \end{cases} \tag{2.15}$$

and continue recursively by putting

$$\begin{cases} t_{2k+1} := \max\big\{\tau \in [t_{2k}, t]; u(\tau) = \max\{u(\sigma); \sigma \in [t_{2k}, t]\}\big\}, k = (0), 1, 2, \dots, \\ t_{2k} := \max\big\{\tau \in [t_{2k-1}, t]; u(\tau) = \min\{u(\sigma); \sigma \in [t_{2k-1}, t]\}\big\}, k = 1, 2, \dots, \\ r_{j+1} := \frac{(-1)^j}{2}\big(u(t_{j+1}) - u(t_j)\big), j = (0), 1, 2, \dots \end{cases} \tag{2.16}$$

until $t_{2k+1} = t$ or $t_{2k} = t$.

One of the following two possibilities occurs.

A. The sequence $\{(t_j, r_j)\}$ is infinite, $u(t) = \lim\limits_{j\to\infty} u(t_j)$, $\lim\limits_{j\to\infty} r_j = 0$;

B. The sequence $\{(t_j, r_j)\}$ is finite, $t = t_n$. In this case we put $r_j := 0$ for $j \geqslant n+1$.

In the sequel, the sequence $\{(t_j, r_j)\}$ is called *memory sequence of* u *at the point* t *with respect to the initial configuration* λ and denoted by $MS_\lambda(u)(t)$.

Proposition 2.5. *Let* $u \in C([0,T])$, $\lambda \in \Lambda_0$, $r > 0$ *and* $t \in [0,T]$ *be given, and let* $MS_\lambda(u)(t) = \{(t_j, r_j)\}$ *be the memory sequence (2.15),(2.16). Then we have*

$$p_r(\lambda, u)(t) = \begin{cases} \lambda(r) & \textit{for} \quad r \geqslant \bar{r}, \\ u(t_j) + (-1)^j r & \textit{for} \quad r \in [r_{j+1}, r_j[,\ j = (0), 1, 2, \dots \end{cases} \tag{2.17}$$

Let us make a remark before proving Proposition 2.5. Formula (2.17) shows that the increasing sequence $\{u(t_{2k})\}$ of local minima and decreasing sequence $\{u(t_{2k+1})\}$ of local maxima is precisely what the system $\{p_r(\lambda, u)(t);\, r > 0\}$ keeps in memory. The

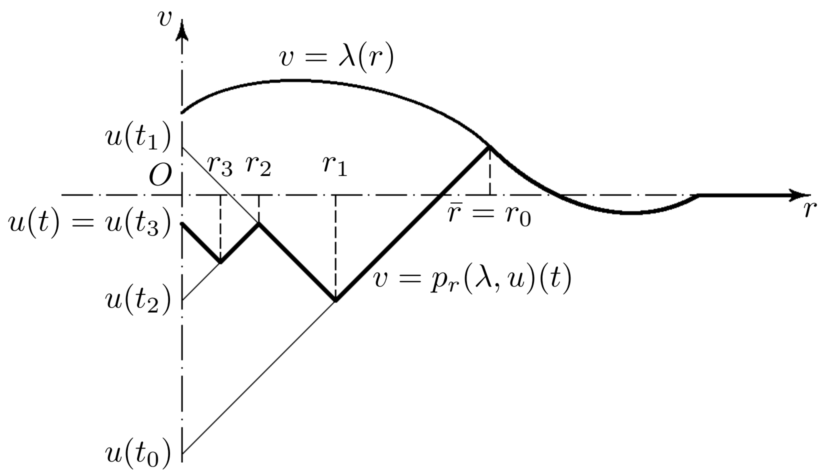


Fig. 8

instantaneous output values are determined by these sequences and the rest of the input history for $\tau \in [0,t] \setminus \{t_j\}$ is irrelevant (see Fig. 8). A similar memory structure can be observed in the vector case for the model of Mróz (see Brokate, Dressler, Krejčí (to appear/a)), but not for vector plays and stops in general.

P r o o f of Proposition 2.5. Assume for instance $\bar{r} = r_1, \bar{t} = t_1$ (the other case is analogous). For $r > 0$ put $\lambda_1(r) := p_r(\lambda, u)(t_1)$. We have $m_\lambda\big(u(t_1)\big) = M(\lambda, u, t) = M(\lambda, u, t_1) = r_1$ and Lemma 2.4 yields

$$\lambda_1(r) = \begin{cases} \lambda(r) & \text{for} \quad r \geqslant r_1, \\ u(t_1) - r & \text{for} \quad r < r_1. \end{cases}$$

We are done if $t = t_1$; otherwise, for $j = 2, 3, \ldots$ put $\lambda_j(r) := p_r(\lambda, u)(t_j)$ and assume

$$\lambda_j(r) = \begin{cases} \lambda(r) & \text{for} \quad r \geqslant r_1, \\ u(t_i) + (-1)^i r & \text{for} \quad r \in [r_{i+1}, r_i[, i = 1, \ldots, j-1, \\ u(t_j) + (-1)^j r & \text{for} \quad r \in [0, r_j[\end{cases} \tag{2.18}$$

for some $j \geqslant 1, t_j < t$. We now prove that (2.18) holds for $j+1$.

Put $u_j(\tau) := u(\tau + t_j)$ for $\tau \in [0, T - t_j]$. By (2.16) we have $0 \leqslant (-1)^j\big(u_j(\tau) - u_j(0)\big) \leqslant 2r_{j+1}$ for $\tau \in [0, t - t_j]$, hence for $r \in [r_{j+1}, r_j[$ it follows from (2.18)

$$|u_j(\tau) - \lambda_j(r)| = |r - (-1)^j\big(u_j(\tau) - u_j(0)\big)| \leqslant r.$$

For $r \in]0, r_{j+1}[$ we similarly obtain $|u_j(t_{j+1} - t_j) - \lambda_j(r)| > r$, hence $M(\lambda_j, u_j, t - t_j) = m_{\lambda_j}\big(u_j(t_{j+1} - t_j)\big) = r_{j+1}$. Lemma 2.4 and the semigroup property I(1.27) then entail

$$\lambda_{j+1}(r) = p_r(\lambda_j, u_j)(t_{j+1} - t_j) = \begin{cases} \lambda_j(r) & \text{for} \quad r \geqslant r_{j+1}, \\ u(t_{j+1}) + (-1)^{j+1} r & \text{for} \quad r \in]0, r_{j+1}[, \end{cases}$$

and the induction argument completes the proof. □

Corollary 2.6. *Let* $\lambda \in \Lambda_R$ *and* $u \in C([0,T])$ *be given. For* $t \in [0,T]$ *and* $r > 0$ *put* $\lambda^t(r) := p_r(\lambda, u)(t)$*. Then for every* $t \in [0,T]$ *we have*

(i) $\lambda^t \in \Lambda_{\hat{R}}$, *where* $\hat{R} = \max\{R, |u|_\infty\}$,
(ii) $\lambda^t(r) = \lambda(r)$ *for* $r \geqslant M(\lambda, u, t)$,
(iii) $\lambda^t(0) = u(t)$,
(iv) $\left|\frac{\partial}{\partial r}\lambda^t(r)\right| = 1$ *for a.e.* $r \in]0, M(\lambda, u, t)[$.

P r o o f. Statements (ii)-(iv) follow immediately from Proposition 2.5. To prove (i) we just note that for $r > \max\{R, |u|_\infty\}$ we have $|u(t) - \lambda(r)| = |u(t)| < r$, hence $M(\lambda, u, t) \leqslant \max\{R, |u|_\infty\}$ for all $t \in [0,T]$. □

In Sect. I.3 (Theorem I.3.14) we proved that the vector play and stop are asymptotically periodic on periodic inputs. The scalar case is again much simpler. Let us denote by C_ω for a given period $\omega > 0$ the space of continuous functions $u : \mathbb{R}^1 \to \mathbb{R}^1$ such that $u(t+\omega) = u(t)$ for all $t \in \mathbb{R}^1$. We immediately see that the function $t \mapsto m_\lambda\big(u(t)\big)$ is ω-periodic in $\mathbb{R}^1$ and $M(\lambda, u, \cdot)$ defined by (2.12) is constant in $[\omega, +\infty[$. We state explicitly the following Corollary of Proposition 2.5.

Corollary 2.7. *Let* $u \in C_\omega$ *and* $\lambda \in \Lambda_0$ *be given. Then for every* $r > 0$ *and* $t \geqslant \omega$ *we have* $p_r(\lambda, u)(t + \omega) = p_r(\lambda, u)(t)$.

Differentiability

Despite the regularity of the play, it is clear that the derivative $\dot{\xi}_r(t)$ of the output $\xi_r = p_r(\lambda, u)$ of the play with input $u \in C([0,T])$ and initial configuration $\lambda \in \Lambda_0$ at a given point $t \in]0, T[$ need not exist even if $\dot{u}(t)$ exists. Indeed, this is not the case if $\dot{u}(t) = 0$, since formula (1.5) then implies $\dot{\xi}_r(t) = 0$. On the other hand, if $\dot{u}(t)$ exists, then the right and left derivatives $\dot{\xi}_r^+(t), \dot{\xi}_r^-(t)$ always exist and can be computed from the memory formula (2.17) in the following way.

Proposition 2.8. *Let* $\lambda \in \Lambda_0$, $u \in C([0,T])$ *and* $t \in]0, T[$ *be given such that* $\dot{u}(t) \neq 0$ *exists. Then there exist* $\varrho_1(t) \geqslant \varrho_0(t) > 0$ *such that*

$$\dot{\xi}_r^-(t) = \begin{cases} 0 & \text{for } r \geqslant \varrho_0(t), \\ \dot{u}(t) & \text{for } r < \varrho_0(t), \end{cases} \qquad \dot{\xi}_r^+(t) = \begin{cases} 0 & \text{for } r > \varrho_1(t), \\ \dot{u}(t) & \text{for } r \leqslant \varrho_1(t). \end{cases} \tag{2.19}$$

P r o o f. It suffices to assume $\dot{u}(t) > 0$; otherwise we use the fact that the operator p_r is odd and pass from (λ, u) to $(-\lambda, -u)$.

Assume first $M(\lambda,u,t) = m_\lambda\big(u(t)\big) =: \bar r$. The assumption $\dot u(t) > 0$ entails $u(t) = \lambda(\bar r) + \bar r$. If the set $A := \{\tau \in [0,t[;\ u(\tau) = u(t)\}$ is nonempty, then we define $t_1 := \max A$, $t_2 := \max\big\{\tau \in [t_1,t];\ u(\tau) = \min\{u(\sigma);\ \sigma \in [t_1,t]\}\big\}$ similarly as in (2.16). We put

$$(2.20)\quad \varrho_0(t) := \begin{cases} \bar r & \text{if } A = \emptyset \\ \frac{1}{2}\big(u(t) - u(t_2)\big) & \text{if } A \neq \emptyset \end{cases}, \qquad \varrho_1(t) := \max\{r > 0;\ u(t) = \lambda(r) + r\}$$

and claim that (2.19) holds.

We have to consider separately several cases.

A. $r > \varrho_1(t)$

There exists $\delta > 0$ sufficiently small such that $|u(\tau) - \lambda(r)| < r$ for all $\tau \in [t, t+\delta[$, hence $M(\lambda,u,\tau) < r$ and Lemma 2.4 yields $\xi_r(\tau) = \lambda(r)$ for $\tau \in [t,t+\delta[$, in particular $\dot\xi_r^+(t) = 0$.

B. $0 < r \leqslant \varrho_1(t)$

We choose $\delta > 0$ such that $0 < u(\tau) - u(t) < 2r$ for all $\tau \in]t, t+\delta[$. We have by hypothesis $u(t) = \lambda\big(\varrho_1(t)\big) + \varrho_1(t)$, hence $u(\tau) > \lambda\big(\varrho_1(t)\big) + \varrho_1(t)$ and the memory sequence $MS_\lambda(u)(\tau) = \big\{(\hat t_j, \hat r_j)\big\}_{j\geqslant 1}$ satisfies $\hat r_1 = M(\lambda,u,\tau) > \varrho_1(t)$, $\hat t_1 \in]t,\tau]$, $\hat r_j < r$ for all $j \geqslant 2$. By Proposition 2.6 we have $\xi_r(\tau) = u(\hat t_1) - r$, $\xi_r(t) = u(t) - r$, hence $\frac{\xi_r(\tau)-\xi_r(t)}{\tau - t} = \frac{u(t_1)-u(t)}{\tau-t}$. Letting δ tend to 0 and using obvious inequalities $\frac{u(t_1)-u(t)}{t_1-t} \geqslant \frac{u(t_1)-u(t)}{\tau-t} \geqslant \frac{u(\tau)-u(t)}{\tau-t}$ we obtain $\dot\xi_r^+(t) = \dot u(t)$.

C. $\varrho_0(t) = \bar r,\ r \geqslant \varrho_0(t)$.

We have $m_\lambda\big(u(\tau)\big) \leqslant \bar r$ for all $\tau \in [0,t]$. Proposition 2.5 implies $\xi_r(\tau) = \lambda(r)$ for $\tau \leqslant t$, hence $\dot\xi_r^-(t) = 0$.

D. $\varrho_0(t) < \bar r, r \geqslant \varrho_0(t)$.

We define auxiliary functions $\lambda_2(\varrho) := \xi_\varrho(t_2)$ for $\varrho > 0, u_2(\tau) := u(t_2+\tau)$ for $\tau \in [0, t-t_2]$. For $\varrho \in [\varrho_0(t), \bar r]$ Proposition 2.5 entails $\lambda_2(\varrho) = u(t_1) - \varrho = u(t) - \varrho$, hence $|u_2(\tau) - \lambda_2\big(\varrho_0(t)\big)| = |\varrho_0(t) - \big(u(t) - u(\tau+t_2)\big)| \leqslant \varrho_0(t)$ for all $\tau \in [0, t-t_2]$. This yields $m_{\lambda_2}\big(u_2(\tau)\big) \leqslant \varrho_0(t)$ and using once more Proposition 2.5 and the semigroup property we obtain $\xi_r(\tau + t_2) = p_r(\lambda_2, u_2)(\tau) = \lambda_2(r)$ with the same conclusion as in C.

E. $r < \varrho_0(t), A = \emptyset, \big|u(\tau) - \lambda\big(\varrho_0(t)\big)\big| < \varrho_0(t)$ for all $\tau \in [0,t[$.

We choose $\tau_1 \in [0,t]$ such that $u(\tau) > \lambda(r)+r$ for all $\tau \in [\tau_1, t]$. Put $\hat r := M(\lambda,u,\tau_1) \in]r, \varrho_0(t)[, \tau_2 := \max\{\tau \in [0,t]; u(\tau) \leqslant \lambda(\hat r) + \hat r\} \in [\tau_1, t[$. We now fix $\tau_3 \in [\tau_2, t[$ such that $0 < u(t) - u(\tau) < 2r$ for all $\tau \in]\tau_3, t[$. For such τ we therefore have $MS_\lambda(u)(\tau) = \{(\hat t_j, \hat r_j)\}_{j\geqslant 1}$ with $\hat t_1 \in]\tau_2, \tau[$, $\hat r_1 \in]\hat r, \varrho_0[$, $\hat r_j < r$ for $j \geqslant 2$. From Proposition 2.5 it follows $\xi_r(\tau) = u(\hat t_1) - r$, $\xi_r(t) = u(t) - r$ and we argue as in B for $\tau_1 \to t$ to obtain $\dot\xi_r^-(t) = \dot u(t)$.

F. $r < \varrho_0(t), A = \emptyset, \exists t_0 < t \ : \ u(t_0) = \lambda(\varrho_0(t)) - \varrho_0(t)$.

Putting $\lambda_0(\varrho) := \xi_\varrho(t_0)$ for $\varrho > 0$ and $u_0(\tau) := u(\tau + t_0)$ for $\tau \in [0, t - t_0]$ we apply the argument of E to u_0 using the semigroup property similarly as in D.

G. $r < \varrho_0(t), A \neq \emptyset$.

We argue as in F for $u_2(\tau) := u(\tau + t_2)$ and $\lambda_2(\varrho) := \xi_\varrho(t_2)$.

To complete the proof, it remains to consider the case $m_\lambda\big(u(t)\big) < M(\lambda, u, t)$. The assumption $\dot{u}(t) > 0$ entails $t = t_{2k+1}$ for some $k \geqslant 0$, where $\{(t_j, r_j)\} = MS_\lambda(u)(t)$. The above argument applied to $u_{2k}(\tau) := u(\tau + t_{2k})$, $\lambda_{2k}(\varrho) := \xi_\varrho(t_{2k})$ gives the assertion for

$$
(2.21) \qquad \varrho_0(t) := \begin{cases} r_{2k+1} & \text{if} \quad A_{2k} = \emptyset, \\ \frac{1}{2}\big(u(t) - u(\hat{t}_{2k+2})\big) & \text{if} \quad A_{2k} \neq \emptyset \end{cases}, \qquad \varrho_1(t) := r_{2k+1},
$$

where we denote $A_{2k} := \{\tau \in [t_{2k}, t[; \ u(\tau) = u(t)\}$, $\hat{t}_{2k+1} := \max A_{2k}$ and $\hat{t}_{2k+2} := \max\big\{\tau \in [\hat{t}_{2k+1}, t]; \ u(\tau) = \min\{u(\sigma); \sigma \in [\hat{t}_{2k+1}, t]\}\big\}$. □

Corollary 2.9. *Let* $u \in W^{1,1}(0,T)$ *and* $\lambda \in \Lambda_0$ *be given and let* $L \subset]0,T[$ *be the set of Lebesgue points of* $\dot{u}$. *Put* $L^* := \{t \in L; \dot{u}(t) \neq 0\}$ *and for* $r > 0$ *denote* $L^*_r := \{t \in L^*; r \in [\varrho_0(t), \varrho_1(t)]\}$, *where* ϱ_0, ϱ_1 *are as in Proposition 2.8. Then we have*

(i) $\operatorname{meas} L^*_r = 0$,

(ii) $\operatorname{meas}\{t \in L^*; \varrho_0(t) < \varrho_1(t)\} = 0$.

P r o o f. Put $\xi_r(t) := p_r(\lambda, u)(t)$ for $t \in [0,T]$ and $r > 0$. According to Proposition 2.8, the set L^*_r has an empty intersection with the set of Lebesgue points of ξ_r and $\xi_r \in W^{1,1}(0,T)$ by Proposition 1.1, hence $\operatorname{meas} L^*_r = 0$. To prove (ii) we denote $\Omega := \{t \in L^*; \ \varrho_0(t) < \varrho_1(t)\}$, $\Omega_n := \{t \in L^*; \ \varrho_1(t) - \varrho_0(t) \geqslant \frac{1}{n}\}$ for $n \in \mathbb{N}$. We have $\Omega_n \subset \bigcup_{k=1}^{\infty} L^*_{\frac{k}{n}}$ and $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$, hence $\operatorname{meas} \Omega = 0$. □

Monotonicity

The trivial inequality (1.4) will have important consequences in the next chapter, where it enables us to use monotonicity techniques for solving partial differential equations with hysteretic constitutive operators. We prove here a less trivial complement to inequality (1.4) which shows that this inequality is strict in a certain sense.

Proposition 2.10. *Let* $\lambda, \mu \in \Lambda_0$ *and* $u, v \in W^{1,1}(0,T)$ *be given such that* $\lambda(0) = u(0)$, $\mu(0) = v(0)$. *For* $r > 0$ *put* $\xi_r := p_r(\lambda, u)$, $\eta_r := p_r(\mu, v)$, $x_r := u - \xi_r$, $y_r := v - \eta_r$. *Then the following three conditions are equivalent.*

(i) *For every* $r > 0$ *we have* $\big(\dot\xi_r(t) - \dot\eta_r(t)\big)\big(x_r(t) - y_r(t)\big) = 0$ *a.e.*
(ii) *For every* $r > 0,\ t \in [0,T]$ *and* $\delta \in [0,1]$ *we have* $p_r\big(\delta\lambda+(1-\delta)\mu, \delta u+(1-\delta)v\big)(t) = \delta\xi_r(t) + (1-\delta)\eta_r(t)$.
(iii) *For every* $t \in [0,T]$ *we have*

$$\xi_r(t) - \eta_r(t) = \begin{cases} \lambda(r) - \mu(r) & \text{for} \quad r \geqslant R(t), \\ \lambda\big(R(t)\big) - \mu\big(R(t)\big) & \text{for} \quad r \in]0, R(t)[, \end{cases}$$

where $R(t) := \max\{M(\lambda,u,t), M(\mu,v,t)\}$.

Remark 2.11. Assertion (iii) of Proposition 2.10 for $r \to 0+$ says that the difference $u(t)-v(t)$ depends only on the value of the nondecreasing function $R(t)$. If in particular both u and v are ω-periodic, then $u(t)-v(t)$ is constant for $t \geqslant \omega$.

The assumption $\lambda(0) = u(0),\ \mu(0) = v(0)$ is not restrictive. If λ, μ, u, v are arbitrarily given, then putting $\lambda_0(r) := p_r(\lambda,u)(0),\ \mu_0(r) := p_r(\mu,v)(0)$ we have $p_r(\lambda,u)(t) = p_r(\lambda_0,u)(t),\ p_r(\mu,v)(t) = p_r(\mu_0,v)(t)$ for all $r > 0$ and $t \in [0,T]$, so we may replace λ, μ with λ_0, μ_0.

P r o o f of Proposition 2.10.
(**i**) $\Rightarrow$ (**ii**):
Using the inequalities $\dot\xi_r(t)(x_r(t)-\varphi) \geqslant 0,\ \dot\eta_r(t)\big(\eta_r(t)-\varphi\big) \geqslant 0$ a.e. for all $\varphi \in [-r,r]$ we infer from (i) for every $r > 0$

$$\dot\xi_r(t)\big(x_r(t) - \eta_r(t)\big) = \dot\eta_r(t)\big(\eta_r(t) - x_r(t)\big) = 0 \qquad \text{a.e.},$$

hence

$$\dot\xi_r(t)\big(\eta_r(t) - \varphi\big) \geqslant 0,\ \dot\eta_r(t)\big(x_r(t) - \varphi\big) \geqslant 0 \quad \text{a.e.} \quad \forall\varphi \in [-r,r].$$

For every $\delta \in [0,1], r > 0$ and $\varphi \in [-r,r]$ we thus have

$$\dot\xi_r(t)\big(\delta x_r(t) + (1-\delta)y_r(t) - \varphi\big) \geqslant 0, \quad \dot\eta_r(t)\big(\delta x_r(t) + (1-\delta)y_r(t) - \varphi\big) \geqslant 0 \qquad \text{a.e.},$$

and in particular

$$\big(\delta\dot\xi_r(t) + (1-\delta)\dot\eta_r(t)\big)\big(\delta x_r(t) + (1-\delta)y_r(t) - \varphi\big) \geqslant 0 \quad \text{a.e.}$$

From (1.1), (1.2), (2.6), (2.7) we directly obtain (ii).

(**ii**) $\Rightarrow$ (**iii**).
For $r \geqslant R(t)$ Corollary 2.6 yields $\xi_r(t) = \lambda(r), \eta_r(t) = \mu(r)$, hence (iii) holds.

Let now $r < R(t)$ be arbitrarily chosen and let us suppose that $\frac{\partial}{\partial r}\xi_r(t)$, $\frac{\partial}{\partial r}\eta_r(t)$ exist and $\frac{\partial}{\partial r}\big(\xi_r(t) - \eta_r(t)\big) \neq 0$. Then for $\delta \in]0,1[$ we obtain from (ii)

$$\Big|\frac{\partial}{\partial r} p_r\big(\delta\lambda + (1-\delta)\mu, \delta u + (1-\delta)v\big)(t)\Big| < 1.$$

Corollary 2.6 (iv) then entails $r > M\big(\delta\lambda + (1-\delta)\mu, \delta u + (1-\delta)v, t\big)$, hence $|\delta u(\tau) + (1-\delta)v(\tau) - \delta\lambda(r) - (1-\delta)\mu(r)| \leqslant r$ for all $\tau \in [0,t]$.

By hypothesis $r < R(t)$ there exists $\tau \in [0,t]$ such that either $|u(\tau) - \lambda(r)| > r$ or $|v(\tau) - \mu(r)| > r$. In the latter case we have for instance

$$\begin{aligned} r &\geqslant |\delta u(\tau) + (1-\delta)v(\tau) - \delta\lambda(r) - (1-\delta)\mu(r)| \\ &\geqslant |v(\tau) - \mu(r)| - \delta|u(\tau) - v(\tau) - \lambda(r) + \mu(r)| \end{aligned}$$

which is a contradiction for δ sufficiently small. We therefore have $\frac{\partial}{\partial r}\xi_r(t) = \frac{\partial}{\partial r}\eta_r(t)$ for a.e. $r \in]0, R(t)[$ and (iii) follows.

(iii) $\Rightarrow$ **(i)**.

Let $r > 0$ be arbitrarily chosen. The function $t \mapsto R(t)$ is nondecreasing in $[0,T]$. Put $A_r := \{t \in [0,T];\ R(t) \geqslant r\}$ and $t_r := \inf A_r$ if $A_r \neq \emptyset$, $t_r = T$ if $A_r = \emptyset$. Then for $t \in [0, t_r[$ we have by (iii) $\xi_r(t) - \eta_r(t) = u(t) - v(t)$, hence $x_r(t) = y_r(t)$, for $t \in]t_r, T[$ we have $\xi_r(t) - \eta_r(t) = \lambda(r) - \eta(r)$, hence $\dot{\xi}_r(t) = \dot{\eta}_r(t)$. In both cases (i) is fulfilled.

Proposition 2.10 is proved. □

Another useful inequality which belongs to this subsection is due to Hilpert (1989) and reads as follows.

Proposition 2.12. *For $\lambda, \mu \in \Lambda_0$, $u, v \in W^{1,1}(0,T)$ and $r > 0$ put $\xi_r := p_r(\lambda, u)$, $\eta_r := p_r(\mu, v)$. Then we have*

$$\big(\dot{\xi}_r(t) - \dot{\eta}_r(t)\big)\,\mathrm{sign}\,\big(u(t) - v(t)\big) \geqslant \frac{\partial}{\partial t}|\xi_r(t) - \eta_r(t)| \quad \text{a.e.} \tag{2.22}$$

P r o o f. Inequality (1.4) has the form $(\dot{\xi}_r - \dot{\eta}_r)(u - v - \xi_r + \eta_r) \geqslant 0$ a.e. The sign function is nondecreasing, therefore $(\dot{\xi}_r - \dot{\eta}_r)\,\mathrm{sign}(u - v) \geqslant (\dot{\xi}_r - \dot{\eta}_r)\,\mathrm{sign}(\xi_r - \eta_r)$, which is nothing but (2.22). □

II.3 Multiyield scalar hysteresis models

We apply here the results of the two previous sections to the analysis of more complex hysteresis operators, namely those of Prandtl-Ishlinskii, Preisach and Della Torre. Main emphasis is put on the structure of memory and analytical properties in the space of continuous functions.

Prandtl-Ishlinskii operators

Rheological constructions of vector-valued Prandtl-Ishlinskii operators were introduced in Sect. I.1. The distinction between operators of stop type and play type plays a substantial role in the study of energy dissipation properties. At this stage we can adopt the following definition which includes both concepts.

Definition 3.1. *Let a constant* $a \geqslant 0$ *and a function* $h \in BV_{\mathrm{loc}}(0,\infty)$ *be given,* $h(0+) = a$. *Put*

$$\varphi(r) := \int_0^r h(s)\, ds \quad \text{for} \quad r > 0. \tag{3.1}$$

Then the operator $\mathcal{F}_\varphi : \Lambda_0 \times C([0,T]) \to C([0,T])$ *defined by the formula*

$$\underset{\varphi}{\mathcal{F}}(\lambda, u) = au + \int_0^\infty p_r(\lambda, u)\, dh(r), \;\; \lambda \in \Lambda_0,\, u \in C([0,T]), \tag{3.2}$$

where p_r *is the play operator (2.6), is called a* Prandtl-Ishlinskii operator *generated by the function* φ *and* φ *is called the* generator *of the operator* $\mathcal{F}_\varphi$.

The Stieltjes integral in (3.2) is finite due to the assumption $\lambda \in \Lambda_0$ and Corollary 2.6(i) which ensure that $p_r(\lambda, u)$ vanishes for r sufficiently large.

From inequality (1.5) it follows that the mapping $t \mapsto \mathcal{F}_\varphi(\lambda, u)(t)$ is continuous. Moreover, the operator $\mathcal{F}_\varphi$ is locally Lipschitz in $\Lambda_0 \times C([0,T])$ in the following sense.

Proposition 3.2. *Let* φ *satisfy the hypotheses of Definition 3.1 and let* $R > 0$ *be given. For* $r > 0$ *put* $V_h(r) := \operatorname*{Var}_{[0,r]} h$. *Then for every* $\lambda, \mu \in \Lambda_R$ *and* $u, v \in C([0,T])$ *such that* $|u|_\infty, |v|_\infty \leqslant R$ *we have*

$$\text{(i)} \quad \big|\underset{\varphi}{\mathcal{F}}(\lambda, u) - \underset{\varphi}{\mathcal{F}}(\mu, v)\big|_\infty \leqslant \int_0^R |\lambda(r) - \mu(r)|\, dV_h(r) + \big(a + V_h(R)\big)|u - v|_\infty. \tag{3.3}$$

If moreover h *is nonnegative and nonincreasing, then*

$$\text{(ii)} \quad \big|\underset{\varphi}{\mathcal{F}}(\lambda, u) - \underset{\varphi}{\mathcal{F}}(\mu, v)\big|_\infty \leqslant h(0+)|\lambda - \mu|_\infty + 2\varphi\big(|u - v|_\infty\big).$$

P r o o f. It suffices to use Lemma 2.3 and the elementary inequality $| \int_0^R f(r)\, dh(r)| \leqslant \int_0^R |f(r)|\, dV_h(r)$ for $f \in C([0,R])$. In the case of h nonincreasing we put $\xi_r := p_r(\lambda, u)$, $\eta_r := p_r(\mu, v)$, $x_r := u - \xi_r$, $y_r := v - \eta_r$. By (2.9),(2.6),(1.2) we have $|x_r - y_r|_\infty \leqslant \min\{2r, |u-v|_\infty + \max\{|\lambda(r) - \mu(r)|, |u-v|_\infty\}\} \leqslant |\lambda(r) - \mu(r)| + 2\min\{r, |u - v|_\infty\}$ and the assertion follows from the inequality

$$| \mathcal{F}_\varphi(\lambda, u) - \mathcal{F}_\varphi(\mu, v)|_\infty \leqslant h(\infty)|u - v|_\infty - \int_0^\infty |x_r - y_r|_\infty dh(r).$$

□

The Prandtl-Ishlinskii operator preserves the memory structure in the following sense.

Proposition 3.3. *Let φ satisfy the hypotheses of Definition 3.1 with $h(r) > 0$ for $r > 0$, $\lim_{r\to\infty} \varphi(r) = +\infty$. For $\lambda \in \Lambda_0$ and $u \in C([0,T])$ put $w := \mathcal{F}_\varphi(\lambda, u)$ and*

$$\mu(s) := -\int_{\varphi^{-1}(s)}^{\infty} \lambda'(r)\, h(r)\, dr \qquad \text{for} \quad s > 0, \tag{3.4}$$

where φ^{-1} is the inverse function to φ and $\lambda' = \frac{d\lambda}{dr}$. Let $t \in [0,T]$ be arbitrarily chosen and let $MS_\lambda(u)(t) = \{(t_j, r_j)\}$ be the corresponding memory sequence.

Then $\mu \in \Lambda_0$, $MS_\mu(w)(t) = \{(t_j, \varphi(r_j))\}$ and

$$p_s(\mu, w)(t) = -\int_{\varphi^{-1}(s)}^{\infty} \frac{\partial}{\partial r} p_r(\lambda, u)(t)\, h(r)\, dr \quad \forall s > 0. \tag{3.5}$$

P r o o f. We first note that an equivalent formula for $w(t)$ reads

$$w(t) = -\int_0^\infty \frac{\partial}{\partial r} p_r(\lambda, u)(t)\, h(r)\, dr \tag{3.6}$$

by Remark V.1.31.

Let $\bar{t}, \bar{r}$ be given by (2.14), i.e. $u(\bar{t}) - \lambda(\bar{r}) = S\bar{r}$, where $S = \operatorname{sign}\big(u(\bar{t}) - \lambda(0)\big)$ and for every $q < \bar{r}$ we have

$$\int_q^{\bar{r}} \big(1 + S\lambda'(r)\big) dr = S\big(u(\bar{t}) - \lambda(q) - Sq\big) > 0. \tag{3.7}$$

Lemma 2.4 yields $w(\bar{t}) = \mu\big(\varphi(\bar{r})\big) + S\varphi(\bar{r})$ and for $q < \bar{r}$ we obtain from (3.4), (3.7)

$$S\Big(w(\bar{t}) - \mu\big(\varphi(q)\big)\Big) = \varphi(q) + \int_q^{\bar{r}} \big(1 + S\lambda'(r)\big)\, h(r)\, dr > \varphi(q),$$

hence $\varphi(\bar{r}) = m_\mu\big(w(\bar{t})\big)$, $\operatorname{sign}\big(w(\bar{t}) - \mu(0)\big) = \operatorname{sign}\big(u(\bar{t}) - \lambda(0)\big)$.

For all $\tau \in [0,t]$ and $r \geqslant \bar r$ we have $p_r(\lambda,u)(\tau) = \lambda(r)$ by Theorem 2.5; formula (3.6) then entails

$$\big|w(\tau) - \mu\big(\varphi(\bar r)\big)\big| = \Big|\int_0^{\bar r} \frac{\partial}{\partial r} p_r(\lambda,u)(\tau)\, h(r)\, dr\Big| \leqslant \varphi(\bar r),$$

consequently $\varphi(\bar r) = M(\mu,w,t)$, the pair $\big(\bar t, \varphi(\bar r)\big)$ is the first point of $MS_\mu(w)(t)$ and $\frac{\partial}{\partial s} p_s(\mu,w)(\bar t)\big|_{s=\varphi(r)} = \frac{\partial}{\partial r} p_r(\lambda,u)(\bar t)$ for a.e. $r > 0$.

Assume now that for some $t_k < t$ we have $MS_\mu(w)(t_k) = \big\{\big(t_j, \varphi(r_j)\big);\, j \leqslant k\big\}$. For $r,s > 0$ and $\tau \in [0, t-t_k]$ put $\lambda_k(r) := p_r(\lambda,u)(t_k)$, $\mu_k(s) := p_s(\mu,w)(t_k)$, $u_k(\tau) := u(\tau + t_k)$, $w_k(\tau) := \mathcal{F}_\varphi(\lambda_k, u_k)(\tau)$. By Proposition 2.5 we have $\mu_k'(s) = \lambda_k'\big(\varphi^{-1}(s)\big)$ for all $s > 0$, hence $\mu_k(s) = -\int_{\varphi^{-1}(s)}^\infty \lambda_k'(r)\, h(r)\, dr$. Applying the above argument to $u_k, w_k, \lambda_k, \mu_k$ in $[0, t-t_k]$ we obtain $MS_\mu(w)(t_{k+1}) = \big\{\big(t_j, \varphi(r_j)\big);\, j \leqslant k+1\big\}$ as a consequence of the semigroup property of the play. A standard induction procedure completes the proof. □

We immediately see that if φ is the identity $\varphi(r) = r$, then $\mathcal{F}_\varphi(\lambda,u) = u$ for all $\lambda \in \Lambda_0$ and $u \in C([0,T])$. The following superposition formula is an immediate consequence of identity (3.5).

Corollary 3.4. *If φ, ψ are functions satisfying the hypotheses of Proposition 3.3, then for all $u \in C([0,T])$ and $\lambda \in \Lambda_0$ we have*

$$\mathop{\mathcal{F}}_{\psi}\big(\mu, \mathop{\mathcal{F}}_{\varphi}(\lambda,u)\big) = \mathop{\mathcal{F}}_{\psi\circ\varphi}(\lambda,u),$$
$$\mathop{\mathcal{F}}_{\varphi^{-1}}\big(\mu, \mathop{\mathcal{F}}_{\varphi}(\lambda,u)\big) = u,$$

where μ is given by (3.4) and $\psi\circ\varphi(r) := \psi\big(\varphi(r)\big)$.

Remarks 3.5.

(i) Play and stop with threshold r_0 belong to the class of Prandtl-Ishlinskii operators for $\varphi(r) = \max\{0, r - r_0\}$, $\varphi(r) = \min\{r, r_0\}$, respectively.

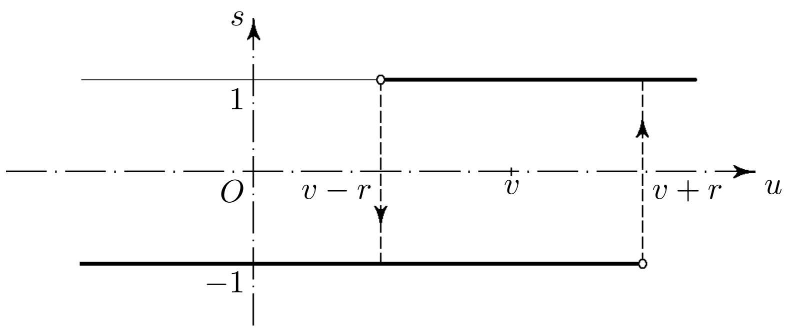
(ii) Superposition and inversion formulas in Corollary 3.4 show that every group Γ with respect to superposition of generators $\varphi : [0,\infty[\to [0,\infty[$ generates a group $\tilde\Gamma = \{\mathcal{F}_\varphi(0,\cdot);\, \varphi \in \Gamma\}$ of Prandtl-Ishlinskii operators $C([0,T]) \to C([0,T])$ which is isomorphic to Γ. The choice $\lambda \equiv 0$ of trivial initial configuration corresponds to the "virginal state", cf. Remark I.3.4.

(iii) The distinction between Prandtl - Ishlinskii operators of stop type and play type can be characterized in terms of generators: a convex function φ generates an operator of play type, a concave φ generates an operator of stop type.

Preisach operator

The classical model of ferromagnetism due to Preisach (1935) is based on the concept of delayed switching element or *relay* with values $+1$ (the switch is "on") and -1 (the switch is "off"). It can be described by an operator $R_{v,r} : \{-1,1\} \times C([0,T]) \to BV(0,T)$ with input u (magnetic field) and output s (magnetization), depending on two parameters $v \in \mathbb{R}^1$ (interaction field) and $r > 0$ (critical field of coercivity) and defined formally as follows (see Fig. 9).

Fig. 9



Let $\mathbb{R}^2_+$ denote the set $\{(v,r) \in \mathbb{R}^2;\, r > 0\}$. For given parameters $(v,r) \in \mathbb{R}^2_+$, input $u \in C([0,T])$, initial magnetization $s_0 \in \{-1,1\}$ and time $t \in [0,T]$ put

$$S(t) := \{\tau \in [0,t];\, |u(\tau) - v| = r\},$$

and $\tau_t := \max S(t)$ provided $S(t) \neq \emptyset$. We then define

$$R_{v,r}(s_0,u)(0) := \begin{cases} +1 & \text{if } \ u(0) \geqslant v + r, \\ -1 & \text{if } \ u(0) \leqslant v - r, \\ s_0 & \text{if } \ u(0) \in]v - r, v + r[, \end{cases} \tag{3.8}$$

$$R_{v,r}(s_0,u)(t) := \begin{cases} R_{v,r}(s_0,u)(0) & \text{if } \ S(t) = \emptyset, \\ \frac{1}{r}\big(u(\tau_t) - v\big) & \text{if } \ S(t) \neq \emptyset. \end{cases} \tag{3.9}$$

The number of switching points $t \in [0,T]$ where the value of $R_{v,r}(s_0,u)$ switches from -1 to $+1$ or vice versa is obviously finite and a similar estimate as in Proposition 1.4 holds. Moreover, $R_{v,r}(s_0,u)$ is right continuous in $[0,T[$.

In applications, it is convenient to use the following representation of the relay by means of the system $\{p_r;\, r > 0\}$ of play operators.

Lemma 3.6. *Let $\lambda \in \Lambda_0$ and $u \in C([0,T])$ be given. For $(v,r) \in \mathbb{R}^2_+$ put $s_\lambda(v,r) := -1$ if $v \geqslant \lambda(r)$, $s_\lambda(v,r) = +1$ if $v < \lambda(r)$. Then for every $t \in [0,T]$*

and $(v,r) \in \mathbb{R}^2_+$, $v \neq p_r(\lambda,u)(t)$ *we have*

$$R_{v,r}\big(s_\lambda(v,r),u\big)(t) = \begin{cases} +1 & \textit{if} \quad v < p_r(\lambda,u)(t), \\ -1 & \textit{if} \quad v > p_r(\lambda,u)(t). \end{cases}$$

For interpreting Lemma 3.6 we can use Fig. 8. At each instant $t \in [0,T]$ the curve $v = p_r(\lambda,u)(t)$ describes the interface in the (v,r)-plane between the region below, where all switches $R_{v,r}$ are on and above, where all switches are off.

P r o o f of Lemma 3.6. We make use of the memory representation of the play in Theorem 2.5. Assume first $m_\lambda\big(u(t)\big) = M(\lambda,u,t) := \bar{r}$. For $r \geqslant \bar{r}$ we have $p_r(\lambda,u)(t) = \lambda(r)$ and $\lambda(r) - r \leqslant u(\tau) \leqslant \lambda(r) + r$ for all $\tau \in [0,t]$. Choosing $v > \lambda(r)$ we thus have $u(\tau) - v < r$ for all $\tau \in [0,t]$ and $s_\lambda(v,r) = -1$, hence $R_{v,r}\big(s_\lambda(v,r),u\big)(\tau) = -1$ for all $\tau \in [0,t]$. For $v < \lambda(r)$ we similarly have $u(\tau) - v > -r$ and $R_{v,r}(s_\lambda(v,r),u)(\tau) = +1$ for all $\tau \in [0,t]$.

The case $r < \bar{r}$ is analogous. Assume for instance $u(t) = \lambda(\bar{r}) + \bar{r} > \lambda(r) + r$. We then have $p_r(\lambda,u)(t) = u(t) - r \geqslant u(\tau) - r$ for all $\tau \in [0,t]$. For $v > p_r(\lambda,u)(t)$ we obtain $v > \lambda(r)$ and $u(\tau) - v < r$ for all $\tau \in [0,t]$, hence $R_{v,r}\big(s_\lambda(v,r),u\big)(t) = -1$ similarly as above. For $v < p_r(\lambda,u)(t)$ we have $u(t) - v > r$ and two cases can occur. If $S(t) = \emptyset$, then $u(0) - v > r$ and if $S(t) \neq \emptyset$, then $u(\tau_t) - v = r$. In both situations we have by definition $R_{v,r}\big(s_\lambda(v,r),u\big)(t) = +1$. We proceed analogously if $u(t) = \lambda(\bar{r}) - \bar{r}$. In the case $m_\lambda\big(u(t)\big) < M(\lambda,u,t)$ we construct the memory sequence $MS_\lambda(u)(t) = \big\{(t_j,r_j)\big\}$ and use the above argument by induction over j with t_{j+1} instead of t and $\lambda_j(r) = p_r(\lambda,u)(t_j)$ instead of $\lambda(r)$ as in the proof of Theorem 2.5.

□

The output $w(t)$ of the Preisach model is formally defined as an average over all elementary switches with a given density function $\psi \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ by the formula (see Krasnosel'skii, Pokrovskii (1983), Visintin (1984), Brokate, Visintin (1989), Mayergoyz (1991))

$$w(t) := \frac{1}{2} \iint_{\mathbb{R}^2_+} R_{v,r}\big(s_\lambda(v,r),u\big)(t)\,\psi(v,r)\,dv\,dr. \tag{3.10}$$

To justify the integration we adopt the following hypotheses.

Assumption 3.7.

(i) *The antisymmetric part* $\psi_a(v,r) := \frac{1}{2}\big(\psi(v,r) - \psi(-v,r)\big)$ *of* ψ *satisfies* $\psi_a \in L^1(\mathbb{R}^2_+)$.

(ii) *The integral in* (3.10) *is considered in the sense of principal value.*

Using Lemma 3.6 and putting

$$g(v,r) := \int_0^v \psi(z,r)\,dz \quad \text{for} \quad (v,r) \in \mathbb{R}^2_+ \tag{3.11}$$

we rewrite formula (3.10) in the form

$$w(t) = C + \int_0^\infty g\big(p_r(\lambda,u)(t),r\big)\,dr, \tag{3.12}$$

with $C = -\int_0^\infty \int_0^\infty \psi_a(v,r)\,dv\,dr$.

Notice that the integral in (3.12) is meaningful independently of Assumption 3.7, since $p_r(\lambda,u)(t) = 0$ for r sufficiently large and $g(0,r) = 0$ for all $r > 0$. Furthermore, by (1.5) and the absolute continuity of Lebesgue's integral the function w in (3.12) is continuous. This justifies the following definition.

Definition 3.8. *Let $\psi \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ be given and let g be defined by (3.11). Then the Preisach operator $\mathcal{W} : \Lambda_0 \times C([0,T]) \to C([0,T])$ generated by the function g is defined by the formula*

$$\mathcal{W}(\lambda,u)(t) := \int_0^\infty g\big(p_r(\lambda,u)(t),r\big)\,dr \tag{3.13}$$

for $\lambda \in \Lambda_0$, $u \in C([0,T])$ and $t \in [0,T]$.

Remark 3.9. It is clear that the Prandtl-Ishlinskii operator (3.2) with $a = 0$ and $h \in W^{1,1}_{\mathrm{loc}}(0,\infty)$ belongs to the class of Preisach operators for $\psi(v,r) = h'(r)$. On the other hand, the Preisach operator can be used in elastoplasticity for modeling nonlinear counterparts to the Prandtl-Ishlinskii model of play type corresponding to the rheological formula $\sum_{r>0} \mathcal{N}_r | \mathcal{R}_r$, where $\mathcal{R}_r$ is the (scalar) rigid-plastic element with constraint $Z_r = [-r,r]$ and $\mathcal{N}_r$ is the nonlinear elastic element with constitutive equation $\varepsilon = g(\sigma,r)$. The thermodynamical admissibility of the element $\mathcal{N}_r$ is ensured by choosing the potential energy

$$\mathcal{U}_r = G(\sigma,r) := \sigma g(\sigma,r) - \int_0^\sigma g(v,r)\,dv. \tag{3.14}$$

The constitutive equation of the model $\sum_{r>0} \mathcal{N}_r | \mathcal{R}_r$ then has the form $\varepsilon = \mathcal{W}(\lambda,\sigma)$ with the Preisach operator (3.13) for a suitably chosen distribution of initial plastic stresses. General rheological principles of Sect. I.1 suggest to define the potential energy associated to the Preisach operator $\mathcal{W}$ by the integral

$$\mathcal{U}(\lambda,u)(t) := \int_0^\infty G\big(p_r(\lambda,u)(t),r\big)\,dr. \tag{3.15}$$

Mathematical consequences of this definition will be given in the next section.

We require in the sequel that the following hypothesis is fulfilled.

Assumption 3.10. *There exist* $\beta_0, \beta_1 \in L^1_{\mathrm{loc}}(0,\infty)$, $\beta_1(r) \geqslant \beta_0(r) \geqslant 0$ *a.e.,* $b_0 := \int_0^\infty \beta_0(r)dr < \infty$ *such that*

$$\beta_1(r) \geqslant \psi(v,r) \geqslant -\beta_0(r) \quad \text{for a.e. } (v,r) \in \mathbb{R}^2_+.$$

For $R > 0$ *put* $b_1(R) := \int_0^R \beta_1(r)\, dr$.

The following continuity result is analogous to Proposition 3.2 and we leave the proof to the reader.

Proposition 3.11. *Let Assumption 3.10 be satisfied and let* $R > 0$ *be given. Then for every* $\lambda, \mu \in \Lambda_R$ *and* $u, v \in C([0,T])$ *such that* $|u|_\infty, |v|_\infty \leqslant R$ *the Preisach operator (3.13) satisfies*

$$|\,\mathcal{W}(\lambda,u) - \mathcal{W}(\mu,v)|_\infty \leqslant \int_0^R |\lambda(r) - \mu(r)|\, \beta_1(r)\, dr + b_1(R)|u-v|_\infty .$$

We now pass to the description of memory related to Preisach operators. We fix a number $b \geqslant b_0$ and define an auxiliary function $f : \mathbb{R}^2_+ \to \mathbb{R}^1$ as the solution of the hyperbolic Cauchy problem

$$\left\{\begin{array}{ll} \frac{\partial^2 f}{\partial r^2} - \frac{\partial^2 f}{\partial v^2} & = g(v,r) \\ f(v,0) & = 0 \\ \frac{\partial f}{\partial r}(v,0) & = bv, \end{array}\right. \tag{3.16}$$

where the "memory" variable $r > 0$ plays the role of "time". We obviously have

$$f(v,r) = brv + \frac{1}{2}\int_0^r \int_{v-r+\varrho}^{v+r-\varrho} g(z,\varrho)\, dz\, d\varrho. \tag{3.17}$$

The main result is Proposition 3.14 below as a counterpart to Proposition 3.3. We start with two lemmas.

Lemma 3.12. *Let Assumption 3.10 hold and let* $\lambda \in \Lambda_0$ *be given. Then the function*

$$\varphi_\lambda(r) := \frac{\partial f}{\partial v}(v,r)\Big|_{v=\lambda(r)} \quad \text{for} \quad r > 0 \tag{3.18}$$

with f, g *satisfying (3.17),(3.11) has the following properties.*

(i) $\varphi_\lambda(0) = 0$,

(ii) $(b - b_0)(r_2 - r_1) \leqslant \varphi_\lambda(r_2) - \varphi_\lambda(r_1) \leqslant \big(b + b_1(r_2)\big)(r_2 - r_1) \qquad$ *for all* $r_2 > r_1$.

P r o o f. We have

$$\varphi_\lambda(r_2) - \varphi_\lambda(r_1) = b(r_2 - r_1) + \frac{1}{2}\int_0^{r_1}\int_{\lambda(r_1)+r_1-\varrho}^{\lambda(r_2)+r_2-\varrho} \psi(v,\varrho)\,dv\,d\varrho + \\ + \frac{1}{2}\int_0^{r_1}\int_{\lambda(r_2)-r_2+\varrho}^{\lambda(r_1)-r_1+\varrho} \psi(v,\varrho)\,dv\,d\varrho + \frac{1}{2}\int_{r_1}^{r_2}\int_{\lambda(r_2)-r_2+\varrho}^{\lambda(r_2)+r_2-\varrho} \psi(v,\varrho)\,dv\,d\varrho$$

and Lemma 3.12 follows from Assumption 3.10. □

Lemma 3.13. *Let the hypotheses of Lemma 3.12 with* $b > b_0$ *be fulfilled and let* $\mu : [0,\infty[\to \mathbb{R}^1$ *be defined by the formula*

$$\mu(s) := \frac{\partial f}{\partial r}(v,r)\Big|_{v=\lambda(r)} + \int_r^\infty g(\lambda(\varrho),\varrho)\,d\varrho \quad \text{for } s > 0,\ r = \varphi_\lambda^{-1}(s). \tag{3.19}$$

Then $\mu \in \Lambda$ *and the implication*

$$|\lambda'(r)| = 1 \quad \Rightarrow \quad \mu'\big(\varphi_\lambda(r)\big) = \lambda'(r) \tag{3.20}$$

holds for all $r > 0$ *such that* $\lambda'(r)$ *exists. If moreover we assume*

$$\int_0^r\int_0^{r-\varrho} \psi_a(v,\varrho)\,dv\,d\varrho = 0 \qquad \text{for } r \text{ sufficiently large,} \tag{3.21}$$

then $\mu \in \Lambda_0$.

P r o o f. For an arbitrary $s > 0$ and $r = \varphi_\lambda^{-1}(s)$ we have

$$\mu(s) + s = \mu(0) + b\big(\lambda(r) + r - \lambda(0)\big) + \int_0^r \big(g(\lambda(r) + r - \varrho,\varrho) - g(\lambda(\varrho),\varrho)\big)\,d\varrho,$$
$$\mu(s) - s = \mu(0) + b\big(\lambda(r) - r - \lambda(0)\big) + \int_0^r \big(g(\lambda(r) - r + \varrho,\varrho) - g(\lambda(\varrho),\varrho)\big)\,d\varrho$$

and Assumption 3.10 entails

$$0 \leqslant (b - b_0)\big(\lambda(r_2) + r_2 - \lambda(r_1) - r_1\big) \leqslant \mu(s_2) + s_2 - \mu(s_1) - s_1 \\ \leqslant \big(b + b_1(r_2)\big)\big(\lambda(r_2) + r_2 - \lambda(r_1) - r_1\big), \tag{3.22}$$

$$0 \leqslant (b - b_0)\big(r_2 - \lambda(r_2) - r_1 + \lambda(r_1)\big) \leqslant s_2 - \mu(s_2) - s_1 + \mu(s_1) \\ \leqslant \big(b + b_1(r_2)\big)\big(r_2 - \lambda(r_2) - r_1 + \lambda(r_1)\big) \tag{3.23}$$

for all $r_2 > r_1$ and $s_i = \varphi_\lambda(r_i)$, $i = 1, 2$.

Using Lemma 3.12 we rewrite inequalities (3.22), (3.23) in the form

$$
\begin{aligned}
(3.24) \qquad 0 &\leqslant \frac{b - b_0}{b + b_1(r_2)} \left(1 \pm \frac{\lambda(r_2) - \lambda(r_1)}{r_2 - r_1}\right) \leqslant 1 \pm \frac{\mu(s_2) - \mu(s_2)}{s_2 - s_1} \\
&\leqslant \frac{b + b_1(r_2)}{b - b_0} \left(1 \pm \frac{\lambda(r_2) - \lambda(r_1)}{r_2 - r_1}\right),
\end{aligned}
$$

hence $\mu \in \Lambda$ and (3.20) holds.

It remains to check that $\mu(s) = 0$ for s sufficiently large provided (3.21) holds. We find $\tilde{r}$ such that for $r > \tilde{r}$ we have $\lambda(r) = 0$. For $s > \varphi_\lambda(\tilde{r})$ and $r = \varphi_\lambda^{-1}(s)$ we have $\mu(s) = \int_0^r \int_0^{r-\varrho} \psi_a(v, \varrho)\, dv\, d\varrho$ and the assertion follows easily. □

We now apply Lemmas 3.12, 3.13 to the time-dependent situation. The result is analogous to Proposition 3.3.

Proposition 3.14. *Let g, f, λ, μ be as in Lemmas 3.12, 3.13 and let $u \in C([0,T])$ and $t \in [0,T]$ be given with memory sequence $MS_\lambda(u)(t) = \{(t_j, r_j)\}$. Put $w := bu + \mathcal{W}(\lambda, u)$, where $\mathcal{W}$ is the Preisach operator (3.13), $\lambda^t(\varrho) := p_\varrho(\lambda, u)(t)$ for $\varrho > 0$ and*

$$
(3.25) \qquad \mu^t(s) := \frac{\partial f}{\partial r}(v, r)\Big|_{v=\lambda^t(r)} + \int_r^\infty g\big(\lambda^t(\varrho), \varrho\big)\, d\varrho \quad \text{for } s > 0,\ r = \varphi_{\lambda^t}^{-1}(s)
$$

Then $MS_\mu(w)(t) = \big\{\big(t_j, \varphi_{\lambda^t}(r_j)\big)\big\}$ and for all $s > 0$ we have $p_s(\mu, w)(t) = \mu^t(s)$.

P r o o f. We first note that the function $t \mapsto \varphi_{\lambda^t}^{-1}(s)$ is continuous for each fixed $s > 0$. Indeed, putting $r := \varphi_{\lambda^t}^{-1}(s)$ for some $t \in [0,T]$ we have for each $\tau \neq t$ by Lemma 3.12

$$
(b - b_0)\ \big|\varphi_{\lambda^\tau}^{-1}(s) - \varphi_{\lambda^t}^{-1}(s)\big| \leqslant |s - \varphi_{\lambda^\tau}(r)| = |\varphi_{\lambda^t}(r) - \varphi_{\lambda^\tau}(r)| \leqslant b_1(r)\, \big|\lambda^t(r) - \lambda^\tau(r)\big|
$$

and it suffices to use the estimate (1.5).

Let now $(\bar{t}, \bar{r})$ be the first point of $MS_\lambda(u)(t)$ and put $\bar{s} := \varphi_\lambda(\bar{r})$. For all $\tau \in [0,t]$ and $r \geqslant \bar{r}$ we have $\lambda^\tau(r) = \lambda(r)$, hence $\varphi_{\lambda^\tau}(\bar{r}) = \bar{s}$. The identities

$$
\begin{aligned}
\mu(\bar{s}) + \bar{s} - w(\tau) &= b\big(\lambda(\bar{r}) + \bar{r} - u(\tau)\big) + \int_0^{\bar{r}} \big(g(\lambda(\bar{r}) + \bar{r} - \varrho, \varrho) - g(\lambda^\tau(\varrho), \varrho)\big)\, d\varrho, \\
\mu(\bar{s}) - \bar{s} - w(\tau) &= b\big(\lambda(\bar{r}) - \bar{r} - u(\tau)\big) + \int_0^{\bar{r}} \big(g(\lambda(\bar{r}) - \bar{r} + \varrho, \varrho) - g(\lambda^\tau(\varrho), \varrho)\big)\, d\varrho
\end{aligned}
$$

yield $\bar{s} = m_\mu(w)(\bar{t})$, $|w(\tau) - \mu(\bar{s})| \leqslant \bar{s}$ for $\tau \in [0,T]$, hence $\big(\bar{t}, \varphi_{\lambda^t}(\bar{r})\big)$ is the first point of $MS_\mu(w)(t)$. For $s \geqslant \bar{s}$ we obviously have $\mu^\tau(s) = \mu(s) = p_s(\mu, w)(\tau)$ for all $\tau \in [0,t]$, for $s < \bar{s}$ implication (3.20) applied to $\lambda^{\bar{t}}, \mu^{\bar{t}}$ entails $\mu^{\bar{t}}(s) = p_s(\mu, w)(\bar{t})$. We now repeat the induction procedure over t_j from the proof of Proposition 3.3. Details are left to the reader. □

Exercise 3.15. Let $\mathcal{F}_\varphi$ be the Prandtl-Ishlinskii operator (3.2) with $h' \in L^1_{\mathrm{loc}}(0,\infty)$ and let $\mathcal{W}$ be the Preisach operator (3.13). Let the function ψ in (3.11) satisfy Assumption 3.10 and identity (3.21) and let f be given by (3.17). Let $\lambda \in \Lambda_0$ be given and let μ be as in Lemma 3.13. Put $\tilde{f}(z,r) := \int_0^z \varphi\big(\frac{\partial f}{\partial v}(v,r)\big)\,dv$, $\tilde{g}(z,r) := \frac{\partial^2 \tilde{f}}{\partial r^2} - \frac{\partial^2 \tilde{f}}{\partial z^2}$. Let $\tilde{\mathcal{W}}$ be the Preisach operator $\tilde{\mathcal{W}}(\lambda,u) := \int_0^\infty \tilde{g}\big(p_r(\lambda,u),r\big)\,dr$.

Prove the superposition formula

$$\mathcal{F}_{\varphi}\big(\mu, bu + \mathcal{W}(\lambda,u)\big) = abu + \tilde{\mathcal{W}}(\lambda,u) \tag{3.26}$$

for all $u \in C([0,T])$.

Hint. Use Proposition 3.14, formula (3.6) and the identities

$$-\int_0^\infty \frac{\partial \mu^t(s)}{\partial s} h(s)\,ds = -\int_0^\infty \Big(\frac{\partial^2 f}{\partial v^2}(v,r) + \frac{\partial^2 f}{\partial r \partial v}(v,r)\frac{\partial \lambda^t(r)}{\partial r}\Big) h\Big(\frac{\partial f}{\partial v}(v,r)\Big)\Big|_{v=\lambda^t(r)} dr,$$

$$\frac{\partial}{\partial r}\Big(\frac{\partial \tilde{f}}{\partial r}(v,r)\Big|_{v=\lambda^t(r)}\Big) = \frac{\partial^2 \tilde{f}}{\partial r^2}(v,r) + \frac{\partial^2 f}{\partial r \partial v}(v,r)\,\frac{\partial \lambda^t(r)}{\partial r}\,h\Big(\frac{\partial f}{\partial v}(v,r)\Big)\Big|_{v=\lambda^t(r)}.$$

It can be shown that the superposition of two Preisach operators $\mathcal{W}_1 \circ \mathcal{W}_2$ is in general not Preisach. Moreover, the inverse of $bI + \mathcal{W}(\lambda,\cdot)$, where I is the identity, is Preisach if and only if $\mathcal{W}$ is Prandtl-Ishlinskii. We do not pursue this question here; an interested reader can find more information in Krejčí (1991/a). On the other hand, we prove below that the conditions in Proposition 3.14 are sufficient for the continuous invertibility of the operator $bI + \mathcal{W}(\lambda,\cdot)$ in $C([0,T])$.

Lemma 3.16. *Let Assumption 3.10 hold and let $\lambda \in \Lambda_0, b > b_0$ be given. For $u_1, u_2 \in C([0,T])$ put $w_i := bu_i + \mathcal{W}(\lambda,u_i)$, $i = 1,2$. Then we have*

$$|u_1 - u_2|_\infty \leqslant \frac{2}{b - b_0}|w_1 - w_2|_\infty.$$

Proof. We choose $t \in [0,T]$ such that for instance $u_1(t) - u_2(t) = |u_1 - u_2|_\infty > 0$ and put

$$\begin{aligned} r^* &:= \inf\{r > 0;\, p_r(\lambda,u_1)(t) \leqslant p_r(\lambda,u_2)(t)\},\\ s^* &:= \frac{\partial f}{\partial v}(v,r^*)\Big|_{v=p_{r^*}(\lambda,u_1)(t)} = \frac{\partial f}{\partial v}(v,r^*)\Big|_{v=p_{r^*}(\lambda,u_2)(t)}, \end{aligned}$$

where f is given by (3.17). For μ defined by (3.19) we obtain from Proposition 3.14

$$p_{s^*}(\mu,w_i)(t) = \frac{\partial f}{\partial r}(v,r^*)\Big|_{v=p_{r^*}(\lambda,u_i)(t)} + \int_{r^*}^\infty g\big(p_r(\lambda,u_i)(t),r\big)\,dr, \quad i = 1,2.$$

The identities

$$w_1(t) - w_2(t) = b\big(u_1(t) - u_2(t)\big) + \int_0^\infty \Big(g\big(p_r(\lambda, u_1)(t), r\big) - g\big(p_r(\lambda, u_2)(t), r\big)\Big)\, dr,$$

$$p_{s^*}(\mu, w_1)(t) - p_{s^*}(\mu, w_2)(t) = \int_{r^*}^\infty \Big(g\big(p_r(\lambda, u_1)(t), r\big) - g\big(p_r(\lambda, u_2)(t), r\big)\Big)\, dr$$

entail

$$\begin{aligned} b\big(u_1(t) - u_2(t)\big) + \int_0^{r^*} \int_{p_r(\lambda,u_2)(t)}^{p_r(\lambda,u_1)(t)} \psi(v, r)\, dv\, dr \leqslant \\ \leqslant w_1(t) - w_2(t) + \big|p_{s^*}(\mu, w_1)(t) - p_{s^*}(\mu, w_2)(t)\big|. \end{aligned}$$

The assertion now follows from Assumption 3.10 and inequality (2.9). □

Theorem 3.17. *Let Assumption 3.10 be fulfilled and let* $\lambda \in \Lambda_0, b > b_0$ *be given. Then the operator* $bI + \mathcal{W}(\lambda, \cdot) : C([0,T]) \to C([0,T])$ *is invertible and its inverse is Lipschitz continuous.*

Theorem 3.17 will follow from Lemma 3.16 if we prove that for every w from a dense subset of $C([0,T])$ there exists $u \in C([0,T])$ such that $w = bu + \mathcal{W}(\lambda, u)$. A suitable candidate seems to be the subspace $C_{pm}([0,T]) \subset C([0,T])$ of continuous piecewise monotone functions. We first investigate the behavior of the Preisach operator on locally monotone inputs.

Lemma 3.18. *Let Assumption 3.10 hold and let* $b \geqslant b_0$ *be given. For* $\lambda \in \Lambda_0$ *and* $v \in \mathbb{R}^1$ *put*

$$\Phi_\lambda(v) := \begin{cases} \int_{\lambda(0)}^{v} \big[b + \int_0^{m_\lambda(z)} \psi(z - r, r)\, dr\big] dz & \text{if } v \geqslant \lambda(0), \\ -\int_v^{\lambda(0)} \big[b + \int_0^{m_\lambda(z)} \psi(z + r, r)\, dr\big] dz & \text{if } v < \lambda(0). \end{cases} \tag{3.27}$$

Then we have

$$(b - b_0)(v_2 - v_1) \leqslant \Phi_\lambda(v_2) - \Phi_\lambda(v_1) \leqslant \big(b + \kappa(v_1, v_2)\big)(v_2 - v_1) \tag{3.28}$$

for all $v_1 < v_2$, *where* $\kappa(v_1, v_2) := \max\Big\{\int_0^{m_\lambda(v_i)} \beta_1(t)\, dr;\ i = 1, 2\Big\}$.

Inequality (3.28) is a straightforward consequence of Assumption 3.10. Lemma 3.19 below shows the connection to the Preisach operator $\mathcal{W}$.

Lemma 3.19. *Let Assumption 3.10 hold and let* $\lambda \in \Lambda_0, b \geqslant b_0$ *and* $u \in C([0,T])$ *be given such that* u *is monotone (nondecreasing or nonincreasing) in an interval* $[\hat{t}, \hat{t}+\delta]$. *Put* $w := bu + \mathcal{W}(\lambda, u)$, $\hat{\lambda}(r) := p_r(\lambda, u)(\hat{t})$ *for* $r > 0$. *Then for all* $t \in [\hat{t}, \hat{t}+\delta]$ *we have*

$$w(t) = w(\hat{t}) + \Phi_{\hat{\lambda}}\big(u(t)\big). \tag{3.29}$$

P r o o f. The semigroup property I(1.27) entails that we are in the situation of Lemma 2.4, i.e. putting $\hat{u}(\tau) := u(\tau + \hat{t})$ for $\tau \in [0,\delta]$ we have $p_r(\lambda, u)(\tau + \hat{t}) = p_r(\hat{\lambda}, \hat{u})(\tau)$, hence

$$p_r(\lambda, u)(t) = \begin{cases} \hat{\lambda}(r) & \text{for } r \geqslant \varrho(t), \\ u(t) - r & \text{for } r < \varrho(t) \quad \text{if } u \text{ increases}, \\ u(t) + r & \text{for } r < \varrho(t) \quad \text{if } u \text{ decreases}, \end{cases}$$

where $\varrho(t) := m_{\hat{\lambda}}(u(t))$. This yields

$$w(t) = w(\hat{t}) + b\big(u(t) - u(\hat{t})\big) + \begin{cases} \int_0^{\varrho(t)} \big(g(u(t) - r, r) - g(\hat{\lambda}(r), r)\big)\, dr & \text{if } u \text{ increases}, \\ \int_0^{\varrho(t)} \big(g(u(t) + r, r) - g(\hat{\lambda}(r), r)\big)\, dr & \text{if } u \text{ decreases}, \end{cases}$$

and (3.29) follows easily. □

We now pass to the proof of Theorem 3.17.

P r o o f of Theorem 3.17. Let $w \in C_{pm}([0,T])$ be monotone in each interval $[t_{j-1}, t_j]$ of the partition $0 = t_0 < t_1 < \ldots < t_N = T$. We construct a function $u \in C_{pm}([0,T])$ successively by putting

$$u(0) := \Phi_{\lambda}^{-1}\left(w(0) - b\lambda(0) - \int_0^{\infty} g\big(\lambda(r), r\big) dr\right),$$
$$u(t) := \Phi_{\lambda_j}^{-1}\big(w(t) - w(t_j)\big) \;\text{ for } t \in]t_j, t_{j+1}],\; j = 0, \ldots, N-1,$$

where $\lambda_j(r) := p_r(\lambda, u)(t_j)$ for $r > 0$. From Lemma 3.19 we infer $w = bu + \mathcal{W}(\lambda, u)$. Since $C_{pm}([0,T])$ is dense in $C([0,T])$, Theorem 3.17 follows from Lemma 3.16. □

The remaining part of this section is devoted to a generalization of the Preisach model for ferromagnetism.

The Della Torre model

The model of Preisach (1935) was originally intended to describe the dependence of the magnetization M in a ferromagnetic medium on the intensity of the magnetic field H. The material is represented as a homogeneous mixture of dipoles with two possible orientations ± 1 obeying the switching rule (3.9) and distributed with a nonnegative density $\psi(v,r)$ as in formula (3.10). We have seen that the Preisach model leads to the constitutive equation $M = \mathcal{W}(\lambda, H)$ in operator form with a given initial configuration $\lambda \in \Lambda_0$.

Della Torre (1966) proposed to include a feedback influence into the Preisach model by assuming a modified implicit constitutive law ("moving model")

$$M = \mathcal{W}(\lambda, H + \alpha M) \tag{3.30}$$

with a real parameter α.

Under the hypotheses of Theorem 3.17 with $\beta_0 = 0$ and $\beta_1 \in L^1(0,\infty), b_1(\infty) := \int_0^\infty \beta_1(r)\, dr < \infty$ we can rewrite identity (3.30) in input-output form by introducing an auxiliary quantity $Z := H + \alpha M$. Then $M = \mathcal{W}(\lambda, Z)$ and $H = Z - \alpha\,\mathcal{W}(\lambda, Z)$.

Theorem 3.17 ensures that for $\alpha < \frac{1}{b_1(\infty)}$ the operator $I - \alpha\,\mathcal{W}(\lambda,\cdot)$ is invertible and its inverse $\big(I - \alpha\,\mathcal{W}(\lambda,\cdot)\big)^{-1}$ is Lipschitz in $C([0,T])$. We conclude that (3.30) is equivalent to

$$M = \underset{\alpha}{\mathcal{W}}(\lambda, H), \tag{3.31}$$

where $\mathcal{W}_\alpha(\lambda,\cdot) := \mathcal{W}(\lambda,\cdot) \circ \big(I - \alpha\,\mathcal{W}(\lambda,\cdot)\big)^{-1}$ is a locally Lipschitz operator in $C([0,T])$.

We immediately see that $\mathcal{W}_\alpha$ is a rate independent operator, but it cannot be represented in general by a Preisach operator (see Brokate (1992)) except for the trivial case where $\mathcal{W}$ is Prandtl-Ishlinskii.

While Preisach operators are thermodynamically consistent due to their rheological structure, this is not obvious for the Della Torre operator $\mathcal{W}_\alpha$. We shall see in the next section (Corollary 4.4) that a suitable choice of potential energy $\mathcal{U}_\alpha$ for the constitutive law (3.31) consists in putting

$$\mathcal{U}_\alpha(\lambda, H) := \mathcal{U}\Big(\lambda, \big(I - \alpha\,\mathcal{W}(\lambda,\cdot)\big)^{-1}(H)\Big) - \frac{\alpha}{2}\big(\underset{\alpha}{\mathcal{W}}(\lambda, H)\big)^2, \tag{3.32}$$

where $\mathcal{U}$ is the Preisach potential energy (3.15).

II.4 Monotonicity and energy inequalities

This section is the most important with respect to applications. The energy estimates and monotonicity relations that we derive here constitute the main tool for solving hyperbolic equations with hysteresis operators in the next chapter. The interesting fact that introducing hysteresis into hyperbolic equations makes the problem easier originates in particular dissipation properties of hysteresis operators. We already noticed in Theorem I.3.16 that besides the rheological potential energy the stop operator admits a second order potential energy. We shall see that in the scalar case, the dissipation of the second order energy is related to the convexity of hysteresis loops in the input-output diagram and we derive a lower bound for the dissipation rate which is proportional to the cube of the input derivative. We first use Proposition 2.8 to clarify how Preisach type operators act on absolutely continuous inputs.

Lemma 4.1. *Let $a \in L^1_{\mathrm{loc}}(\mathbb{R}^2_+)$ be given such that the following conditions are fulfilled.*

(i) *The function $v \mapsto a(v,r)$ is continuous for a.e. $r > 0$,*

(ii) *there exist $c \in L^\infty_{\mathrm{loc}}(\mathbb{R}^1)$ and $\beta_1 \in L^1_{\mathrm{loc}}(0,\infty)$ such that for a.e. $r > 0$ and all $v \in \mathbb{R}^1$ we have*

$$|a(v,r)| \leqslant c(v)\beta_1(r).$$

Let further p_r be the play defined by (2.6). For $\lambda \in \Lambda_0$, $u \in W^{1,1}(0,T)$, $r > 0$ and $t \in [0,T]$ put $\xi_r(t) := p_r(\lambda,u)(t)$ and $w(t) := \int_0^\infty \int_0^{\xi_r(t)} a(v,r)\,dv\,dr$. Then $w \in W^{1,1}(0,T)$ and for a.e. $t \in]0,T[$ we have

$$\dot w(t) = \int_0^\infty \dot\xi_r(t)\, a(\xi_r(t),r)\,dr. \tag{4.1}$$

P r o o f. With the notation of Corollary 2.9 put

$$L_\lambda(u) := \big\{t \in L;\ \dot u(t) = 0\big\} \cup \big\{t \in L^*;\ \varrho_0(t) = \varrho_1(t)\big\}. \tag{4.2}$$

Then $\mathrm{meas}\big(]0,T[\backslash L_\lambda(u)\big) = 0$ and for every $t \in L_\lambda(u)$ we can pass to the limit as $\delta \to 0$ in the identity

$$\begin{aligned}\frac{1}{\delta}\big(w(t+\delta) - w(t)\big) = \int_0^\infty \frac{1}{\delta}\int_{\xi_r(t)}^{\xi_r(t+\delta)} \big(a(v,r) - a\big(\xi_r(t),r\big)\big)\,dv\,dr\,+ \\ + \int_0^\infty \frac{1}{\delta}\big(\xi_r(t+\delta) - \xi_r(t)\big)\,a\big(\xi_r(t),r\big)\,dr\end{aligned}$$

using Lebesgue's Dominated Convergence Theorem (Proposition V.1.13) and estimate (1.5). □

In the sequel we restrict the class of Preisach operators (3.13) by requiring more regularity. In addition to Assumption 3.10 we assume

$$
\text{(4.3)} \qquad \begin{array}{ll} \text{(i)} & \dfrac{\partial \psi}{\partial v} \in L^\infty_{\mathrm{loc}}(\mathbb{R}^2_+) \\ \text{(ii)} & \psi(v,r) \geqslant 0 \quad \text{a.e.,} \end{array}
$$

We immediately see that under the hypothesis (4.3)(i) the Preisach operator is locally Lipschitz with respect to the norm in $W^{1,1}(0,T)$ in the following sense.

Proposition 4.2. *Let* ψ *be a function fulfilling (4.3)(i) and Assumption 3.10. For a given* $R>0$ *put* $C_R := \sup\mathrm{ess}\,\{\frac{\partial \psi}{\partial v}(v,r);\ |v|+r \leqslant R\}$. *Then for every* $\lambda_1,\lambda_2 \in \Lambda_R$ *and* $u_1,u_2 \in W^{1,1}(0,T)$ *such that* $\max\left\{|u_i|_\infty, \int_0^T |\dot u_i(t)|dt;\ i=1,2\right\} \leqslant R$ *the outputs* $w_i := \mathcal{W}(\lambda_i,u_i)$, $i=1,2$ *of the Preisach operator (3.13) satisfy*

$$
\text{(4.4)} \qquad \int_0^T |\dot w_1(t) - \dot w_2(t)|dt \leqslant b_1(R)\int_0^T |\dot u_1(t) - \dot u_2(t)|\,dt + \\ + \big(b_1(R) + R^2 C_R\big)\big(|\lambda_1 - \lambda_2|_\infty + |u_1-u_2|_\infty\big).
$$

P r o o f. Putting $\xi^i_r := p_r(\lambda_i,u_i)$ for $r>0, i=1,2$ we obtain from Lemma 4.1

$$
|\dot w_1(t) - \dot w_2(t)| \leqslant \int_0^R \left|\dot\xi^1_r(t) - \dot\xi^2_r(t)\right| \psi(\xi^1_r(t),r)\,dr + \\ + \int_0^R \left|\dot\xi^2_r(t)\right| \left|\psi\big(\xi^1_r(t),r\big) - \psi\big(\xi^2_r(t),r\big)\right| dr
$$

and the assertion follows from Proposition 1.1 and Assumption 3.10. □

Thermodynamical consistency

We now give a rigorous proof of the thermodynamical consistency of the Preisach model which, as it was mentioned in Remark 3.9, formally follows from the rheological construction. Recall that the Preisach potential energy $\mathcal{U}$ is given by (3.15), i.e.

$$
\text{(4.5)} \qquad \mathcal{U}(\lambda,u) := \int_0^\infty G\big(p_r(\lambda,u),r\big)\,dr,
$$

where $G(v,r) = vg(v,r) - \int_0^v g(z,r)dz = \int_0^v z\psi(z,r)dz$. We further introduce the *dissipation operator*

$$
\text{(4.6)} \qquad \mathcal{D}(\lambda,u) := \int_0^\infty rg\big(p_r(\lambda,u),r\big)\,dr.
$$

Theorem 4.3. *Let the Preisach operator* $\mathcal{W}$ *satisfy (4.3)(i),(ii) and Assumption 3.10 and let* $R > 0$ *be given. For arbitrary* $\lambda \in \Lambda_R$ *and* $u \in W^{1,1}(0,T)$ *such that* $|u|_\infty \leqslant R$ *put* $w := \mathcal{W}(\lambda, u)$, $U := \mathcal{U}(\lambda, u)$, $D := \mathcal{D}(\lambda, u)$. *Then we have*

(i) $U(t) \geqslant \frac{1}{2b_1(R)} w^2(t) \qquad \forall t \in [0,T]$,

(ii) $\dot{w}(t)u(t) - \dot{U}(t) = |\dot{D}(t)|$ a.e.

P r o o f. For a.e. $(v,r) \in \mathbb{R}^2_+$ we have by Assumption 3.10 $vg(v,r) \leqslant v^2\beta_1(r)$, consequently

$$\operatorname{sign}(v) g(v,r)\psi(v,r) \leqslant \beta_1(r)|v|\psi(r,v).$$

Integration with respect to v of this last inequality yields $\frac{1}{2}g^2(v,r) \leqslant \beta_1(r)G(v,r)$ a.e., and using Hölder's inequality

$$w^2(t) \leqslant \Big(\int_0^R \frac{1}{\beta_1(r)} g^2\big(p_r(\lambda,u)(t),r\big)dr\Big)\Big(\int_0^R \beta_1(r)dr\Big) \leqslant 2U(t)b_1(R)$$

we obtain (i).

Formula (ii) immediately follows from Lemma 4.1 and identity (1.8). Note that for each $t \in L_\lambda(u)$ (cf. (4.2)) all nonzero derivatives $\dot{\xi}_r(t)$ have the same sign independent of r as a consequence of Proposition 2.8. □

Theorem 4.3 enables us to prove that also the Della Torre model (3.30) - (3.32) is thermodynamically consistent.

Corollary 4.4. *Let the hypotheses of Theorem 4.3 hold and assume* $\alpha_1 := \frac{1}{b_1(\infty)} > 0$. *For arbitrary* $0 \leqslant \alpha < \alpha_1$, $\lambda \in \Lambda_0$ *and* $u \in W^{1,1}(0,T)$ *put*

$$\begin{aligned} w_\alpha &:= \underset{\alpha}{\mathcal{W}}(\lambda,u) = \mathcal{W}\big(\lambda, (I - \alpha\,\mathcal{W}(\lambda,\cdot))^{-1}(u)\big),\\ U_\alpha &:= \underset{\alpha}{\mathcal{U}}(\lambda,u) = \mathcal{U}\big(\lambda, (I - \alpha\,\mathcal{W}(\lambda,\cdot))^{-1}(u)\big) - \frac{\alpha}{2}w_\alpha^2,\\ D_\alpha &:= \underset{\alpha}{\mathcal{D}}(\lambda,u) = \mathcal{D}\big(\lambda, (I - \alpha\,\mathcal{W}(\lambda,\cdot))^{-1}(u)\big), \end{aligned}$$

where $\mathcal{U}, \mathcal{D}$ *are defined by (4.5), (4.6). Then we have*

(i) $U_\alpha(t) \geqslant \frac{\alpha_1 - \alpha}{2} w_\alpha^2(t) \qquad \forall t \in [0,T]$,

(ii) $\dot{w}_\alpha(t)u(t) - \dot{U}_\alpha(t) = |\dot{D}_\alpha(t)|$ a.e.

P r o o f. By Theorem 3.17 the operator $I - \alpha\,\mathcal{W}(\lambda,\cdot)$ is invertible and its inverse is Lipschitz in $C([0,T])$. Put $z_\alpha := \big(I - \alpha\,\mathcal{W}(\lambda,\cdot)\big)^{-1}(u)$. Similarly as in Remark 1.3 we obtain from Lemma 3.16 the estimate

$$|z_\alpha(t) - z_\alpha(s)| \leqslant \frac{2\alpha_1}{\alpha_1 - \alpha}\int_s^t |\dot{u}(\tau)|d\tau \quad \text{for all} \quad 0 < s < t < T$$

analogous to (1.5). We have in particular $z_\alpha \in W^{1,1}(0,T)$ and using z_α as input in the identities $w_\alpha = \mathcal{W}(\lambda, z_\alpha)$, $u = z_\alpha - \alpha w_\alpha$, $U_\alpha = \mathcal{U}(\lambda, z_\alpha) - \frac{\alpha}{2}w_\alpha^2$, $D_\alpha = \mathcal{D}(\lambda, z_\alpha)$ we obtain the assertion directly from Theorem 4.3. □

Remark 4.5. We see that for the models of Preisach and Della Torre the dissipation rate $\dot{q}$ in formula I(1.2) is given in terms of hysteresis operators of the same kind. For the sake of completeness we give explicit energy dissipation formulas for Prandtl-Ishlinskii operators (3.2). We restrict ourselves to the physically natural case where φ is convex (operators of play type) or concave (operators of stop type).

Proposition 4.6. *Let R, λ, u be as in Theorem 4.3 and let $h \in BV_{\mathrm{loc}}(0,\infty)$ be a given nonnegative function. For $r > 0$ put $\xi_r := p_r(\lambda, u)$, $x_r := u - \xi_r$. Then the following two cases are thermodynamically consistent.*

A. (Operators of play type) *Assume that h is nondecreasing and put*

$$w := h(0)u + \int_0^\infty \xi_r \, dh(r), \;\; U := \frac{1}{2}h(0)u^2 + \frac{1}{2}\int_0^\infty \xi_r^2 \, dh(r), \;\; D := \int_0^\infty r\xi_r \, dh(r).$$

Then we have

$$\dot{w}(t)u(t) - \dot{U}(t) = |\dot{D}(t)| \quad \text{a.e.} \tag{4.8}$$

B. (Operators of stop type) *Let h be nonincreasing, and put*

$$w := h(\infty)u - \int_0^\infty x_r \, dh(r), \;\; U := \frac{1}{2}h(\infty)u^2 - \frac{1}{2}\int_0^\infty x_r^2 \, dh(r), \;\; D := -\int_0^\infty r\xi_r \, dh(r).$$

Then we have

$$w(t)\dot{u}(t) - \dot{U}(t) = |\dot{D}(t)| \quad \text{a.e.} \tag{4.9}$$

P r o o f. Lemma 4.1 is applicable here due to the linear dependence of the operator on the play system. The rest of the proof is a special case of Theorem 4.3. □

Remark 4.7. Notice that the convexity of φ is not necessary for the thermodynamical consistency of the Prandtl-Ishlinskii model. From Proposition 2.8 we easily derive the necessary and sufficient condition in the form $\int_0^R r \, dh(r) \geqslant 0$ for operators of play type and $\int_0^R r \, dh(r) \leqslant 0$ for operators of stop type for every $R > 0$.

Thermodynamically consistent “Preisach models of stop type” correspond for instance to operators of the form $I - \alpha \mathcal{W}(\lambda, \cdot)$ or $\big(I + \alpha \mathcal{W}(\lambda, \cdot)\big)^{-1}$ with W, α satisfying the assumptions of Corollary 4.4. The derivation of explicit energy inequalities in these cases is left to the reader.

Monotonicity

It is obvious that hysteresis operators are never monotone with respect to the scalar product in L^2. On the other hand, Proposition 4.8 below states that Preisach operators are *locally monotone* ("piecewise monotone" in the terminology of Visintin (1994)) under natural assumptions.

Proposition 4.8. *Let the Preisach operator W satisfy (4.3)(i) and Assumption 3.10. Let $b \geqslant b_0$, $R > 0$, $\lambda \in \Lambda_R$ and $u \in W^{1,1}(0,T)$ be given such that $|u|_\infty \leqslant R$. Put $w := bu + \mathcal{W}(\lambda, u)$. Then*

$$(4.10) \qquad (b - b_0)\dot{u}^2(t) \leqslant \dot{w}(t)\dot{u}(t) \leqslant \big(b + b_1(R)\big)\dot{u}^2(t) \quad \text{a.e.}$$

Proof. By I(3.22)(ii) we have $0 \leqslant \dot{\xi}_r(t)\dot{u}(t) \leqslant \dot{u}^2(t)$ and (4.10) follows from Lemma 4.1. □

We immediately check that Della Torre's operator is locally monotone in the above sense. Another important concept of monotonicity based on inequality (1.4) is typical for Prandtl-Ishlinskii operators and cannot be extended to more general Preisach-type models. In fact, as a straightforward consequence of inequality (1.4) and Lemma 4.1 we have

Theorem 4.9. *Let $h : [0,\infty[\to [0,\infty[$ be a monotone function. For $u_1, u_2 \in W^{1,1}(0,T)$, $\lambda_1, \lambda_2 \in \Lambda_0$ and $r > 0$ put $\xi_r^{(i)} := p_r(\lambda_i, u_i)$, $x_r^{(i)} := u_i - \xi_r^{(i)}$, $w_i := \mathcal{F}_\varphi(\lambda_i, u_i) = h(0)u_i + \int_0^\infty \xi_r^{(i)}\, dh(r)$, $i = 1,2$, $\tilde{u} := u_1 - u_2$, $\tilde{w} := w_1 - w_2$, $\tilde{\xi}_r := \xi_r^{(1)} - \xi_r^{(2)}$, $\tilde{x}_r := x_r^{(1)} - x_r^{(2)}$. Then*

$$(4.11) \quad \dot{\tilde{w}}(t)\tilde{u}(t) \geqslant \frac{1}{2}\frac{d}{dt}\Big[h(0)\tilde{u}^2(t) + \int_0^\infty \tilde{\xi}_r^2(t)\, dh(r)\Big] \quad \text{a.e.} \quad \textit{if } h \textit{ is nondecreasing,}$$

$$(4.12) \quad \tilde{w}(t)\dot{\tilde{u}}(t) \geqslant \frac{1}{2}\frac{d}{dt}\Big[h(\infty)\tilde{u}^2(t) - \int_0^\infty \tilde{x}_r^2(t)\, dh(r)\Big] \quad \text{a.e.} \quad \textit{if } h \textit{ is nonincreasing.}$$

The case distinction in Theorem 4.9 corresponds again to operators of play type and stop type, respectively. We now prove that for strictly convex (strictly concave) generators φ inequalities (4.11), (4.12) are "almost strict".

Theorem 4.10. *Let the hypotheses of Theorem 4.9 be fulfilled and let h be strictly monotone (increasing or decreasing). Suppose that the identity*

(4.13) (i) $\displaystyle\int_0^T \dot{\tilde{w}}(t)\tilde{u}(t)\,dt = \frac{1}{2}h(0)\big(\tilde{u}^2(T) - \tilde{u}^2(0)\big) + \frac{1}{2}\int_0^\infty \big(\tilde{\xi}_r^2(T) - \tilde{\xi}_r^2(0)\big)\,dh(r)$
if h increases,

(ii) $\displaystyle\int_0^T \dot{\tilde{u}}(t)\tilde{w}(t)\,dt = \frac{1}{2}h(\infty)\big(\tilde{u}^2(T) - \tilde{u}^2(0)\big) - \frac{1}{2}\int_0^\infty \big(\tilde{x}_r^2(T) - \tilde{x}_r^2(0)\big)\,dh(r)$
if h decreases

holds. Then

(4.14) $$u_1(t) - u_2(t) = \lambda_1^0\big(R_0(t)\big) - \lambda_2^0\big(R_0(t)\big),$$

where $\lambda_i^0(r) := \xi_r^{(i)}(0)$, $i = 1,2$ *and* $R_0(t) := \max\{M(\lambda_i^0, u_i, t);\ i = 1,2\}$.

P r o o f. Assumption (4.13) and inequalities (4.11), (4.12) yield respectively either $\dot{\tilde{w}}(t)\tilde{u}(t) = \frac{1}{2}\frac{d}{dt}\big[h(0)\tilde{u}^2(t) + \int_0^\infty \tilde{\xi}_r^2(t)\,dh(r)\big]$ or $\dot{\tilde{u}}(t)\tilde{w}(t) = \frac{1}{2}\frac{d}{dt}\big[h(\infty)\tilde{u}^2(t) - \int_0^\infty \tilde{x}_r^2(t)\,dh(r)\big]$ for a.e. $t \in]0,T[$, hence

(4.15) $$\int_0^\infty \big(\dot{\xi}_r^{(1)}(t) - \dot{\xi}_r^{(2)}(t)\big)\big(x_r^{(1)}(t) - x_r^{(2)}(t)\big)dh(r) = 0 \quad \text{a.e.}$$

Let now $r > 0$ be arbitrarily fixed. Analogously as in Corollary 2.9 we denote by L the set of Lebesgue points of both $\dot{u}_1, \dot{u}_2$, and define the sets $L_i^* := \{t \in L;\ \dot{u}_i(t) \neq 0\}$, $L_r^{*i} := \big\{t \in L_i^*;\ r \in \big[\varrho_0^{(i)}(t), \varrho_1^{(i)}(t)\big]\big\}$, $i = 1,2$. For each $t \in L \setminus (L_r^{*1} \cup L_r^{*2})$ such that (4.15) holds we find $\varepsilon(t) > 0$ such that $\dot{\xi}_\varrho^{(i)}(t) = \dot{\xi}_r^{(i)}(t)$, $i = 1,2$ for all $\varrho \in]r - \varepsilon(t), r + \varepsilon(t)[$. Inequality (1.4) and the strict monotonicity of h then imply

$$\big(\dot{\xi}_r^{(1)}(t) - \dot{\xi}_r^{(2)}(t)\big)\big(x_r^{(1)}(t) - x_r^{(2)}(t)\big) = 0 \quad \text{a.e.}$$

We thus verified that condition (i) of Proposition 2.10 is fulfilled and (4.14) follows from Proposition 2.10 (iii) and Remark 2.11. □

Corollary 4.11. *Let u_1, u_2 be absolutely continuous ω-periodic functions and let w_1, w_2 be as in Theorem 4.10. Assume that h is strictly monotone and*

$$\int_\omega^{2\omega} \big(\dot{u}_1(t) - \dot{u}_2(t)\big)\big(w_1(t) - w_2(t)\big)dt = 0.$$

Then $u_1(t) - u_2(t) = \text{const.}$ *for all* $t \in \mathbb{R}^1$.

P r o o f. The assertion follows from Corollary 2.7 and Theorem 4.10, where $R_0(t)$ is constant in $[\omega, \infty[$ by definition (2.12) of $M(\lambda, u, t)$. □

The Prandtl-Ishlinskii operator exhibits a *two-level monotonicity* on periodic functions: under the hypotheses of Corollary 4.11 with $\lambda_1 = \lambda_2$ we have

$$\big(w_1(t) - w_2(t)\big)\big(u_1(t) - u_2(t)\big) \geqslant 0 \quad \forall t \in [\omega, \infty[.$$

We formulate this result more precisely in the following way.

Proposition 4.12. *Let $\lambda \in \Lambda_R$ and a Prandtl-Ishlinskii operator $\mathcal{F}_\varphi$ satisfying the hypotheses of Theorem 4.10 be given. Let u be an absolutely continuous ω-periodic function and for $c \in \mathbb{R}^1$ put $\vartheta(u,c) := \mathcal{F}_\varphi(\lambda, u(\cdot)+c)(\omega) - \mathcal{F}_\varphi(\lambda, u)(\omega)$. Then for every $c \in \mathbb{R}^1$ and $t \in [\omega, \infty[$ we have $\mathcal{F}_\varphi(\lambda, u(\cdot)+c)(t) - \mathcal{F}_\varphi(\lambda, u)(t) = \vartheta(u,c)$ and putting $\hat{R} := \max\big\{R, |u|_\infty + \max\{|c_1|, |c_2|\}\big\}$ we have for all $c_1 < c_2$*

$$(4.16)\quad \text{(i)}\quad 2\varphi\Big(\frac{c_2 - c_1}{2}\Big) \leqslant \vartheta(u,c_2) - \vartheta(u,c_1) \leqslant h(\hat{R})(c_2 - c_1) \quad \text{if } h \text{ increases,}$$

$$\text{(ii)}\quad 2\varphi\Big(\frac{c_2 - c_1}{2}\Big) \geqslant \vartheta(u,c_2) - \vartheta(u,c_1) \geqslant h(\hat{R})(c_2 - c_1) \quad \text{if } h \text{ decreases.}$$

Proof. For $c \in \mathbb{R}^1$ put $u_c(t) := u(t) + c$, $\xi_r^c := p_r(\lambda, u_c)$. From inequality (1.4) we obtain $\frac{\partial}{\partial t}|\xi_r^{c_2}(t) - \xi_r^{c_1}(t)|^2 \leqslant 0$ a.e. for all $r > 0$ and $c_2 > c_1$. The functions ξ_r^c are ω-periodic for $t \geqslant \omega$ by Corollary 2.7, hence $q(r,c) := \xi_r^c(t) - \xi_r^0(t)$ is independent of t for $t \geqslant \omega$. By (2.6), (2.7) we have $0 \leqslant \xi_r^{c_2}(0) - \xi_r^{c_1}(0) \leqslant c_2 - c_1$, hence $0 \leqslant q(r,c_2) - q(r,c_1) \leqslant c_2 - c_1$ for all $r > 0$, $c_2 > c_1$. On the other hand, by Corollary 2.2 we have $\xi_r^{c_2}(t) - \xi_r^{c_1}(t) \geqslant c_2 - c_1 - 2r$ for $r \in [0, \frac{c_2 - c_1}{2}]$ and inequalities (4.16) follow from the identity $\vartheta(u,c) := h(0)c + \int_0^\infty q(r,c)\, dh(r)$. □

The Preisach operator is in general not monotone in the sense (4.11) due to the nonlinear dependence on the play system. We nevertheless mention a weaker result.

Proposition 4.13. *Let $\mathcal{W}$ be a Preisach operator (3.13) satisfying (4.3) and Assumption 3.10. For given $u_1, u_2 \in W^{1,1}(0,T)$ and $\lambda_1, \lambda_2 \in \Lambda_0$ put $\xi_r^{(i)} := p_r(\lambda_i, u_i)$, $w_i := \mathcal{W}(\lambda_i, u_i) = \int_0^\infty g(\xi_r^{(i)}, r)\, dr$, $i = 1, 2$. Then for a.e. $t \in]0, T[$ we have*

$$\big(\dot{w}_1(t) - \dot{w}_2(t)\big)\big(u_1(t) - u_2(t)\big) \geqslant \int_0^\infty \big(\xi_r^{(1)}(t) - \xi_r^{(2)}(t)\big)\frac{\partial}{\partial t}\big(g(\xi_r^{(1)}(t), r) - g(\xi_r^{(2)}(t), r)\big) dr.$$

Proof. Put $x_r^{(i)} := u_i - \xi_r^{(i)}$ for $r > 0$. We have $\frac{\partial}{\partial t} g(\xi_r^{(1)}, r)(x_r^{(1)} - x_r^{(2)}) \geqslant 0$, $\frac{\partial}{\partial t} g(\xi_r^{(2)}, r)(x_r^{(2)} - x_r^{(1)}) \geqslant 0$ a.e. by (4.3)(ii) and (1.1)(ii). The assertion then follows from Lemma 4.1. □

Second order energy inequalities

The concept of second order potential energy refers to geometrical properties of hysteresis loops rather than to the rheological structure of concrete hysteresis operators. We therefore derive the energy inequalities for general rate-independent operators characterized by the property I(1.28). We first represent rate-independent operators locally by superposition operators.

Proposition 4.14. *Let $F : C([0,T]) \to C([0,T])$ be a rate-independent operator and let $u \in C([0,T])$ be a function which is monotone (nonincreasing or nondecreasing) in $[t_1, t_2] \subset [0,T]$, $u(t_i) = u_i$, $i = 1,2$. Then there exists a continuous function $\Phi : \mathrm{Conv}\{u_1, u_2\} \to \mathbb{R}^1$ such that $F(v)(t) = \Phi\big(v(t)\big)$ for all $t \in [t_1, t_2]$ and for every function $v \in C([0,T])$ which is monotone in $[t_1, t_2]$ and $v(t) = u(t)$ for $t \in [0,T] \setminus]t_1, t_2[$. If moreover F maps $W^{1,1}(0,T)$ into $W^{1,1}(0,T)$ and is locally monotone, then Φ is nondecreasing and absolutely continuous.*

P r o o f. For $u_1 = u_2$ we have $u(t) = u\big(\beta(t)\big)$ for every $t \in [t_1, t_2]$ and every nondecreasing mapping β of $[t_1, t_2]$ onto $[t_1, t_2]$, hence $F(u)$ is constant in $[t_1, t_2]$. For $u_1 \neq u_2$ put

$$\hat{u}(s) := \begin{cases} u(s) & \text{for} \quad s \in [0, t_1] \cup [t_2, T], \\ u_1 + (s - t_1)\frac{u_2 - u_1}{t_2 - t_1} & \text{for} \quad s \in]t_1, t_2[, \end{cases}$$

$$\alpha(t) := \begin{cases} t & \text{for} \quad t \in [0, t_1] \cup [t_2, T], \\ t_1 + (u(t) - u_1)\frac{t_2 - t_1}{u_2 - u_1} & \text{for} \quad t \in]t_1, t_2[, \end{cases}$$

$$\hat{w}(s) := F(\hat{u})(s) \qquad \text{for} \quad s \in [0,T].$$

Then $u(t) = \hat{u}\big(\alpha(t)\big)$ and we easily check that Proposition 4.14 holds for $\Phi(v) := \hat{w}\big(t_1 + (v - u_1)\frac{t_2 - t_1}{u_2 - u_1}\big)$. □

Remark 4.15. The function Φ in Proposition 4.14 will be called *trajectory of F along u in* $[t_1, t_2]$. If moreover F is causal (cf.I(1.29)), then the assertion of Proposition 4.14 can be strengthened in the following way. For every $u \in C([0,T])$ and $t_1 \in [0, T[$ we can find a function $\Phi : \mathbb{R}^1 \to \mathbb{R}^1$ such that $\Phi(u(t_1)) = F(u)(t_1)$ and if a function $v \in C([0,T])$ satisfies $v\big|_{[0,t_1]} = u\big|_{[0,t_1]}$, v monotone in $[t_1, t_1 + \delta]$ for some $\delta > 0$, then for every $t \in [t_1, t_1 + \delta]$ we have $F(v)(t) = \Phi(v(t))$. This is the case of the trajectories Φ_λ of the Preisach operator given in Lemmas 3.18, 3.19.

We now recall some elementary results on monotone functions.

Lemma 4.16. *Let $]a, b[\subset \mathbb{R}^1$ be a given interval and let $h \in L^\infty_{\mathrm{loc}}(a,b)$ be a given function. Then h is nondecreasing if and only if for every nonnegative function $\eta \in$*

$W^{1,1}(a,b)$ *which vanishes outside a compact interval* $[\bar a,\bar b]\subset]a,b[$ *we have*

$$\int_a^b h(v)\,\eta'(v)\,dv \leqslant 0. \tag{4.17}$$

P r o o f. (i) Let h be nondecreasing and let η be given. For an arbitrary partition $\bar a = v_0 < v_1 < \ldots < v_N = \bar b$ we define a piecewise linear approximation

$$\tilde h(v) := h(v_{i-1}) + (v - v_{i-1})\frac{h(v_i)-h(v_{i-1})}{v_i - v_{i-1}},\ v\in[v_{i-1},v_i],\ i=1,\ldots,N. \tag{4.18}$$

Inequality (4.17) holds for $\tilde h$ and refining the partition we obtain an equibounded sequence of nondecreasing functions which converges to h at each point of continuity, so we may pass to the limit in (4.17).

(ii) Let (4.17) hold and let $v_1, v_2 \in]a,b[,\ v_1 < v_2$ be arbitrary Lebesgue points of h. For $0<\varepsilon<\min\{\frac12(v_2-v_1), v_1-a, b-v_2\}$ put

$$\eta'(v) := \begin{cases} \frac{1}{2\varepsilon} & \text{for} \quad v\in]v_1-\varepsilon, v_1+\varepsilon[,\\ -\frac{1}{2\varepsilon} & \text{for} \quad v\in]v_2-\varepsilon, v_2+\varepsilon[,\\ 0 & \text{otherwise.}\end{cases}$$

Then (4.17) yields $\frac{1}{2\varepsilon}\int_{v_1-\varepsilon}^{v_1+\varepsilon} h(v)\,dv \leqslant \frac{1}{2\varepsilon}\int_{v_2-\varepsilon}^{v_2+\varepsilon} h(v)\,dv$ for ε sufficiently small, hence $h(v_1)\leqslant h(v_2)$ and the proof is complete. □

Proposition 4.17. *Let* $]a,b[\subset \mathbb{R}^1$ *be a bounded interval and let* $f\in L^\infty(a,b)$, $\eta\in W^{1,1}(a,b)$ *be given functions,* $\eta(v)\geqslant 0$ *for all* $v\in[a,b]$.

(i) *Assume that the function* $f(v)-Kv$ *is nondecreasing for some* $K\geqslant 0$. *Then*

$$\int_a^b f(v)\,\eta'(v)\,dv \leqslant f(b-)\eta(b) - f(a+)\eta(a) - K\int_a^b \eta(v)\,dv, \tag{4.19}$$

$$\int_a^b \frac{\eta'(v)}{f(v)}\,dv \geqslant \frac{\eta(b)}{f(b-)} - \frac{\eta(a)}{f(a+)} + K\int_a^b \frac{\eta(v)}{f^2(v)}\,dv, \quad \textit{provided } f(a+)>0. \tag{4.20}$$

(ii) *Assume that the function* $f(v)+Kv$ *is nonincreasing for some* $K\geqslant 0$. *Then*

$$\int_a^b f(v)\,\eta'(v)\,dv \geqslant f(b-)\eta(b) - f(a+)\eta(a) + K\int_a^b \eta(v)\,dv, \tag{4.21}$$

$$\int_a^b \frac{\eta'(v)}{f(v)}\,dv \leqslant \frac{\eta(b)}{f(b-)} - \frac{\eta(a)}{f(a+)} - K\int_a^b \frac{\eta(v)}{f^2(v)}\,dv, \quad \textit{provided } f(b-)>0. \tag{4.22}$$

P r o o f. Part (ii) is obtained from (i) by symmetry. It therefore suffices to assume that the function

$$h(v) := \begin{cases} f(v) - Kv, & v \in]a,b[, \\ f(b-) - Kb, & v \geqslant b, \\ f(a+) - Ka, & v \leqslant a \end{cases} \tag{4.23}$$

is nondecreasing in $\mathbb{R}^1$, and inequality (4.19) directly follows from (4.17) with η piecewise linearly extended to a function satisfying the hypotheses of Lemma 4.16 in a sufficiently large interval. A similar argument can be applied to (4.20) provided we prove that the function $g(v) := \frac{1}{f(v)} + K\int_a^v \frac{ds}{f^2(s)}$ is nonincreasing in $]a,b[$. This can be done in the following way. Let h be given by (4.23). We construct piecewise linear approximations $\tilde{h}$ in $[a,b]$ as in (4.18) and put $\tilde{f}(v) := \tilde{h}(v) + Kv$, $\tilde{g}(v) := \frac{1}{\tilde{f}(v)} + K\int_a^v \frac{ds}{\tilde{f}^2(s)}$. We have $\tilde{f}'(v) \geqslant K$ a.e., hence $\tilde{g}'(v) \geqslant 0$ a.e. and passing to the limit as in the proof of Lemma 4.16 we obtain the assertion. □

We are now ready to give a precise formulation of the second order energy inequalities. Similarly as in Proposition 4.6, we consider separately the operators of "stop type" and "play type" in Theorems 4.18, 4.19, respectively.

Theorem 4.18. *Let $F : C([0,T]) \to C([0,T])$ be a continuous rate independent operator. Assume that there exist constants $R > 0$, $b_R > a_R \geqslant 0$, $K_R \geqslant 0$ such that for every $u \in C([0,T])$, $|u|_\infty \leqslant R$ the trajectory Φ of F along u in a monotonicity interval $[t_1, t_2]$ has the following properties.*

(i) *Φ is absolutely continuous in $J := \mathrm{Conv}\{u(t_1), u(t_2)\}$, $a_R \leqslant \Phi'(v) \leqslant b_R$ for a.e. $v \in \mathrm{Int}\, J$,*
(ii) *if u is nondecreasing in $[t_1,t_2]$, then $\Phi(v) + \frac{1}{2}K_R v^2$ is concave in J,*
(iii) *if u is nonincreasing in $[t_1,t_2]$, then $\Phi(v) - \frac{1}{2}K_R v^2$ is convex in J.*

Then for every $u \in W^{2,1}(0,T)$, $|u|_\infty \leqslant R$ we have

(4.24) (i) $w := F(u) \in W^{1,\infty}(0,T)$,

(ii) *the function* $P(t) := \dfrac{1}{2}\dot{w}(t)\dot{u}(t)$ *belongs to* $BV(0,T)$ *and*

$$\frac{a_R}{2}\dot{u}^2(t) \leqslant P(t) \leqslant \frac{b_R}{2}\dot{u}^2(t) \quad \text{a.e.},$$

(iii) $$\int_s^t \dot{w}(\tau)\ddot{u}(\tau)d\tau - P(t) + P(s) \geqslant \frac{1}{2}K_R \int_s^t |\dot{u}(\tau)|^3 d\tau$$

for almost all $0 < s < t < T$.

Theorem 4.19. *Let $F : C([0,T]) \to C([0,T])$ be a continuous rate independent operator. Assume that there exist constants $R > 0$, $b_R > a_R \geqslant 0$, $K_R \geqslant 0$ such that* *for every* $u \in C([0,T])$, $|u|_\infty \leqslant R$ *the trajectory* Φ *of* F *along* u *in a monotonicity interval* $[t_1, t_2]$ *has the following properties.*

(i) Φ *is absolutely continuous in* $J := \mathrm{Conv}\{u(t_1), u(t_2)\}$, $a_R \leqslant \Phi'(v) \leqslant b_R$ *for a.e.* $v \in \mathrm{Int}\, J$,

(ii) *if* u *is nondecreasing in* $[t_1, t_2]$, *then* $\Phi(v) - \frac{1}{2}K_R v^2$ *is convex in* J,

(iii) *if* u *is nonincreasing in* $[t_1, t_2]$, *then* $\Phi(v) + \frac{1}{2}K_R v^2$ *is concave in* J.

Let $u \in W^{1,\infty}(0,T)$ *be a given function such that* $|u|_\infty \leqslant R$ *and* $w := F(u) \in W^{2,1}(0,T)$. *Then (4.24)(ii) holds and*

$$\int_s^t \ddot{w}(\tau)\dot{u}(\tau)d\tau - P(t) + P(s) \geqslant \frac{1}{2}K_R \int_s^t |\dot{u}(\tau)|^3 d\tau \tag{4.25}$$

for almost all $0 < s < t < T$.

Remark 4.20.

(i) Trajectories of a hysteresis operator is precisely what we observe on a hysteresis diagram. A hysteresis loop is formed by one trajectory along an increasing input and one along a decreasing input. Theorems 4.18, 4.19 concern the situation where the part of the plane contained in the interior of each sufficiently small closed hysteresis loop is a *convex set* (see Fig. 10). The two cases differ only by the orientation of the boundary analogously as on Fig. 3. In another context Krasnosel'skii and Pokrovskii (1983) similarly introduce *hysterons with positive or negative spin.*

(ii) We observe an analogy between (4.24)(iii) and (4.9), and between (4.25) and (4.8). This is why we call (4.24)(iii), (4.25) "second order energy inequalities" and their right-hand side term $\frac{1}{2}K_R|\dot{u}(t)|^3$ "lower bound for the dissipation rate".

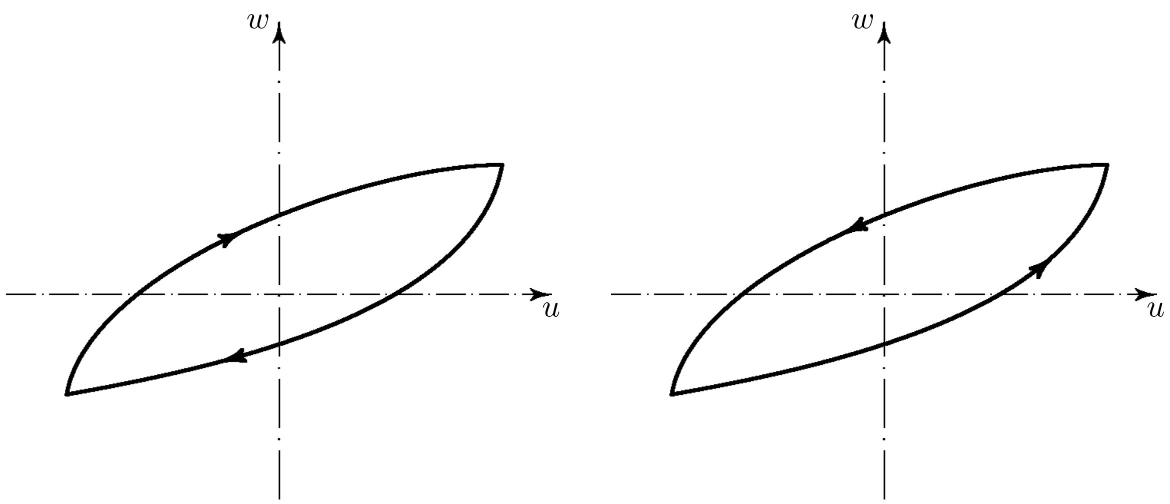


Fig. 10: *operators of stop type* *operators of play type*

We postpone the proof of Theorems 4.18, 4.19 and verify first that they can be applied to hysteresis operators introduced in Sect. II.3.

Proposition 4.21. *Let* $h \in BV_{\rm loc}(0,\infty)$ *be a given nonnegative function and let* $F := \mathcal{F}_\varphi(\lambda_0,\cdot)$ *be the Prandtl-Ishlinskii operator (3.2) for some* $R > 0$ *and* $\lambda_0 \in \Lambda_R$. *Put*

$$H_+(R) := \sup\left\{\frac{h(b)-h(a)}{b-a};\ 0<a<b<R\right\},$$
$$H_-(R) := \inf\left\{\frac{h(b)-h(a)}{b-a};\ 0<a<b<R\right\}.$$

If $H_+(R) \leqslant 0$, *then the hypotheses of Theorem 4.18 are satisfied for* $K_R := -\frac{1}{2}H_+(R)$. *If* $H_-(R) \geqslant 0$, *then the hypotheses of Theorem 4.19 are satisfied for* $K_R := \frac{1}{2}H_-(R)$.

Proof. Let $u \in C([0,T])$ be given, $|u|_\infty \leqslant R$. At each time t the configuration $\lambda(r) := p_r(\lambda_0,u)(t)$ satisfies $\lambda \in \Lambda_0$, $\lambda(r) = 0$ for $r \geqslant R$. We are in the situation of Lemmas 3.18, 3.19, hence all trajectories of $\mathcal{F}_\varphi(\lambda_0,\cdot)$ have the form analogous to (3.27)

$$(4.26)\qquad \Phi(v) = \Phi\big(\lambda(0)\big) + \begin{cases} \int_{\lambda(0)}^{v} h\big(m_\lambda(s)\big)\,ds & \text{for } v \geqslant \lambda(0),\\ -\int_{v}^{\lambda(0)} h\big(m_\lambda(s)\big)\,ds & \text{for } v < \lambda(0), \end{cases}$$

and we immediately see that Φ satisfies condition (i) of Theorems 4.18, 4.19. Assume now $H_-(R) \geqslant 0$ and put $K_R := \frac{1}{2}H_-(R)$. We have to prove

$$(4.27)\qquad \begin{cases} \Psi_+(v) := \Phi(v) - \frac{1}{2}K_R v^2 & \text{is convex in } [\lambda(0),R],\\ \Psi_-(v) := \Phi(v) + \frac{1}{2}K_R v^2 & \text{is concave in } [-R,\lambda(0)]. \end{cases}$$

The functions $h(r) - 2K_R r$, $r - \lambda(r)$, $r + \lambda(r)$ are nondecreasing, hence also

$$\zeta_+(r) := (h(r) - 2K_R r) + K_R\big(r - \lambda(r)\big),$$
$$\zeta_-(r) := (h(r) - 2K_R r) + K_R\big(r + \lambda(r)\big).$$

are nondecreasing and we have $\Psi'_\pm(v) = \zeta_\pm\big(m_\lambda(v)\big)$ a.e. Since m_λ is increasing in $[\lambda(0),\infty[$ and decreasing in $]-\infty,\lambda(0)]$, we immediately obtain (4.27). The proof can easily be adapted to the case $H_+(R) \leqslant 0$. We leave the details to the reader. □

Proposition 4.22. *Let* $\mathcal{W}$ *be a Preisach operator satisfying Assumption 3.10 and (4.3). Assume that there exists* $\varrho > 0$ *such that*

$$(4.28)\qquad A_\varrho := \inf\operatorname{ess}\{\psi(v,r);\ |v| + r \leqslant \varrho\} > 0.$$

Then there exists $R > 0$ *such that for every* $\lambda_0 \in \Lambda_R$ *and* $b \geqslant 0$ *the operator* $bI + \mathcal{W}(\lambda_0,\cdot)$ *and the Della Torre operator* $bI + \mathcal{W}_\alpha(\lambda_0,\cdot)$ *for* $\alpha > 0$ *sufficiently small satisfy the hypotheses of Theorem 4.19.*

P r o o f. We choose $R \in]0, \varrho[$ sufficiently small such that

$$K_R := \frac{1}{2} A_R - R C_R > 0,$$

where C_R is defined in Proposition 4.2. Lemmas 3.18, 3.19 for $b_0 = 0$ and Assumption 3.10 entail that the trajectories of $bI + \mathcal{W}$ satisfy condition (i) of Theorem 4.19 with $a_R = b$ and $b_R = b + b_1(R)$. The proof for $bI + \mathcal{W}$ will be complete if we prove that for every $\lambda \in \Lambda_R$ the trajectory (3.27) has the property (4.27).

We have $\Phi'_\lambda(v) - K_R v = \tilde{\zeta}_+\big(m_\lambda(v)\big)$ for a.e. $v > \lambda(0)$, $\Phi'_\lambda(v) + K_R v = \tilde{\zeta}_-\big(m_\lambda(v)\big)$ for a.e. $v < \lambda(0)$, where

$$\begin{aligned} \tilde{\zeta}_+(s) &:= b + \int_0^s \psi\big(\lambda(s) + s - r, r\big) dr - K_R\big(s + \lambda(s)\big), \\ \tilde{\zeta}_-(s) &:= b + \int_0^s \psi\big(\lambda(s) - s + r, r\big) dr - K_R\big(s - \lambda(s)\big). \end{aligned}$$

For $0 < s_1 < s_2 < R$ we have by hypothesis

$$\begin{aligned} \tilde{\zeta}_+(s_2) - \tilde{\zeta}_+(s_1) = & \int_{s_1}^{s_2} \psi\big(\lambda(s_2) + s_2 - r, r\big) dr - K_R\big(\lambda(s_2) + s_2 - \lambda(s_1) - s_1\big) + \\ & + \int_0^{s_1} \Big[\psi\big(\lambda(s_2) + s_2 - r, r\big) - \psi\big(\lambda(s_1) + s_1 - r, r\big)\Big] dr \\ & \geqslant \big(A_R - 2(RC_R + K_R)\big)(s_2 - s_1) \geqslant 0 \end{aligned}$$

and analogously for $\tilde{\zeta}_-$, hence $\tilde{\zeta}_\pm$ are nondecreasing and we argue as in the proof of Theorem 4.21.

We similarly prove that the trajectories of the operator $I - \alpha\,\mathcal{W}(\lambda, \cdot)$ are uniformly concave if u increases and uniformly convex if u decreases. Since the trajectories of the inverse operator are obtained by inversion of trajectories and superposition of operators corresponds to superposition of trajectories, we obtain the assertion for the Della Torre operator. □

The rest of this section is devoted to the proofs of Theorems 4.18, 4.19.

P r o o f of Theorem 4.18. For every piecewise linear approximation $\tilde{u}$ of u and almost every $\tau \in]0, T[$ we have by Proposition 4.14 $|\dot{F(\tilde{u})}(\tau)| \leqslant b_R |\dot{\tilde{u}}(\tau)|$. Passing to the limit we obtain $|w(\beta) - w(\alpha)| \leqslant b_R \int_\alpha^\beta |\dot{u}(t)|\, d\tau$ for every $0 < \alpha < \beta < T$, hence w is Lipschitz. The properties of Φ imply the upper and lower bound for $P(t)$. On the other hand, if we prove (4.24)(iii), then P is the sum of one nonincreasing and one absolutely continuous function, hence $P \in BV(0, T)$. It remains to check that (4.24)(iii) holds.

Let $s < t$ be Lebesgue points of $\dot{w}$ and put $A_0 := \{\tau \in [s, t];\ \dot{u}(\tau) = 0\}$. The set A_0 is closed, since $\dot{u}$ is absolutely continuous, and its complement $A_1 :]s, t[\setminus A_0$ is a countable disjoint union $A_1 = \bigcup_{k=1}^{\infty}]\alpha_k, \beta_k[$ of open intervals. We have $\dot w(\tau) = 0$ for a.e. $\tau \in A_0$, hence

$$\begin{cases} \int_s^t \dot w(\tau)\ddot u(\tau)d\tau & = \sum_{k=1}^{\infty} \int_{\alpha_k}^{\beta_k} \dot w(\tau)\ddot u(\tau)d\tau, \\ \int_s^t |\dot u(\tau)|^3 d\tau & = \sum_{k=1}^{\infty} \int_{\alpha_k}^{\beta_k} |\dot u(\tau)|^3 d\tau. \end{cases} \tag{4.29}$$

In $[\alpha_k, \beta_k]$ the function u is strictly monotone, hence w has the form $w(t) = \Phi_k\big(u(t)\big)$, where Φ_k is a trajectory of F.

a) Let u increase in $[\alpha_k, \beta_k]$. By hypothesis, the function $\Phi_k(v) + \frac{1}{2}K_R v^2$ is concave and after substitution we obtain

$$I_k := \int_{\alpha_k}^{\beta_k} \dot w(\tau)\ddot u(\tau)d\tau = \frac{1}{2}\int_{u(\alpha_k)}^{u(\beta_k)} \Phi_k'(v)\frac{d}{dv}\left(\dot u\big(u^{-1}(v)\big)\right)^2 dv.$$

We now apply Proposition 4.17 for $f(v) = \Phi_k'(v)$, $\eta(v) = \frac{1}{2}\left(\dot u\big(u^{-1}(v)\big)\right)^2$, $K = K_R$. From (4.21) we infer

$$I_k \geqslant \frac{1}{2}\Phi_k'\big(u(\beta_k)-\big)\dot u^2(\beta_k) - \frac{1}{2}\Phi_k'\big(u(\alpha_k)+\big)\dot u^2(\alpha_k) + \frac{1}{2}K_R\int_{\alpha_k}^{\beta_k} |\dot u(\tau)|^3 d\tau. \tag{4.30}$$

b) Let u decrease in $[\alpha_k, \beta_k]$. Then $\Phi_k(v) - \frac{1}{2}K_R v^2$ is convex and

$$I_k = \frac{1}{2}\int_{u(\beta_k)}^{u(\alpha_k)} \Phi_k'(v)\frac{d}{dv}\left(\dot u\big(u^{-1}(v)\big)\right)^2 dv.$$

Using inequality (4.19) for f, η as above we obtain

$$I_k \geqslant \frac{1}{2}\Phi_k'\big(u(\beta_k)+\big)\dot u^2(\beta_k) - \frac{1}{2}\Phi_k'\big(u(\alpha_k)-\big)\dot u^2(\alpha_k) + \frac{1}{2}K_R\int_{\alpha_k}^{\beta_k} |\dot u(\tau)|^3 d\tau. \tag{4.31}$$

We have $\dot u(\alpha_j) = \dot u(\beta_k) = 0$ for all j, k except possibly for the single case $\alpha_j = s, \beta_k = t$. Combining (4.30) with (4.31) we obtain

$$\sum_{k=1}^{\infty} I_k \geqslant \frac{1}{2}\dot w(t)\dot u(t) - \frac{1}{2}\dot w(s)\dot u(s) + \frac{1}{2}K_R\sum_{k=1}^{\infty}\int_{\alpha_k}^{\beta_k} |\dot u(\tau)|^3 d\tau$$

and identities (4.29) complete the proof. □

P r o o f of Theorem 4.19. Let $0 < s < t < T$ be Lebesgue points of $\dot u$. We proceed analogously as in the proof of Theorem 4.18. The upper and lower bound for $P(t)$ in (4.24)(ii) is less obvious now; we use the fact that sequences of smooth inputs which are strongly convergent in $W^{1,p}$ generate weakly convergent sequences of outputs and pass to the limit.

To prove (4.25) put $A_0 := \{\tau \in [s,t];\ \dot w(\tau) = 0\}$, and let $]\alpha_k, \beta_k[$ be an arbitrary component of the set $]s,t[\setminus A_0$. Both u and w are increasing in $]\alpha_k,\beta_k[$, $w(\tau) = \Phi_k\big(u(\tau)\big)$ for $\tau \in [\alpha_k, \beta_k]$.

a) Let u increase in $[\alpha_k, \beta_k]$. Then $\Phi_k(v) - \frac{1}{2}K_R v^2$ is convex and by substitution we obtain

$$I_k := \int_{\alpha_k}^{\beta_k} \ddot w(\tau)\dot u(\tau)d\tau = \int_{u(\alpha_k)}^{u(\beta_k)} \frac{d}{dv}\left(\dot w\big(u^{-1}(v)\big)\right)\dot u\big(u^{-1}(v)\big)dv.$$

The function $\varrho(v) := \dot w\big(u^{-1}(v)\big)$ is nonnegative and absolutely continuous in the interval $[u(\alpha_k), u(\beta_k)]$, $\varrho(v) = \Phi_k'(v)\dot u\big(u^{-1}(v)\big)$ a.e. We can use inequality (4.20) for $f(v) = \Phi_k'(v)$, $\eta(v) = \frac{1}{2}\varrho^2(v)$, $K = K_R$ to obtain

$$I_k \geqslant \frac{\dot w^2(\beta_k)}{\Phi_k'\big(u(\beta_k)-\big)} - \frac{\dot w^2(\alpha_k)}{\Phi_k'\big(u(\alpha_k)+\big)} + \frac{1}{2}K_R\int_{\alpha_k}^{\beta_k}|\dot u(\tau)|^3 d\tau.$$

b) Let u decrease in $[\alpha_k, \beta_k]$. Then $\Phi_k(v) + \frac{1}{2}K_R v^2$ is concave and we have

$$I_k = \int_{u(\beta_k)}^{u(\alpha_k)} \frac{d}{dv}\left(\dot w\big(u^{-1}(v)\big)\right)\dot u\big(u^{-1}(v)\big)dv.$$

Inequality (4.22) for f, η as above entails

$$I_k \geqslant \frac{\dot w^2(\beta_k)}{\Phi_k'\big(u(\beta_k)+\big)} - \frac{\dot w^2(\alpha_k)}{\Phi_k'\big(u(\alpha_k)-\big)} + \frac{1}{2}K_R\int_{\alpha_k}^{\beta_k}|\dot u(\tau)|^3 d\tau.$$

and we argue as in the proof of Theorem 4.18. □

For the sake of completeness we mention the following variant of Theorems 4.18, 4.19.

Corollary 4.23. *Let the operator F satisfy the hypotheses of Theorem 4.19. Then for every $u \in W^{2,1}(0,T)$, $|u|_\infty \leqslant R$ assertions (4.24)(i),(ii) hold and (4.24)(iii) is replaced with*

$$-\int_s^t \dot w(\tau)\ddot u(\tau)d\tau + P(t) - P(s) \geqslant \frac{1}{2}K_R\int_s^t |\dot u(\tau)|^3 d\tau \tag{4.32}$$

for almost all $0 < s < t < T$.

P r o o f. We exactly follow the argument of the proof of Theorem 4.18, where inequality (4.19) is applied in case a) and (4.21) in case b). □

Remark 4.24. In Theorems 4.18, 4.19 we always have $b_R - a_R \geqslant 2RK_R$. Indeed, for a monotone input u such that $u(0) = -R$, $u(T) = R$ the hypotheses yield either $b_R - RK_R \geqslant \Phi'(R) - RK_R \geqslant \Phi'(-R) + RK_R \geqslant a_R + RK_R$ or $b_R - RK_R \geqslant \Phi'(-R) - RK_R \geqslant \Phi'(R) + RK_R \geqslant a_R + RK_R$.

II.5 Models for fatigue and damage

We discuss here briefly two rate-independent models for the accumulation of fatigue in elastoplastic materials. Both admit a hysteretic interpretation and can be combined with hyperbolic equations of motion.

The first model consists in a rheological combination of elasto-brittle-plastic elements and leads to a generalization of the Preisach model with similar analytical properties and energy inequalities. The fatigue is manifested by the decrease of the elasticity modulus as a result of large amplitude loading.

The second model is based on the idea that the accumulation of fatigue due to a large number of bounded amplitude oscillations obeys the same mathematical rules as the accumulation of dissipated energy. In fact, it was shown in Brokate, Dressler, Krejčí (to appear/b) that the *rainflow method* of evaluation leads to a damage functional in the form of total variation of the output of a Preisach operator. We have seen in Theorem 4.3 that the same holds for the total energy dissipation which has the form $\operatorname*{Var}_{[0,T]} \mathcal{D}(\lambda, u)$, where $\mathcal{D}$ is the dissipation operator. We assume that the elasticity modulus is a decreasing function of the dissipated energy and we observe that a singularity due to the accumulated fatigue may occur in a finite time.

We do not deal with the rainflow method itself which has no connection to hyperbolic equations. An interested reader can find a good information in Brokate, Sprekels (to appear).

A nonlinear elasto-brittle-plastic model

The basic rheological elements for the construction of a nonlinear elasto-brittle-plastic model are the *nonlinear elastic element* from Remark 3.9

$\mathcal{N} \,:\, \varepsilon = g(\sigma),\ U = G(\sigma) = \sigma g(\sigma) - \int_0^\sigma g(v)\,dv,$

the *brittle element* from Example I.1.6

$\mathcal{B}_h \,:\, \varepsilon(t) H(h - \|\sigma\|_{[0,t]}) = 0,\ \sigma(t)\big[1 - H(h - \|\sigma\|_{[0,t]})\big] = 0,\ U = 0$

with a *fragility parameter* $h > 0$, and the *rigid-plastic element* from Example I.1.4

$\mathcal{R}_r \,:\, \sigma \in [-r, r],\ \dot\varepsilon(\sigma - \tilde\sigma) \geqslant 0 \quad \forall \tilde\sigma \in [-r, r], U = 0$

with a *yield point* $r > 0$.

Let us assume now that a system $\{\mathcal{N}_{h,r};\, h, r \geqslant 0\}$ of nonlinear elastic elements is given with a constitutive law $\varepsilon = g(\sigma, h, r)$ and potential energy $U_{h,r} = G(\sigma, h, r) := \sigma g(\sigma, h, r) - \int_0^\sigma g(v, h, r)\,dv$. We define the rheological combination (see Sect. I.1)

$$\underset{0}{\mathcal{M}} - \sum_{r>0} \underset{r}{\mathcal{M}} \,|\, \underset{r}{\mathcal{R}}, \tag{5.1}$$

where M_r is the nonlinear multibrittle element

$$\underset{r}{\mathcal{M}} = \underset{0,r}{\mathcal{N}} - \sum_{h>0} \underset{h,r}{\mathcal{N}} \,|\, \underset{h}{\mathcal{B}}\,. \tag{5.2}$$

Assuming that the constitutive function g fulfils the condition

(5.3) (i) g is continuous in $\mathbb{R}^1 \times [0,\infty[^2$ together with its derivatives $\dfrac{\partial g}{\partial\sigma}, \dfrac{\partial^2 g}{\partial\sigma^2}$,

(ii) $\dfrac{\partial g}{\partial\sigma}(\sigma,h,r) \geqslant 0 \qquad \forall(\sigma,h,r) \in \mathbb{R}^1\times]0,\infty[^2$,

(iii) $\sigma\cdot g(\sigma,h,r) > 0 \qquad \forall\sigma \neq 0,\ h,r \geqslant 0$,

we argue similarly as in Example I.1.7 to derive the constitutive relation for a single elastobrittle element in the form

$$\underset{h,r}{\mathcal{N}} \,|\, \underset{h}{\mathcal{B}} \,:\, \varepsilon(t) = g\left(\sigma(t)\left[1 - H(h - \|\sigma\|_{[0,t]})\right], h, r\right), \tag{5.4}$$

consequently

$$\underset{r}{\mathcal{M}} \,:\, \varepsilon(t) = g\big(\sigma(t),0,r\big) + \int_0^{\|\sigma\|_{[0,t]}} g\big(\sigma(t),h,r\big)dh. \tag{5.5}$$

The constitutive relations for the model $\mathcal{M}_r \,|\, \mathcal{R}_r$ therefore read

$$\begin{cases} \sigma = \sigma_{\mathcal{M}} + \sigma_{\mathcal{R}},\ \sigma_{\mathcal{R}} \in [-r,r],\ \dot\sigma_{\mathcal{M}}(\sigma_{\mathcal{R}} - \tilde\sigma) \geqslant 0 \quad \forall\tilde\sigma \in [-r,r], \\ \varepsilon(t) = g\big(\sigma_{\mathcal{M}}(t),0,r\big) + \int_0^{\|\sigma\|_{[0,t]}} g\big(\sigma_{\mathcal{M}}(t),h,r\big)dh. \end{cases} \tag{5.6}$$

Let us choose for the sake of simplicity the virgin initial configuration $\lambda = 0$ for the stress $\sigma_{\mathcal{M}}$. Then $\sigma_{\mathcal{M}} = p_r(0,\sigma)$, where p_r is the play operator (2.6). From Proposition 2.5 we infer $\|\sigma_{\mathcal{M}}\|_{[0,t]} = \|p_r(0,\sigma)\|_{[0,t]} = \max\{0, \|\sigma\|_{[0,t]} - r\}$, and for the model (5.1) we obtain a constitutive relation in operator form $\varepsilon = \mathcal{G}(\sigma)$, where

$$\begin{aligned} \mathcal{G}(\sigma)(t) := {} & g\big(\sigma(t),0,0\big) + \int_0^{\|\sigma\|_{[0,t]}} g\big(\sigma(t),h,0\big)dh + \\ & + \int_0^\infty g\big(p_r(0,\sigma)(t),0,r\big)dr + \iint\limits_{\Omega(\|\sigma\|_{[0,t]})} g\big(p_r(0,\sigma)(t),h,r\big)dh\,dr, \end{aligned} \tag{5.7}$$

with the notation $\Omega(R) := \big\{(h,r) \in]0,\infty[^2;\ h + r < R\big\}$.

We analogously derive the corresponding formula for the potential energy

$$\begin{aligned} \mathcal{U}(\sigma)(t) := {} & G\big(\sigma(t),0,0\big) + \int_0^{\|\sigma\|_{[0,t]}} G\big(\sigma(t),h,0\big)dh + \\ & + \int_0^\infty G\big(p_r(0,\sigma)(t),0,r\big)dr + \iint\limits_{\Omega(\|\sigma\|_{[0,t]})} G\big(p_r(0,\sigma)(t),h,r\big)dh\,dr, \end{aligned} \tag{5.8}$$

The operator $\mathcal{G}$ is obviously rate independent and maps $C([0,T])$ into $C([0,T])$ and $W^{1,p}(0,T)$ into $W^{1,p}(0,T)$, $1 \leqslant p \leqslant \infty$. Its properties are analogous to those of the Preisach operator. We have in particular

Theorem 5.1. *Let g satisfy (5.3) and let $R > 0$ be given. Then*

(i) *there exists a constant $c_0(R) > 0$ such that $|\mathcal{G}(\sigma_1) - \mathcal{G}(\sigma_2)|_\infty \leqslant c_0(R)|\sigma_1 - \sigma_2|_\infty$ for every $\sigma_1, \sigma_2 \in C([0,T])$, $|\sigma_1|_\infty, |\sigma_2|_\infty \leqslant R$;*

(ii) *there exists a constant $c_1(R) > 0$ such that $|\mathcal{G}(\sigma_1) - \mathcal{G}(\sigma_2)|_{1,1} \leqslant c_1(R)|\sigma_1 - \sigma_2|_{1,1}$ for every $\sigma_1, \sigma_2 \in W^{1,1}(0,T)$, $|\sigma_1|_{1,1}, |\sigma_2|_{1,1} \leqslant R$.*

(iii) *Assume that $g(-v,h,r) = -g(v,h,r)$ for every $(v,h,r) \in \mathbb{R}^1 \times [0,\infty[^2$. Then for every $\sigma \in W^{1,1}(0,T)$ the energy dissipation law*

$$\sigma(t)\frac{d}{dt}\mathcal{G}(\sigma)(t) - \frac{d}{dt}\mathcal{U}(\sigma)(t) = \frac{d}{dt}Y(\|\sigma\|_{[0,t]}) + \Big|\frac{d}{dt}\mathcal{D}(\sigma)(t)\Big| \quad \text{a.e.} \tag{5.9}$$

holds with a fatigue function

$Y(x) = \int_{|\sigma(0)|}^{x} \big[\int_0^y g(v,y,0)dv + \int_0^y \int_0^{y-r} g(v,y-r,r)dv\,dr\big]dy$

which is nondecreasing in $[|\sigma(0)|, |\sigma|_\infty]$ and a dissipation operator

$\mathcal{D}(\sigma)(t) = \int_0^\infty rg(p_r(0,\sigma)(t),0,r)dr + \iint\limits_{\Omega(\|\sigma\|_{[0,t]})} rg\big(p_r(0,\sigma)(t),h,r\big)dh\,dr.$

(iv) *Assume that there exist $\varrho > 0$, $A_\varrho > 0$ such that for $|v| + h + r < \varrho$ we have $\frac{\partial g}{\partial v}(v,h,r) > A_\varrho$ and $\frac{\partial^2 g}{\partial v^2}(v,h,0) = 0$. Then the trajectories of $\mathcal{G}$ satisfy the hypotheses of Theorem 4.19.*

P r o o f. The inequalities in (i) and (ii) are easy consequences of Proposition 1.1. In (iii), a similar computation as in Lemma 4.1 yields for a.e. t

$$\begin{aligned}\dot q(t) := \sigma(t)\frac{d}{dt}\mathcal{G}(\sigma)(t) - \frac{d}{dt}\mathcal{U}(\sigma)(t) = \int_0^\infty r\frac{\partial g}{\partial v}\big(\xi_r(t),0,r\big)|\dot\xi_r(t)|dr + \\ + \iint\limits_{\Omega(\|\sigma\|_{[0,t]})} r\frac{\partial g}{\partial v}\big(\xi_r(t),h,r\big)|\dot\xi_r(t)|\,dh\,dr + \frac{d}{dt}(\|\sigma\|_{[0,t]})\Big[\int_0^{\sigma(t)} g(v,\|\sigma\|_{[0,t]},0)dv + \\ + \int_0^{\|\sigma\|_{[0,t]}} \big(g(\xi_r(t),\|\sigma\|_{[0,t]}-r,r)\sigma(t) - G(\xi_r(t),\|\sigma\|_{[0,t]}-r,r)\big)dr\Big],\end{aligned}$$

where we denote $\xi_r := p_r(0,\sigma)$. We see that (5.9) holds if $\frac{d}{dt}(\|\sigma\|_{[0,t]}) = 0$.

Assume now that $\frac{d}{dt}(\|\sigma\|_{[0,t]}) > 0$. Then $\sigma(t) = \pm\|\sigma\|_{[0,t]}$, $\xi_r(t) = \pm\big(\|\sigma\|_{[0,t]} - r\big)$ for $r < \|\sigma\|_{[0,t]}$, and

$$\begin{aligned}\Big|\frac{d}{dt}\iint\limits_{\Omega(\|\sigma\|_{[0,t]})} rg\big(\xi_r(t),h,r\big)\,dh\,dr\Big| = \iint\limits_{\Omega(\|\sigma\|_{[0,t]})} r\frac{\partial g}{\partial v}\big(\xi_r(t),h,r\big)|\dot\xi_r(t)|\,dh\,dr + \\ + \frac{d}{dt}(\|\sigma\|_{[0,t]})\int_0^{\|\sigma\|_{[0,t]}} g\big(\|\sigma\|_{[0,t]}-r, \|\sigma\|_{[0,t]}-r, r\big)dr.\end{aligned}$$

We therefore have

$$\dot q(t) = \Big|\frac{d}{dt}\,\mathcal{D}(\sigma)(t)\Big| + \frac{d}{dt}(\|\sigma\|_{[0,t]})\Big[\int_0^{\|\sigma\|_{[0,t]}} g(v,\|\sigma\|_{[0,t]},0)\,dv + \\ + \int_0^{\|\sigma\|_{[0,t]}}\int_0^{\|\sigma\|_{[0,t]}-r} g\big(v,\|\sigma\|_{[0,t]}-r,r\big)\,dv\,dr\Big]$$

and (5.9) follows.

To verify (iv) we proceed as in the proof of Proposition 4.22. Putting $b(s) := \int_0^s \frac{\partial g}{\partial v}(0,h,0)dh$, $a := \frac{\partial g}{\partial v}(0,0,0)$ we have

$$\text{(5.10)}\quad \mathcal{G}(\sigma)(t) = \big(a + b(\|\sigma\|_{[0,t]})\big)\sigma(t) + \int_0^\infty g(\xi_r(t),0,r)dr + \iint\limits_{\Omega(\|\sigma\|_{[0,t]})} g(\xi_r(t),h,r)dh\,dr.$$

Let σ be chosen in such a way that $\|\sigma\|_\infty \leqslant R$, where R is to be specified and assume that σ increases in $[t_0,t_1]$. For $\lambda(r) := p_r(0,\sigma)(t_0)$ and $t\in[t_0,t_1]$ Lemma 2.4 yields

$$\text{(5.11)}\quad \mathcal{G}(\sigma)(t) = \mathcal{G}(\sigma)(t_0) + a\big(\sigma(t)-\sigma(t_0)\big) + b(\|\sigma\|_{[0,t]})\sigma(t) - b(\|\sigma\|_{[0,t_0]})\sigma(t_0) + \\ + \int_0^{m_\lambda(\sigma(t))}\Big(g\big(\sigma(t)-r,0,r\big) - g\big(\lambda(r),0,r\big)\Big)\,dr + \\ + \int_0^{m_\lambda(\sigma(t))}\int_0^{\|\sigma\|_{[0,t]}-r}\Big(g\big(\sigma(t)-r,h,r\big) - g\big(\lambda(r),h,r\big)\Big)\,dh\,dr.$$

Put $\sigma_0 := \|\sigma\|_{[0,t]} \in \big[\sigma(t_0),R\big]$. The trajectory Φ for $v\in[\sigma(t_0),R]$ has the form

$$\Phi(v) = \Phi\big(\sigma(t_0)\big) + a\big(v-\sigma(t_0)\big) + \int_0^v\int_0^{m_\lambda(s)}\frac{\partial g}{\partial v}(s-r,0,r)dr\,ds + \\ + \begin{cases} b(\sigma_0)\big(v-\sigma(t_0)\big) & + \int_0^v\int_0^{m_\lambda(s)}\int_0^{\sigma_0-r}\frac{\partial g}{\partial v}(s-r,h,r)dh\,dr\,ds \quad \text{for } v\leqslant\sigma_0, \\ b(v)v - b(\sigma_0)\sigma(t_0) & + \int_0^v\int_0^{m_\lambda(s)}\int_0^{v-r}\frac{\partial g}{\partial v}(s-r,h,r)dh\,dr\,ds \quad \text{for } v>\sigma_0. \end{cases}$$

Similarly as in the proof of Proposition 4.22 we show that if $R>0$ and $K_R>0$ are chosen sufficiently small, then there exists a nondecreasing function ζ such that $\Phi'(v) - K_R v = \zeta\big(m_\lambda(v)\big)$ for $v\in]\sigma(t_0),R[$, and the assertion follows. The case where σ decreases is analogous. □

Remark 5.2. For the constitutive law $\varepsilon = \mathcal{G}(\sigma)$ given by (5.10) the elasticity modulus is equal to $\frac{1}{a+b(\|\sigma\|_{[0,t]})}$, hence it decreases with increasing value of $\|\sigma\|_{[0,t]}$.

A DIFFERENTIAL MODEL OF FATIGUE

The model of fatigue in Theorem 5.1 does not explain the phenomenon of *cyclic fatigue*, where a large number of bounded amplitude oscillations may produce singularities (cracks). The analysis of the rainflow method of damage evaluation in Brokate, Dressler, Krejčí (to appear/b) suggests that the dissipated energy might be considered as a measure of fatigue in such cases. A more general physical discussion on this point can be found in Chapter 7.3 of Lemaitre and Chaboche (1985).

The model presented here is a modification of a one-yield elastoplastic model with kinematic hardening as on Fig. 4, where the elasticity modulus is a decreasing function of the dissipated energy. We shall see that the corresponding constitutive operator is rate independent with locally convex/concave trajectories like in Theorem 4.19 and, moreover, with a possibly finite lifetime as a result of material fatigue.

Let us consider the constitutive equation

$$\varepsilon(t) = \frac{1}{E}\big(1 + \alpha q^2(t)\big)\sigma(t) + A p_r(\lambda, \sigma)(t), \tag{5.12}$$

where E, α, A, r are given positive constants, $p_r(\lambda, \cdot)$ is the play operator with initial configuration $\lambda \in \Lambda$ and $q(t)$ is the energy dissipated during the interval $[0, t]$. A natural choice of potential energy

$$U(t) := \frac{1}{2E}\big(1 + \alpha q^2(t)\big)\sigma^2(t) + \frac{A}{2} p_r^2(\lambda, v)(t) \tag{5.13}$$

leads formally to a differential equation for q, namely

$$\dot{q} = \dot{\varepsilon}\sigma - \dot{U} = \frac{\alpha}{E} q\dot{q}\sigma^2 + Ar|\dot{\xi}_r|, \tag{5.14}$$

where $\xi_r := p_r(\lambda, \sigma)$.

We assume in the sequel that the material is initially for $t = 0$ in an undeformed and undamaged state, i.e.

$$\lambda = 0,\ q(0) = 0. \tag{5.15}$$

Equation (5.14) has the form

$$\dot{q} = \frac{a}{1 - cq\sigma^2}|\dot{\xi}_r|, \tag{5.16}$$

where $c := \frac{\alpha}{E} > 0,\ a := Ar > 0$.

We immediately see that identity 5.12 defines a thermodynamically consistent rate independent constitutive law provided q is a solution of (5.16) and $1 - cq(t)\sigma^2(t) > 0$. A singularity occurs as soon as $1 - cq(t-)\sigma^2(t-) = 0$.

Our goal is to express (5.12) in terms of a continuous constitutive operator in the space of *continuous functions.*

For $\sigma \in C([0,T])$ put

$$V(\sigma)(t) := a \operatorname*{Var}_{[0,t]}(\xi_r) = \frac{a}{r} \int_0^t \big(\sigma(\tau) - \xi_r(\tau)\big) d\xi_r(\tau). \tag{5.17}$$

We infer from Propositions 1.1, I.4.11 and Theorem V.1.26 that $V(\sigma)$ is a nondecreasing continuous function and $V : C([0,T]) \to C([0,T])$ is a continuous operator. Equation (5.16) can be rewritten in the form

$$q(t) = \int_0^t \frac{dV(\sigma)(\tau)}{1 - cq(\tau)\sigma^2(\tau)}. \tag{5.18}$$

Proposition 5.3. *Let $\sigma \in C([0,T])$ be given. Put $D := \{(t,q) \in [0,T[\times[0,\infty[;\ 1 - cq\sigma^2(t) > 0\}$. Then for each $(t_0, q_0) \in D$ there exists $t_1 > t_0$ and a unique solution $q \in C([t_0,t_1])$ of the equation*

$$q(t) = q_0 + \int_{t_0}^t \frac{dV(\sigma)(\tau)}{1 - cq(\tau)\sigma^2(\tau)}, \quad t \in [t_0, t_1]. \tag{5.19}$$

Proof. Put $\delta := \frac{1}{2}\big(1 - cq_0\sigma^2(t_0)\big) > 0$. We find $t_1 > t_0$ such that

$$\delta c q_0 |\sigma^2(t) - \sigma^2(t_0)| + c|\sigma|_\infty^2 \big(V(\sigma)(t) - V(\sigma)(t_0)\big) < \delta^2 \tag{5.20}$$

for all $t \in [t_0, t_1]$. Let $Z_\delta \subset C([t_0,t_1])$ be the (convex) closed set

$$Z_\delta := \big\{u \in C([t_0,t_1]);\ u(t_0) = q_0,\ 1 - cu(t)\sigma^2(t) \geqslant \delta \quad \forall t \in [t_0,t_1]\big\}$$

and let $\Gamma : Z_\delta \to C([t_0,t_1])$ be the operator

$$\Gamma(u)(t) := q_0 + \int_{t_0}^t \frac{dV(\sigma)(\tau)}{1 - cu(\tau)\sigma^2(\tau)}, \quad t \in [t_0,t_1].$$

Using (5.20) we easily check that Z_δ is nonempty (the constant function $u(t) \equiv q_0$ belongs to Z_δ) and that Γ is a contraction which maps Z_δ into Z_δ. The assertion now follows from a standard fixed point argument. □

Corollary 5.4. *For every $\sigma \in C([0,T])$ there exists a unique maximal solution $q : [0,T^*[\to [0,\infty[$ of equation (5.18). This solution is continuous and nondecreasing in $[0,T^*[$ and $\big(T^*, q(T^*-)\big) \in \partial D$.*

Corollary 5.4 immediately follows from Proposition 5.3. The following result on continuous dependence of q on σ plays a substantial role in the sequel.

Theorem 5.5. *Let* $\sigma \in C([0,T])$ *be given and let* $q : [0,T^*[\to [0,\infty[$ *be the maximal solution of (5.18). For an arbitrary* $\gamma \in]0,T^*[$ *put*

$$\delta := \tfrac{1}{2} \min_{[0,T^*-\gamma]} \big(1 - cq(t)\sigma^2(t)\big) > 0.$$

Let $\{\sigma_n; n \in \mathbb{N}\} \subset C([0,T])$ *be a sequence,* $\lim_{n\to\infty} |\sigma_n - \sigma|_\infty = 0$ *and let* $q_n : [0,T_n^*[\to [0,\infty[$ *be the corresponding maximal solutions of (5.18). Then there exists* $n_0 > 0$ *such that for all* $n \geqslant n_0$ *we have*

(i) $T_n^* \geqslant T^* - \gamma$,

(ii) $1 - cq_n(t)\sigma_n^2(t) \geqslant \delta \quad \forall t \in [0,T^*-\gamma]$,

(iii) $\lim_{n\to\infty} \|q_n - q\|_{[0,T^*-\gamma]} = 0$.

The proof of Theorem 5.5 relies on Gronwall's inequality in the following form.

Lemma 5.6. *Let* w, ζ *be nonnegative continuous functions in* $[0,T]$, $\zeta(0) = 0$, ζ *nondecreasing, and let* M, N *be nonnegative constants. Assume that*

$$w(t) \leqslant M + N \int_0^t w(\tau)\,d\zeta(\tau) \quad \forall t \in [0,T].$$

Then $w(t) \leqslant Me^{N\zeta(t)} \quad \forall t \in [0,t]$.

P r o o f of Lemma 5.6. From elementary identities

$$\int_0^s e^{-N\zeta(t)} d\left(\int_0^t w(\tau)\,d\zeta(\tau)\right) = \int_0^s e^{-N\zeta(t)} w(t)\,d\zeta(t),$$
$$\int_0^s \left(\int_0^t w(\tau)\,d\zeta(\tau)\right) de^{-N\zeta(t)} = -N\int_0^s e^{-N\zeta(t)} \left(\int_0^t w(\tau)\,d\zeta(\tau)\right) d\zeta(t)$$

and from the integration-by-parts formula V(1.22) we obtain

$$\int_0^s e^{-N\zeta(t)} w(t)\,d\zeta(t) = N\int_0^s e^{-N\zeta(t)} \left(\int_0^t w(\tau)\,d\zeta(\tau)\right) d\zeta(t) + e^{-N\zeta(s)} \int_0^s w(t)\,d\zeta(t)$$

and Lemma 5.6 follows easily. □

P r o o f of Theorem 5.5. For $t \in [0,T^*-\gamma]$ and $n \in \mathbb{N}$ put

$$M_n(t) := \Big| \int_0^t \frac{d\big(V(\sigma_n) - V(\sigma)\big)(\tau)}{1 - cq(\tau)\sigma^2(\tau)} \Big| + \frac{c}{2\delta^2} \int_0^t q(\tau)|\sigma_n^2(\tau) - \sigma^2(\tau)|\,dV(\sigma_n)(\tau),$$
$$N := \frac{c}{2\delta^2} \sup\{|\sigma_n|_\infty^2;\ n \in \mathbb{N}\}.$$

For each $n \in \mathbb{N}$ we find a minimal $\gamma_n \in]0,T_n^*[$ such that

$$1 - cq_n(t)\sigma_n^2(t) \geqslant \delta \quad \forall t \in [0,T_n^* - \gamma_n].$$

The quantities M_n, N were chosen in such a way that

$$|q_n(t) - q(t)| \leqslant M_n(t) + N \int_0^t |q_n(\tau) - q(\tau)|\, dV(\sigma_n)(\tau)$$

holds for all $0 \leqslant t \leqslant \min\{T^* - \gamma, T_n^* - \gamma_n\}$ and Lemma 5.6 yields

$$|q_n(t) - q(t)| \leqslant \|M_n\|_{[0,T^*-\gamma]} e^{NV(\sigma_n)(t)}. \tag{5.21}$$

We have $1 - cq(t)\sigma^2(t) \geqslant 2\delta$ for $\delta \in [0, T^* - \gamma]$ and $\lim\limits_{n\to\infty} \|M_n\|_{[0,T^*-\gamma]} = 0$; from (5.21) we infer $T_n^* - \gamma_n \geqslant T^* - \gamma$ for n sufficiently large and the assertion follows easily. □

The right-hand side of (5.12) defines an operator F with domain $C([0,T])$

$$F(\sigma)(t) := \frac{1}{E}\big(1 + \alpha q^2(t)\big)\sigma(t) + Ap_r(0,\sigma)(t), \tag{5.22}$$

where q is the solution of equation (5.18).

From Theorem 5.5 we derive the following properties of the operator F.

Corollary 5.7. *Let F be the operator (5.22). Then for every $\sigma \in C([0,T])$ there exists* a critical time $T^* \in]0,T]$ *such that $F(\sigma)$ is continuous in $[0,T^*[$. Moreover, if $\{\sigma_n;\, n \in \mathbb{N}\} \subset C([0,T])$ is a sequence such that $\lim\limits_{n\to\infty} |\sigma_n - \sigma|_\infty = 0$ and $T_n^* \in]0,T]$ is the critical time corresponding to σ_n, then*

(i) $\liminf\limits_{n\to\infty} T_n^* \geqslant T^*$,

(ii) *$F(\sigma_n) \to F(\sigma)$ locally uniformly in $[0,T^*[$.*

We conclude this section with the formulation of sufficient conditions for the convexity/concavity of the trajectories of F.

Proposition 5.8. *Let F be the operator (5.22) and let $\sigma \in C([0,T])$ be given such that $|\sigma|_\infty \leqslant 2r$. Assume that σ is monotone (nonincreasing or nondecreasing) in an interval $[t_1,t_2] \subset [0,T^*[$, where T^* is the critical time. Then there exists a bounded interval $]v_1,v_2[\supset$ Conv$\{\sigma(t_1),\sigma(t_2)\}$ and a Lipschitz continuous function $\Phi : [v_1,v_2] \to \mathbb{R}^1$ such that*

(i) *$\Phi'(v) \geqslant \frac{1}{E}$ for a.e. $v \in]v_1,v_2[$,*

(ii) *Φ is convex in $[\sigma(t_1),v_2]$ and concave in $[v_1,\sigma(t_1)]$,*

(iii) *$F(\sigma)(t) = \Phi\big(\sigma(t)\big)$ for all $t \in [t_1,t_2]$.*

P r o o f. The operator F is odd. It suffices therefore to assume that σ increases in $[t_1, t_2]$. Then $\xi_r(t) = p_r(0, \sigma)(t)$ is given by formula (1.6), i.e. $\xi_r(t) = \max\{\xi_r(t_1), \sigma(t) - r\}$ for $t \in [t_1, t_2]$. We find $\bar{t} \in [t_1, t_2]$ such that

$$\xi_r(t) = \begin{cases} \xi_r(t_1), & t \in [t_1, \bar{t}], \\ \sigma(t) - r, & t \in]\bar{t}, t_2]. \end{cases}$$

For $\bar{t} = t_2$ we trivially have $q = \text{const}.$ in $[t_1, t_2]$, hence Φ is affine; otherwise q has the form $q(t) = X\big(\sigma(t)\big)$, where X is the solution of the problem

$$\text{(5.23)} \qquad \begin{aligned} &\text{(i)} \quad X(v) = q(t_1) \quad \text{for } v \in [\sigma(t_1), \sigma(\bar{t})], \\ &\text{(ii)} \quad \frac{dX}{dv} = \frac{a}{1 - cv^2 X} \quad \text{for } v > \sigma(\bar{t}). \end{aligned}$$

The solution of equation (5.23)(ii) blows up for a finite value $v \to v^*-$; we choose v_2 arbitrarily in the interval $[\sigma(t_2), v^*[$.

We have $\sigma(\bar{t}) = \xi_r(t_1) + r$ and Corollary 2.6 entails $|\xi_r(t_1)| = |\xi_r(t_1) - \xi_{|u|_\infty}(t_1)| \leqslant |u|_\infty - r$, hence $\sigma(\bar{t}) \geqslant 0$. The function Φ has the form

$$\text{(5.24)} \qquad \Phi(v) = \frac{1}{E}\big(1 + \alpha X^2(v)\big)v + A\max\{\sigma(\bar{t}), v\} - Ar$$

for $v \in [\sigma(\bar{t}), v_2]$ and we can check by a straightforward differentiation that Φ is convex. The case of σ nonincreasing is obtained by symmetry. □

Remark 5.9. The situation is not the same here as in Theorem 4.19 because of singularities which may occur in a finite time. On the other hand, one can easily formulate a sufficient condition in terms of the constants a, c which guarantees that the solution of (5.18) blows up before $|\sigma(t)|$ attains the value $2r$, so that the condition $|\sigma(t)| < 2r$ is automatically satisfied in $[0, T^*[$, see Krejčí (1994).

III. Hyperbolic equations with hysteretic constitutive laws

The problem of coupling equation of motion I(0.1) with an elastoplastic constitutive law is not new. The existence and uniqueness of solutions for the Prandtl-Reuss model with a single yield surface has been established in Duvaut, Lions (1972), a multiyield Prandtl-Ishlinskii model was considered by Visintin (1987). In both cases, the solution is constructed via penalization method for semilinear hyperbolic variational inequalities which corresponds to an approximation of the rate independent plasticity by a rate dependent visco-plasticity. This technique strictly requires that the elastic part of the constitutive law is linear.

We present here an alternative approach which consists in transforming the semilinear variational inequality into a *quasilinear* equation with a hysteresis operator. This enlarges considerably the variety of problems which can be solved, especially in situations, where some knowledge of the memory structure is available. The strong energy inequalities for scalar hysteresis operators derived in the preceding chapter enable us to treat the following questions related either to uniaxial problems or to multiaxial problems with a componentwise scalar hysteretic constitutive law:

- stability with respect to quasilinear perturbations,
- global boundedness of solutions (nonresonance),
- asymptotic decay of solutions,
- existence of periodic solutions,
- asymptotic stability of periodic solutions

by methods which have been developed essentially for the theory of semilinear equations, such as compactness and monotonicity methods of Lions (1969) based on Galerkin-type or discrete approximations and classical methods in the theory of periodic solutions of Vejvoda et al. (1981)

III.1 Construction of solutions

We show here two typical examples of equations of motion I(0.1) with a hysteretic constitutive operator which can be solved by classical functional-analytic methods. For constitutive operators with a specific monotonicity property one can use Minty's trick to establish the solvability of an initial-boundary value problem in the general vector case. In scalar (uniaxial) models, no monotonicity is required any more and strong a priori estimates for solutions of the space-discretized system enable us to pass to the limit. In both situations, the main tool is the second order energy inequality I(3.31), II(4.25) for hysteresis operators. The hyperbolicity of the wave equation with hysteresis is confirmed by the boundedness of the speed of propagation.

Monotonicity method

Similarly as in Sect.I.1 we denote by $\mathbb{T}$ the space of symmetric tensors $N \times N$ endowed with the scalar product $\langle \xi, \eta \rangle_{\mathbb{T}} := \sum_{i,j=1}^{N} \xi_{ij} \eta_{ij}$ and norm $|\xi|_{\mathbb{T}} := \langle \xi, \xi \rangle_{\mathbb{T}}^{1/2}$. The symbol $\langle \cdot, \cdot \rangle$ is used for the scalar product $\langle u, v \rangle := \sum_{i=1}^{N} u^i v^i$ in $\mathbb{R}^N$.

We consider the system of the type I(0.1) with a normalized density $\varrho \equiv 1$

$$
\begin{aligned}
&\text{(i)} \quad u_{tt} = D^* \sigma + q(x,t), \\
&\text{(ii)} \quad \varepsilon = Du, \\
&\text{(iii)} \quad \sigma = F(\varepsilon),
\end{aligned} \tag{1.1}
$$

$(x,t) \in \Omega \times]0,T[$, where $\Omega \subset \mathbb{R}^N$ is a given open bounded set with a smooth boundary, $T > 0$ is a given number, $D : W^{1,2}(\Omega, \mathbb{R}^N) \to L^2(\Omega, \mathbb{T})$, $D^* : W^{1,2}(\Omega; \mathbb{T}) \to L^2(\Omega, \mathbb{R}^N)$ are differential operators given by the formulae

$$
(Du)_{ij} := \frac{1}{2} \left(\frac{\partial u^i}{\partial x_j} + \frac{\partial u^j}{\partial x_i} \right), \ (D^* \sigma)_i := \sum_{j=1}^{N} \frac{\partial \sigma_{ij}}{\partial x_j}, \tag{1.2}
$$

$q(x,t)$ is a given function and F is a constitutive operator whose properties are specified later. For the sake of simplicity we prescribe homogeneous Dirichlet conditions

$$
u(x,t) = 0 \quad \text{for} \quad (x,t) \in \partial\Omega \times]0,T[\tag{1.3}
$$

on the boundary of Ω and initial conditions

$$
u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), \ x \in \Omega. \tag{1.4}
$$

To simplify the notation, we introduce the spaces $H^0 := L^2(\Omega; \mathbb{R}^N)$, $H^0_{\mathbb{T}} := L^2(\Omega; \mathbb{T})$, $H^k := W^{k,2}(\Omega; \mathbb{R}^N)$ for $k \geqslant 1$, $\overset{\circ}{H}{}^1 := \overset{\circ}{W}{}^{1,2}(\Omega; \mathbb{R}^N)$, $L := \{\varepsilon \in L^2(0,T; H^0_{\mathbb{T}}); \varepsilon_t \in L^2(0,T; H^0_{\mathbb{T}})\}$. The standard L^2-norms in $H^0, H^0_{\mathbb{T}}$ are denoted by $|\cdot|_{H^0}, |\cdot|_{H^0_{\mathbb{T}}}$, respectively. By Korn's inequality (see Nečas, Hlaváček (1981)) we can define in $H^1, \overset{\circ}{H}{}^1$ equivalent norms $|w|_{H^1} := \left(|w|^2_{H^0} + |Dw|^2_{H^0_{\mathbb{T}}} \right)^{1/2}$, $|w|_{\overset{\circ}{H}{}^1} := |Dw|_{H^0_{\mathbb{T}}}$.

The general regularity theory for scalar elliptic equations (see Chap. 8 of Gilbarg, Trudinger (1983)) is applicable without modification to the vector elliptic operator $D^* D$

and we can define in H^2 an equivalent norm $|w|_{H^2} := (|w|^2_{H^0} + |Dw|^2_{H^0_{\mathbb{T}}} + |D^*Dw|^2_{H^0})^{1/2}$. The norm in L is chosen in a natural way

$$|\varepsilon|_L := \left(|\varepsilon(\cdot,0)|^2_{H^0_{\mathbb{T}}} + \int_0^T |\varepsilon_t(\cdot,t)|^2_{H^0_{\mathbb{T}}} dt \right)^{1/2}.$$

The operator F is assumed to act on L according to the formula

$$F(\varepsilon)(x,t) := f\big(x, \varepsilon(x,\cdot)\big)(t), \ (x,t) \in \Omega \times [0,T], \tag{1.5}$$

where $f(x,\cdot) : W^{1,2}(0,T;\mathbb{T}) \to W^{1,2}(0,T;\mathbb{T})$ for $x \in \Omega$ is a causal (see I(1.29)) operator with the following properties.

Assumption 1.1. *There exist constants $a, b, c > 0$ and a causal operator $\mathcal{V} : \bar{\Omega} \times \big(W^{1,2}(0,T;\mathbb{T})\big)^2 \to W^{1,1}(0,T)$ such that for every $(x, y, \varepsilon, \vartheta) \in \Omega \times \Omega \times \big(W^{1,2}(0,T;\mathbb{T})\big)^2$ we have*

$$\begin{aligned}
(1.6)\ \text{(i)} \quad & |f(x,\varepsilon)_t(t)|_{\mathbb{T}} \leqslant b|\dot{\varepsilon}(t)|_{\mathbb{T}} \quad \text{a.e.}, \\
\text{(ii)} \quad & |f(x,\varepsilon)(T) - f(y,\vartheta)(T)|^2_{\mathbb{T}} \leqslant c\Big(|x-y|^2 + |\varepsilon(0) - \vartheta(0)|^2_{\mathbb{T}} + \\
& \qquad + \int_0^T |\dot{\varepsilon}(t) - \dot{\vartheta}(t)|^2_{\mathbb{T}} dt\Big), \\
\text{(iii)} \quad & \big\langle f(x,\varepsilon)(t) - f(x,\vartheta)(t), \dot{\varepsilon}(t) - \dot{\vartheta}(t)\big\rangle_{\mathbb{T}} \geqslant \frac{d}{dt}\mathcal{V}(x,\varepsilon,\vartheta)(t) \quad \text{a.e.}, \\
\text{(iv)} \quad & \mathcal{V}(x,\varepsilon,\vartheta)(t) = \mathcal{V}(x,\vartheta,\varepsilon)(t) \geqslant 0, \ \mathcal{V}(x,\varepsilon,\varepsilon)(t) = 0 \quad \forall t \in [0,T], \\
\text{(v)} \quad & |\mathcal{V}(x,\varepsilon_1,\vartheta)(0) - \mathcal{V}(y,\varepsilon_2,\vartheta)(0)| \leqslant c \max\{|\varepsilon_1(0)|_{\mathbb{T}}, |\varepsilon_2(0)|_{\mathbb{T}}, |\vartheta(0)|_{\mathbb{T}}\} \cdot \\
& \qquad \cdot (|x-y| + |\varepsilon_1(0) - \varepsilon_2(0)|_{\mathbb{T}}), \\
\text{(vi)} \quad & \textit{if } \varepsilon \in W^{2,2}(0,T;\mathbb{T}), \textit{ then } \int_0^T \langle f(x,\varepsilon)_\tau, \ddot{\varepsilon}(\tau)\rangle_{\mathbb{T}}\, d\tau \geqslant \frac{a}{2}|\dot{\varepsilon}(T)|^2_{\mathbb{T}} - \frac{b}{2}|\dot{\varepsilon}(0)|^2_{\mathbb{T}}.
\end{aligned}$$

Using Proposition I.3.9, Remark I.3.10 and Theorem I.3.16 we check that Assumption 1.1 is fulfilled for instance for

$$f(x,\varepsilon) = (E-\gamma)\varepsilon + \gamma\,\mathcal{S}\left(\varphi\big(x,\varepsilon(0)\big), \varepsilon\right) \tag{1.7}$$

for some $E > \gamma > 0$, where $\mathcal{S}$ is the stop operator I(3.11) with a Lipschitz continuous initial configuration $\varphi : \bar{\Omega} \times \mathbb{T} \to Z$. Other examples of operators f can be constructed in the class of Prandtl-Ishlinskii operators of stop type I(1.44).

The existence and uniqueness result reads as follows.

Theorem 1.2. *Let* $q \in L^2(0,T;H^0)$, $u_0 \in H^2 \cap \overset{\circ}{H}{}^1$, $u_1 \in \overset{\circ}{H}{}^1$ *be given such that* $q_t \in L^1(0,T;H^0)$ *and let Assumption 1.1 be fulfilled. Then there exists a unique* $u \in L^\infty(0,T;\overset{\circ}{H}{}^1)$ *such that* $u_{tt} \in L^\infty(0,T;H^0)$, $Du_t \in L^\infty(0,T;H^0_{\mathbb{T}})$, *initial conditions (1.4) are satisfied a.e. and the identity*

$$\int_\Omega \big[\, \langle u_{tt}(x,t) - q(x,t), w(x)\rangle + \langle F(Du)(x,t), Dw(x)\rangle_{\mathbb{T}} \,\big] dx = 0 \tag{1.8}$$

holds for every $w \in \overset{\circ}{H}{}^1$ *and a.e.* $t \in]0,T[$.

Identity (1.8) is a weak formulation of (1.1) based on the integration formula for regular functions

$$\int_\Omega [\langle \sigma, Dw\rangle_{\mathbb{T}} + \langle D^*\sigma, w\rangle]\, dx = \sum_{i,j=1}^{N} \int_{\partial\Omega} w^i \sigma_{ij}\nu_j dx, \tag{1.9}$$

where $\nu = (\nu_1,\dots,\nu_N)$ is the unit outward normal vector to $\partial\Omega$. Before proving Theorem 1.2 we verify that the integral in (1.8) is meaningful.

Lemma 1.3. *Let Assumption 1.1 hold. Then the operator* F *defined by (1.5) maps* L *into* L *and for every* $\varepsilon, \vartheta \in L$ *we have*

$$\sup_{t\in[0,T]} |F(\varepsilon)(\cdot,t) - F(\vartheta)(\cdot,t)|_{H^0_{\mathbb{T}}} \leqslant c|\varepsilon - \vartheta|_L. \tag{1.10}$$

P r o o f of Lemma 1.3. Let $\varepsilon \in L$ be given and let $\Omega^* \subset \Omega$ be the set of all $x \in \Omega$ such that $\varepsilon(x,\cdot) \in W^{1,2}(0,T;\mathbb{T})$, meas $(\Omega \setminus \Omega^*) = 0$, and for $x \in \Omega^*$ put $\sigma(x,\cdot) := f(x,\varepsilon(x,\cdot)) \in W^{1,2}(0,T;\mathbb{T})$. Let $\{\varepsilon^n; n \in \mathbb{N}\} \subset C^2(\bar\Omega \times [0,T];\mathbb{T})$ be a sequence such that $\lim_{n\to\infty} |\varepsilon^n - \varepsilon|_L = 0$ and put $\sigma^n := F(\varepsilon^n)$. By (1.6) we have

$$\begin{aligned}|\sigma^n(x,t) - \sigma^n(y,s)|_{\mathbb{T}} &\leqslant |\sigma^n(x,t) - \sigma^n(x,s)|_{\mathbb{T}} + |\sigma^n(x,s) - \sigma^n(y,s)|_{\mathbb{T}} \leqslant \\ &\leqslant \big(b(t-s) + c(x-y)\big)(1+T)\big(1 + |\varepsilon^n|_{C^2(\bar\Omega\times[0,T];\mathbb{T})}\big),\end{aligned}$$

hence $\sigma^n \in C(\bar\Omega \times [0,T];\mathbb{T})$. From (1.6)(ii) and the causality of f we further infer for $x \in \Omega^*$ and $t \in [0,T]$

$$|\sigma(x,t) - \sigma^n(x,t)|_{\mathbb{T}} \leqslant c\left(|\varepsilon(x,0) - \varepsilon^n(x,0)|^2_{\mathbb{T}} + \int_0^t |\varepsilon_\tau(x,\tau) - \varepsilon^n_\tau(x,\tau)|^2_{\mathbb{T}} d\tau\right)^{1/2},$$

consequently σ is measurable, $\sigma \in L$ and $\lim_{n\to\infty} \sup_{t\in[0,T]} |\sigma(\cdot,t) - \sigma^n(\cdot,t)|_{H^0_{\mathbb{T}}} = 0$.

We now choose a sequence $\{\vartheta^n\} \subset C^2(\bar\Omega \times [0,T];\mathbb{T})$ such that $|\vartheta^n - \vartheta|_L \to 0$. Inequality (1.10) for $\varepsilon^n, \vartheta^n$ follows from (1.6)(ii) and passing to the limit we obtain the assertion. □

P r o o f of Theorem 1.2.
Uniqueness. Let u, v be two solutions of (1.8), (1.4). We subtract the identities (1.8) for u and v and put $w(x) := u_t(x,t) - v_t(x,t)$. Integrating with respect to t we obtain

$$\frac{1}{2}\int_\Omega |u_t - v_t|^2(x,t)\,dx + \int_\Omega \int_0^t \langle F(Du) - F(Dv), Du_\tau - Dv_\tau\rangle (x,\tau)\,dx\,d\tau = 0,$$

hence $u = v$ by (1.6)(iii),(iv).

Existence. Let $\{e_k; k \in \mathbb{N}\}$ be a complete orthonormal system in H^0 of eigenfunctions of the operator D^*D in Ω with homogeneous Dirichlet boundary conditions, i.e.

$$-D^*De_k = \lambda_k e_k,\ \lambda_k > 0,\ e_k \in H^2 \cap \overset{\circ}{H}{}^1 . \tag{1.11}$$

For a fixed $n \in \mathbb{N}$ we define the vector $\mathbf{v} = (v_1, \ldots, v_n) : [0,T] \to \mathbb{R}^n$ as the solution of the system for $k = 1, \ldots, n$

$$\ddot v_k(t) = \int_\Omega \Big[-\big\langle F(Du^{(n)})(x,t), De_k(x)\big\rangle_{\mathbb{T}} + \big\langle q(x,t), e_k(x)\big\rangle\Big]\,dx, \tag{1.12}$$

$$v_k(0) = \int_\Omega \langle u_0(x), e_k(x)\rangle\,dx, \quad \dot v_k(0) = \int_\Omega \langle u_1(x), e_k(x)\rangle\,dx, \tag{1.13}$$

$$u^{(n)}(x,t) = \sum_{k=1}^n v_k(t) e_k(x) \quad \text{for } (x,t) \in \Omega \times]0,T[. \tag{1.14}$$

Putting $\mathbf{y} = (y_1, \ldots, y_n) := (\dot v_1, \ldots, \dot v_n)$ we can rewrite system (1.12) in the form

$$\begin{cases} \dot{\mathbf{y}} = & G(\mathbf{v}) + \mathbf{q} \\ \dot{\mathbf{v}} = & \mathbf{y} \end{cases} \tag{1.15}$$

with a Lipschitz continuous causal operator $G : C^1([0,T];\mathbb{R}^n) \to C([0,T];\mathbb{R}^n)$ and a given vector $\mathbf{q} \in W^{1,1}(0,T;\mathbb{R}^n)$. Let $K > 0$ be a constant such that $|G(\mathbf{u}) - G(\mathbf{v})|_\infty \leqslant K(|\mathbf{u} - \mathbf{v}|_\infty + |\dot{\mathbf{u}} - \dot{\mathbf{v}}|_\infty)$. We define successive approximations $\mathbf{y}_j^\ell, \mathbf{v}_j^\ell$ in the following way. For $\ell = 0, 1, \ldots$ put $\kappa := \min\left\{\frac{1}{2}, \frac{1}{2K}\right\}$, $t_\ell := \kappa\ell$, $\mathbf{v}_0^1(t) := \mathbf{v}(0)$, $\mathbf{y}_0^1(t) := \mathbf{y}(0)$ for $t \in [0,T]$, and

$$\begin{cases} \mathbf{y}_{j+1}^1(t) & := \mathbf{y}(0) + \int_0^{\min\{t,t_1\}} \big[G(\mathbf{v}_j^1)(\tau) + \mathbf{q}(\tau)\big]\,d\tau, \\ \mathbf{v}_j^1(t) & := \mathbf{v}(0) + \int_0^t \mathbf{y}_j^1(\tau)d\tau, \end{cases} \quad j = 0, 1, \ldots. \tag{1.16}$$

We have $|\mathbf{y}_{j+1}^1 - \mathbf{y}_j^1|_\infty \leqslant t_1 K(1 + t_1)|\mathbf{y}_j^1 - \mathbf{y}_{j-1}^1|_\infty \leqslant \frac{3}{4}|\mathbf{y}_j^1 - \mathbf{y}_{j-1}^1|_\infty$, hence $\{\mathbf{y}_j^1, \mathbf{v}_j^1;\ j = 0, 1, \ldots\}$ are uniformly convergent sequences in $C([0,T];\mathbb{R}^n)$ and their limits $\mathbf{y}^1, \mathbf{v}^1$ satisfy (1.15) in $[0, t_1]$. Repeating the procedure in $[t_\ell, t_{\ell+1}]$ with $\mathbf{y}_j^{\ell+1}(\mathrm{t}) := \mathbf{y}^\ell(\mathrm{t})$,

$\mathbf{v}_{\rm j}^{\ell+1}({\rm t}) := \mathbf{v}^{\ell}({\rm t})$ for $t \in [0, t_\ell]$, $\ell = 1, 2, \dots$ until $t_{\ell+1} \geqslant T$ we construct by induction a continuously differentiable solution $\mathbf{y}, \mathbf{v}$ of (1.15). Notice that the causality of G plays a substantial role here. Coming back to system (1.12), (1.13) we conclude that it admits a global solution $\mathbf{v} \in W^{3,1}(0, T; \mathbb{R}^n)$.

We now derive estimates for the sequence $\{u^{(n)}\}$ defined by (1.14) which enable us to pass to the limit as $n \to \infty$. For an arbitrary $t \in [0, T]$ we differentiate (1.12), multiply by $\ddot{v}_k(t)$, sum over k and integrate $\int_0^t$. Assumption (1.6)(vi) yields

$$(1.17) \qquad |u_{tt}^{(n)}(\cdot, t)|_{H^0}^2 + a|u_t^{(n)}(\cdot, t)|_{\overset{\circ}{H}{}^1}^2 \leqslant |u_{tt}^{(n)}(\cdot, 0)|_{H^0}^2 + b|u_t^{(n)}(\cdot, 0)|_{\overset{\circ}{H}{}^1} + \\ + 2\int_0^t \int_\Omega \big\langle q_\tau(x, \tau), u_{\tau\tau}^{(n)}(x, \tau)\big\rangle dx\, d\tau.$$

Upper bounds for the right-hand side of this last inequality can be found using special properties of the basis $\{e_k\}$. Putting $\varepsilon_k := \frac{1}{\sqrt{\lambda_k}} De_k$ we have by (1.9) $\int_\Omega \langle \varepsilon_k(x), \varepsilon_\ell(x)\rangle_{\mathbb{T}}\, dx = \delta_{k\ell}$, where $\delta_{k\ell}$ is the Kronecker symbol and $Du_t^{(n)}(x, 0) = \sum_{k=1}^n \big(\int_\Omega \big\langle Du^1(y), \varepsilon_k(y)\big\rangle_{\mathbb{T}} dy\big)\, \varepsilon_k(x)$, hence $\int_\Omega \big\langle Du_t^{(n)}(x, 0) - Du^1(x), Du_t^{(n)}(x, 0)\big\rangle_{\mathbb{T}} dx = 0$. This yields

$$(1.18) \qquad |u_t^{(n)}(\cdot, 0)|_{\overset{\circ}{H}{}^1} \leqslant |u_1|_{\overset{\circ}{H}{}^1} \quad \text{independently of } n.$$

The L^2-norm of $u_{tt}^{(n)}(\cdot, 0)$ will be estimated using equation (1.12). The operator F is causal; this means that there exists a function $\varphi : \Omega \times \mathbb{T}$ such that $F(\varepsilon)(x, 0) = \varphi\big(x, \varepsilon(x, 0)\big)$ and by (1.6)(ii) φ is Lipschitz. We have by (1.12), (1.9)

$$|u_{tt}^{(n)}(\cdot, 0)|_{H^0}^2 = \sum_{k=0}^n |\ddot{v}_k(0)|^2 \leqslant 2\left(|D^*\varphi(\cdot, Du^{(n)}(\cdot, 0)|_{H^0}^2 + |q(\cdot, 0)|_{H^0}^2\right) \\ \leqslant c_1 + c_2|u^{(n)}(\cdot, 0)|_{H^2},$$

where c_1, c_2 are constants independent of n. We estimate $|u^{(n)}(\cdot, 0)|_{H^0} \leqslant |u_0|_{H^0}$, $|Du^{(n)}(\cdot, 0)|_{H^0_{\mathbb{T}}}^2 = \sum_{k=1}^n \lambda_k \left|\int_\Omega \langle u_0(x), e_k(x)\rangle\, dx\right|^2 = \sum_{k=1}^n \left|\int_\Omega \langle Du_0(x), \varepsilon_k(x)\rangle_{\mathbb{T}}\, dx\right|^2 \leqslant |Du_0|_{H^0_{\mathbb{T}}}^2$, $|D^*Du^{(n)}(\cdot, 0)|_{H^0}^2 = \sum_{k=1}^n \lambda_k^2 |\int_\Omega \langle u_0(x), e_k(x)\rangle\, dx|^2 \leqslant |D^*Du_0|_{H^0}$, consequently

$$(1.19) \qquad |u_{tt}^{(n)}(\cdot, 0)|_{H^0}^2 \leqslant c_1 + c_2|u_0|_{H^2}.$$

Furthermore, $\int_0^t \int_\Omega \big\langle q_\tau(x, \tau), u_{\tau\tau}^{(n)}(x, \tau)\big\rangle dx\, d\tau \leqslant \sup_{\tau\in[0,t]} |u_{\tau\tau}^{(n)}(\cdot, \tau)|_{H^0} \int_0^t |q_\tau(\cdot, \tau)|_{H^0} d\tau$. By (1.17)-(1.19) there exists therefore a constant $M > 0$ which depends only on the data u_0, u_1, q such that

$$(1.20) \qquad \sup_{t\in[0,T]} |u_{tt}^{(n)}(\cdot, t)|_{H^0} \leqslant M, \quad \sup_{t\in[0,T]} |u_t^{(n)}(\cdot, t)|_{\overset{\circ}{H}{}^1} \leqslant M$$

and obviously also

$$\sup_{t\in[0,T]} |u^{(n)}(\cdot,t)|_{\overset{\circ}{H}{}^1} \leqslant M, \quad |F(Du^{(n)})|_L \leqslant M. \tag{1.21}$$

There exists a subsequence of $\{u^{(n)}\}$ (still denoted by $u^{(n)}$) and functions $u \in L^\infty(0,T;\overset{\circ}{H}{}^1)$, $\sigma \in L$ such that $u_t \in L^\infty(0,T;\overset{\circ}{H}{}^1), u_{tt} \in L^\infty(0,T;H^0)$ and $u_{tt}^{(n)} \to u_{tt}$ in $L^\infty(0,T;H^0)$ weakly-star, $Du^{(n)} \to Du$, $Du_t^{(n)} \to Du_t$ in $L^\infty(0,T;H^0_{\mathbb{T}})$ weakly-star, $F(Du^{(n)}) \to \sigma$ in L weakly and by Theorem V.2.3, $u^{(n)} \to u, u_t^{(n)} \to u_t$ in $L^2(0,T;H^0)$ strongly. The limit function u satisfies initial conditions (1.4). Indeed, from Hölder's inequality and (1.20) we infer $|u_t^{(n)}(\cdot,t)-u_t^{(n)}(\cdot,0)|_{H^0} \leqslant Mt$ for $t\in[0,T]$ and $n\in\mathbb{N}$. Passing to the limit as $n\to\infty$ we conclude $|u_t(\cdot,t)-u_1|_{H^0} \leqslant Mt$ a.e. and similarly $|u(\cdot,t)-u_0|_{H^0} \leqslant Mt$ a.e.

We now pass to the limit in (1.12) as $n\to\infty$. For an arbitrary $\gamma \in L^2(0,T)$ the limit functions u,σ satisfy

$$\int_0^T \int_\Omega \big[\,\langle u_{tt}(x,t)-q(x,t),\gamma(t)e_k(x)\rangle + \langle \sigma(x,t),\gamma(t)De_k(x)\rangle_{\mathbb{T}}\,\big]dx\,dt = 0 \tag{1.22}$$

for every $k\in\mathbb{N}$. The set of finite linear combinations of functions of the form $\gamma(t)e_k(x)$ is dense in $L^2(0,T;\overset{\circ}{H}{}^1)$, consequently

$$\int_0^T \int_\Omega [\langle u_{tt}(x,t)-q(x,t),w(x,t)\rangle + \langle \sigma(x,t),Dw(x,t)\rangle_{\mathbb{T}}]\,dx\,dt = 0 \tag{1.23}$$

for every $w\in L^2(0,T;\overset{\circ}{H}{}^1)$. On the other hand, from (1.12) it follows for a.e. t

$$\int_\Omega \Big[\big\langle u_{tt}^{(n)}(x,t)-q(x,t),u_t^{(n)}(x,t)\big\rangle + \big\langle F(Du^{(n)})(x,t),Du_t^{(n)}(x,t)\big\rangle_{\mathbb{T}}\Big]\,dx = 0. \tag{1.24}$$

Putting $w=u_t$ in (1.23) we obtain using (1.24)

$$\lim_{n\to\infty}\int_0^T \int_\Omega \big\langle F(Du^{(n)}),Du_t^{(n)}\big\rangle_{\mathbb{T}} dx\,dt = \int_0^T \int_\Omega \big\langle \sigma,Du_t\big\rangle_{\mathbb{T}} dx\,dt. \tag{1.25}$$

For an arbitrary $w\in\overset{\circ}{H}{}^1$, $\gamma\in L^2(0,T)$ and $\delta>0$ put $z(x,t) := u(x,t) - \delta w(x)\int_0^t \gamma(\tau)d\tau$. The monotonicity (1.6)(iii) enables us to use Minty's trick in the inequality

$$\int_0^T \int_\Omega \big\langle F(Du^{(n)})-F(Dz),Du_t^{(n)}-Dz_t\big\rangle_{\mathbb{T}} dx\,dt \geqslant \\ \geqslant -\int_\Omega \mathcal{V}\left(x,Du^{(n)}(x,\cdot),Dz(x,\cdot)\right)(0)\,dx. \tag{1.26}$$

Passing to the limit as $n \to \infty$ we obtain from (1.25), (1.26) and (1.6)(v),(iv)

$$\int_0^T \int_\Omega \langle \sigma(x,t) - F(Dz)(x,t), Dw(x) \rangle_{\mathbb{T}} \gamma(t)\, dx\, dt \geqslant 0 \tag{1.27}$$

and Lemma 1.3 yields for $\delta \to 0$

$$\int_\Omega \langle \sigma(x,t) - F(Du)(x,t), Dw(x) \rangle_{\mathbb{T}} dx = 0 \quad \text{a.e.} \tag{1.28}$$

Identity (1.8) now follows from (1.23) and (1.28). Theorem 1.2 is proved. □

Remark 1.4. The results of Theorem 1.2 are comparable to those obtained by Duvaut, Lions (1972) or Visintin (1987) by penalization method for variational inequalities in the case of constitutive operators of the form (1.7) or Prandtl-Ishlinskii operators of stop type. The hysteresis approach, however, enables us to treat more general classes of constitutive operators. This is particularly convincing in uniaxial models, where the assumptions on the constitutive operator are formulated in terms of geometrical properties of its trajectories without referring to variational inequalities.

Compactness method

A one-dimensional version of system (1.1) will be considered here in the form

$$\begin{cases} v_t &= \sigma_x + q, \\ \varepsilon_t &= v_x, \\ \varepsilon &= F(\sigma), \end{cases} \tag{1.29}$$

for $(x,t) \in]0,1[\times]0,T[$, where F is a given constitutive operator and q is a given forcing. Formally, system (1.29) is equivalent to the wave equation in displacements

$$u_{tt} - F^{-1}(u_x)_x = q \tag{1.30}$$

analogous to (1.8) provided the inverse F^{-1} exists.

Substituting the electric field E for v, magnetic field H for σ and magnetic induction $B = \mu H + M$ for ε, where M is the magnetization and μ is the permeability, we can interpret system (1.29) with a hysteretic constitutive operator $M = \mathcal{W}(H)$ as one-dimensional Maxwell's equations in a ferromagnetic medium.

For (1.29) we prescribe initial and boundary conditions

$$v(x,0) = v^0(x),\ \sigma(x,0) = \sigma^0(x) \qquad \text{for } x \in]0,1[, \tag{1.31}$$

$$v(0,t) = \sigma(1,t) = 0 \qquad \text{for } t \in]0,T[. \tag{1.32}$$

We assume again that operator F has the form

$$F(\sigma)(x,t) := f\big(x, \sigma(x,\cdot)\big)(t) \tag{1.33}$$

for inputs σ such that $\sigma(x,\cdot) \in C([0,T])$ for all $x \in [0,1]$, where

(1.34) (i) $f(x,\cdot) : C([0,T]) \to C([0,T])$ is a hysteresis (i.e. rate-independent and causal) operator for every $x \in [0,1]$,

(ii) $f : [0,1] \times C([0,T]) \to C([0,T])$ is continuous and the trajectories of $f(x,\cdot)$ introduced in Remark II.4.15 fulfil the following hypothesis.

Assumption 1.5. *There exist constants* $R > 0$, $b_R > a_R > 0$, $K_R \geqslant 0$ *such that for every* $x \in [0,1]$ *and* $\sigma \in C([0,T])$, $|\sigma|_\infty \leqslant R$ *the trajectory* $\varphi(x,\cdot)$ *of* $f(x,\cdot)$ *along* σ *in a monotonicity interval* $[t_1, t_2]$ *is absolutely continuous and satisfies*

(1.35) (i) $\varphi \in C([0,1] \times J)$, $J = \operatorname{Conv}\{\sigma(t_1), \sigma(t_2)\}$,

(ii) $a_R \leqslant \frac{\partial}{\partial \varrho}\varphi(x,\varrho) \leqslant b_R$ *for a.e.* $\varrho \in \operatorname{Int} J$,

(iii) *if* σ *is nondecreasing in* $[t_1,t_2]$, *then* $\varrho \mapsto \varphi(x,\varrho) - \frac{1}{2}K_R\varrho^2$ *is convex in* J,

(iv) *if* σ *is nonincreasing in* $[t_1,t_2]$, *then* $\varrho \mapsto \varphi(x,\varrho) + \frac{1}{2}K_R\varrho^2$ *is concave in* J.

We now state a global existence theorem for system (1.29), (1.31), (1.32). The results of Sections II.3, II.4 show that operators F of the form (1.33) with $f(x,\cdot) := aI + \mathcal{W}(\lambda(x,\cdot),\cdot)$, where a is positive and $\mathcal{W}$ is a Prandtl-Ishlinskii operator II(3.2), a Preisach operator II(3.13), a generalized Preisach operator with fatigue II(5.7) or a Della Torre operator II(3.31) with initial configurations $\lambda \in C([0,1];\Lambda_R)$, where Λ_R is endowed with the sup-norm, as well as perturbations of these operators of the form $f(x,u) = au + \mathcal{W}(\lambda,u) + \delta g(u)$, where $g : \mathbb{R}^1 \to \mathbb{R}^1$ is a smooth function and $\delta > 0$ is sufficiently small satisfy the hypotheses above. The case of the operator II(5.22) is more delicate and we refer the reader to Krejčí (1994).

Theorem 1.6. *Let the constitutive operator* F *satisfy (1.34) and Assumption 1.5 and let* $v^0, \sigma^0 \in W^{1,2}(0,1)$, $q \in L^1(0,T;L^2(0,1))$ *be given such that* $v^0(0) = \sigma^0(1) = 0$, $q_t \in L^1\big(0,T;L^2(0,1)\big)$ *and the inequality*

$$\int_0^1 \big[3q^2(x,0) + 2|\sigma_x^0(x)|^2 + \frac{1}{a_R}|v_x^0(x)|^2\big]dx + \\ + 4\left(\int_0^T \left(\int_0^1 q_\tau^2(x,\tau)dx\right)^{1/2} d\tau\right)^2 \leqslant \frac{R^2}{16} \tag{1.36}$$

is fulfilled. Then there exists at least one solution $(v,\sigma,\varepsilon) \in \big[C([0,1]\times[0,T])\big]^3$ *of (1.29), (1.31), (1.32) such that* $v_t, v_x, \sigma_t, \sigma_x, \varepsilon_t \in L^\infty\big(0,T;L^2(0,1)\big), |\sigma|_\infty \leqslant \frac{R}{2}$ *and (1.29) holds almost everywhere in* $]0,1[\times]0,T[$.

The restriction on the size of the data is related to the boundedness of the convexity domains of the operators $f(x,\cdot)$. We see from (1.36) that for operators satisfying Assumption 1.5 for every $R>0$ and $a_R \geqslant a_0 > 0$ we have global existence for any regular data.

Nothing is known about uniqueness of solutions in the general case. In the next section we prove uniqueness results for equations with Preisach or Prandtl-Ishlinskii constitutive operators.

The solution will be constructed by discretization in space. For $n\in\mathbb{N}$ and $j = 0,\dots,n$ we denote $F_j := f\left(\frac{j}{n},\cdot\right)$ and consider the system of ODE's for $t\in]0,T[$,

$$\text{(1.37)} \qquad \begin{aligned} &\text{(i)} && \dot v_j(t) = \Delta_j\sigma(t) + q_j(t),\\ &\text{(ii)} && \dot\varepsilon_j(t) = \Delta_{j-1}v(t),\\ &\text{(iii)} && \varepsilon_j(t) = F_j(\sigma_j)(t), \end{aligned}$$

where $\Delta_j\sigma := n(\sigma_{j+1}-\sigma_j)$, $\Delta_{j-1}v := n(v_j - v_{j-1})$, $q_j(t) := n\int_{\frac{j}{n}}^{\frac{j+1}{n}} q(x,t)dx$, $j = 1,\dots,n-1$. We prescribe initial and "boundary" conditions

$$\text{(1.38)} \qquad v_j(0) = v^0\big(\frac{j}{n}\big),\ \sigma_j(0) = \sigma^0\big(\frac{j}{n}\big),\ j=0,\dots,n,\ v_0(t) = \sigma_n(t) = 0.$$

The global solvability of (1.37), (1.38) for a fixed $n\in\mathbb{N}$ will be established in the following two lemmas.

Lemma 1.7. *Let the hypotheses of Theorem 1.6 hold and let* $t_0\in[0,T[$, $V>0$ *be given. Assume that* $\{(v_j,\sigma_j,\varepsilon_j); j=1,\dots,n-1\}$ *are absolutely continuous functions satisfying (1.37), (1.38) for* $t\in[0,t_0]$, $\max_j \|\sigma_j\|_{[0,t_0]} \leqslant \frac{R}{2}$, $\max_j |v_j(t_0)| \leqslant V$. *Then the solution* $\{(v_j,\sigma_j,\varepsilon_j)\}$ *can be extended to the interval* $[0,t_0+s]\cap[0,T]$ *in such a way that* $\max_j \|\sigma_j\|_{[0,t_0+s]} \leqslant R$, *where* $s := \frac{Ra_R}{4n}\big(V + 2nTR + n\int_0^T\int_0^1 |q(x,t)|dx\,dt\big)^{-1}$.

P r o o f. The hypotheses of Lemma 1.7 are automatically satisfied for $t_0 = 0$ and $V := \frac{R\sqrt{a_R}}{4}$. Initial conditions for ε_j follow from the assumption of causality of the operators $f(x,\cdot)$ which entails that there exists a continuous function $\eta : [0,1]\times\mathbb{R}^1\to\mathbb{R}^1$ such that $\varepsilon_j(0) = \eta\big(\frac{j}{n},\sigma_j(0)\big)$.

For an arbitrary $t_0\in[0,T[$ satisfying the hypothesis we denote

$$\sigma_j^0(t) := \begin{cases} \sigma_j(t) & \text{for } t\in[0,t_0],\\ \sigma_j(t_0) & \text{for } t\in]t_0,t_0+s] \end{cases}$$

and for $k \in \mathbb{N}$ we define the sequences $\{(v_j^k, \sigma_j^k, \varepsilon_j^k); k \in \mathbb{N}\}$ recursively by the formulae

$$(1.39)\qquad \begin{aligned} &\text{(i)} \quad v_j^k(t) := v_j(t_0) + \int_{t_0}^t \big(\Delta_j \sigma^k(\tau) + q_j(\tau)\big) d\tau, \\ &\text{(ii)} \quad \varepsilon_j^{k+1}(t) := \varepsilon_j(t_0) + \int_{t_0}^t \Delta_{j-1} v^k(\tau) d\tau \end{aligned}$$

for $t \in [0, t_0 + s]$. The transition $\varepsilon_j^{k+1} \mapsto \sigma_j^{k+1}$ will be performed in the following way. Let $R_{k+1} : C([t_0, t_0 + s]) \to C([t_0, t_0 + s])$ be the linearization operator

$$(1.40)\qquad R_{k+1}(u)(t) := u(t_i) + \frac{t - t_i}{t_{i+1} - t_i}\big(u(t_{j+1}) - u(t_i)\big) \qquad \text{for } t \in [t_i, t_{i+1}],$$

where $t_i := t_0 + \frac{is}{k+1}$, $i = 0, \ldots, k+1$, and put

$$\tilde{\varepsilon}_j^{k+1}(t) := \begin{cases} \varepsilon_j(t) & \text{for } t \in [0, t_0], \\ R_{k+1}\big(\varepsilon_j^{k+1}\big)(t) & \text{for } t \in [t_0, t_0 + s]. \end{cases}$$

Assume that for some $k \in \mathbb{N} \cup \{0\}$ we have

$$(1.41)\qquad \|\sigma_j^k\|_{[0,t_0+s]} \leqslant R \quad \text{for all} \quad j = 1, \ldots, n-1.$$

From (1.39) we infer $|v_j^k(t)| \leqslant V + 2nTR + n\int_0^T \int_0^1 |q(x,t)| dx\, dt$ and $|\dot{\varepsilon}_j^{k+1}(t)| \leqslant 2n \max_j |v_j^k(t)| \leqslant \frac{Ra_R}{2s}$ for a.e. $t \in]0, t_0 + s[$, therefore also $|\dot{\tilde{\varepsilon}}_j^{k+1}(t)| \leqslant \frac{Ra_R}{2s}$ a.e.

The function $\tilde{\varepsilon}_j^{k+1}$ is monotone in $[t_{i-1}, t_i]$, $i = 1, \ldots, k+1$. For $t \in [0, t_0]$ we define $\sigma_j^{k+1}(t) := \sigma_j(t)$ and continue by induction over i similarly as in the proof of Theorem II.3.17: assuming that σ_j^{k+1} is defined in $[0, t_i]$; the causality of F_j entails that its trajectory $\varphi_j^i : [-R, R] \to \mathbb{R}^1$ along σ_j^{k+1} in $[t_i, t_{i+1}]$ is independent of the values of $\sigma_j^{k+1}|_{]t_i, t_{i+1}]}$. We therefore can put

$$(1.42)\qquad \sigma_j^{k+1}(t) := \big(\varphi_j^i\big)^{-1}\big(\tilde{\varepsilon}_j^{k+1}(t)\big) \qquad \text{for } t \in [t_i, t_{i+1}],\ i = 0, \ldots, k.$$

Formula (1.42) is meaningful provided (1.41) holds for σ_j^{k+1}. From the construction and assumption (1.35)(ii) it follows $|\dot{\sigma}_j^{k+1}(t)| \leqslant \frac{1}{a_R}|\dot{\tilde{\varepsilon}}_j^{k+1}(t)| \leqslant \frac{R}{2s}$ a.e., hence $|\sigma_j^{k+1}(t)| \leqslant |\sigma_j^{k+1}(t_0)| + \int_{t_0}^t |\dot{\sigma}_j^{k+1}(\tau)| d\tau \leqslant R$.

We thus have equibounded equicontinuous sequences $\{v_j^k, \sigma_j^k, \varepsilon_j^k, \tilde{\varepsilon}_j^k;\ k \in \mathbb{N}\}$ in $C([0, t_0 + s])$ satisfying (1.39) and (1.41), $\tilde{\varepsilon}_j^k = F_j(\sigma_j^k), |\tilde{\varepsilon}_j^k - \varepsilon_j^k|_\infty \leqslant \max\{|\varepsilon_j^k(t) - \varepsilon_j^k(\tau)|; |t - \tau| \leqslant \frac{s}{k}\} \leqslant \frac{Ra_R}{2k}$. By Arzelà-Ascoli Theorem V.2.1 there exist uniformly convergent subsequences in $[0, t_0 + s]$ (still indexed by k) such that their limits $v_j := \lim_{k\to\infty} v_j^k$, $\sigma_j := \lim_{k\to\infty} \sigma_j^k$, $\varepsilon_j := \lim_{k\to\infty} \varepsilon_j^k = \lim_{k\to\infty} \tilde{\varepsilon}_j^k$ fulfil Lemma 1.7. □

Lemma 1.8. *Let inequality (1.36) hold and let* $\{v_j, \sigma_j, \varepsilon_j;\ j = 1, \dots, n-1\}$ *satisfy system (1.37), (1.38) in an interval* $[0, t_0] \subset [0, T]$, $\|\sigma_j\|_{[0,t_0]} \leqslant R$. *Put* $C_1 := 4\left[\int_0^1 (q^2(x,0) + |\sigma_x^0(x)|^2 + \frac{1}{2a_R}|v_x^0(x)|^2)dx + \left(\int_0^{t_0}\left(\int_0^1 q_t^2(x,t)dx\right)^{1/2} dt\right)^2\right]$. *Then*

$$\|\sigma_j\|_{[0,t_0]} \leqslant \frac{R}{2}, \quad \|v_j\|_{[0,t_0]} \leqslant \sqrt{b_R C_1} \quad \forall j = 1, \dots, n-1. \tag{1.43}$$

P r o o f. We differentiate (1.37)(i),(ii) $\frac{d}{dt}$ and multiply by $\dot v_j, \dot\sigma_j$, respectively. Summing and integrating $\int_0^t$ we obtain

$$\sum_{j=1}^{n-1}\left[\frac{1}{2}\dot v_j^2(t) + \int_0^t \ddot\varepsilon_j(\tau)\dot\sigma_j(\tau)d\tau\right] = \sum_{j=1}^{n-1}\left[\frac{1}{2}\dot v_j^2(0) + \int_0^t \dot q_j(\tau)\dot v_j(\tau)d\tau\right]. \tag{1.44}$$

The operators F_j satisfy the hypotheses of Theorem II.4.19 and $\varepsilon_j \in W^{2,1}(0,T)$, $\sigma_j \in W^{1,\infty}(0,T)$ for all $j = 1, \dots, n-1$, hence

$$\min\left\{b_R\dot\sigma_j^2(t), \frac{1}{a_R}\dot\varepsilon_j^2(t)\right\} \geqslant \dot\varepsilon_j(t)\dot\sigma_j(t) \geqslant \max\left\{a_R\dot\sigma_j^2(t), \frac{1}{b_R}\dot\varepsilon_j^2(t)\right\} \quad \text{a.e.},$$
$$\int_0^t \ddot\varepsilon_j(\tau)\dot\sigma_j(\tau)d\tau \geqslant \frac{1}{2}\dot\varepsilon_j(t)\dot\sigma_j(t) - \frac{1}{2a_R}\dot\varepsilon_j^2(0) + \frac{K_R}{2}\int_0^t |\dot\sigma_j(\tau)|^3 d\tau \quad \text{a.e.}$$

Combining this last inequality with (1.44) yields

$$\sum_{j=1}^{n-1}\left[\dot v_j^2(t) + \dot\varepsilon_j(t)\dot\sigma_j(t) + K_R\int_0^t |\dot\sigma_j(\tau)|^3 d\tau\right] \leqslant \tag{1.45}$$
$$\leqslant \sum_{j=1}^{n-1}\left[\dot v_j^2(0) + \frac{1}{a_R}\dot\varepsilon_j^2(0) + 2\int_0^t \dot q_j(\tau)\dot v_j(\tau)d\tau\right] \text{ a.e.}$$

From (1.45), (1.37), (1.36) and from the inequalities

$$\frac{1}{n}\sum_{j=1}^{n-1}\int_0^t \dot q_j(\tau)\dot v_j(\tau)d\tau \leqslant \left\|\frac{1}{n}\sum_{j=1}^{n-1}\dot v_j^2\right\|_{[0,t]}^{1/2}\int_0^t\left(\frac{1}{n}\sum_{j=1}^{n-1}\dot q_j^2(\tau)\right)^{1/2}d\tau \tag{1.46}$$
$$\leqslant \left\|\frac{1}{n}\sum_{j=1}^{n-1}\dot v_j^2\right\|_{[0,t]}^{1/2}\int_0^t\left(\int_0^1 q_\tau^2(x,\tau)dx\right)^{1/2}d\tau,$$

$$\frac{1}{n}\sum_{j=1}^{n-1}\dot v_j^2(0) \leqslant \frac{2}{n}\sum_{j=1}^{n-1}\left[n^2(\sigma_{j+1}-\sigma_j)^2 + q_j^2\right](0) \leqslant 2\int_0^1\left[|\sigma_x^0(x)|^2 + q^2(x,0)\right]dx, \tag{1.47}$$

$$\frac{1}{n}\sum_{j=1}^{n-1}\dot\varepsilon_j^2(0) \leqslant \int_0^1 |v_x^0(x)|^2 dx, \tag{1.48}$$

$$\left\|\int_0^1 q^2(x,\cdot)dx\right\|_{[0,t_0]} \leqslant 2\int_0^1 q^2(x,0)dx + 4\left(\int_0^{t_0}\left(\int_0^1 q_t^2(x,t)dx\right)^{1/2}dt\right)^2 \tag{1.49}$$

we obtain for a.e. $t \in]0, t_0[$

$$(1.50) \qquad \text{(i)} \quad \frac{1}{n}\sum_{j=1}^{n-1}\left[\dot v_j^2(t) + a_R \dot\sigma_j^2(t)\right] \leqslant C_1,$$

$$\text{(ii)} \quad \frac{1}{n}\sum_{j=1}^{n-1}|\Delta_j\sigma(t)|^2 \leqslant C_2 := 2\big(C_1 + \int_0^1 q^2(x,t)dx\big) \leqslant \frac{R^2}{4},$$

$$\text{(iii)} \quad \frac{1}{n}\sum_{j=1}^{n-1}|\Delta_{j-1}v(t)|^2 = \frac{1}{n}\sum_{j=1}^{n-1}\dot\varepsilon_j^2(t) \leqslant b_R C_1,$$

$$\text{(iv)} \quad |\sigma_j(t)| \leqslant \sum_{i=j}^{n-1}|\sigma_{i+1}(t) - \sigma_i(t)| \leqslant \frac{R}{2},$$

$$\text{(v)} \quad |v_j(t)| \leqslant \sum_{i=1}^{j}|v_i(t) - v_{i-1}(t)| \leqslant \sqrt{b_R C_1}$$

and Lemma 1.8 is proved. □

P r o o f of Theorem 1.6. From Lemmas 1.7, 1.8 it follows that system (1.37), (1.38) has a classical solution in [0,T] for every $n \in \mathbb{N}$ and estimates (1.50) hold for a.e. $t \in]0, T[$. For $x \in [\frac{j}{n}, \frac{j+1}{n}[$, $j = 0, \ldots, n-1$ and $t \in [0, T]$ we define the functions

$$(1.51) \qquad \begin{cases} \sigma^{(n)}(x,t) := \sigma_{j+1}(t), & \tilde\sigma^{(n)}(x,t) := \sigma_j(t) + \big(x - \frac{j}{n}\big)\,\Delta_j\sigma(t),\\ v^{(n)}(x,t) := v_j(t), & \tilde v^{(n)}(x,t) := v_j(t) + \big(x - \frac{j}{n}\big)\,\Delta_j v(t),\\ \varepsilon^{(n)}(x,t) := \varepsilon_{j+1}(t), & q^{(n)}(x,t) := q_j(t) \end{cases}$$

continuously extended to $x = 1$.

Estimates (1.50) show that functions $\sigma_t^{(n)}, v_t^{(n)}, \varepsilon_t^{(n)}, \tilde\sigma_t^{(n)}, \tilde v_t^{(n)}, \tilde\sigma_x^{(n)}, \tilde v_x^{(n)}$ are bounded in $L^\infty\big(0,T; L^2(0,1)\big)$ independently of n, $|\sigma^{(n)}(x,t) - \tilde\sigma^{(n)}(x,t)|^2 \leqslant \sum_{j=1}^{n-1}|\sigma_{j+1}(t) - \sigma_j(t)|^2 \leqslant \frac{C_2}{n}$, $|v^{(n)}(x,t) - \tilde v^{(n)}(x,t)|^2 \leqslant \frac{b_R C_1}{n}$. The space $W^{1,\mathbf{p}}(]0,1[\times]0,T[)$ for $\mathbf{p} = \big((\infty,2),(\infty,2)\big)$ is compactly embedded in $C([0,1]\times[0,T])$ by Corollary V.2.5; there exist therefore functions $v, \sigma \in C([0,1]\times[0,T])$ and $\xi \in L^\infty\big(0,T;L^2(0,1)\big)$ such that $v_t, v_x, \sigma_t, \sigma_x \in L^\infty\big(0,T;L^2(0,1)\big)$ and subsequences of the sequences above (still indexed by n) such that $v^{(n)} \to v, \sigma^{(n)} \to \sigma, \tilde v^{(n)} \to v, \tilde\sigma^{(n)} \to \sigma$ uniformly, $v_t^{(n)} \to v_t, \tilde v_t^{(n)} \to v_t, \tilde\sigma_t^{(n)} \to \sigma_t, \sigma_t^{(n)} \to \sigma_t, \varepsilon_t^{(n)} \to \xi, \tilde v_x^{(n)} \to v_x, \tilde\sigma_x^{(n)} \to \sigma_x$ weakly-star in $L^\infty\big(0,T;L^2(0,1)\big)$.

System (1.37) has the form

$$(1.52) \qquad \text{(i)} \quad v_t^{(n)} = \tilde\sigma_x^{(n)} + q^{(n)},$$

$$\text{(ii)} \quad \varepsilon_t^{(n)} = \tilde v_x^{(n)},$$

$$\text{(iii)} \quad \varepsilon^{(n)} = F^{(n)}\big(\sigma^{(n)}\big),$$

where $F^{(n)}$ is the operator $F^{(n)}(u)(x,t) := f\left(\frac{j+1}{n}, u(x,\cdot)\right)(t)$ for $x \in \left[\frac{j}{n}, \frac{j+1}{n}\right], j = 0,\dots,n$ and arbitrary u such that $u(x,\cdot) \in C([0,T])$ for all $x \in [0,1]$.

For each $n \in \mathbb{N}$ and $t \in [0,T]$ we have

$$\int_0^1 |q^{(n)}(x,t) - q(x,t)|dx \leqslant n \sum_{j=1}^{n-1} \int_{\frac{j}{n}}^{\frac{j+1}{n}} \int_{\frac{j}{n}}^{\frac{j+1}{n}} |q(x,t) - q(\xi,t)|dx\, d\xi,$$

hence $q^{(n)} \to q$ in L^1 by Mean Continuity Theorem (Proposition V.1.14). Furthermore, assumption (1.34)(ii) yields $\left|F^{(n)}(\sigma^{(n)})(x,t) - F(\sigma)(x,t)\right| = \left|f\left(\frac{j+1}{n}, \sigma^{(n)}(x,\cdot)\right)(t) - f\big(x,\sigma(x,\cdot)\big)(t)\right| \to 0$ uniformly in $[0,1]\times[0,T]$ as $n \to \infty$, hence $\varepsilon^{(n)}$ converge uniformly to $\varepsilon = F(\sigma)$ and $\varepsilon_t = \xi$. Passing to the limit in (1.52) we obtain (1.29). We moreover have $|\sigma^{(n)}(y,0) - \sigma^0(y)|^2 \leqslant \frac{1}{n}\int_0^1 |\sigma_x^0(x)|^2 dx$, $|v^{(n)}(y,0) - v^0(y)| \leqslant \frac{1}{n}\int_0^1 |v_x^0(x)|^2 dx$ for every $y \in [0,1]$, $\tilde{\sigma}^{(n)}(1,t) = \tilde{v}^{(n)}(0,t) = 0$. Conditions (1.31), (1.32) then follow from the uniform convergence. Theorem 1.6 is proved. □

Let us mention here an additional regularity result.

Proposition 1.9. *Let v, σ, ε be as in Theorem 1.6. Then the functions $v_x, \sigma_x : [0,T] \to L^2(0,1)$ are weakly continuous.*

P r o o f. The argument is standard (see Arosio (1981)). Let $t_n \to t_0 \in [0,T]$ be an arbitrary sequence and let $\delta > 0$ be given. For each $\psi \in L^2(0,1)$ we find $\tilde{\psi} \in \mathring{W}^{2,1}(0,1)$ such that $|\psi - \tilde{\psi}|_2 < \delta$. Then

$$\Big|\int_0^1 \big(\sigma_x(x,t_n) - \sigma_x(x,t_0)\big)\psi(x)dx\Big| \leqslant 2\delta \sup_t \Big(\int_0^1 |\sigma_x^2(x,t)|dx\Big)^{1/2} + \\ + \Big|\int_0^1 \big(\sigma(x,t_n) - \sigma(x,t_0)\big)\tilde{\psi}'(x)dx\Big|$$

and similarly for v_x. The assertion now follows from the estimates (1.50) and continuity of v and σ. □

To conclude this section we prove under a natural energy condition that system (1.29) is hyperbolic in the sense of bounded speed of propagation.

Proposition 1.10. *Let the hypotheses of Theorem 1.6 be fulfilled. Assume that there exists a potential energy operator $U : [0,1] \times W^{1,\infty}(0,T) \to W^{1,\infty}(0,T)$ and a constant $c > 0$ such that for every $u \in W^{1,\infty}(0,T)$*

(1.53)
$$\text{(i)} \quad U(x,u)(t) \geqslant \frac{1}{2c^2}u^2(t) \qquad \forall (x,t) \in [0,1]\times[0,T],$$
$$\text{(ii)} \quad u(t)\frac{\partial}{\partial t} f(x,u)(t) \geqslant \frac{\partial}{\partial t} U(x,u)(t) \text{ a.e.},$$
$$\text{(iii)} \quad U(x,u)(0) = 0 \quad \textit{if} \quad u(0) = 0.$$

Let there exist an interval $[a,b] \subset]0,1[$ *such that the data* σ^0, v^0, q *satisfy* $\sigma^0(x) = v^0(x) = 0$ *for* $x \in [a,b]$, $q(x,t) = 0$ *for* $(x,t) \in \Omega := \{(x,t) \in [0,1] \times [0,T];\ a + ct < x < b - ct\}$. *Then every solution* (v,σ) *of (1.29) vanishes in* $\bar{\Omega}$.

P r o o f (cf. Courant, Hilbert (1937)). Put $\mathcal{U}(\sigma)(x,t) := U\big(x, \sigma(x,\cdot)\big)(t)$. For $(x,t) \in \Omega$ we have $\big(\frac{1}{2}v^2 + \mathcal{U}(\sigma)\big)_t - (v\sigma)_x \leqslant 0$. For an arbitrary $\tau \in \big[0, \frac{b-a}{2c}\big]$ we denote $\Omega_\tau := \{(x,t) \in \Omega;\ t < \tau\}$. A straightforward integration yields

$$0 \geqslant \iint_{\Omega_\tau} \Big[\big(\frac{1}{2}v^2 + \mathcal{U}(\sigma)\big)_t - (v\sigma)_x\Big] dx\, dt = \int_{a+c\tau}^{b-c\tau} \Big(\frac{1}{2}v^2 + \mathcal{U}(\sigma)\Big)(x,\tau) dx +$$
$$+ \int_a^{a+c\tau} \Big(\frac{1}{2}v^2 + \mathcal{U}(\sigma) + \frac{1}{c}v\sigma\Big)\Big(x, \frac{x-a}{c}\Big) dx +$$
$$+ \int_{b-c\tau}^{b} \Big(\frac{1}{2}v^2 + \mathcal{U}(\sigma) - \frac{1}{c}v\sigma\Big)\Big(x, \frac{b-x}{c}\Big) dx,$$

and the assertion follows from (1.53)(i). □

Remark 1.11. The expressions $\Phi := \frac{1}{2}v^2 + \mathcal{U}(\sigma), \Psi := v\sigma$ are the *energy density* and *energy flow density*, respectively. Note that the hypothesis is fulfilled for the operator $f(x,\cdot) = \frac{1}{c^2}I + \mathcal{W}\big(\lambda(x,\cdot),\cdot\big)$, where $\mathcal{W}$ is a Preisach operator II(3.13) with initial configuration $\lambda(x,\cdot) = 0$ for $x \in [0,1]$, a Della Torre operator II(3.31) or a Preisach operator with fatigue II(5.7).

III.2 Uniqueness and asymptotics

Further investigation of qualitative properties of global solutions to system (1.29), (1.31), (1.32) constitutes the objective of this section. In general, the problem of uniqueness is open. Besides the easy case, where F is monotone as in Theorem 1.2, we explicitly formulate a uniqueness condition if F is a Preisach operator.

The main part is devoted to the asymptotic behavior of solutions as $t \to \infty$. In the previous section we proved that the convexity of loops of the constitutive operator implies that shocks do not occur. Here, we prove that *strict convexity* of loops implies nonresonance and decay of solutions.

Uniqueness

Theorem 2.1. *Let* $R_0 > 0$ *and a function* $\lambda \in C([0,1]; \Lambda_{R_0})$ *with* Λ_{R_0} *endowed with the sup-norm be given. Let the constitutive operator* F *be of the form (1.33),* *where* $f(x,\cdot) := \mathcal{F}_\varphi\big(\lambda(x,\cdot),\cdot\big)$ *is the Prandtl-Ishlinskii operator II(3.2) and let* h *be nondecreasing,* $a = h(0+) > 0$. *Then for every* $v^0, \sigma^0 \in W^{1,2}(0,1)$, $q \in L^1(]0,1[\times]0,T[)$ *such that* $v^0(0) = \sigma^0(1) = 0$, $q_t \in L^1\big(0,T;L^2(0,1)\big)$ *there exists a unique solution* $(v,\sigma,\varepsilon) \in \big(C[0,1]\times[0,T]\big)\big)^3$ *of (1.29), (1.31), (1.32) such that* $v_t, v_x, \sigma_t, \sigma_x, \varepsilon_t \in L^\infty\big(0,T;L^2(0,1)\big)$ *and (1.29) holds a.e.*

P r o o f. Existence follows from Theorem 1.6, since $R > 0$ can be chosen arbitrarily large and $a_R \geqslant a > 0$. Uniqueness follows from inequality II(4.11) similarly as in the proof of Theorem 1.2. $\square$

Theorem 2.2. *Let* $\mathcal{W}$ *be a Preisach operator satisfying the hypotheses of Theorem II.4.22 and let* F *be of the form (1.33) with* $f(x,\cdot) = aI + \mathcal{W}\big(\lambda(x,\cdot),\cdot\big)$, $a > 0$, $\lambda \in C([0,1];\Lambda_R)$. *Let* q, v^0, σ^0 *satisfy condition (1.36) and let* $(v^{(i)}, \sigma^{(i)}, \varepsilon^{(i)}) \in (C[0,1]\times[0,T])^3$, $i = 1,2$, *be two solutions of (1.29), (1.31), (1.32) such that* $\sigma_t^{(i)} \in L^1\big(0,T;L^\infty(0,1)\big)$, $i = 1,2$. *Then* $v^{(1)} = v^{(2)}, \sigma^{(1)} = \sigma^{(2)}, \varepsilon^{(1)} = \varepsilon^{(2)}$.

We see that the regularity $\sigma_t \in L^\infty\big(0,T;L^2(0,1)\big)$, $|\sigma|_\infty \leqslant \frac{R}{2}$ obtained in Theorem 1.6 is not sufficient in Theorem 2.2. To obtain uniqueness we have to assume that the solution σ is more regular with respect to x. However, the problem whether more regular data guarantee more regular solutions remains open.

P r o o f of Theorem 2.2. Put $w^{(i)} := \mathcal{W}(\lambda, \sigma^{(i)})$, $i = 1,2$. From (1.29) we infer in a standard way that

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \big[(v^{(1)}-v^{(2)})^2 + a(\sigma^{(1)}-\sigma^{(2)})^2\big](x,t)dx + \int_0^1 (w_t^{(1)}-w_t^{(2)})(\sigma^{(1)}-\sigma^{(2)})dx = 0 \quad \text{a.e.}$$

For $\xi_r^{(i)} := p_r(\lambda, \sigma^{(i)})$, $r > 0$, $i = 1,2$ it follows from Proposition II.4.13

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \Big[(v^{(1)}-v^{(2)})^2 + a(\sigma^{(1)}-\sigma^{(2)})^2\Big](x,t)dx + \\ + \int_0^\infty\int_0^1 (\xi_r^{(1)}-\xi_r^{(2)})\frac{\partial}{\partial t}\big(g(\xi_r^{(1)},r) - g(\xi_r^{(2)},r)\big)(x,t)dx\,dr \leqslant 0 \quad \text{a.e.,}$$

hence for a.e. $t \in [0,T[$ we have

$$(2.1)\ \frac{1}{2}\frac{d}{dt}\int_0^1 \Big[(v^{(1)}-v^{(2)})^2 + a(\sigma^{(1)}-\sigma^{(2)})^2 + \int_0^\infty \psi(\xi_r^{(1)},r)(\xi_r^{(1)}-\xi_r^{(2)})^2 dr\Big](x,t)dx \leqslant \\ \leqslant \int_0^\infty\int_0^1 \Big[\frac{1}{2}(\xi_r^{(1)}-\xi_r^{(2)})^2\frac{\partial\psi}{\partial v}(\xi_r^{(1)},r)\frac{\partial}{\partial t}\xi_r^{(1)}\Big](x,t)dx\,dt - \\ - \int_0^\infty\int_0^1 \Big[(\xi_r^{(1)}-\xi_r^{(2)})\big(\psi(\xi_r^{(1)},r)-\psi(\xi_r^{(2)},r)\big)\frac{\partial}{\partial t}\xi_r^{(2)}\Big](x,t)dx\,dr.$$

Put $k(t) := \operatorname{supess}\left\{|\sigma_t^{(i)}(x,t)|;\ x \in]0,1[,\ i = 1,2\right\}$. Then we have $k \in L^1(0,T)$ and $\left|\frac{\partial}{\partial t}\xi_r^{(i)}(x,t)\right| \leqslant k(t)$ for a.e. $(r,x,t) \in]0,\infty[\times]0,1[\times]0,T[$. From (2.1) we obtain

$$\frac{1}{2}\frac{d}{dt}\int_0^1 \left[(v^{(1)} - v^{(2)})^2 + a(\sigma^{(1)} - \sigma^{(2)})^2 + A_R\int_0^R (\xi_r^{(1)} - \xi_r^{(2)})^2 dr\right](x,t)dx \leqslant$$
$$\leqslant \frac{3}{2}C_R k(t)\int_0^1\int_0^R (\xi_r^{(1)} - \xi_r^{(2)})^2(x,t)dr\,dx \quad \text{a.e.},$$

where A_R, C_R are positive constants from Propositions II.4.22, II.4.2. The assertion now follows in a standard way from a Gronwall type argument like Lemma II.5.6. □

Exercise 2.3. Let the hypotheses of Theorem 2.1 be fulfilled and let the number $H_-(R)$ from Proposition II.4.21 be strictly positive for all $R > 0$. Use inequality II(4.11) to prove that the solution of problem (1.29), (1.31), (1.32) is stable with respect to quasilinear perturbations of the form $\varepsilon = F(\sigma) + \delta g(\sigma)$, where g is a smooth increasing function and $\delta \to 0+$.

Nonresonance

We have seen that the convexity of hysteresis loops of the constitutive operator prevents system (1.29) from the formation of shocks. We now show that if the hysteresis loops are strictly convex and the right-hand side q is bounded, then the solution remains globally bounded due to the hysteretic dissipation of energy. This phenomenon is called *nonresonance* and the precise statement reads as follows.

Theorem 2.4. *Let the hypotheses of Theorem 1.6 be fulfilled with $K_R > 0$ and $q, q_t \in L^\infty\big(0,\infty; L^2(0,1)\big)$. Let condition (1.36) be replaced with*

$$\text{(2.2)} \qquad \text{(i)}\quad Q_1 K_R b_R^{-\frac{1}{2}} \leqslant 1,$$
$$\text{(ii)}\quad 6\max\{E_0, YQ_1\} + Q_0^2 \leqslant \frac{1}{4}R^2,$$

where we put $E_0 := \frac{1}{2}\int_0^1 \big(|\sigma_x^0(x) + q(x,0)|^2 + \frac{1}{a_R}|v_x^0(x)|^2\big)dx$, $Y := 6b_R^{\frac{3}{2}}K_R^{-1} + \frac{1}{6}b_R Q_1$, $Q_0 :=$ supess $\left\{\big(\int_0^1 q^2(x,t)dx\big)^{\frac{1}{2}};\ t > 0\right\}$, $Q_1 :=$ supess $\left\{\big(\int_0^1 q_t^2(x,t)dx\big)^{\frac{1}{2}};\ t > 0\right\}$.

Then system (1.29), (1.31), (1.32) admits a solution $(v,\sigma,\varepsilon) \in \bigcup_{T>0} C([0,1]\times[0,T])$ such that $|\sigma(x,t)| \leqslant \frac{R}{2}$, $|v(x,t)| \leqslant \big(6b_R\max\{E_0, YQ_1\}\big)^{\frac{1}{2}}$ for all $(x,t) \in [0,1]\times[0,\infty[$, v_t, v_x, σ_t, σ_x, $\varepsilon_t \in L^\infty\big(0,\infty; L^2(0,1)\big)$.

Conditions (2.2) express the requirement that the solution does not leave the convexity domain of F. If the operator F is globally convex, i.e. for every $R > 0$ there exist $b_R > a_R \geqslant a > 0$ and $K_R > 0$ such that condition (1.35) holds and if moreover we assume

$$\lim_{R\to\infty} K_R b_R^{-\frac{1}{2}} = 0, \qquad \lim_{R\to\infty} \frac{b_R^{\frac{3}{2}}}{R^2 K_R} = 0, \tag{2.3}$$

then assumption (2.2) is automatically satisfied without any restriction on the size of the data for R sufficiently large. Let us note that operators satisfying (2.3) exist. It suffices to consider a Prandtl-Ishlinskii operator II(3.2) with $h(r) := a + 2cr^{\alpha}$ with $a, c > 0$ and $\alpha \in]0,1]$. Indeed, by Proposition II.4.21 we have $b_R = h(R), K_R = c\alpha R^{\alpha-1}$.

Proof of Theorem 2.4. It suffices to check that the assertion of Lemma 1.8 holds for every $t_0 > 0$ with $C_1 = 6\max\{E_0, YQ_1\}$. Lemma 1.7 then guarantees that system (1.37), (1.38) has a globally bounded solution in $[0,\infty[$. The argument of the proof of Theorem 1.6 then shows that for every fixed $T > 0$ the sequence (1.46) contains a subsequence which converges uniformly in $[0,1]\times[0,T]$ to a solution of (1.29), (1.31), (1.32). Let $\{v^{(n,1)}, \sigma^{(n,1)}, \varepsilon^{(n,1)}\}$ denote the corresponding convergent subsequence for $T = 1$. By induction we construct for $\ell \in \mathbb{N}$ subsequences $\{v^{(n,\ell)}, \sigma^{(n,\ell)}, \varepsilon^{(n,\ell)}\}$ of $\{v^{(n,\ell-1)}, \sigma^{(n,\ell-1)}, \varepsilon^{(n,\ell-1)}\}$ which converge in $[0,\ell]$. To each $\ell \in \mathbb{N}$ we find n_ℓ such that for $m, n \geqslant n_\ell$ we have $|v^{(m,\ell)} - v^{(n,\ell)}| + |\sigma^{(m,\ell)} - \sigma^{(n,\ell)}| + |\varepsilon^{(m,\ell)} - \varepsilon^{(n,\ell)}| < \frac{1}{\ell}$. The sequence $\{v^{(n_\ell,\ell)}, \sigma^{(n_\ell,\ell)}, \varepsilon^{(n_\ell,\ell)}; \ell \in \mathbb{N}\}$ then converges locally uniformly to a global solution of (1.29), (1.31), (1.32) satisfying the assertion of Theorem 2.4.

To derive estimates (1.43) we proceed analogously as in the proof of Lemma 1.8. The counterpart of (1.44)

$$\sum_{j=1}^{n-1} \Big[\frac{1}{2}\big(\dot v_j^2(t) - \dot v_j^2(s)\big) + \int_s^t \ddot\varepsilon_j(\tau)\dot\sigma_j(\tau)d\tau\Big] = \sum_{j=1}^{n-1} \int_s^t \dot q_j(\tau)\dot v_j(\tau)d\tau \tag{2.4}$$

holds for every $t > s > 0$. To simplify the computation we introduce the notation

$$\begin{aligned}
E(t) &:= \frac{1}{2n}\sum_{j=1}^{n-1}\big(\dot v_j^2(t) + \dot\varepsilon_j(t)\dot\sigma_j(t)\big),\\
S(t) &:= \frac{1}{n}\sum_{j=1}^{n-1}|\dot\sigma_j(t)|^3,\\
V(t) &:= \frac{1}{n}\sum_{j=1}^{n-1}\dot v_j^2(t),\\
Z(t) &:= -\frac{1}{n}\sum_{j=1}^{n-1} v_j(t)\dot v_j(t)
\end{aligned}$$

and choose an arbitrary function $\varrho \in W^{1,\infty}(0,\infty)$ such that

$$\varrho(t) > \Big(\int_0^1 q_t^2(x,t)dx\Big)^{\frac{1}{4}} \quad \forall t > 0. \tag{2.5}$$

For almost all $t > s > 0$ we obtain from (2.4), Theorem II.4.19 and equation (1.37)

$$E(t) - E(s) + \frac{K_R}{2}\int_s^t S(\tau)d\tau \leqslant \frac{1}{n}\sum_{j=1}^{n-1}\int_s^t \dot q_j(\tau)\dot v_j(\tau)d\tau, \tag{2.6}$$

$$\dot Z(t) + V(t) = \frac{1}{n}\sum_{j=1}^{n-1}\big[\dot\varepsilon_j(t)\dot\sigma_j(t) - v_j(t)\dot q_j(t)\big]. \tag{2.7}$$

Using Hölder's inequality for sums and the inequalities

$$\begin{cases} |v_j(t)| \leqslant \frac{1}{n}\sum_{j=1}^{n-1}|\Delta_{j-1}v(t)| \leqslant \big(\frac{1}{n}\sum_{j=1}^{n-1}|\dot\varepsilon_j(t)|^2\big)^{1/2} & \text{a.e.,}\\ |\dot\varepsilon_j(t)| \leqslant b_R|\dot\sigma_j(t)|, \quad \frac{1}{n}\sum_{j=1}^{n-1}\dot q_j^2(t) \leqslant \int_0^1 q_t^2(x,t)dx & \text{a.e.}\end{cases} \tag{2.8}$$

we infer from (2.6), (2.7)

$$E(t) - E(s) + \frac{K_R}{2}\int_s^t S(\tau)d\tau \leqslant \int_s^t \varrho^2(\tau)V^{\frac{1}{2}}(\tau)d\tau, \tag{2.9}$$

$$\dot Z(t) + E(t) + \frac{1}{2}V(t) \leqslant b_R\Big(\frac{3}{2}S^{\frac{2}{3}}(t) + \varrho^2(t)S^{\frac{1}{3}}(t)\Big) \leqslant b_R\Big(3S^{\frac{2}{3}}(t) + \frac{1}{6}\varrho^4(t)\Big). \tag{2.10}$$

We now fix a constant $C > 0$ which will be specified later and put $B(t) := \frac{K_R}{4C\varrho(t)}$. The inequalities

$$\varrho^2(\tau)V^{\frac{1}{2}}(\tau) \leqslant \frac{1}{2}C\varrho(\tau)V(\tau) + \frac{1}{2C}\varrho^3(\tau),$$
$$3b_RS^{\frac{2}{3}}(t) \leqslant \frac{b_R^3}{B^2(t)} + 2B(t)S(t)$$

combined with (2.9), (2.10) yield for a.e. $t > s > 0$

$$\begin{aligned} E(t) - E(s) + \int_s^t C\varrho(\tau)\big(\dot Z(\tau) + E(\tau)\big)d\tau \leqslant \\ \leqslant \int_s^t \Big[\frac{1}{2C} + \frac{16b_R^3C^3}{K_R^2} + \frac{b_RC}{6}\varrho^2(\tau)\Big]\varrho^3(\tau)d\tau.\end{aligned} \tag{2.11}$$

We can choose in particular $\varrho(\tau) \equiv Q_1^{\frac{1}{2}}$ and put $C := \frac{1}{2}K_R^{\frac{1}{2}}b_R^{-\frac{3}{4}}$, $\hat E(t) := E(t) + CQ_1^{\frac{1}{2}}Z(t)$. By (2.2)(i) and (2.8) we have $|CQ_1^{\frac{1}{2}}Z(t)| \leqslant \frac{1}{2}E(t)$ a.e., hence

$$\frac{1}{2}E(t) \leqslant \hat E(t) \leqslant \frac{3}{2}E(t) \quad \text{a.e.} \tag{2.12}$$

Inequality (2.11) for $\varrho(\tau) = Q_1^{\frac{1}{2}}$ then yields

$$\hat{E}(t) - \hat{E}(s) + \frac{2}{3}CQ_1^{\frac{1}{2}} \int_s^t \hat{E}(\tau)d\tau \leqslant YCQ_1^{\frac{3}{2}}(t-s) \quad \text{a.e.} \tag{2.13}$$

The function $f(t) := \hat{E}(t) + \frac{2}{3}CQ_1^{\frac{1}{2}} \int_s^t \hat{E}(\tau)d\tau - YCQ_1^{\frac{3}{2}}t$ is nonincreasing in $]0,\infty[$ by (2.13); from inequality II(4.21) for $\eta(t) = e^{\frac{2}{3}CQ_1^{\frac{1}{2}}t}$ and $K = 0$ it follows

$$\begin{aligned}\hat{E}(t-) &\leqslant \hat{E}(s+)e^{\frac{2}{3}CQ_1^{\frac{1}{2}}(s-t)} + \frac{3}{2}YQ_1(1 - e^{\frac{2}{3}CQ_1^{\frac{1}{2}}(s-t)}) \\ &\leqslant \frac{3}{2}\max\{E(s+), YQ_1\} \quad \text{for all} \quad t > s \geqslant 0,\end{aligned}$$

consequently

$$E(t-) \leqslant 3\max\{E(0+), YQ_1\} \quad \forall t > 0. \tag{2.14}$$

We have $E(s) \leqslant \frac{1}{n}\sum_{j=1}^{n-1}\big(\dot{v}_j^2(s) + \frac{1}{a_R}\dot{\varepsilon}_j^2(s)\big)$ for a.e. $s > 0$, hence $E(0+) \leqslant E_0$ and

$$E(t) \leqslant 3\max\{E_0, YQ_1\} \quad \text{a.e.} \tag{2.15}$$

We have by (1.37) and (2.8)

$$\begin{cases} |v_j(t)| \leqslant \Big(\frac{1}{n}\sum_{j=1}^{n-1}|\dot{\varepsilon}_j(t)|^2\Big)^{\frac{1}{2}} & \leqslant \big(2b_R E(t)\big)^{\frac{1}{2}} \quad \text{a.e.}, \\ |\sigma_j(t)| \leqslant \frac{1}{n}\sum_{j=1}^{n-1}|\Delta_j\sigma(t)| & \leqslant \frac{1}{n}\sum_{j=1}^{n-1}|\dot{v}_j(t) + q_j(t)| \\ & \leqslant \Big(2E(t) + \int_0^1 q^2(x,t)dx\Big)^{\frac{1}{2}} \quad \text{a.e.} \end{cases} \tag{2.16}$$

and estimates (1.43) with $C_1 = 6\max\{E_0, YQ_1\}$ follow from (2.15) and (2.2)(ii). Theorem 2.4 is proved. ☐

Decay of solutions

A natural question about the decay of solutions of (1.29), (1.31), (1.32) as $t \to \infty$ if the right-hand side decays to 0 can be answered in the following way.

Theorem 2.5. *Let the hypotheses of Theorem 2.4 hold and assume that the function $\varrho \in W^{1,\infty}(0,\infty)$ in (2.5) can be chosen in such a way that*

$$\lim_{t\to\infty} \varrho(t) = 0, \quad 0 \leqslant -\dot{\varrho}(t) \leqslant M\varrho^2(t) \quad \text{a.e.} \tag{2.17}$$

for some constant $M > 0$. Then there exist constants $A > 0$ and $t_0 \geqslant 0$ such that each global solution (v, σ, ε) of (1.29), (1.31), (1.32) satisfies

$$\begin{cases} |v(x,t)| \leqslant \sqrt{2b_R A}\,\varrho(t), \\ |\sigma(x,t)| \leqslant \big(2A\varrho^2(t) + \int_0^1 q^2(\xi,t)d\xi\big)^{\frac{1}{2}} \end{cases} \quad \forall (x,t) \in [0,1] \times [t_0, \infty[. \tag{2.18}$$

The best estimate of the decay rate for $q \equiv 0$ that Theorem 2.5 can yield is $\frac{1}{t}$. Example 2.6 and Remark 2.10(i) below show that it cannot be improved.

Proof of Theorem 2.5. Using inequalities (2.16) and the locally uniform convergence of the approximate solutions we see that it suffices to derive from (2.11) the inequality

$$E(t) \leqslant A\varrho^2(t) \quad \text{for a.e.} \quad t > t_0. \tag{2.19}$$

for suitable $A > 0$ and $t_0 \geqslant 0$.

We fix some $\delta \in \left]0, \frac{1}{9}\right[$ and put

$$t_0 := \sup\left\{t > 0;\ M\varrho(t_0)b_R^{\frac{1}{2}} \geqslant \delta\right\}. \tag{2.20}$$

In inequality (2.11) we denote

$$C := \frac{4M}{1-5\delta}, \qquad L := \frac{1}{2C} + \frac{16b_R^3C^3}{K_R^2} + \frac{1}{6}b_R C\varrho^2(t_0). \tag{2.21}$$

We have $|M\varrho(t)Z(t)| \leqslant \delta E(t)$ for a.e. $t > t_0$ and the function $\hat{E}(t) := E(t) + C\varrho(t)Z(t)$ satisfies

$$\frac{1-9\delta}{1-5\delta}E(t) \leqslant \hat{E}(t) \leqslant \frac{1-\delta}{1-5\delta}E(t) \quad \text{for a.e. } t > t_0. \tag{2.22}$$

We rewrite inequality (2.11) for a.e. $t > s > t_0$ in the form

$$\hat{E}(t) - \hat{E}(s) + C\int_s^t \big(\varrho(\tau)E(\tau) - \dot{\varrho}(\tau)Z(\tau)\big)d\tau \leqslant L\int_s^t \varrho^3(\tau)d\tau. \tag{2.23}$$

Hypothesis (2.17) yields

$$\begin{aligned}\varrho(\tau)E(\tau) - \dot{\varrho}(\tau)Z(\tau) &\geqslant \varrho(\tau)\big(E(\tau) - |M\varrho(\tau)Z(\tau)|\big)\\ &\geqslant (1-\delta)\varrho(\tau)E(\tau) \geqslant (1-5\delta)\varrho(\tau)\hat{E}(\tau) \text{ a.e.}\end{aligned}$$

and from (2.23) we conclude

$$\hat{E}(t) - \hat{E}(s) + 4M\int_s^t \varrho(\tau)\hat{E}(\tau)d\tau \leqslant L\int_s^t \varrho^3(\tau)d\tau \quad \text{for a.e. } t > s > t_0. \tag{2.24}$$

Put $P(t) := \int_{t_0}^t \varrho(\tau)d\tau$ for $t \geqslant t_0$. Similarly as in the proof of Theorem 2.4 we use inequality II(4.21) for $f(t) := \hat{E}(t) + 4M\int_{t_0}^t \varrho(\tau)\hat{E}(\tau)d\tau - L\int_{t_0}^t \varrho^3(\tau)d\tau$ and $\eta(t) := e^{4MP(t)}$. We obtain for a.e. $t > s > t_0$.

$$e^{4MP(t)}\hat{E}(t) - e^{4MP(s)}\hat{E}(s) \leqslant L\int_s^t \varrho^3(\tau)e^{4MP(\tau)}d\tau \tag{2.25}$$

From hypothesis (2.17) it follows $\frac{d}{d\tau}\big(\varrho^2(\tau)e^{4MP(\tau)}\big) \geqslant 2M\varrho^3(\tau)e^{4MP(\tau)} > 0$, hence

$$(2.26)\qquad \begin{cases} e^{-4MP(t)} & \leqslant \frac{\varrho^2(t)}{\varrho^2(t_0)}, \\ \int_s^t \varrho^3(\tau)e^{4MP(\tau)}d\tau & \leqslant \frac{1}{2M}\Big(\varrho^2(t)e^{4MP(t)} - \varrho^2(s)e^{4MP(s)}\Big). \end{cases}$$

Combining (2.26) with (2.25) we obtain

$$e^{4MP(t)}\left(\hat{E}(t) - \frac{L}{4M}\varrho^2(t)\right) \leqslant \hat{E}(t_0+) - \frac{L}{4M}\varrho^2(t_0)$$

and either $\hat{E}(t_0+) \leqslant \frac{L}{4M}\varrho^2(t_0)$ and $\hat{E}(t) \leqslant \frac{L}{4M}\varrho^2(t)$ for a.e. $t > t_0$ or $\hat{E}(t_0+) > \frac{L}{4M}\varrho^2(t_0)$ and $\hat{E}(t) \leqslant \frac{\hat{E}(t_0)}{\varrho^2(t_0)}\varrho^2(t)$. Inequality (2.19) now follows from (2.22) with $A = \frac{1-\delta}{1-9\delta}\max\left\{\frac{L}{4M}, \frac{\hat{E}(t_0+)}{\varrho^2(t_0)}\right\}$ and the proof is complete. □

The rest of this section is devoted to the problem of optimality of estimates (2.18) for $q \equiv 0$.

Example 2.6. Let us consider the system of ODE's

$$(2.27)\qquad \begin{cases} \dot{v} & = -\sigma, \\ \dot{u} & = v, \\ \sigma & = \mathcal{F}_\varphi(0,u), \end{cases}$$

$$(2.28)\qquad u(0) = u_1 > 0,\ v(0) = 0,$$

describing the oscillations of an elastoplastic spring-mass system, where $\mathcal{F}_\varphi(0,\cdot)$ is the Prandtl-Ishlinskii operator II(3.2) with zero initial memory configuration. We assume

$$(2.29)\qquad h \text{ is nonincreasing in }]0,\infty[, \quad 0 \leqslant h(r) < h(0+) \quad \text{for all } r > 0.$$

By Proposition II.4.6 B every solution of (2.27) satisfies

$$(2.30)\qquad \frac{d}{dt}\left(\frac{1}{2}v^2(t) + U(t)\right) = -|\dot{D}(t)|,$$

where

$$(2.31)\qquad \begin{cases} U(t) & = \int_0^\infty \big(\xi_r(t) - u(t)\big)\frac{\partial}{\partial r}\xi_r(t)\,h(r)\,dr, \\ D(t) & = \int_0^\infty \frac{\partial}{\partial r}\big(r\xi_r(t)\big)\,h(r)\,dr, \end{cases}$$

with $\xi_r(t) := p_r(0,u)(t)$.

The technique of construction of the solution to system (1.15) is applicable here and we conclude that system (2.27), (2.28) admits a unique global classical solution. Our aim is to derive the following properties of the solution.

Proposition 2.7.

(i) *There exists a sequence* $0 = t_0 < t_1 < \dots$ *such that* $(-1)^k u$ *is increasing in* $[t_{k-1}, t_k]$, $\lim_{k\to\infty} t_k - t_{k-1} = \frac{\pi}{\sqrt{h(0+)}}$.

(ii) *There exists a decreasing positive function* $\Gamma :]0,\infty[\to]0,\infty[$ *and positive constants* κ_i, $i = 1,2,3,4$ *such that* $\lim_{t\to\infty} \Gamma(t) = \lim_{t\to\infty} \frac{\log\Gamma(t)}{t} = 0$ *and*

$$\kappa_1\Gamma(\kappa_2 t) \leqslant |\sigma(t)| + |v(t)| \leqslant \kappa_3\Gamma(\kappa_4 t) \quad \forall t > 0 \tag{2.32}$$

Statement (ii) means that the solution decays to 0, but the rate of decay is not exponential. In the proof we find an explicit formula for Γ in terms of the generator φ of the operator $\mathcal{F}_\varphi$ and we show that under the hypotheses of Theorem 2.5 we have $\Gamma(t) \approx \frac{1}{t}$. The proof of Proposition 2.7 is based on Lemmas 2.8, 2.9 below.

Lemma 2.8. *Let* (u, v, σ) *be a solution of (2.27) and assume that for some* $s_0 \geqslant 0$ *we have* $v(s_0) = 0$, $\sigma(s_0) \neq 0$. *Put* $\lambda(r) := p_r(0,u)(s_0)$. *Then there exists* $s_1 > s_0$ *such that* u, σ *are strictly monotone in* $[s_0, s_1]$, $v(s_1) = 0$, $\sigma(s_1)\sigma(s_0) < 0$ *and putting* $S := -\operatorname{sign}\big(\sigma(s_0)\big)$, $r^*(t) := m_\lambda\big(u(t)\big)$ *we have for* $t \in [s_0, s_1]$

$$\begin{aligned} \frac{1}{2}v^2(t) = {} & \big(\lambda(0) - \lambda\big(r^*(t)\big) - Sr^*(t)\big)\,\sigma(s_0) - \\ & - \int_0^{r^*(t)} \big(S + \lambda'(r)\big)\,\big(\lambda\big(r^*(t)\big) - \lambda(r) + S\big(r^*(t) - r\big)\big)\,h(r)dr, \end{aligned} \tag{2.33}$$

$$\sigma(t) = \sigma(s_0) + \int_0^{r^*(t)} \big(S + \lambda'(r)\big)h(r)dr. \tag{2.34}$$

P r o o f of Lemma 2.8. There exists $\delta > 0$ such that $\operatorname{sign}\dot v(t) = \operatorname{sign} v(t) = S$ for $t \in]s_0, s_0+\delta[$. Put $s_1 := \inf\{t > s_0;\ Sv(t) \leqslant 0\}$ with the convention $\inf\emptyset = +\infty$. In $]s_0, s_1[$ we have $S\dot D(t) \geqslant 0$ and energy identity (2.30) entails

$$\frac{1}{2}v^2(t) + U(t) + SD(t) = U(s_0) + SD(s_0). \tag{2.35}$$

Using formula (2.31) for $\xi_r(s_0) = \lambda(r)$, $\xi_r(t) = \lambda(r)$ for $r \geqslant r^*(t)$, $\xi_r(t) = u(t) - Sr$ for $r \in]0, r^*(t)[$ we obtain (2.33) directly from the identity $u(t) = \lambda\big(r^*(t)\big) + Sr^*(t)$ and (2.35). Formula (2.34) follows immediately from the definition II(3.2).

Assume that $s_1 = +\infty$. Then $Sv = S\dot{u} > 0$, hence $S\dot{\sigma} = -S\ddot{v} \geqslant 0$ in $]s_0, +\infty[$ by Proposition II.4.8. This yields $S\dot{v} = S\ddot{u} \geqslant 0$ in $]s_0, +\infty[$, $Su(+\infty) = +\infty$. We have $\lambda \in \Lambda_R$ for $R = \|u\|_{[0,s_0]}$ by Corollary II.2.6, hence $r^*(t) = |u(t)|$ and $\sigma(t) = S\varphi\big(|u(t)|\big)$ for t sufficiently large. This means in particular $S\sigma(t) = -S\dot{v}(t) > 0$ for t sufficiently large which is a contradiction. We therefore have $s_1 < \infty$, $S\sigma(s_1) = -S\dot{v}(s_1) > 0$ and Lemma 2.8 is proved. $\square$

Lemma 2.8 enables us to define the sequence $0 = t_0 < t_1 < \dots$ such that $(-1)^k u$ increases in $[t_{k-1}, t_k]$, $v(t_k) = 0$, $(-1)^k\sigma(t_k) > 0$. Put $u_k := u(t_k)$, $\sigma_k := \sigma(t_k)$, $a_k := |\sigma_k|$, $\lambda_k(r) = p_r(0,u)(t_k)$ for $r > 0$, $r_0 := u_0$, $r_{k+1} := m_{\lambda_k}(u_{k+1})$ for $k \geqslant 0$. For $r > 0$ we define auxiliary functions

$$\begin{aligned}
\Phi(r) &:= \int_0^r \varphi(\varrho)d\varrho,\\
\alpha(r) &:= 2\Big(\frac{2\Phi(r)}{r} - \varphi(r)\Big) = -\frac{2}{r}\int_0^r \varrho(r-\varrho)dh(\varrho),\\
\beta(r) &:= \frac{2\Phi(r)}{r} = \frac{2}{r}\int_0^r (r-\varrho)\,h(\varrho)\,d\varrho,\\
\gamma(r) &:= \int_r^{r_0} \frac{\beta'(\varrho)}{\alpha(\varrho)}d\varrho,\\
\mu(r) &:= h(0+) - h(r).
\end{aligned}$$

From hypothesis (2.29) we easily derive the following properties of the above functions.

(2.36) (i) $\alpha(0+) = 0,\ 0 < \alpha'(r) < 2\mu(r)$ for $r > 0$,

(ii) $\beta(0+) = 0,\ \beta(+\infty) = 2\varphi(+\infty),\ h(r) < \beta'(r) < h(0+)$ for $r > 0$,

(iii) $\gamma(r) \geqslant \dfrac{h(r)}{2\mu(r)}\log\dfrac{r_0}{r},\ \gamma'(r) < 0$ for $r > 0$.

Lemma 2.9. *For every* $k \in \mathbb{N}$ *we have* $a_{k-1} > \varphi(r_k) > a_k$, $a_{k-1} = \beta(r_k)$, $a_k = 2\varphi(r_k) - a_{k-1}$, $r_k = \frac{1}{2}|u_k - u_{k-1}|$.

P r o o f of Lemma 2.9. We proceed by induction. We have $\sigma_0 = \varphi(r_0)$, $S = -1$ in $]0, t_1[$, $\lambda_0(r) = \max\{0, r_0 - r\}$ and the function

$$f_0(\varrho) := \big(u_0 - \lambda_0(\varrho) + \varrho\big)\sigma_0 + \int_0^\varrho \big(1 - \lambda_0'(r)\big)\big(\lambda_0(\varrho) - \lambda_0(r) - \varrho + r\big)\,h(r)\,dr$$

satisfies $f_0(0) = 0$, $f_0(r_0) = -r_0\alpha(r_0) < 0$, $f_0'(\varrho) > 0$ for $\varrho \in \big]0, \varphi^{-1}\big(\frac{\sigma_0}{2}\big)\big[$, $f_0'(\varrho) < 0$ for $\varrho > \varphi^{-1}\big(\frac{\sigma_0}{2}\big)$. From (2.33) we infer that $\varrho = r_1$ is the unique positive root of the equation $f_0(\varrho) = 0$, hence $r_1 \in]0, r_0[$, $r_1 = \frac{1}{2}(u_0 - u_1)$ and $\beta(r_1) = \sigma_0$. By (2.34) we have $\sigma_1 = \sigma_0 - 2\varphi(r_1)$ and the assertion for $k = 1$ follows.

Assume now that for some $k \in \mathbb{N}$ the assertion of Lemma 2.9 holds. We analogously define the function

$$f_k(\varrho) := \big(u_k - \lambda_k(\varrho) + S\varrho\big)\sigma_k + \int_0^{\varrho} \big(S - \lambda_k'(r)\big)\big(\lambda_k(\varrho) - \lambda_k(r) - S(\varrho - r)\big)\, h(r)\, dr,$$

where $S = (-1)^k$, and using the formula $\lambda_k(r) = u_k - (-1)^k r$ for $r \in]0, r_k[$, $\lambda_k(r) = \lambda_{k-1}(r)$ for $r > r_k$ and the induction hypothesis we obtain $f_k(0) = 0$, $f_k(r_k) = 2r_k a_k - 4\Phi(r_k) = -2r_k\alpha(r_k) < 0$, $f_k'(\varrho) > 0$ for $\varrho \in \big]0, \varphi^{-1}\big(\frac{a_k}{2}\big)\big[$, $f_k'(\varrho) < 0$ for $\varrho > \varphi^{-1}\big(\frac{a_k}{2}\big)$. From (2.33) we again infer that $\varrho = r_{k+1}$ is the unique positive root of the equation $f_k(\varrho) = 0$, hence $r_{k+1} \in]0, r_k[$, $r_{k+1} = \frac{S}{2}(u_k - u_{k+1})$ and $\beta(r_{k+1}) = a_k$. Identity $a_{k+1} = 2\varphi(r_{k+1}) - a_k$ follows from (2.34). We obviously have $a_k - \varphi(r_{k+1}) = \frac{1}{2}\alpha(r_{k+1}) > 0$, hence $a_{k+1} - \varphi(r_{k+1}) < 0$ and the induction step is complete. $\square$

Proof of Proposition 2.7. We rewrite the identities in Lemma 2.9 in the form

$$a_{k-1} = \beta(r_k), \ \beta(r_k) - \beta(r_{k+1}) = \alpha(r_k). \tag{2.37}$$

Both sequences $\{a_k\}, \{r_k\}$ are decreasing, $\lim\limits_{k\to\infty} a_k = \lim\limits_{k\to\infty} r_k = 0$. The difference $t_{k+1} - t_k$ can be directly estimated using the formula

$$t_{k+1} - t_k = \int_{u_k}^{u_{k+1}} (u^{-1})'(x)\, dx = \int_{u_k}^{u_{k+1}} \frac{dx}{v(u^{-1}(x))}, \tag{2.38}$$

where $u^{-1} : \operatorname{Conv}\{u_k, u_{k+1}\} \to [t_k, t_{k+1}]$ is the inverse function to $u|_{[t_k, t_{k+1}]}$. By (2.33) we have

$$\frac{1}{2}v^2(t) = |u(t) - u_k|a_k - 4\Phi\big(\frac{1}{2}|u(t) - u_k|\big)$$

and (2.38) yields

$$t_{k+1} - t_k = \sqrt{\frac{r_{k+1}}{2}} \int_0^{r_{k+1}} \frac{dr}{\sqrt{r\Phi(r_{k+1}) - r_{k+1}\Phi(r)}} = \tag{2.39}$$
$$= \frac{r_{k+1}}{\sqrt{2}} \int_0^1 \frac{ds}{\sqrt{s\Phi(r_{k+1}) - \Phi(sr_{k+1})}}.$$

From the integral formula $s\Phi(r) - \Phi(sr) = s\int_0^r \int_{s\varrho}^{\varrho} h(z)\, dz\, d\varrho$ we infer

$$\frac{r_{k+1}^2}{2} s(1-s)h(r_{k+1}) \leqslant s\Phi(r_{k+1}) - \Phi(sr_{k+1}) \leqslant \frac{r_{k+1}^2}{2} s(1-s)h(0+),$$

hence

$$\frac{\pi}{\sqrt{h(0+)}} \leqslant t_{k+1} - t_k \leqslant \frac{\pi}{\sqrt{h(r_{k+1})}} \tag{2.40}$$

and statement (i) of Proposition 2.7 is proved.

The function Γ in (ii) is defined as

$$\Gamma(t) := \gamma^{-1}(t) \quad \text{for} \quad t > 0, \tag{2.41}$$

where $\gamma^{-1} :]0, \infty[\to]0, r_0[$ is the inverse function to $\gamma|_{]0,r_o[}$. By (2.36)(iii) we have

$$\lim_{t\to\infty} \Gamma(t) = 0, \ \lim_{t\to\infty} \frac{\log \Gamma(t)}{t} = \lim_{r\to 0+} \frac{\log r}{\gamma(r)} = 0.$$

Furthermore, from the Mean Value Theorem it follows for every $k \in \mathbb{N}$

$$\frac{\gamma(r_{k-1}) - \gamma(r_k)}{\beta(r_{k-1}) - \beta(r_k)} = -\frac{1}{\alpha(\varrho_k)} \qquad \text{for some } \varrho_k \in [r_k, r_{k-1}],$$

and (2.37) yields

$$1 \leqslant \gamma(r_k) - \gamma(r_{k-1}) \leqslant \frac{\alpha(r_{k-1})}{\alpha(r_k)} \quad \forall k \in \mathbb{N}. \tag{2.42}$$

By (2.37),(2.36) we have $\frac{\alpha(r_{k-1})-\alpha(r_k)}{\alpha(r_{k-1})} = \frac{\alpha(r_{k-1})-\alpha(r_k)}{\beta(r_{k-1})-\beta(r_k)} \leqslant \frac{2\mu(r_{k-1})}{h(r_{k-1})}$, and similarly $\frac{r_{k-1}-r_k}{r_{k-1}} \leqslant \frac{\beta(r_{k-1})-\beta(r_k)}{r_{k-1}h(r_{k-1})} \leqslant \frac{2\mu(r_{k-1})}{h(r_{k-1})}$, hence

$$\lim_{k\to\infty} \frac{\alpha(r_{k-1})}{\alpha(r_k)} = 1, \quad \lim_{k\to\infty} \frac{r_k}{r_{k-1}} = 1. \tag{2.43}$$

Using (2.40), (2.42) we find positive constants c_i, $i = 1, 2, \ldots$ independent of k such that $c_1 k \geqslant \gamma(r_k) \geqslant k$, $c_2 k \geqslant t_k \geqslant c_3 k$ and

$$\Gamma\big(\frac{1}{c_2} t_k\big) \leqslant r_k \leqslant \Gamma\big(\frac{c_1}{c_3} t_k\big) \quad \forall k \in \mathbb{N}. \tag{2.44}$$

Using once more identities (2.33), (2.34) in $[t_{k-1}, t_k]$ with $r^*(t) = \frac{1}{2}|u(t) - u(t_{k-1})|$ we obtain

$$\sigma^2(t) + \frac{\varphi\big(r^*(t)\big)}{r^*(t)} v^2(t) = \sigma_{k-1}^2 - 2\alpha\big(r^*(t)\big)\varphi\big(r^*(t)\big),$$

consequently

$$\sigma_k^2 \leqslant \sigma^2(t) + \frac{\varphi\big(r^*(t)\big)}{r^*(t)} v^2(t) \leqslant \sigma_{k-1}^2 \quad \forall t \in]t_{k-1}, t_k[. \tag{2.45}$$

By (2.37), (2.43) we have $\sigma_{k-1}^2 \leqslant c_4 r_k^2$, $\sigma_k^2 \geqslant c_5 r_{k-1}^2$, and (2.44), (2.45) entail

$$c_5 \Gamma^2\Big(\frac{1}{c_2} t\Big) \leqslant c_5 r_{k-1}^2 \leqslant \sigma^2(t) + \frac{\varphi\big(r^*(t)\big)}{r^*(t)} v^2(t) \leqslant c_4 r_k^2 \leqslant c_4 \Gamma^2\Big(\frac{c_1}{c_3} t\Big).$$

The assertion now follows from the fact that $r^*(t) \in]0, r_k[$ and $h(0+) > \frac{\varphi(r^*(t))}{r^*(t)} \geqslant c_6 > 0$ for all $t \in]t_{k-1}, t_k[$. A suitable choice of constants $\kappa_1, \kappa_2, \kappa_3, \kappa_4$ now completes the proof. □

Remarks 2.10.

(i) It is easy to check that the estimate $\frac{1}{t}$ of the order of decay in the context of Theorem 2.5 is optimal. By Proposition II.4.21, the hypotheses of Theorem 2.5 are fulfilled if there exist constants $0 < k_1 < k_2$ such that

(2.46) $$k_1(r-s) \leqslant h(s) - h(r) \leqslant k_2(r-s) \quad \text{for all} \quad r_0 > r > s > 0.$$

In this case we have $\frac{k_1}{3} r^2 \leqslant \alpha(r) \leqslant \frac{k_2}{3} r^2$, hence $\frac{3h(r_0)}{k_2}\big(\frac{1}{r} - \frac{1}{r_0}\big) \leqslant \gamma(r) \leqslant \frac{3h(0+)}{k_1}\big(\frac{1}{r} - \frac{1}{r_0}\big)$ for $r \in]0, r_0[$. The definition of Γ then yields

(2.47) $$\frac{3r_0 h(r_0)}{1 + k_2 r_0 t} \leqslant \Gamma(t) \leqslant \frac{3r_0 h(0+)}{1 + k_1 r_0 t} \quad \text{for} \quad t > 0.$$

(ii) While the stress σ and velocity v vanish as $t \to \infty$, the displacement u in Example 2.6 tends to a positive value u_∞ which corresponds to a remanent deformation of the spring. This follows from the formula $r_k = \frac{(-1)^k}{2}(u_k - u_{k-1})$, $r_0 = u_0$ which entails $u_\infty = \sum_{k=0}^\infty (-1)^k (r_k - r_{k+1})$. By (2.37) we have $\beta(r_{k-1}) - 2\beta(r_k) + \beta(r_{k+1}) = \alpha(r_{k-1}) - \alpha(r_k) > 0$, hence $\beta(r_k) < \frac{1}{2}\big(\beta(r_{k-1}) + \beta(r_{k+1})\big)$. The function β is increasing and concave, hence $r_k < \frac{1}{2}(r_{k-1} + r_{k+1})$ for all $k \in \mathbb{N}$. We thus conclude $u_\infty = \sum_{k=0}^\infty (r_{2k} - 2r_{2k+1} + r_{2k+2}) > 0$.

(iii) We similarly prove that the energy $E(t) = \frac{1}{2} v^2(t) + U(t)$ does not vanish as $t \to \infty$. Putting $E_k = U_k := U(t_k)$ we obtain from (2.31)

$$E_0 = r_0 \varphi(r_0) - \Phi(r_0),\ E_k = E_{k-1} - r_k \alpha(r_k) \quad \text{for } k \in \mathbb{N}.$$

Lemma 2.9 and formula (2.37) then yield

$$E_\infty = E_0 - \sum_{k=1}^\infty r_k \alpha(r_k) = E_0 - \sum_{k=0}^\infty \varphi^{-1}\Big(\frac{a_k + a_{k+1}}{2}\Big)(a_k - a_{k+1}).$$

The function φ^{-1} is strictly convex in its domain of definition. We therefore have $\varphi^{-1}\Big(\frac{a_k + a_{k+1}}{2}\Big)\ (a_k - a_{k+1}) < \int_{a_{k+1}}^{a_k} \varphi^{-1}(s)ds$ and $E_\infty > E_0 - \int_0^{a_0} \varphi^{-1}(s)ds = 0$.

We see that a positive part of the initial energy is stored in the remanent deformation of the spring, the rest is dissipated in the form of heat.

III.3 Periodic solutions

The nonresonance property of wave equations with strictly convex hysteretic constitutive operators which was proved in Sect. III.2 is manifested from another viewpoint by the fact that time-periodic forcing terms imply the time-periodicity of solutions. For a general hysteretic constitutive operator we prove the existence of a periodic solution in its convexity domain by Galerkin method. In the case of Prandtl-Ishlinskii operators we apply the Minty-Browder method to prove the existence and uniqueness of periodic solutions even in the multidimensional case and, in dimension one, we prove that this solution is asymptotically stable.

Compactness method

We pursue here the study of the scalar system (1.29) coupled with boundary conditions

(3.1) $$v(0,t) = \sigma\big(\frac{\pi}{2}, t\big) = 0 \quad \text{for} \quad t > 0$$

and time-periodicity condition

(3.2) $$v(x,t) = v(x,t+\omega),\ \sigma(x,t) = \sigma(x,t+\omega) \qquad \text{for} \quad (x,t) \in]0, \frac{\pi}{2}[\times]0,\infty[,$$

where $\omega > 0$ is a given period.

We introduce the spaces of ω-periodic functions

$$L^p_\omega := \big\{u \in L^p_{\mathrm{loc}}(0,\infty);\ u(t+\omega) = u(t) \quad \text{for a.e.} \quad t > 0\big\}$$

endowed with the norm of $L^p(0,\omega)$, and C_ω as in Corollary II.2.7. For the sake of simplicity we write $C_\omega([0,\frac{\pi}{2}])$ instead of $C([0,\frac{\pi}{2}];C_\omega)$ and $L^p_\omega(0,\frac{\pi}{2})$ instead of $L^p(0,\frac{\pi}{2};L^p_\omega)$. The corresponding L^p-norms are still denoted by $|\cdot|_p$, since confusion is unlikely.

Theorem 3.1. *Let the hypotheses of Theorem 1.6 be fulfilled with $K_R > 0$ and $q, q_t \in L^2_\omega(0,\frac{\pi}{2})$. Let condition (1.36) be replaced with*

(3.3) $$c^0_R\big|q\big|_2 + c^1_R\big|q_t\big|_2^{\frac{1}{2}} \leqslant \gamma R,$$

where $c^0_R := 5\sqrt{\frac{2}{3}} + 7\Big(1 + \sqrt{\frac{3}{2}}(1 + \frac{1}{b_R})\Big)$, $c^1_R := K_R^{-\frac{1}{2}}\left[\frac{10}{3} + 2\big(\omega\pi b^3_R\big)^{\frac{1}{4}}\Big(\frac{31}{3} + \frac{7}{b_R}\Big)\right]$ *and* $\gamma := \frac{1}{2}\big(\frac{\pi}{2}\big)^{-\frac{1}{5}}\Big(\frac{1}{\omega}\big(\frac{\pi}{2}\big)^{\frac{7}{5}} + 1\Big)^{-\frac{1}{2}}$. *Let the operator F satisfy the periodicity condition*

(3.4) $$F(u)(x,t+\omega) = F(u)(x,t) \qquad \forall u \in C_\omega\big([0,\frac{\pi}{2}]\big),\ \forall (x,t) \in \big[0,\frac{\pi}{2}\big] \times [\omega,\infty[.$$

Then there exists at least one solution $(v,\sigma,\varepsilon) \in \left(C_\omega\left([0,\frac{\pi}{2}]\right)\right)^3$ *to (1.29), (3.1), (3.2) such that* $v_t, \sigma_x \in L^2_\omega\left(0,\frac{\pi}{2}\right)$, $\varepsilon_t, \sigma_t, v_x \in L^3_\omega\left(0,\frac{\pi}{2}\right)$, $|\sigma|_\infty \leqslant R$, $|v|_\infty \leqslant b_R R$ *and identities (1.29) hold for* a.e. $(x,t) \in]0,\frac{\pi}{2}[\times]\omega,\infty[$.

From Corollary II.2.7 we immediately see that assumption (3.4) is satisfied for Prandtl-Ishlinskii, Preisach and Della Torre operators as well as for the fatigue operator II(5.7). Condition (3.3) represents again the restriction to the convexity domain of F. Similarly as in previous cases, for Prandtl-Ishlinskii operators II(3.2) generated by functions $\varphi(r) = ar + \frac{2c}{\alpha+1} r^{\alpha+1}$ with $a,c > 0$ and $\alpha \in]0,1]$ we have $b_R = a + 2cR^\alpha$, $K_R = c\alpha R^{\alpha-1}$, hence (3.3) is automatically satisfied for every q and for R sufficiently large.

The uniqueness of periodic solutions is an open problem in general except for the case where F is a Prandtl-Ishlinskii operator. In Theorems 3.4, 3.9 below we show how the two-level monotonicity established in Theorem II.4.9 and Proposition II.4.12 implies uniqueness and stability of periodic solutions.

P r o o f of Theorem 3.1. The solution will be constructed by Galerkin method. Let $\mathbb{Z}$ denote the set of all integers. We define basis functions $\{e_j; j \in \mathbb{Z}\}$ by the formula

$$e_j(t) := \begin{cases} \sin \frac{2\pi}{\omega} jt & \text{for} \quad j > 0 \\ \cos \frac{2\pi}{\omega} jt & \text{for} \quad j \leqslant 0 \end{cases} \tag{3.5}$$

and for a fixed $n \in \mathbb{N}$ we solve the algebraic system for $j = -n, \dots, n$, $k = 0, \dots, n$

$$\begin{aligned} &\text{(i)} && \int_0^\omega \int_0^{\frac{\pi}{2}} \left(v_t^{(n)} - \sigma_x^{(n)} - q\right) e_j(t) \sin(2k+1)x \; dx\, dt = 0 \\ &\text{(ii)} && \int_\omega^{2\omega} \int_0^{\frac{\pi}{2}} \left(F(\sigma^{(n)})_t - v_x^{(n)}\right) e_j(t) \cos(2k+1)x \; dx\, dt = 0 \\ &\text{(iii)} && v^{(n)}(x,t) := \sum_{j=-n}^{n} \sum_{k=0}^{n} v_{jk}\, e_j(t) \sin(2k+1)x, \\ &\text{(iv)} && \sigma^{(n)}(x,t) := \sum_{j=-n}^{n} \sum_{k=0}^{n} \sigma_{jk}\, e_j(t) \cos(2k+1)x \end{aligned} \tag{3.6}$$

where $\{v_{jk}, \sigma_{jk}; j = -n, \dots, n, k = 0, \dots, n\}$ are to be found.

Instead of solving directly system (3.6)(i),(ii) we consider the following modified system

$$\begin{aligned} &\text{(i)} && \int_0^\omega \int_0^{\pi 2} \left(v_t^{(n)} - \sigma_x^{(n)} - \alpha q\right) e_j(t) \sin(2k+1)x \; dx\, dt = 0, \\ &\text{(ii)} && \int_\omega^{2\omega} \int_0^{\frac{\pi}{2}} \left(F_\alpha(\sigma^{(n)})_t - v_x^{(n)}\right) e_j(t) \cos(2k+1)x \; dx\, dt = 0, \end{aligned} \tag{3.7}$$

$j = -n, \dots, n$, $k = 0, \dots, n$ with a parameter $\alpha \in [0,1]$ and operators F_α have the form $F_\alpha(u)(x,t) = (1-\alpha)\,\mathcal{F}_\varphi(0, u(x,\cdot)(t)) + \alpha F(u)(x,t)$, where $\mathcal{F}_\varphi(0,\cdot)$ is the Prandtl-Ishlinskii operator II(3.2) with initial configuration $\lambda = 0$ and generator $\varphi(r) = a_R r + K_R r^2$ for $r > 0$. By Proposition II.4.21 and Remark II.4.24 operators F_α, $\alpha \in [0,1]$ satisfy the hypotheses of Theorem 3.1, in particular Assumption 1.5.

Let $V \in X := \mathbb{R}^{2(n+1)(2n+1)}$ be the vector with components v_{jk}, σ_{jk}; $j = -n, \dots, n$, $k = 0, \dots, n$. System (3.7) has the form

$$P(\alpha, V) = 0, \tag{3.8}$$

where $P : [0,1] \times X \to X$ is a continuous mapping such that $P(0,\cdot) : X \to X$ is odd. We endow the space X with the norm

$$\|V\| := \max\Big\{\big|\sigma^{(n)}\big|_\infty, \frac{1}{b_R}\big|v^{(n)}\big|_\infty\Big\}. \tag{3.9}$$

To establish the existence of a solution $V = V_\alpha$ of (3.8) for every $\alpha \in [0,1]$ it suffices to prove the implication

$$P(\alpha, V) = 0 \Rightarrow \|V\| \neq R. \tag{3.10}$$

Indeed, the Brouwer degree $d\big(P(\alpha,\cdot), B_R(0), 0\big)$ of the mapping $P(\alpha,\cdot)$ with respect to the set $B_R(0) = \{V \in X;\ \|V\| < R\}$ and the origin $0 \in X$ (see Fučík, Kufner (1980)) is then independent of α; the degree $d\big(P(0,\cdot), B_R(0), 0\big)$ of the odd mapping $P(0,\cdot)$ is odd, hence in particular $d\big(P(\alpha,\cdot), B_R(0), 0\big) \neq 0$ and for every $\alpha \in [0,1]$ there exists a solution $V_\alpha \in B_R(0)$ of (3.8).

We now prove implication (3.10). Let $v^{(n)}, \sigma^{(n)}$ satisfy (3.7) for some $\alpha \in [0,1]$, $\|V\| \leqslant R$. Using the fact that $\dot e_j(t) = \frac{2\pi}{\omega} j e_{-j}(t)$ for every $j \in \mathbb{Z}$ we multiply (3.7)(i) by $\big(\frac{2\pi}{\omega} j\big)^2 v_{jk}$ and (ii) by $\big(\frac{2\pi}{\omega} j\big)^2 \sigma_{jk}$. Summing up we obtain

$$\int_\omega^{2\omega}\int_0^{\frac{\pi}{2}} \Big[F_\alpha\big(\sigma^{(n)}\big)_t \sigma^{(n)}_{tt} - \alpha q_t v^{(n)}_t\Big]\,dx\,dt = 0 \tag{3.11}$$

Similarly, multiplying (3.7)(i) by $\frac{2\pi}{\omega} j v_{-j,k}$ and (ii) by $\frac{2\pi}{\omega} j \sigma_{-j,k}$ yields

$$\int_\omega^{2\omega}\int_0^{\frac{\pi}{2}} \big(F_\alpha(\sigma^{(n)})_t \sigma^{(n)}_t - (v^{(n)}_t)^2 + \alpha q v^{(n)}_t\big)\,dx\,dt = 0. \tag{3.12}$$

We now apply Corollary II.4.23 to identities (3.11), (3.12). From II(4.32) and II(4.24)(ii) it follows

$$\begin{cases} \frac{1}{2}K_R \int_0^\omega \int_0^{\frac{\pi}{2}} |\sigma^{(n)}_t|^3\,dx\,dt & \leqslant \int_0^\omega \int_0^{\frac{\pi}{2}} |q_t|\,|v^{(n)}_t|\,dx\,dt, \\ \int_0^\omega \int_0^{\frac{\pi}{2}} |v^{(n)}_t|^2\,dx\,dt & \leqslant \int_0^\omega \int_0^{\frac{\pi}{2}} \big(b_R|\sigma^{(n)}_t|^2 + |q|\,|v^{(n)}_t|\big)\,dx\,dt. \end{cases} \tag{3.13}$$

Put $S := |\sigma_t^{(n)}|_3$, $Y := |v_t^{(n)}|_2$, $Q_1 := |q_t|_2$, $Q_0 := |q|_2$. Using Hölder's inequality we rewrite (3.13) in the form

$$\begin{cases} K_R S^3 & \leqslant 2Q_1 Y, \\ Y^2 & \leqslant b_R\big(\frac{\omega\pi}{2}\big)^{\frac{1}{3}} S^2 + Q_0 Y \end{cases} \tag{3.14}$$

Put $M_R := \big(\frac{\omega\pi b_R^3}{K_R^2}\big)^{\frac{1}{4}}$. From (3.14) we infer $Y^2 \leqslant (2M_R)^{\frac{4}{3}} Q_1^{\frac{2}{3}} Y^{\frac{2}{3}} + Q_0^2$, hence

$$\begin{cases} Y^2 & \leqslant 4M_R^2 Q_1 + \frac{3}{2}Q_0^2 \leqslant \big(2M_R Q_1^{\frac{1}{2}} + \sqrt{\frac{3}{2}}Q_0\big)^2, \\ S & \leqslant \frac{2}{3}\Big(Y + \big(\frac{Q_1}{K_R}\big)^{\frac{1}{2}}\Big) \leqslant \frac{2}{3}\big(K_R^{-\frac{1}{2}} + 2M_R\big)Q_1^{\frac{1}{2}} + \sqrt{\frac{2}{3}}Q_0 \end{cases} \tag{3.15}$$

From (3.7) we directly obtain

$$|\sigma_x^{(n)}|_2 \leqslant Y + Q_0. \tag{3.16}$$

On the other hand, for an arbitrary function $w \in \operatorname{Lin}\big\{e_j(t)\cos(2k+1)x;\ j = -n,\dots,n,\ k = 0,\dots,n\big\}$ equation (3.7)(ii) yields

$$\int_0^\omega \int_0^{\frac{\pi}{2}} v_x^{(n)} w \, dx\, dt \leqslant |F_\alpha(\sigma^{(n)})_t|_3\, |w|_{\frac{3}{2}} \leqslant b_R S |w|_{\frac{3}{2}}. \tag{3.17}$$

By density, inequality (3.17) holds for all $w \in L_\omega^{\frac{3}{2}}(0, \frac{\pi}{2})$. We therefore have

$$|v_x^{(n)}|_3 \leqslant b_R S. \tag{3.18}$$

The embedding theorem V.2.4 enables us to estimate the sup-norm of $\sigma^{(n)}$, $v^{(n)}$ using boundary conditions (3.1). In order to fulfil condition V(2.5) we find integers ℓ_1, ℓ_2 such that $\frac{1}{\omega}\big(\frac{\pi}{2}\big)^{\frac{5}{7}} \leqslant \ell_1 < \frac{1}{\omega}\big(\frac{\pi}{2}\big)^{\frac{5}{7}} + 1$, $\frac{1}{\omega}\big(\frac{\pi}{2}\big)^{\frac{7}{5}} \leqslant \ell_2 < \frac{1}{\omega}\big(\frac{\pi}{2}\big)^{\frac{7}{5}} + 1$ and put $T_1 := \ell_1\omega$, $T_2 := \ell_2\omega$. For arbitrary $(x,t) \in [0, \frac{\pi}{2}] \times [0, \omega]$ formula V(2.6) for $p_0 = q_0 = 3$, $p_1 = q_1 = 2$ and estimate (3.16) yield

$$\begin{aligned} \big|\sigma^{(n)}(x,t)\big| &= \big|\sigma^{(n)}(x,t) - \sigma^{(n)}(\frac{\pi}{2}, t)\big| \\ &\leqslant 2\big(\frac{\pi}{2}\big)^{\frac{1}{7}} \Big[5\Big(\int_0^{\frac{\pi}{2}}\int_0^{T_1} |\sigma_t^{(n)}|^3 dt\, dx\Big)^{\frac{1}{3}} + 7\Big(\int_0^{\frac{\pi}{2}}\int_0^{T_1} |\sigma_x^{(n)}|^2 dt\, dx\Big)^{\frac{1}{2}}\Big] \\ &= 2\big(\frac{\pi}{2}\big)^{\frac{1}{7}} \big[5\ell_1^{\frac{1}{3}} S + 7\ell_1^{\frac{1}{2}}(Y + Q_0)\big] \end{aligned}$$

and similarly

$$\big|v^{(n)}(x,t)\big| \leqslant 2\big(\frac{\pi}{2}\big)^{\frac{1}{5}} \big[5\ell_2^{\frac{1}{3}} b_R S + 7\ell_2^{\frac{1}{2}} Y\big].$$

For $c := \Big(\frac{1}{\omega}\big(\frac{\pi}{2}\big)^{\frac{7}{5}}+1\Big)^{\frac{1}{2}}$ we thus obtain from estimate (3.15) and assumption (3.3)

$$\|V\| < 2c\Big(\frac{\pi}{2}\Big)^{\frac{1}{5}}\Big(5S+7\big(Q_0+(1+\frac{1}{b_R})Y\big)\Big) \leqslant R.$$

This estimate ensures that implication (3.10) holds, hence for every $n\in\mathbb{N}$ system (3.6) has a solution such that the sequences $\big\{|\sigma_t^{(n)}|_3, |\sigma_x^{(n)}|_2, |v_x^{(n)}|_3, |v_t^{(n)}|_2, |\sigma^{(n)}|_\infty, |v^{(n)}|_\infty;$ $n\in\mathbb{N}\big\}$ are bounded. By Corollary V.2.5 there exist functions $\sigma, v \in C_\omega(0,\frac{\pi}{2})$ and subsequences (still indexed by n) such that $\sigma^{(n)} \to \sigma$, $v^{(n)} \to v$, $F(\sigma^{(n)}) \to F(\sigma)$ uniformly in $C_\omega\big(0,\frac{\pi}{2}\big)$, $\sigma_t^{(n)} \to \sigma_t$, $F(\sigma^{(n)})_t \to F(\sigma)_t$, $v_x^{(n)} \to v_x$ weakly in $L^3_\omega\big(0,\frac{\pi}{2}\big)$, $\sigma_x^{(n)} \to \sigma_x$, $v_t^{(n)} \to v_t$ weakly in $L^2_\omega\big(0,\frac{\pi}{2}\big)$. We pass to the limit in (3.6) and Theorem 3.1 is proved. □

Monotonicity method

To illustrate the method we consider the scalar equation

$$u_{tt} - \div F(\nabla u) = q, \quad (x,t)\in\Omega\times]0,\infty[\tag{3.19}$$

with homogeneous Dirichlet boundary condition

$$u(x,t) = 0 \quad \text{for} \quad (x,t)\in\partial\Omega\times]0,\infty[\tag{3.20}$$

and time-periodicity condition

$$u(x,t+\omega) = u(x,t) \quad \text{for} \quad (x,t)\in\Omega\times]0,\infty[, \tag{3.21}$$

where $\Omega\subset\mathbb{R}^N$ is an open bounded set with a Lipschitzian boundary, $\omega>0$ is a given number, ∇u is the gradient vector $\nabla u := (\partial_1 u,\dots,\partial_N u)$, $\partial_i u := \frac{\partial u}{\partial x_i}$ and the constitutive operator F has a special diagonal form

$$\big(F(\nabla u)\big)_i := F_i(\partial_i u),\ i=1,\dots,N, \tag{3.22}$$

where $F_1,\dots,F_N$ are scalar Prandtl-Ishlinskii operators of the form II(3.2), i.e.

$$F_i(\partial_i u)(x,t) = \mathcal{F}_{\varphi_i}\big(\lambda_i(x,\cdot),\partial_i u(x,\cdot)\big)(t) \tag{3.23}$$

satisfying for every $i=1,\dots,N$ the following hypotheses.

Assumption 3.2.

(i) $R_0 > 0$ *and* $\lambda_i \in C(\bar{\Omega}; \Lambda_{R_0})$ *for* $i = 1, \dots, N$ *are given;*

(ii) $\varphi_i(r) = \int_0^r h_i(s)\,ds$, *where* $h_i \in W^{1,\infty}(0,\infty)$, $i = 1, \dots, N$ *are given functions such that* $h_i(\infty) = 0$ *and there exist* $b > a > 0$, $\alpha_i \in]0,1[$ *such that*

$$
a\big[\max\{r, R_0\}\big]^{\alpha_i - 2} \leqslant -h_i'(r) \leqslant b r^{\alpha_i - 2} \quad \text{for a.e. } r > 0. \tag{3.24}
$$

From Assumption 3.2(ii) we immediately derive the following properties of φ_i.

$$
\begin{aligned}
&\text{(i)} \quad h_i(r) \geqslant \frac{a}{1-\alpha_i} r^{\alpha_i - 1} \quad \text{for} \quad r \geqslant R_0,\\
&\text{(ii)} \quad \varphi_i(r) \leqslant \frac{b}{\alpha_i(1-\alpha_i)} r^{\alpha_i} \quad \text{for} \quad r \geqslant 0.
\end{aligned} \tag{3.25}
$$

We shall deal with anisotropic spaces defined in Appendix V.2. Let us first mention the following easy result.

Lemma 3.3. *Operators* F_i *map* $L^p(\Omega; C_\omega)$ *into* $L^{\frac{p}{\alpha_i}}(\Omega, C_\omega)$ *for* $p \in [1, \infty[$, $i = 1, \dots, N$ *and for every* $v, w \in L^p(\Omega, C_\omega)$ *we have*

$$
|F_i(v) - F_i(w)|_{(\frac{p}{\alpha_i}, \infty)} \leqslant \frac{2b}{\alpha_i(1-\alpha_i)} |v - w|_{(p,\infty)}. \tag{3.26}
$$

P r o o f. By II(3.3)(ii) and (3.25) we have for every $v, w \in C(\bar{\Omega}, C_\omega)$ and $x \in \Omega$

$$
|F_i(v)(x,\cdot) - F_i(w)(x,\cdot)|_\infty \leqslant \frac{2b}{\alpha_i(1-\alpha_i)} |v(x,\cdot) - w(x,\cdot)|_\infty^{\alpha_i},
$$

hence (3.26) holds. The functions $F_i(v)(x,\cdot)$, $F_i(w)(x,\cdot)$ are ω-periodic for $t \geqslant \omega$ and we can assume that they belong to $C(\bar{\Omega}; C_\omega)$. The assertion follows from the density of $C(\bar{\Omega}, C_\omega)$ in $L^p(\Omega, C_\omega)$. $\square$

We now fix multiindices $\mathbf{p} = (p_i)$, $\mathbf{p}' = (p_i')$, where $p_i = 1 + \alpha_i$, $p_i' = 1 + \frac{1}{\alpha_i}$ (note that we have $\frac{1}{p_i} + \frac{1}{p_i'} = 1$) for $i = 1, \dots, N$ and put $\alpha_0 := \min\{\alpha_i;\ i = 1, \dots, N\}$. Our basic functional framework consists of the space

$$
\begin{aligned}
Z := \big\{ u \in L^{1+\alpha_0}(\Omega; L^2_\omega);\ u_t \in L^2(\Omega; L^2_\omega), \partial_i u \in L^{p_i}(\Omega; C_\omega),\\
\partial_i u_t \in L^{p_i}(\Omega; L^3_\omega), i = 1, \dots, N \big\}
\end{aligned} \tag{3.27}
$$

endowed with the natural norm

$$
|u|_Z := |u|_{(1+\alpha_0, 3)} + |u_t|_2 + |\nabla u|_{(\mathbf{p},\infty)} + |\nabla u_t|_{(\mathbf{p},3)} \tag{3.28}
$$

with the notation of Appendix V.2.

Let $\{u_k;\, k \in \mathbb{N}\} \subset L^2(\Omega)$ be the complete orthonormal system of eigenfunctions of the Laplacian

$$-\Delta u_k = \lambda_k u_k, \quad u_k \in \overset{\circ}{W}{}^{1,2}(\Omega),\ \lambda_k > 0. \tag{3.29}$$

We define a system of basis functions

$$w^{k\ell} := \begin{cases} \sin \frac{2\pi}{\omega} \ell t\ u_k(x) & \text{for} \quad \ell = 1, 2, \dots,\ k = 1, 2, \dots, \\ \cos \frac{2\pi}{\omega} \ell t\ u_k(x) & \text{for} \quad \ell = 0, -1, -2, \dots,\ k = 1, 2, \dots \end{cases} \tag{3.30}$$

and subspaces of Z

$$\begin{aligned} Z_0 :& \quad \text{closure in } Z \text{ of } \operatorname{Lin}\{w^{k\ell};\, \ell \in \mathbb{Z},\, k \in \mathbb{N}\}, \\ Z_1 :& \quad \text{closure in } Z \text{ of } \operatorname{Lin}\{w^{k\ell};\, \ell \neq 0,\, k \in \mathbb{N}\}, \\ Z_2 :& \quad \text{closure in } Z \text{ of } \operatorname{Lin}\{w^{k0};\, k \in \mathbb{N}\}, \end{aligned}$$

We now state the main existence and uniqueness theorem.

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^N$ be an open bounded set with a Lipschitzian boundary and let the operator F given by (3.22), (3.23) satisfy Assumption 3.2. Let $Q \in L^{\mathsf{p}'}(\Omega; C_\omega)$ be given such that $Q_{tt} \in L^{\mathsf{p}'}(\Omega; L^{\frac{3}{2}}_\omega)$. Then there exists a unique $u \in Z_0$ such that for every $z^0 \in Z_0$ we have*

$$\int_\Omega \int_\omega^{2\omega} -u_t z_t^0 + \langle F(\nabla u) + Q, \nabla z^0 \rangle dt\, dx = 0. \tag{3.31}$$

The integral in (3.31) is meaningful, since by Lemma 2.3 the operator F maps (continuously) $L^{\mathsf{p}}(\Omega; C_\omega)$ into $L^{\mathsf{p}'}(\Omega, C_\omega)$. The method of proof consists in splitting the unknown function into two components $u = v + w,\ v \in Z_1,\ w \in Z_2$ following the idea of Prodi (1966), cf. also Lions (1969), Sect.7.1 of Chap.4. We consider two auxiliary problems.

Auxiliary Problem I. *Find $v \in Z_1$ such that*

$$\int_\Omega \int_\omega^{2\omega} -v_t z_t^1 + \langle F(\nabla v) + Q, \nabla z^1 \rangle\, dt\, dx = 0 \quad \forall z^1 \in Z_1. \tag{3.32}$$

Assume first that we are able to solve Auxiliary Problem I. It remains to determine $w \in Z_2$ such that $u := v + w$ satisfies identity (3.31). We have indeed $u_t = v_t$ and $z^0_t = z^1_t$ for every $z^0 = z^1 + z^2$, $z^j \in Z_j$, $j = 0, 1, 2$. It is therefore natural to require

$$\int_\Omega \int_\omega^{2\omega} \Big(\big\langle F(\nabla v + \nabla w) - F(\nabla v), \nabla z^0 \big\rangle + \big\langle Q + F(\nabla v), \nabla z^2 \big\rangle \Big) dt\, dx = 0. \tag{3.33}$$

By Proposition II.4.12 the function $F(\nabla v + \nabla w) - F(\nabla v)$ is independent of t for $t \geqslant \omega$; we denote by ϑ_i the function corresponding to F_i by Proposition II.4.12 and for $i = 1, \dots, N$, $\zeta \in \mathbb{R}^N$ put

$$\big(\Theta(\nabla v(x, \cdot), \zeta)\big)_i := \vartheta_i\big(\partial_i v(x, \cdot), \zeta_i\big). \tag{3.34}$$

We then have for a.e. $x \in \Omega$

$$F(\nabla v + \nabla w)(x, t) - F(\nabla v)(x, t) = \Theta\big(\nabla v(x, \cdot), \nabla w(x)\big) \tag{3.35}$$

and (3.33) is equivalent to

$$\int_\Omega \big\langle \Theta\big(\nabla v(x, \cdot), \nabla w(x)\big) + \hat{Q}(x), \nabla z^2(x) \big\rangle dx \quad \forall z^2 \in Z_2, \tag{3.36}$$

where

$$\hat{Q}(x) := \frac{1}{\omega} \int_\omega^{2\omega} \big(Q + F(\nabla v)\big)(x, t) dt. \tag{3.37}$$

We therefore state

Auxiliary Problem II. *Let v be a solution of Auxiliary Problem I. Find $w \in Z_2$ such that identity (3.36) holds.*

Before solving Auxiliary Problems I, II we mention an elementary, but useful property of periodic functions.

Lemma 3.5. *Let $\varrho \in \mathcal{D}(\mathbb{R}^1)$ be an odd function. Then for each $f \in L^2_\omega$ we have*

$$\int_0^\omega \int_{-\infty}^\infty \varrho(s - t)\, f(s)\, f(t)\, dt\, ds = 0.$$

P r o o f. The assertion follows from obvious integral identities

$$\int_0^\omega \int_{-\infty}^\infty \varrho(s - t)\, f(s)\, f(t)\, dt\, ds = \int_0^\infty \varrho(\tau) \int_0^\omega \big(f(s) f(s - \tau) - f(s + \tau) f(s) \big)\, ds\, d\tau,$$

$$\int_0^\omega f(s)\, f(s - \tau)\, ds = \int_0^\omega f(s + \tau)\, f(s)\, ds \qquad \forall \tau \in \mathbb{R}^1.$$

□

Lemma 3.6. *Let the hypotheses of Theorem 3.4 be fulfilled. Then there exists a unique solution v of Auxiliary Problem I.*

P r o o f. *Uniqueness.* Let $v, \tilde{v} \in Z_1$ be two solutions of (3.32) and let $\gamma \in \mathcal{D}(\mathbb{R}^1)$ be a nonnegative *even* function, $\int_{-\infty}^{\infty} \gamma(s)ds = 1$. For $m \in \mathbb{N}$ put $z^{(m)}(x,t) := m\int_{-\infty}^{\infty} \gamma\big(m(s-t)\big)\big(v_t(x,s) - \tilde{v}_t(x,s)\big)ds$. We have $z^{(m)} \in Z_1$ and (3.32) yields for $z^1 = z^{(m)}$

$$(3.38) \qquad \int_\Omega \int_\omega^{2\omega} \big((\tilde{v}_t - v_t)z_t^{(m)} + \big\langle F(\nabla v) - F(\nabla \tilde{v}), \nabla z^{(m)}\big\rangle\big)dt\,dx = 0.$$

By Lemma 3.5 we have for a.e. $x \in \Omega$

$$\int_\omega^{2\omega} (v_t - \tilde{v}_t)z_t^{(m)}dt = m^2 \int_0^\omega \int_{-\infty}^{\infty} \gamma'\big(m(s-t)\big)(v_t - \tilde{v}_t)(x,t)(v_t - \tilde{v}_t)(x,s)\,ds\,dt = 0$$

and passing to the limit as $m \to \infty$ in (3.38) we obtain using the Mean Continuity Theorem (Proposition V.1.14)

$$(3.39) \qquad \int_\Omega \int_\omega^{2\omega} \big\langle F(\nabla v - \nabla \tilde{v}), \nabla v_t - \nabla \tilde{v}_t\big\rangle\,dt\,dx = 0.$$

Using the fact that $\int_0^\omega v(x,t)dt = \int_0^\omega \tilde{v}(x,t)dt = 0$ a.e., we conclude from (3.39) and Corollary II.4.11 $\nabla v = \nabla \tilde{v}$, hence $v = \tilde{v}$.

Existence. We proceed by Galerkin method analogously as in the proof of Theorem 3.2. For a fixed $n \in \mathbb{N}$ we consider the system of equations

$$(3.40) \qquad \int_\Omega \int_\omega^{2\omega} -v_t^{(n)} w_t^{k\ell} + \big\langle F(\nabla v^{(n)}) + Q, \nabla w^{k\ell}\big\rangle\,dt\,dx = 0$$

for $k = 1, \ldots, n$, $\ell = -n, \ldots, n$, $\ell \neq 0$, where

$$(3.41) \qquad v^{(n)}(x,t) := \sum_{k=1}^{n} \sum_{\substack{\ell=-n \\ \ell \neq 0}}^{n} v_{k\ell}\, w^{k\ell}(x,t)$$

and $v_{k\ell} \in \mathbb{R}^1$ are to be determined from (3.40).

We now derive a priori estimates which imply the existence of a solution $\{v_{k\ell}\}$ to (3.40) and enable us to pass to the limit as $n \to \infty$ in the same way as in the proof of Theorem 3.2.

Assume that (3.40) holds. Multiplying (3.40) by $\big(\frac{2\pi}{\omega}\ell\big)^3 v_{k,-\ell}$ we obtain

$$(3.42) \qquad \int_\Omega \int_\omega^{2\omega} \langle F(\nabla v^{(n)})_t, \nabla v_{tt}^{(n)}\rangle dt\,dx = \int_\Omega \int_\omega^{2\omega} \langle Q_{tt}, \nabla v_t^{(n)}\rangle dt\,dx.$$

Proposition II.4.21 and Theorem II.4.18 applied to F_i yield

$$\int_\Omega \int_\omega^{2\omega} F_i(\partial_i v^{(n)})_t \, \partial_i v_{tt}^{(n)} dt\, dx \geqslant \frac{1}{2} \int_\Omega K_i^{(n)}(x) \big|\partial_i v_t(x,\cdot)\big|_3^3 dx \tag{3.43}$$

with $K_i^{(n)}(x) = \frac{1}{2}\inf\{-h_i'(r);\, 0 < r < R_i^{(n)}(x)\}$, $R_i^{(n)}(x) = \max\{R_0, |\partial_i v^{(n)}(x,\cdot)|_\infty\}$. We have $\int_0^\omega \partial_i v^{(n)}(x,t)dt = 0$, hence $|\partial_i v^{(n)}(x,\cdot)|_\infty \leqslant \omega^{\frac{2}{3}} |\partial_i v_t^{(n)}(x,\cdot)|_3$ for a.e. $x \in \Omega$. Assumption 3.2(ii) then implies

$$K_i^{(n)}(x) \geqslant \frac{a}{2}\big(\max\{R_0, \omega^{\frac{2}{3}} \big|\partial_i v_t^{(n)}(x,\cdot)\big|_3\}\big)^{\alpha_i - 2}. \tag{3.44}$$

We define the sets $M_+^i := \{x \in \Omega; |\partial_i v_t^{(n)}(x,\cdot)|_3 \geqslant R_0 \omega^{-\frac{2}{3}}\}$, $M_-^i := \Omega \setminus M_+^i$. In the estimates below we denote by $c_1, c_2, \ldots$ suitable positive constants depending only on $a, b, \omega, \Omega, R_0$ and Q. From (3.44) it follows

$$\begin{aligned} \int_\Omega K_i^{(n)}(x) \big|\partial_i v_t^{(n)}(x,\cdot)\big|_3^3 dx &= \Big(\int_{M_+^i} + \int_{M_-^i}\Big) K_i^{(n)}(x) \big|\partial_i v_t^{(n)}(x,\cdot)\big|_3^3 dx \\ &\geqslant c_1 \int_{M_+^i} |\partial_i v_t^{(n)}(x,\cdot)|_3^{1+\alpha_i} dx \geqslant c_1 \int_\Omega |\partial_i v_t^{(n)}(x,\cdot)|_3^{1+\alpha_i} dx - c_2 \end{aligned}$$

and using (3.43), (3.42) and Hölder's inequality we conclude

$$\big|\nabla v^{(n)}\big|_{(\mathbf{p},\infty)} \leqslant \omega^{\frac{2}{3}} \big|\nabla v_t^{(n)}\big|_{(\mathbf{p},\infty)} \leqslant c_3. \tag{3.45}$$

Inequality (3.26) is valid in particular for $w = 0$, consequently

$$\big|F(\nabla v^{(n)})\big|_{(\mathbf{p}',\infty)} \leqslant c_4. \tag{3.46}$$

A second estimate is obtained by multiplying equation (3.40) by $v_{k\ell}$ which yields

$$\begin{cases} \big|v_t^{(n)}\big|_2^2 & \leqslant \int_\Omega \int_\omega^{2\omega} |\nabla v^{(n)}|\big(|F(\nabla v^{(n)})| + |Q|\big) dt\, dx \leqslant c_5, \\ \big|v^{(n)}\big|_{(2,\infty)} & \leqslant \omega^{\frac{1}{2}} |v_t^{(n)}|_2 \leqslant c_6. \end{cases} \tag{3.47}$$

Estimates (3.45) - (3.47) and the Brouwer degree theory entail similarly as in the proof of Theorem 3.2 that system (3.40) has a solution $\{v_{k\ell}\}$ for every $n \in \mathbb{N}$; moreover, there exist subsequences (still indexed by n) and functions $\sigma \in L^{\mathbf{p}'}(\Omega; L_\omega^\infty)$, $v \in Z_1$ such that $\nabla v^{(n)} \to \nabla v$ in $L^{\mathbf{p}}(\Omega; L_\omega^\infty)$, $F(\nabla v^{(n)}) \to \sigma$ in $L^{\mathbf{p}'}(\Omega; L_\omega^\infty)$ and $v^{(n)} \to v$ in $L^2(\Omega; L_\omega^\infty)$ weakly-star, $\nabla v_t^{(n)} \to \nabla v_t$ in $L^{\mathbf{p}}(\Omega; L_\omega^3)$ and $v_t^{(n)} \to v_t$ in $L^2(\Omega; L_\omega^2)$ weakly. Passing to the limit in (3.40) as $n \to \infty$ we obtain

$$\int_\Omega \int_\omega^{2\omega} \big(- v_t z_t^1 + \langle \sigma + Q, \nabla z^1 \rangle\big) dt\, dx = 0 \quad \forall z^1 \in Z_1. \tag{3.48}$$

The monotonicity of F enables us to use Minty's trick similarly as in the proof of Theorem 1.2. Putting $z^1(x,t) := m\int_{-\infty}^{\infty}\gamma(m(s-t))v_t(x,s)ds$ in (3.48) we obtain for $m\to\infty$ in the same way as in (3.38) for the same choice of γ

$$\int_\Omega\int_\omega^{2\omega}\langle\sigma+Q,\nabla v_t\rangle dt\,dx = 0. \tag{3.49}$$

On the other hand, multiplying equation (3.40) by $\frac{2\pi}{\omega}\ell v_{k,-\ell}$ we have

$$\int_\Omega\int_\omega^{2\omega}\langle F(\nabla v^{(n)})+Q,\nabla v_t^{(n)}\rangle dt\,dx = 0 \quad \forall n\in\mathbb{N}, \qquad \text{hence}$$

$$\begin{aligned}\lim_{n\to\infty}\int_\Omega\int_\omega^{2\omega}\langle F(\nabla v^{(n)}),\nabla v_t^{(n)}\rangle dt\,dx &= -\int_\Omega\int_\omega^{2\omega}\langle Q,\nabla v_t\rangle dt\,dx\\ &= -\int_\Omega\int_\omega^{2\omega}\langle\sigma,\nabla v_t\rangle dt\,dx\end{aligned} \tag{3.50}$$

by (3.49). Let now $z^1\in Z_1$ be arbitrary. For $\delta>0$ we define an element $z^{(\delta)}\in Z_1$ by the formula

$$z^{(\delta)}(x,t) := v(x,t)-\delta\Big(\int_0^t z^1(x,s)ds+\frac{1}{\omega}\int_0^\omega s z^1(x,s)ds\Big).$$

Theorem II.4.9 yields for all $n\in\mathbb{N}$ and $\delta>0$

$$\int_\Omega\int_\omega^{2\omega}\langle F(\nabla v^{(n)})-F(\nabla z^{(\delta)}),\nabla v_t^{(n)}-\nabla z_t^{(\delta)}\rangle dt\,dx \geqslant 0 \tag{3.51}$$

and combining (3.50), (3.51) we obtain

$$\int_\Omega\int_\omega^{2\omega}\langle\sigma-F(\nabla z^{(\delta)}),\nabla z^1\rangle dt\,dx \geqslant 0. \tag{3.52}$$

The operator F is continuous by (3.26); for $\delta\to 0+$ we infer from inequality (3.52) $\int_\Omega\int_\omega^{2\omega}\langle\sigma-F(\nabla v),\nabla z^1\rangle dt\,dx = 0$ for every $z^1\in Z_1$ and identity (3.48) completes the proof. □

Lemma 3.7. *Let the hypotheses of Theorem 3.4 be fulfilled and let $v\in Z_1$ be the solution of Auxiliary Problem I. Then there exists a unique solution $w\in Z_2$ of Auxiliary Problem II.*

P r o o f. The space Z_2 is reflexive and continuously embedded into the Sobolev space $\mathring{W}^{1,1+\alpha_0}(\Omega)$, hence we may define an equivalent norm $|\cdot|_{Z_2}$ as $|w|_{Z_2} := |\nabla w|_{\mathbf{p}}$.

We denote by Z_2^* the dual space of Z_2 and by $((\cdot,\cdot))$ the duality pairing between Z_2 and Z_2^*. Let $T : Z_2 \to Z_2^*$ be the mapping

$$((Tw, z_2)) := \int_\Omega \big\langle \Theta\big(\nabla v(x,\cdot), \nabla w(x)\big) + \hat{Q}(x), \nabla z_2(x)\big\rangle dx, \tag{3.53}$$

$w, z_2 \in Z_2$, where $\Theta, \hat{Q}$ are given by (3.34), (3.37). By Browder's Theorem (Fučík, Kufner (1980), Thm. 29.5) existence of a unique solution $w \in Z_2$ to Auxiliary Problem II is ensured provided T is

(a) *demicontinuous*: $\big|w_n - w\big|_{Z_2} \to 0 \Rightarrow ((Tw_n - Tw, z_2)) \to 0 \quad \forall z_2 \in Z_2$;
(b) *bounded*: if B is a bounded subset of Z_2, then $T(B)$ is a bounded subset of Z_2^*;
(c) *strictly monotone*: $((Tw_1 - Tw_2, w_1 - w_2)) > 0$ for $w_1 \neq w_2$;
(d) *coercive*: $\lim\limits_{|w|_{Z_2} \to \infty} \frac{((Tw,w))}{|w|_{Z_2}} = \infty$.

Properties (a) - (c) are obvious consequences of inequalities II(4.16)(ii) for each of the functions ϑ_i, $i = 1, \dots, N$. To verify the coerciveness of T we introduce for every $w \in Z_2$ the sets

$$\begin{aligned}
M_1^i(w) &:= \big\{x \in \Omega;\ |\partial_i w(x)| \geqslant \max\{R_0, |\partial_i v(x,\cdot)|_\infty\}\big\},\\
M_2^i(w) &:= \big\{x \in \Omega; |\partial_i w(x)| \leqslant |\partial_i v(x,\cdot)|_\infty\big\},\\
M_3^i(w) &:= \Omega \setminus (M_1^i \cup M_2^i).
\end{aligned}$$

By $c_1, c_2, \dots$ we denote again suitable positive constants independent of w. For $x \in M_1^i$ inequalities II(4.16)(ii) and (3.25)(i) yield

$$\begin{aligned}
\vartheta_i\big(\partial_i v(x,\cdot), \partial_i w(x)\big)\partial_i w(x) &\geqslant h_i\big(|\partial_i v(x,\cdot)|_\infty + |\partial_i w(x)|\big)|\partial_i w(x)|^2\\
&\geqslant h_i\big(2|\partial w(x)|\big)|\partial_i w(x)|^2 \geqslant c_1|\partial_i w(x)|^{1+\alpha_i}.
\end{aligned}$$

For $x \in M_2^i$ we obtain from II(4.16)(ii) and (3.25)(ii)

$$\begin{aligned}
\vartheta_i\big(\partial_i v(x,\cdot), \partial_i w(x)\big)\partial_i w(x) &\leqslant 2|\partial_i v(x,\cdot)|_\infty\, \varphi_i\big(\frac{1}{2}|\partial_i v(x,\cdot)|_\infty\big)\\
&\leqslant c_2|\partial_i v(x,\cdot)|_\infty^{1+\alpha_i}
\end{aligned}$$

and for a.e. $x \in M_3^i$ we trivially have $\vartheta_i\big(\partial_i v(x,\cdot), \partial_i w(x)\big)\partial_i w(x) \leqslant c_3$.

This yields

$$\begin{aligned}
((Tw, w)) \geqslant{}& c_1 \sum_{i=1}^N \int_\Omega \big|\partial_i w(x)\big|^{1+\alpha_i} dx\\
&- c_4 \sum_{i=1}^N \int_\Omega \big|\partial_i v(x,\cdot)\big|_\infty^{1+\alpha_i} dx - c_5 - \big|\hat{Q}\big|_{\mathbf{p}'}\big|\nabla w\big|_{\mathbf{p}}\\
\geqslant{}& c_6\big|w\big|_{Z_2}^{1+\alpha_0} - c_7 - c_8\big|w\big|_{Z_2},
\end{aligned}$$

hence T is coercive and Lemma 3.7 is proved. □

Proof of Theorem 3.4. Put $u := v + w$, where v, w are solutions of Auxiliary Problems I, II, respectively. The assertion follows from Lemmas 3.6, 3.7 and identities (3.33)-(3.37). □

The same technique can be applied to the equation of motion (1.1) of an elastoplastic continuum with a constitutive operator of diagonal type $\sigma_{ij} = F_{ij}(\varepsilon_{ij})$, where F_{ij} are scalar Prandtl-Ishlinskii operators satisfying Assumption 3.2. The coercivity of the mapping T defined by (3.53) then follows from the generalized Korn inequality in $W^{1,p}(\Omega)$ proved by Nečas (1966).

Asymptotic stability

To conclude this section we consider again the scalar hyperbolic initial-boundary value problem (1.29), (1.31), (1.32) with an ω-periodic right-hand side q and a constitutive operator F of Prandtl-Ishlinskii type. Theorem 3.2 gives sufficient conditions for the existence of ω-periodic solutions to (1.29), (1.32). Here, we prove by Ficken-Fleishman method (see Vejvoda et al. (1981) for further references) that under natural assumptions the periodic solution is unique and asymptotically stable.

Assumption 3.8.
(i) *$h \in W^{1,\infty}_{\text{loc}}(0,\infty)$ is an increasing function, $a := h(0) > 0$ and for $R > 0$ we denote $b_R := h(R)$, $K_R := \frac{1}{2} \inf \operatorname{ess}\{h'(r); 0 < r < R\}$;*
(ii) *$F : C([0,1];\Lambda_R) \times C([0,1]\times[0,T]) \to C([0,1]\times[0,T])$ for arbitrary $R > 0$ and $T > 0$ is an operator of the form*

$$F(\lambda,\sigma)(x,t) := \mathop{\mathcal{F}}_{\varphi}\big(\lambda(x,\cdot),\sigma(x,\cdot)\big)(t),$$

where $\mathcal{F}_\varphi$ is the Prandtl-Ishlinskii operator II(3.2) and Λ_R is endowed with the sup-norm.

To simplify the notation we introduce the space S of pairs (v,σ) of functions defined for $(x,t) \in [0,1]\times[0,\infty[$

$$\begin{aligned} S := \Big\{(v,\sigma) \in \big(L^\infty(]0,1[\times]0,\infty[)\big)^2;\ & v_t, v_x, \sigma_t, \sigma_x \in L^\infty\big(0,\infty;L^2(0,1)\big), \\ & v(0,t) = \sigma(1,t) = 0 \quad \forall t \geqslant 0\Big\}, \end{aligned}$$

endowed with norm $\big|(v,\sigma)\big|_S := |v|_\infty + |\sigma|_\infty + \big|\int_0^1 (|v_t|^2+|v_x|^2+|\sigma_t|^2+|\sigma_x|^2)(x,\cdot)dx\big|_\infty^{\frac{1}{2}}$. Theorem V.2.4 entails that for $(v,\sigma) \in S$ both v and σ are $\frac{1}{2}$-Hölder continuous in $[0,1]\times[0,\infty[$.

For $(v,\sigma) \in S$, $q \in L^\infty\big(0,\infty; L^2(0,1)\big)$ and $\lambda \in C([0,1];\Lambda_R)$ we further denote

$$\mathcal{E}(\lambda, v, \sigma, q) := \begin{pmatrix} v_t - \sigma_x - q \\ F(\lambda,\sigma)_t - v_x \end{pmatrix} \tag{3.54}$$

The main result reads as follows.

Theorem 3.9. *Let F fulfil Assumption 3.8 and let $q \in L^\infty\big(0,\infty; L^2(0,1)\big)$ be a given function such that $q_t \in L^\infty\big(0,\infty; L^2(0,1)\big)$, $q(x,t+\omega) = q(x,t)$ for a.e. $(x,t) \in]0,1[\times]0,\infty[$. Assume that the set*

$$J_R := \big\{(v^0,\sigma^0) \in \big(W^{1,2}(0,1)\big)^2;\ v^0(0) = \sigma^0(1) = 0 \ \text{and conditions (2.2) hold}\big\}$$

is nonempty for some $R > 0$. Then there exists a unique element $(v^*,\sigma^*) \in S \cap \big(C([0,1];C_\omega)\big)^2$ such that

(i) *$\mathcal{E}(\lambda, v^*, \sigma^*, q) = 0$ for every $\lambda \in C([0,1];\Lambda_R)$ and a.e. $(x,t) \in]0,1[\times]\omega,\infty[$,*

(ii) *for every $(v^0,\sigma^0) \in J_R$ and $\lambda \in C([0,1];\Lambda_R)$ the solution $(v,\sigma) \in S$ of the equation*

$$\mathcal{E}(\lambda, v, \sigma, q) = 0 \tag{3.55}$$

satisfying initial conditions (1.31) has the property

$$\lim_{t\to\infty} \Big(\big|v(\cdot,t) - v^*(\cdot,t)\big|_\infty + \big|\sigma(\cdot,t) - \sigma^*(\cdot,t)\big|_\infty\Big) = 0. \tag{3.56}$$

P r o o f. For $(v,\sigma), (\hat v,\hat\sigma) \in S$ and $\lambda, \hat\lambda \in C([0,1];\Lambda_R)$ we define the functional

$$\begin{aligned} V(v,\hat v,\sigma,\hat\sigma,\lambda,\hat\lambda)(t) := \int_0^1 \Big[& a(\sigma-\hat\sigma)^2(x,t) + (v-\hat v)^2(x,t) + \\ & + \int_0^\infty \Big(p_r\big(\lambda(x,\cdot),\sigma(x,\cdot)\big)(t) - p_r\big(\hat\lambda(x,\cdot),\hat\sigma(x,\cdot)\big)\Big)^2(t)\,dh(r)\Big]dx. \end{aligned}$$

If now equation (3.55) is satisfied for both v,σ,λ and $\hat v,\hat\sigma,\hat\lambda$, then Theorem II.4.9 yields

$$\frac{d}{dt} V(v,\hat v,\sigma,\hat\sigma,\lambda,\hat\lambda)(t) \leqslant 0 \quad \text{a.e.} \tag{3.57}$$

Inequality (3.57) provides a sufficient tool for proving the uniqueness of periodic solutions to (i). Assume that for $(v^i,\sigma^i) \in S \cap \big(C([0,1],C_\omega)\big)^2$ and $\lambda_i \in C([0,1];\Lambda_R)$ we have $\mathcal{E}(\lambda^i, v^i, \sigma^i, q) = 0$, $i = 1,2$. Then $V(v^1,v^2,\sigma^1,\sigma^2,\lambda_1,\lambda_2)$ is ω-periodic for $t \geqslant \omega$, hence $\frac{d}{dt}V(v^1,v^2,\sigma^1,\sigma^2,\lambda_1,\lambda_2)(t) = 0$ for a.e. $t > \omega$. From Theorems II.4.9, II.4.10 we obtain $\sigma_t^1(x,t) - \sigma_t^2(x,t) = 0$ for a.e. $t > \omega$.

For $r > 0$ and $x \in [0,1]$ put $\xi_r^i(x,t) := p_r\big(\lambda_i(x,\cdot),\sigma^i(x,\cdot)\big)(t)$, $i = 1,2$. From inequality II(1.4) it follows $\frac{\partial}{\partial t}|\xi_r^1(x,t) - \xi_r^2(x,t)|^2 \leqslant 0$ a.e.; since ξ_r^i are periodic for $t \geqslant \omega$, we conclude $F(\lambda_1,\sigma^1)_t = F(\lambda_2,\sigma^2)_t$ for $t > \omega$, hence $v^1 = v^2$ and $\sigma^1 = \sigma^2$. We see in particular that it suffices to prove statement (i) for one special $\lambda^* \in C([0,1];\Lambda_R)$.

In order to construct (v^*,σ^*) with the required properties we consider arbitrary $\lambda \in C([0,1];\Lambda_R)$ and $(v^0,\sigma^0) \in J_R$. By Theorems 2.3, 2.1 there exists a unique solution $(v,\sigma) \in S$ to system (3.55), (1.31), (1.32). Let $\big\{(v^{(n)},\sigma^{(n)});\, n \in \mathbb{N}\big\} \subset S$ be the sequence

$$v^{(n)}(x,t) := v(x,t+n\omega),\ \sigma^{(n)}(x,t) := \sigma(x,t+\omega);\ (x,t) \in [0,1] \times [0,\infty[.$$

The semigroup property I(1.27) applied to the play operator reads

$$p_r\big(\lambda(x,\cdot),\sigma(x,\cdot)\big)(t+n\omega) = p_r\big(\lambda_n(x,\cdot),\sigma^{(n)}(x,\cdot)\big)(t) \quad \text{for} \quad t \geqslant 0, \tag{3.58}$$

where $\lambda_n(x,r) := p_r\big(\lambda(x,\cdot),\sigma(x,\cdot)\big)(n\omega) \in C([0,1];\Lambda_R)$. For every $n \in \mathbb{N}$ we can rewrite (3.55) in the form

$$\mathcal{E}(\lambda_n, v^{(n)}, \sigma^{(n)}, q) = 0 \quad \text{a.e.} \tag{3.59}$$

The sequence $\big\{(v^{(n)},\sigma^{(n)})\big\}$ is equibounded in S. By Theorem V.2.4 there exists a subsequence $\{n_k;\, k \in \mathbb{N}\} \subset \mathbb{N}$ and an element $(v^*,\sigma^*) \in S$ such that $(v^{(n_k)},\sigma^{(n_k)}) \to (v^*,\sigma^*)$ in S weakly-star, $v^{(n_k)} \to v^*, \sigma^{(n_k)} \to \sigma^*$ locally uniformly in $[0,1]\times[0,\infty[$. Inequality II(2.9) yields $|\lambda_n(x,r)-\lambda_n(y,r)| \leqslant \max\{|\lambda(x,r)-\lambda(y,r)|, |\sigma(x,\cdot)-\sigma(y,\cdot)|_\infty\}$, hence $\{\lambda_n\}$ is an equicontinuous sequence in $C([0,1];\Lambda_R)$. Since Λ_R is compact in $C([0,R])$, we can assume using Arzelà-Ascoli Theorem V.2.1 that there exists $\lambda^* \in C([0,1];\Lambda_R)$ such that $\lambda_{n_k} \to \lambda^*$ uniformly. From (3.57) it follows

$$\big|V(v^{(n_k)},v^*,\sigma^{(n_k)},\sigma^*,\lambda_{n_k},\lambda^*)\big|_\infty \leqslant V(v^{(n_k)},v^*,\sigma^{(n_k)},\sigma^*,\lambda_{n_k},\lambda^*)(0),$$

hence

$$\lim_{k\to\infty}\Big|\int_0^1 \big(|v^{(n_k)}+v^*|^2(x,\cdot) - |\sigma^{(n_k)}-\sigma^*|^2(x,\cdot)\big)dx\Big|_\infty = 0. \tag{3.60}$$

We now prove that v^*,σ^* are ω-periodic. Put $v^{**}(x,t) := v^*(x,t+\omega)$, $\sigma^{**}(x,t) := \sigma^*(x,t+\omega)$, $\lambda^{**}(x,r) := p_r\big(\lambda^*(x,\cdot),\sigma^*(x,\cdot)\big)(\omega)$. Passing to the limit in (3.59) as $k \to \infty$ we obtain for a.e. $(x,t) \in]0,1[\times]0,\infty[$

$$\begin{aligned} &\text{(i)} \quad \mathcal{E}(\lambda^*,v^*,\sigma^*,q) = 0,\\ &\text{(ii)} \quad \mathcal{E}(\lambda^{**},v^{**},\sigma^{**},q) = 0. \end{aligned} \tag{3.61}$$

Put $\beta := \lim_{k\to\infty} V(v^{(1)}, v, \sigma^{(1)}, \sigma, \lambda_1, \lambda)(t) \geqslant 0$. For every $t \geqslant 0$ we have

$$\begin{aligned}\beta &= \lim_{k\to\infty} V(v^{(n_k+1)}, v^{(n_k)}, \sigma^{(n_k+1)}, \sigma^{(n_k)}, \lambda_{n_k+1}, \lambda_{n_k})(t)\\ &= V(v^{**}, v^{*}, \sigma^{**}, \sigma^{*}, \lambda^{**}, \lambda^{*})(t),\end{aligned} \tag{3.62}$$

hence

$$\begin{aligned}0 = \frac{1}{2}\frac{d}{dt} V(v^{**}, v^{*}, \sigma^{**}, \sigma^{*}, \lambda^{**}, \lambda^{*})(t) = \int_0^1 \Big(\frac{1}{2}\frac{\partial}{\partial t}\big[a(\sigma^{**}-\sigma^{*})^2(x,t)+ \\ + \int_0^\infty \big(p_r(\lambda^{**},\sigma^{*}) - p_r(\lambda^{*},\sigma^{*})\big)^2(x,t)dh(r)\big]- \\ - \big(F(\lambda^{**},\sigma^{*}) - F(\lambda^{*},\sigma^{*})\big)_t(\sigma^{**}-\sigma^{*})(x,t)\Big)dx.\end{aligned}$$

Theorems II.4.9, II.4.10 yield $\sigma^{**}(x,t) - \sigma^{*}(x,t) = \lambda^{**}\big(x, R_0(x,t)\big) - \lambda^{*}\big(x, R_0(x,t)\big)$, where $R_0(x,t) := \max\big\{M(\lambda^{**}(x,\cdot), \sigma^{**}(x,\cdot), t),\ M(\lambda^{*}(x,\cdot), \sigma^{*}(x,\cdot), t)\big\}$. For every $x \in [0,1]$ the function $R_0(x,t)$ is monotone; there exists therefore the limit $s(x) := \lim_{t\to\infty} \sigma^{**}(x,t) - \sigma^{*}(x,t)$. Since σ^{*} is bounded, we necessarily have $s(x) \equiv 0$.

From Proposition II.2.10 we similarly infer that for every x and r there exists the limit $z(x,r) := \lim_{t\to\infty} p_r\big(\lambda^{**}(x,\cdot), \sigma^{**}(x,\cdot)\big)(t) - p_r\big(\lambda^{*}(x,\cdot), \sigma^{*}(x,\cdot)\big)(t)$ and that $z(x,r) \equiv 0$.

Let $\delta > 0$ be given. Since σ^{*} is uniformly continuous, there exists $T_0 > 0$ such that

$$\big|\sigma^{*}(\cdot, t+\omega) - \sigma^{*}(\cdot, t)\big|_\infty < \delta \quad \text{for all} \quad t \geqslant T_0. \tag{3.63}$$

By (3.60) there exists $\ell \in \mathbb{N}$ such that

$$\Big|\int_0^1 |\sigma^{*}(x,\cdot) - \sigma^{(n_k)}(x,\cdot)|^2 dx\Big|_\infty < \delta^2 \quad \text{for} \quad k \geqslant \ell. \tag{3.64}$$

Put $T_1 := T_0 + n_\ell\omega$. For $s \geqslant T_1$ we have $s - n_\ell\omega \geqslant T_0$, hence

$$\begin{aligned}\big|\sigma(\cdot, s+\omega) - \sigma(\cdot, s)\big|_2 \leqslant \big|\sigma(\cdot, s+\omega) - \sigma^{*}(\cdot, s - n_\ell\omega + \omega)\big|_2 + \\ + \big|\sigma^{*}(\cdot, s - n_\ell\omega + \omega) - \sigma^{*}(\cdot, s - n_\ell\omega)\big|_2 + \big|\sigma^{*}(\cdot, s - n_\ell\omega) - \sigma(\cdot, s)\big|_2 \leqslant 3\delta.\end{aligned} \tag{3.65}$$

Let now $t \geqslant 0$ be arbitrary. We fix $k \geqslant \ell$ such that $t + n_k\omega \geqslant T_1$. Then

$$\begin{aligned}\big|\sigma^{*}(\cdot, t+\omega) - \sigma^{*}(\cdot, t)\big|_2 \leqslant \big|\sigma^{*}(\cdot, t+\omega) - \sigma^{n_k}(\cdot, t+\omega)\big|_2 + \\ + \big|\sigma^{*}(\cdot, t) - \sigma^{(n_k)}(\cdot, t)\big|_2 + \big|\sigma(\cdot, t + n_k\omega + \omega) - \sigma(\cdot, t + n_k\omega)\big|_2 \leqslant 5\delta.\end{aligned}$$

Since $\delta > 0$ was arbitrary, we conclude from this last inequality that $\sigma^{*} = \sigma^{**}$, i.e. $\sigma^{*} \in C([0,1]; C_\omega)$. By (3.61), v_t^{*} is ω-periodic and v_x^{*} is ω-periodic for $t \geqslant \omega$. We thus have $v^*(x,t+\omega) - v^*(x,t) = v^*(x,\omega) - v^*(x,0)$, hence $v^*_x(x,\omega) = v^*_x(x,0)$ a.e. and $v^{**} = v^* \in C([0,1];C_\omega)$. Passing to the limit as $t \to \infty$ in (3.62) we obtain $\beta = 0$, hence $\lambda^{**} = \lambda^*$.

Let now $\{d_n\}$ be the sequence

$$d_n := V(v^{(n)}, v^*, \sigma^{(n)}, \sigma^*, \lambda_n, \lambda^*)(0).$$

By (3.57) we have $d_{n+1} \leqslant d_n$ for every n and $\lim_{k\to\infty} d_{n_k} = 0$, hence $\lim_{n\to\infty} d_n = 0$ and (3.57) yields

$$\Big| \int_0^1 (|v^{(n)} - v^*|^2 + |\sigma^{(n)} - \sigma^*|^2)(x,\cdot)dx \Big|_\infty \to 0 \quad \text{as} \quad n \to \infty.$$

The whole sequence $\{(v^{(n)}, \sigma^{(n)}); n \in \mathbb{N}\}$ therefore converges locally uniformly to (v^*, σ^*) in $[0,1] \times [0,\infty[$.

We fix again an arbitrary $\delta > 0$ and find n_0 such that for every $n \geqslant n_0$ and $(x,t) \in [0,1] \times [0,\omega]$ we have

$$|v^{(n)}(x,t) - v^*(x,t)| + |\sigma^{(n)}(x,t) - \sigma^*(x,t)| < \delta.$$

For each $t > n_0\omega$ we find $n \geqslant n_0$ such that $t - n\omega \in [0,\omega[$. Then

$$\begin{aligned} &\big|v(\cdot,t) - v^*(\cdot,t)\big|_\infty + \big|\sigma(\cdot,t) - \sigma^*(\cdot,t)\big|_\infty = \\ &= \big|v^{(n)}(\cdot,t-n\omega) - v^*(\cdot,t-n\omega)\big|_\infty + \big|\sigma^{(n)}(\cdot,t-n\omega) - \sigma^*(\cdot,t-n\omega)\big|_\infty < \delta \end{aligned}$$

and Theorem 3.9 is proved. □

IV. The Riemann problem

We illustrate here the connection between hysteresis and hyperbolic equations from another point of view. We consider the Riemann problem for a system of the form III(1.29) with a non-hysteretic constitutive operator F which is generated by a single-valued not necessarily monotone scalar constitutive function g. This is different from the approach of Keyfitz (1986), where a nonmonotone constitutive law is replaced with a hysteretic one. We assume no hysteresis in the data and transform the Riemann problem for self-similar solutions into a boundary-value problem for a singular first-order ordinary differential equation. One observes the following facts.

- If the constitutive function is nonlinear, then even smooth data admit infinitely many solutions.
- The Second Principle of Thermodynamics does not guarantee uniqueness of solutions if and only if the constitutive function has at least one inflection point.
- The Lax (1957) entropy condition does not guarantee uniqueness of solutions if the constitutive function has at least two inflection points.

The investigation of monotone solutions separately for forward and backward waves shows that they can be represented by their trajectories along the graph of the constitutive function, where shocks correspond to straight segments connecting two points on the constitutive graph. These trajectories are convex if the solution increases and concave if the solution decreases (see Fig. 14 on page 167). The solutions themselves therefore exhibit a hysteretic behavior which thus appears as an intrinsic property of quasilinear hyperbolic equations.

We obtain existence and uniqueness in the Riemann problem by splitting the solution into the backward and forward part with an auxiliary transition condition which is to be found. Each of the two parts is then subjected to a new form of the maximal dissipation principle which selects the solution with minimal L^2-norm, or equivalently the monotone solution with the minimal convex (maximal concave) trajectory along the convex hull of the constitutive graph similarly to the idea of Leibovich (1974), see Fig. 15 on page 170. We prove that this selection rule is compatible with the shock admissibility criteria of Lax (1957), Liu (1981) and with the vanishing viscosity criterion, but not with the Dafermos (1973) maximal entropy rate criterion in general.

We concentrate our attention to particular aspects of the Riemann problem; a more complete information can be found for instance in the recent monograph by Chang and Hsiao (1989).

IV.1 Weak self-similar solutions

Quasilinear hyperbolic systems with regular data may exhibit singularities in a finite time. To be able to continue the solution, one has to pass to a generalized concept of weak solutions where discontinuities are allowed. Self-similar solutions then naturally arise as the limit case when we magnify the scale of observation of a solution in a neighborhood of an isolated discontinuity. We show that weak solutions are in general not uniquely determined by the data and further physically motivated conditions have to be prescribed.

Nonexistence of smooth solutions

We start with a modification of an example of John (1976).

Example 1.1. Let us consider the system

$$\begin{cases} v_t = c^2(\varepsilon)\varepsilon_x \\ \varepsilon_t = v_x \end{cases} \tag{1.1}$$

analogous to III(1.29) with constitutive law $\sigma = g(\varepsilon)$, $g'(\varepsilon) = c^2(\varepsilon)$.

We prescribe initial conditions

$$\varepsilon(x,0) = \Phi(x),\ v(x,0) = \int_0^{\Phi(x)} c(s)ds, \tag{1.2}$$

where $c, \Phi : \mathbb{R}^1 \to \mathbb{R}^1$ are given smooth functions. We further assume that there exist constants $\alpha, \beta > 0$, $c_0, \varphi_0 \in \mathbb{R}^1$ and an interval $[x_1, x_2] \subset \mathbb{R}^1$ such that

$$\sup\{|\Phi'(x)|; x \in \mathbb{R}^1\} \leqslant \alpha, \quad \sup\{|c'(s)|; s \in \mathbb{R}^1\} \leqslant \beta, \tag{1.3}$$

$$c(s) = c_0 + \beta s \quad \text{for} \quad s \in [s_1, s_2], \tag{1.4}$$

$$\Phi(x) = \varphi_0 + \alpha x \quad \text{for} \quad x \in [x_1, x_2], \tag{1.5}$$

where $s_i := \Phi(x_i)$, $i = 1, 2$. Put $t_0 := \frac{1}{\alpha\beta}$, $x_0 := -t_0(c_0 + \beta\varphi_0)$.

According to the classical general theory of Courant, Hilbert (1937), problem (1.1),(1.2) has a unique local smooth solution. In fact, this solution can be directly found.

Using the Banach Contraction Principle we define the functions $\varepsilon, v : \mathbb{R}^1 \times [0, t_0[\to \mathbb{R}^1$ implicitly by

$$\varepsilon = \Phi(x + tc(\varepsilon)), \quad v(x,t) := \int_0^{\varepsilon(x,t)} c(s)\, ds. \tag{1.6}$$

The elementary identity $\varepsilon_t(x,t) = c(\varepsilon(x,t))\varepsilon_x(x,t)$ implies that ε, v solve (1.1), (1.2) for $t < t_0$.

We now show that the limits of $\varepsilon(x,t), v(x,t)$ as $(x,t) \to (x_0,t_0)$ do not exist. Let λ_p for $p \in \mathbb{R}^1$ denote the segment

$$\lambda_p := \{(x,t) \in \mathbb{R}^1 \times [0,t_0[;\ x = p - tc(\Phi(p))\}.$$

For $(x,t) \in \lambda_p$ we have $\varepsilon(x,t) = \Phi\big(p + t(c(\varepsilon(x,t)) - c(\Phi(p)))\big)$, consequently $\varepsilon(x,t) = \Phi(p)$. For $p \in [x_1,x_2]$ the equation of λ_p reads $x - x_0 = (p - x_0)(1 - \frac{t}{t_0})$, hence all λ_p's intersect each other at the point (x_0,t_0).

Weak solutions

Example 1.1 suggests that an appropriate functional framework for describing the global behavior of solutions to quasilinear systems should include discontinuous functions.

An alternative approach to systems of the type (1.1) consists in a formal transformation into a single quasilinear wave equation

$$u_{tt} = g(u_x)_x \tag{1.7}$$

for $u(x,t) := \int_0^x \varepsilon(\xi,t)d\xi$, $(x,t) \in \mathbb{R}^2_+ := \mathbb{R}^1 \times]0,\infty[$, where g is a function defined in an (unbounded or bounded) interval $]a,b[\subset \mathbb{R}^1$ with values in another (unbounded or bounded) interval $]c,d[\subset \mathbb{R}^1$. Throughout this chapter we assume only that

$$\begin{aligned} &\text{(i)} \quad g :]a,b[\to]c,d[\quad \text{is locally Lipschitz,} \\ &\text{(ii)} \quad g(a+) = c,\ g(b-) = d. \end{aligned} \tag{1.8}$$

The fact that the function g is *not necessarily monotone* (so that equation (1.7) may change type) plays here a less important role than the fact that g is allowed to be *nonlinear*.

We prescribe initial conditions

$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x), \tag{1.9}$$

where $\varphi, \psi : \mathbb{R}^1 \to \mathbb{R}^1$ are given functions.

A suitable function space for the superposition operator generated by the function g is L^∞; we thus define a weak solution to (1.7) as a function u such that

$$\begin{aligned} &\text{(i)} \quad u \in W^{1,\infty}(\mathbb{R}^2_+),\ u_x(x,t) \in\ ]a,b[\text{ a.e.},\ g(u_x) \in L^\infty(\mathbb{R}^2_+), \\ &\text{(ii)} \iint_{\mathbb{R}^2_+} \big(u_t\varrho_t - g(u_x)\varrho_x\big)dx\,dt = 0 \quad \forall \varrho \in \mathcal{D}(\mathbb{R}^2_+). \end{aligned} \tag{1.10}$$

According to this definition we require

$$\varphi(0)=0,\ \varphi\in W^{1,\infty}(\mathbb{R}^1),\ \varphi'(x)\in]a,b[\text{ a.e.},\ g(\varphi')\in L^\infty(\mathbb{R}^1),\ \psi\in L^\infty(\mathbb{R}^1). \tag{1.11}$$

We have to interpret condition (1.9) which is not well defined in L^∞. If g is linear (say $g(v)=k^2v$ for some $k>0$), then the solution to (1.10), (1.9) is given by the formula

$$u(x,t)=\frac{1}{2}(\varphi(x-kt)+\varphi(x+kt))+\frac{1}{2k}\int_{x-kt}^{x+kt}\psi(\xi)d\xi, \tag{1.12}$$

and under the hypothesis (1.11) we have

$$u_t,u_x\in C([0,\infty[;L^2_{\text{loc}}(\mathbb{R}^1)). \tag{1.13}$$

One could equivalently choose a different L^p_{loc}-space with an exponent $p\in[1,\infty[$. We shall see that (1.13) with $p=2$ expresses the fact that the energy is continuous.

Conditions (1.9) can therefore be understood as the limit as $t\to 0+$ with respect to the metric in the Fréchet space $L^2_{\text{loc}}(\mathbb{R}^1)$.

In the nonlinear case, we consider (1.13) as a *prescribed regularity in addition to* (1.10)(i).

In general, the problem of existence of solutions of (1.10), (1.9) is open, except for the special case, where the function g is increasing and has suitable convexity properties. The solution can then be constructed by compensated compactness method, see DiPerna (1983), Serre (1986). Here, we do not relax the assumptions (1.8) and concentrate our attention to local properties of weak solutions.

To derive some necessary conditions for the local behavior of isolated discontinuities, we assume

(1.14) (i) $\varphi(0)=0$ and there exist the limits $\varphi'(0\pm)=V_\pm$, $\psi(0\pm)=D_\pm$;

(ii) there exists $\delta>0$ and a local solution defined in $\Omega_\delta:=]-\delta,\delta[\times]0,\delta[$ such that $u\in W^{1,\infty}(\Omega_\delta)$, $u_t,u_x\in C([0,\delta];L^2(-\delta,\delta))$, identity (1.10)(ii) holds for all $\varrho\in\mathcal{D}(\Omega_\delta)$ and conditions (1.9) are satisfied for a.e. $x\in]-\delta,\delta[$;

(iii) for all $(x,t)\in\Omega_\delta$ there exists the limit $\bar{u}(x,t):=\lim_{\gamma\to\infty}\gamma u(\frac{x}{\gamma},\frac{t}{\gamma})$ such that $\bar{u}_x=\lim_{\gamma\to\infty}u_x(\frac{\cdot}{\gamma},\frac{\cdot}{\gamma})$, $\bar{u}_t=\lim_{\gamma\to\infty}u_t(\frac{\cdot}{\gamma},\frac{\cdot}{\gamma})$ are strong limits in the Banach space $C([0,\delta];L^2(-\delta,\delta))$.

We immediately see that the limit function $\bar{u}$ can be extended to $(x,t)\in\mathbb{R}^2_+$, belongs to $W^{1,\infty}(\mathbb{R}^2_+)$ and satisfies the *self-similarity condition*

$$\bar{u}(x,t)=\gamma\bar{u}\Big(\frac{x}{\gamma},\frac{t}{\gamma}\Big)\quad\text{for all}\quad(x,t)\in\mathbb{R}^2_+\text{ and }\gamma>0. \tag{1.15}$$

Let us define an auxiliary function

$$f(z) := \bar{u}(z,1) \quad \text{for} \quad z \in \mathbb{R}^1. \tag{1.16}$$

Then f is Lipschitz and (1.15) entails

$$\bar{u}(x,t) = tf\Big(\frac{x}{t}\Big) \quad \text{for all} \quad (x,t) \in \mathbb{R}^2_+. \tag{1.17}$$

Passing to the limit as $\gamma \to +\infty$ we easily check that $\bar{u}$ satisfies equation (1.10) (ii) with initial conditions

$$\bar{u}(x,0) = \begin{cases} xV_+ & \text{for } x \geqslant 0 \\ xV_- & \text{for } x < 0 \end{cases}, \quad \bar{u}_t(x,0) = \begin{cases} D_+ & \text{for } x > 0 \\ D_- & \text{for } x < 0 \end{cases}. \tag{1.18}$$

We now reformulate problem (1.10),(1.18) for self-similar solutions by introducing a new unknown function

$$\theta(z) = \frac{df}{dz}(z) \quad \text{for } z \in \mathbb{R}^1, \tag{1.19}$$

where f is defined by (1.16).

Proposition 1.2. *Let (1.8) hold and let $V_\pm \in \,]a,b[$, $D_\pm \in \mathbb{R}^1$ be given. A function $\bar{u}$ satisfies conditions (1.10), (1.13), (1.17), (1.18) if and only if the function θ defined by (1.19) has the following properties:*

(1.20) (i) $\theta(z) \in \,]a,b[$ *a.e.*, $\theta, g(\theta) \in L^\infty(\mathbb{R}^1)$,

(ii) *the function* $z \mapsto z^2\theta(z) - g(\theta(z))$ *is Lipschitz in* $\mathbb{R}^1$,

(iii) $\dfrac{d}{dz}\Big(z^2\theta(z) - g\big(\theta(z)\big)\Big) = 2z\theta(z)$ *a.e.*,

(iv) $\theta(\pm\infty) = V_\pm$,

(v) $\displaystyle\int_{-\infty}^{\infty} (\theta(z) - P_0(z))dz = D_+ - D_-,$ *where* $P_0(z) := \begin{cases} V_+ & \textit{for } z > 0 \\ V_- & \textit{for } z < 0 \end{cases}.$

Equation (1.10) with initial conditions (1.18) constitute the *Riemann problem.* System (1.20) represents its equivalent formulation for self-similar solutions. However, the question of existence of non-self-similar solutions to the Riemann problem seems to be open.

Before proving Proposition 1.2 we state an auxiliary result.

Lemma 1.3. *Let θ satisfy conditions (1.20)(i)-(iv). Then there exists a constant $R > 0$ such that for $|z| > R$ we have $\theta(z) = P_0(z)$.*

P r o o f of Lemma 1.3. We choose an arbitrary open bounded interval $J \subset \mathbb{R}^1$ such that $V_\pm \in J \subset \bar{J} \subset]a, b[$ and put

$$L := \sup\Big\{\Big|\frac{g(r) - g(s)}{r - s}\Big| \; ; \; r, s \in J, \; r \neq s\Big\}.$$

We find $R > \sqrt{L}$ sufficiently large such that $\theta(z) \in J$ for $|z| \geqslant R$ and put $\delta := R^2 - L > 0$.

Integrating equation (1.20)(iii) $\int_R^\xi dz$ for $\xi > R$ we obtain

$$\xi^2(\theta(\xi) - \theta(R)) - g(\theta(\xi)) + g(\theta(R)) = \int_R^\xi 2z(\theta(z) - \theta(R))dz, \tag{1.21}$$

hence $\delta|\theta(\xi) - \theta(R)| \leqslant \int_R^\xi 2z|\theta(z) - \theta(R)|dz$ and Gronwall's inequality (Lemma II.5.6) yields $\theta(\xi) = \theta(R)$ for all $\xi \geqslant R$. The argument for $z < -R$ is analogous. □

P r o o f of Proposition 1.2. Let θ be an arbitrary solution to (1.20). We choose arbitrarily $f(0) \in \mathbb{R}^1$ and define $\bar{u}$ by (1.17). Let $\varrho \in \mathcal{D}(\mathbb{R}^2_+)$ be an arbitrary test function. For $z \in \mathbb{R}^1$ put $\eta(z) := \int_0^\infty \varrho(zt, t)dt$. Then $\eta \in \mathcal{D}(\mathbb{R}^1)$ and using the identities $\int_0^\infty t\varrho_x(zt, t)dt = \frac{d\eta}{dz}(z)$, $\int_0^\infty t\varrho_t(zt, t)dt = -\frac{d}{dz}(z\eta(z))$, $\frac{d}{dz}(zf(z)\eta(z)) = z\theta(z)\eta(z) + f(z)\frac{d}{dz}(z\eta(z))$ we obtain from (1.20)(iii)

$$\begin{aligned} 0 &= \int_{-\infty}^\infty [(z^2\theta(z) - g(\theta(z))\frac{d\eta}{dz} + 2z\theta(z)\eta(z)]dz \\ &= \iint_{R^2_+} [(f(z) - z\theta(z))t\varrho_t(zt, t) - g(\theta(z))t\varrho_x(zt, t)]dz\, dt, \end{aligned} \tag{1.22}$$

hence $\bar{u}$ is a solution of (1.10).

Conversely, let $\bar{u}$ satisfy (1.10)(ii) and let $\eta \in \mathcal{D}(\mathbb{R}^1)$ be na arbitrary test function. Putting $\varrho(x, t) := \eta(\frac{x}{t})\mu(t)$ for some $\mu \in \mathcal{D}(]0, \infty[), \int_0^\infty \mu(t)dt = 1$ we conclude analogously as in (1.22) that (1.20)(ii),(iii) hold.

We now prove that the initial condition (1.18) is equivalent to (1.20)(iv),(v). Assume first that (1.18) is fulfilled. Then for each $K > 0$ we have

$$0 = \lim_{t \to 0+} \int_{-K}^K |\bar{u}_x(x, t) - P_0(x)|^2 dx = \lim_{\xi \to \infty} \frac{1}{\xi} \int_{-\xi}^\xi |\theta(z) - P_0(z)|^2 dz. \tag{1.23}$$

Formula (1.21) for a fixed $R>0$ and arbitrary $\xi>R$ entails $\xi^2(\theta(\xi)-V_+)-g(\theta(\xi))+g(\theta(R))=\int_R^\xi 2z(\theta(z)-V_+)dz+R^2(\theta(R)-V_+)$, hence

$$\begin{aligned}|\theta(\xi)-V_+| &\leqslant \frac{1}{\xi^2}|g(\theta(\xi))-g(\theta(R))|+\frac{R^2}{\xi^2}|\theta(R)-V_+|+\frac{2}{\xi}\int_R^\xi|\theta(z)-V_+|dz\\ &\leqslant \frac{2}{\xi^2}(|g(\theta)|_\infty+R^2|\theta|_\infty)+2\Big(\frac{1}{\xi}\int_0^\xi|\theta(z)-V_+|^2dz\Big)^{1/2}.\end{aligned}$$

Combining the last inequality with (1.23) and (1.10)(i) we obtain $V_+=\lim\limits_{\xi\to+\infty}\theta(\xi)$ and similarly $V_-=\lim\limits_{\xi\to-\infty}\theta(\xi)$.

We further have for each $K>0$

$$\lim_{t\to0+}\int_{-K}^K|\bar u_t(x,t)-P_1(x)|^2dx=0,\quad \text{where}\quad P_1(x):=\begin{cases}D_+ & \text{for } x>0\\ D_- & \text{for } x<0\end{cases}.$$

Choosing R as in Lemma 1.3 we infer from elementary computations

$$\begin{aligned}0&=\lim_{\xi\to+\infty}\frac{1}{\xi}\int_{-\xi}^{\xi}|f(z)-z\theta(z)-P_1(z)|^2dz\\ &=\lim_{\xi\to+\infty}\frac{1}{\xi}\int_R^\xi|f(0)+\int_0^R(\theta(s)-V_+)ds-D_+|^2dz+\\ &+\lim_{\xi\to+\infty}\frac{1}{\xi}\int_{-R}^R|f(z)-z\theta(z)-P_1(z)|^2dz+\\ &+\lim_{\xi\to+\infty}\frac{1}{\xi}\int_{-\xi}^{-R}|f(0)-\int_{-R}^0(\theta(s)-V_-)ds-D_-|^2dz,\end{aligned}$$

consequently

$$0=f(0)+\int_0^R(\theta(z)-V_+)dz-D_+=f(0)-\int_{-R}^0(\theta(z)-V_-)dz-D_-, \tag{1.24}$$

and condition (1.20)(v) follows again from Lemma 1.3.

The proof of the converse, namely that conditions (1.20)(iv),(v) imply (1.18), follows immediately from Lemma 1.3 provided that $f(0)$ is chosen according to (1.24).

Proposition 1.2 is proved. □

A naive approach to problem (1.20) consists in a formal differentiation of equation (1.20)(iii), i.e.

$$\theta'(z)(z^2-g'(\theta(z)))=0, \tag{1.25}$$

where prime denotes derivative. Identity (1.25) suggests that there are two types of solutions, namely the *constant states* $\theta'(z) = 0$ and *rarefaction waves* $\theta(z) = (g')^{-1}(z^2)$. We shall not try to justify this procedure which can be useful in concrete examples, but in general it leads to serious difficulties, since both θ and g' are in principle arbitrary bounded measurable functions, so that equation (1.25) is unmanageable. We proceed in a different way which will be explained in detail in the next sections. The rest of this section is devoted to examples and counterexamples related to the problem (1.20).

Multiplicity of weak solutions

We first mention the following classical result which is an immediate consequence of the continuity condition (1.20)(ii).

Lemma 1.4. (Rankine-Hugoniot condition).
Let θ be a solution of (1.20) and let there exist two sequences $z_n \to z, \hat{z}_n \to z$ such that $\lim\limits_{n\to\infty} \theta(z_n) = \theta_1 \neq \theta_2 = \lim\limits_{n\to\infty} \theta(\hat{z}_n)$. Then $z^2 = \frac{g(\theta_1)-g(\theta_2)}{\theta_1-\theta_2}$.

We have already noticed that constant functions always solve equation (1.20)(iii). Lemma 1.4 gives us a tool for constructing *piecewise constant* solutions of the form

$$\theta(z) = \theta_i \quad \text{for} \quad z \in \,]z_{i-1}, z_i[, \; i = 1, \dots, N \tag{1.26}$$

corresponding to a partition

$$-\infty = z_0 < z_1 < \ldots < z_N = +\infty \tag{1.27}$$

and to a sequence $\{\theta_1, \dots, \theta_n\} \subset]a, b[, \; \theta_i \neq \theta_{i+1}$ for all $i = 1, \dots, N-1$.

The criterion is obvious and can be expressed in the following way.

Proposition 1.5. *A function θ of the form (1.26) is a solution of (1.20) if and only if the following conditions are fulfilled.*

$$\begin{aligned} &\text{(i)} && \theta_1 = V_-, \; \theta_N = V_+, \\ &\text{(ii)} && z_i^2 = \frac{g(\theta_{i+1}) - g(\theta_i)}{\theta_{i+1} - \theta_i}, \; i = 1, \dots, N-1, \\ &\text{(iii)} && \sum_{i=1}^{N-1} z_i(\theta_i - \theta_{i+1}) = D_+ - D_-. \end{aligned} \tag{1.28}$$

Remark 1.6. The case where g is a linear function of the form $g(v) = k^2 v$ with $k > 0$ is trivial. We immediately see that the piecewise constant function

$$\theta(z) = \begin{cases} V_- & \text{for } z < -k \\ V_+ & \text{for } z > k \\ V_0 & \text{for } z \in \,]-k, k[\end{cases}$$

with $V_0 = \frac{1}{2k}(D_+ - D_- + k(V_+ + V_-))$ is the unique solution of (1.20).

In the nonlinear case, even smooth data do not ensure the uniqueness of weak solutions. The exact statement reads as follows.

Proposition 1.7. *Let g be a nonlinear function satisfying (1.8). Let $V_+ = V_- \in \,]a, b[$ and $D_+ = D_- \in \mathbb{R}^1$ be given. Then there exist infinitely many distinct piecewise constant solutions to (1.20).*

Problem (1.20) with $V_+ = V_-$ and $D_+ = D_-$ obviously admits the trivial constant (i.e. smooth) solution. The construction of non-smooth solutions is based on the following lemma.

Lemma 1.8. *Let $]\hat{a}, \hat{b}[,]\hat{c}, \hat{d}[\subset \mathbb{R}^1$ be given intervals, $0 \in \,]\hat{a}, \hat{b}[\cap]\hat{c}, \hat{d}[$, and let $\hat{g} : \,]\hat{a}, \hat{b}[\to]\hat{c}, \hat{d}[$ be a nonlinear locally Lipschitz function such that $\hat{g}(r)r > 0$ for all $r \neq 0$. Then there exist $\hat{a} < q < 0 < p < \hat{b}$ such that*

$$\text{(1.29)} \qquad \textit{either} \quad \text{(i)} \quad \frac{\hat{g}(p) - \hat{g}(q)}{p - q} > \frac{\hat{g}(p) - \hat{g}(r)}{p - r} \qquad \forall r \in \,]q, 0]$$

$$\textit{or} \quad \text{(ii)} \quad \frac{\hat{g}(p) - \hat{g}(q)}{p - q} > \frac{\hat{g}(s) - \hat{g}(q)}{s - q} \qquad \forall s \in [0, p[.$$

P r o o f of Lemma 1.8. Let us assume that for every $\hat{a} < q < 0 < p < \hat{b}$ both sets

$$A_+(p, q) := \Big\{ s \in [0, p[;\ \frac{\hat{g}(p) - \hat{g}(q)}{p - q} \leqslant \frac{\hat{g}(s) - \hat{g}(q)}{s - q} \Big\},$$

$$A_-(p, q) := \Big\{ r \in \,]q, 0];\ \frac{\hat{g}(p) - \hat{g}(q)}{p - q} \leqslant \frac{\hat{g}(p) - \hat{g}(r)}{p - r} \Big\}$$

are non-empty. Put $\bar{r} := \max A_-(p, q)$, $\bar{s} := \min A_+(p, q)$ and assume for instance $\bar{r} < 0$. By hypothesis, the set $A_-(p, \bar{r})$ is a non-empty subset of $A_-(p, q)$ which contradicts the definition of $\bar{r}$. We therefore have $\bar{r} = 0$ and similarly $\bar{s} = 0$. The inequalities $\frac{\hat{g}(p) - \hat{g}(q)}{p - q} \leqslant \frac{\hat{g}(p)}{p}$, $\frac{\hat{g}(p) - \hat{g}(q)}{p - q} \leqslant \frac{\hat{g}(q)}{q}$ combined with the elementary identity

$$\text{(1.30)} \qquad \frac{\hat{g}(p) - \hat{g}(q)}{p - q} - \frac{\hat{g}(q)}{q} = \frac{p}{q}\Big(\frac{\hat{g}(p) - \hat{g}(q)}{p - q} - \frac{\hat{g}(p)}{p} \Big)$$

yield $\frac{\hat{g}(p)}{p} = \frac{\hat{g}(q)}{q}$ for all $\hat{a} < q < 0 < p < \hat{b}$. We conclude that $\hat{g}$ is linear in $]\hat{a}, \hat{b}[$, which is a contradiction. □

P r o o f of Proposition 1.7. Put $V := V_+ = V_-$, $\hat{g}(r) := g(r+V) - g(V)$ for $r \in]\hat{a}, \hat{b}[:=]a-V, b-V[$. We distinguish four cases (see Fig. 11)

A. $\hat{g}(r)r > 0$ for $r \in]\hat{a}, \hat{b}[\setminus\{0\}$ and (1.29)(i) holds for some $\hat{a} < q < 0 < p < \hat{b}$.

For some $r \in]q, 0[$ which will be specified later we define

$$(1.31)\quad z_1 := -\sqrt{\frac{\hat{g}(q)}{q}},\ z_2 := -\sqrt{\frac{\hat{g}(p)-\hat{g}(q)}{p-q}},\ z_3 := -\sqrt{\frac{\hat{g}(p)-\hat{g}(r)}{p-r}},\ z_4 := \sqrt{\frac{\hat{g}(r)}{r}},$$

and

$$(1.32)\qquad \theta(z) := \begin{cases} V & \text{for} \quad z < z_1, \\ V+q & \text{for} \quad z \in]z_1, z_2[, \\ V+p & \text{for} \quad z \in]z_2, z_3[, \\ V+r & \text{for} \quad z \in]z_3, z_4[, \\ V & \text{for} \quad z > z_4. \end{cases}$$

Lemma 1.8 and identity (1.30) ensure that we have $z_1 < z_2 < z_3 < z_4$ and θ defined by (1.32) is a solution to (1.20) according to Proposition 1.5 provided condition (1.28)(iii) holds. Here it reads

$$(1.33)\qquad -\sqrt{\hat{g}(q)q} + \sqrt{(\hat{g}(p)-\hat{g}(q))(p-q)} - \sqrt{(\hat{g}(p)-\hat{g}(r))(p-r)} - \sqrt{\hat{g}(r)r} = 0.$$

Let us denote by $h(r)$ the left-hand side of equation (1.33). We have $h(0) > 0, h(q) < 0$, hence (1.33) is satisfied for a suitable $r \in]q, 0[$.

B. $\hat{g}(r)r > 0$ for $r \in]\hat{a}, \hat{b}[\setminus\{0\}$ and (1.29)(ii) holds for some $\hat{a} < q < 0 < p < \hat{b}$.

Analogously as above we define for $s \in]0, p[$

$$(1.34)\quad z_1 := -\sqrt{\frac{\hat{g}(p)}{p}},\ z_2 := -\sqrt{\frac{\hat{g}(p)-\hat{g}(q)}{p-q}},\ z_3 := -\sqrt{\frac{\hat{g}(s)-\hat{g}(q)}{s-q}},\ z_4 := \sqrt{\frac{\hat{g}(s)}{s}}$$

and

$$(1.35)\qquad \theta(z) := \begin{cases} V & \text{for} \quad z < z_1, \\ V+p & \text{for} \quad z \in]z_1, z_2[, \\ V+q & \text{for} \quad z \in]z_2, z_3[, \\ V+s & \text{for} \quad z \in]z_3, z_4[, \\ V & \text{for} \quad z > z_4. \end{cases}$$

Similarly as in the case A we check that θ solves (1.20) provided $s \in]0, p[$ is a solution of the equation

$$(1.36)\qquad \sqrt{p\hat{g}(p)} - \sqrt{(\hat{g}(p)-\hat{g}(q))(p-q)} + \sqrt{(\hat{g}(s)-\hat{g}(q))(s-q)} + \sqrt{s\hat{g}(s)} = 0.$$

Denoting by $\tilde{h}(s)$ the left-hand side of equation (1.36) we easily obtain $\tilde{h}(0) < 0$, $\tilde{h}(p) > 0$, hence (1.36) holds for some $s \in \,]0, p[$.

C. There exists $p > 0$ such that $\hat{g}(p) = 0$. We then put $r_0 := \min\{r \leqslant 0; \hat{g}(r) \geqslant 0\}$ and fix some $q_0 \in \,]\hat{a}, r_0[$.

For an arbitrary $\gamma \in \left]0, \frac{\hat{g}(q_0)}{q_0 - p}\right[$ put $q := \max\{u \in [q_0, r_0]; \frac{\hat{g}(u)}{u-p} = \gamma\}$. By Proposition 1.5, the function θ defined by (1.31), (1.32) for some $r \in \,]q, r_0[$ is a solution to (1.20) provided condition (1.33) holds. For the auxiliary function $h(r)$ as in (1.33) we have $h(r_0) > 0$, $h(q) < 0$ with the same conclusion as above.

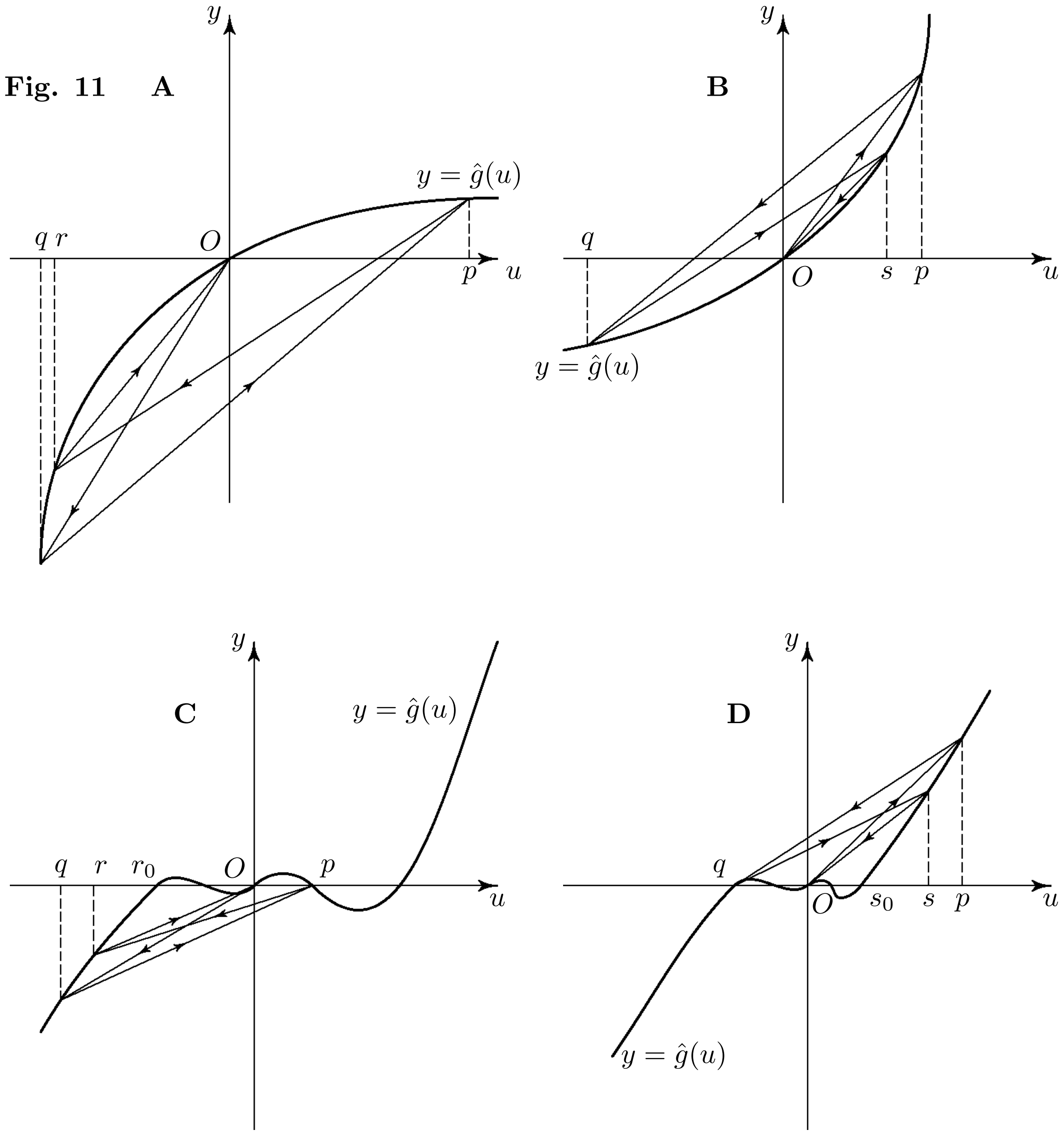


Fig. 11

D. There exists $q < 0$ such that $\hat{g}(q) = 0$. We put $s_0 := \max\{r \geqslant 0; \hat{g}(r) \leqslant 0\}$ and fix some $p_0 \in]s_0, \hat{b}[$. For a fixed $\gamma \in]0, \frac{\hat{g}(p_0)}{p_0 - q}[$ put $p := \min\{s \in [s_o, p_o]; \frac{\hat{g}(s)}{s-q} = \gamma\}$.

For $s \in]s_0, p[$ we define the function θ by formulas (1.34),(1.35). Similarly as in the previous cases we choose s such that equation (1.36) is satisfied.

It remains to check that there exist in fact infinitely many solutions of the form above. This is obvious in the cases C and D, where for each γ we obtain a different solution. In the situation A we similarly find a continuum of solutions parametrized by $\gamma \in]\frac{\hat{g}(p)}{p}, \frac{\hat{g}(p)-\hat{g}(q)}{p-q}[$ given by formulas (1.31),(1.32) with q replaced with $q_\gamma := \max\{u \in]q, 0[; \frac{\hat{g}(p)-\hat{g}(u)}{p-u} = \gamma\}$ and with a suitable r. Case B is analogous. This completes the proof of Proposition 1.7. □

Our task now is to find convincing arguments for the exclusion of pathological solutions described in the proof of Proposition 1.7. The first attempt in Sect. IV.2 will be the dissipation condition deduced from the 2nd Principle of Thermodynamics. We shall see in Proposition 2.3 that the solutions above violate the dissipation condition if g is monotone; this need not be the case if nonmonotonicities are allowed.

IV.2 Dissipation of energy

In the preceding section we observed that the Riemann problem in the form (1.20) may admit in general infinitely many solutions. To reduce the multiplicity, we impose, in addition to (1.20), a condition based on the 2nd Principle of Thermodynamics which states that the dissipation rate is nonnegative. We shall see that this condition ensures existence and uniqueness for system (1.20) if and only if g is globally convex or globally concave in $]a, b[$. In other words, to obtain existence and uniqueness in the general case, the dissipation condition has to be strengthened. This will be done in Section IV.3.

Dissipation condition

Let us come back to equation (1.7). We associate to each weak solution u the functions

(2.1) $$\mathcal{E}(u) := \frac{1}{2}u_t^2 + G(u_x), \quad \mathcal{F}(u) := u_t g(u_x)$$

called *energy density* and *energy flow density*, respectively (cf. Remark III.1.11), where G is a primitive function to g

(2.2) $$G(v) := \int_V^v g(u)du \qquad \text{for } v \in]a, b[$$

with an arbitrarily fixed $V \in]a, b[$.

Smooth solutions satisfy the Energy Conservation Law

$$\frac{\partial}{\partial t}\mathcal{E}(u) = \frac{\partial}{\partial x}\mathcal{F}(u). \tag{2.3}$$

For weak solutions, one cannot ensure that the energy is preserved even if the nonlinearity g is monotone and regular. This will follow from Theorem 3.15 in Sect. IV.3. Instead of (2.3), according to the 2^{nd} Principle of Thermodynamics we require that the energy dissipation rate is nonnegative, i.e.

$$\frac{\partial}{\partial t}\mathcal{E}(u) - \frac{\partial}{\partial x}\mathcal{F}(u) \leqslant 0 \qquad \text{in the sense of distributions.} \tag{2.4}$$

For self-similar solutions we can rewrite condition (2.4) in the following way.

Proposition 2.1. *Let the hypotheses of Proposition 1.2 hold. Then the solution* $\bar{u}$ *of (1.10), (1.18) satisfies condition (2.4) if and only if the corresponding solution* θ *of (1.20) satisfies the* dissipation condition

(2.5) The function $z \mapsto G(\theta(z)) - \theta(z)g(\theta(z)) + \dfrac{z^2}{2}\theta^2(z) - \displaystyle\int_0^z \zeta\theta^2(\zeta)d\zeta$ is nondecreasing for $z > 0$ and nonincreasing for $z < 0$.

P r o o f. Condition (2.4) means

$$\iint_{R^2_+} \Big[\Big(\frac{1}{2}\bar{u}_t^2 + G(\bar{u}_x)\Big)\varrho_t(x,t) - \bar{u}_t g(\bar{u}_x)\varrho_x(x,t)\Big]dx\,dt \geqslant 0 \quad \forall \varrho \in \mathcal{D}(\mathbb{R}^2_+),\ \varrho \geqslant 0. \tag{2.6}$$

Analogously as in the proof of Proposition 1.2 we rewrite inequality (2.6) in the form

$$\int_{-\infty}^{\infty} \Big[\Big(\frac{1}{2}(f - z\theta)^2 + G(\theta)\Big)\frac{d}{dz}(z\eta(z)) + (f - z\theta)g(\theta)\frac{d}{dz}\eta(z)\Big]dz \leqslant 0 \tag{2.7}$$

for every $\eta \in \mathcal{D}(\mathbb{R}^1), \eta \geqslant 0$.

The identity

$$\frac{d}{dz}\big((z^2\theta - g(\theta))f\eta\big) = (z^2\theta - g(\theta))f\frac{d\eta}{dz} + +\big((z^2\theta - g(\theta))\theta + 2z\theta f\big)\eta$$

combined with (2.7) entails

$$\int_{-\infty}^{\infty} \Big[\Big(G(\theta) - \theta g(\theta) + \frac{z^2}{2}\theta^2\Big)\frac{d}{dz}(z\eta(z)) + z^2\theta^2\eta(z)\Big]dz \leqslant 0 \quad \forall \eta \in \mathcal{D}(\mathbb{R}^1), \eta \geqslant 0,$$

or equivalently

$$(2.8) \qquad \begin{cases} \int_0^\infty \Big[\Big(G(\theta)-\theta g(\theta)+\frac{z^2}{2}\theta^2\Big)\frac{d\xi_1}{dz}+z\theta^2\xi_1\Big]dz \leqslant 0, \\ \int_{-\infty}^0 \Big[\Big(G(\theta)-\theta g(\theta)+\frac{z^2}{2}\theta^2\Big)\frac{d\xi_2}{dz}+z\theta^2\xi_2\Big]dz \geqslant 0 \end{cases}$$

for every $\xi_1 \in \mathcal{D}(]0,\infty[)$, $\xi_2 \in \mathcal{D}(]-\infty,0[)$, $\xi_1,\xi_2 \geqslant 0$. The assertion now follows from Lemma II.4.16. □

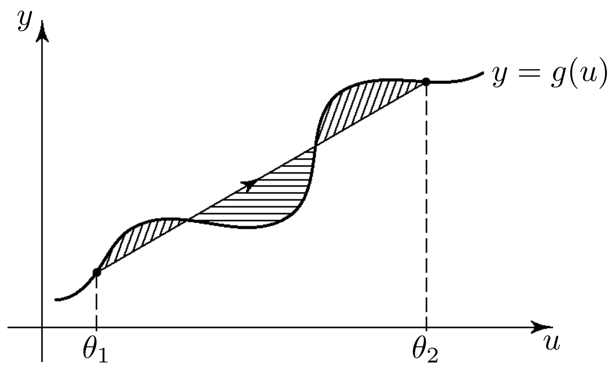
Remark 2.2. For a discontinuity of the first kind, i.e. such that the limits $\theta_1 := \theta(z-)$, $\theta_2 := \theta(z+)$ exist and are not equal, condition (2.5) and Lemma 1.4 entail

$$(2.9) \qquad z\Big[G(\theta_2)-G(\theta_1)-\frac{1}{2}(\theta_2-\theta_1)(g(\theta_2)+g(\theta_1))\Big] \geqslant 0,$$

where the left-hand side expresses the energy dissipation across the jump.

Inequality (2.9) has a clear geometrical meaning: the bracketted expression represents the signed area between the graph of the constitutive function g and the straight segment with slope z^2 between the points $(\theta_1, g(\theta_1))$ and $(\theta_2, g(\theta_2))$ (see Fig. 12)

Fig. 12



We can try to apply condition (2.9) to the situation described in Proposition 1.7. The result reads as follows.

Proposition 2.3. *Let the hypotheses of Proposition 1.7 hold and let θ be a solution of (1.20) satisfying condition (2.5). If g is nondecreasing, then θ is constant.*

P r o o f. Put $V := V_+ = V_-, D := D_+ = D_-$. For $u \in]a,b[$ and $z \in \mathbb{R}^1$ we define auxiliary functions

$$\tilde{G}(u) := \int_V^u (g(r)-g(V))dr,$$
$$f(z) := D - \int_0^\infty (\theta(\xi)-V)d\xi + \int_0^z \theta(\xi)d\xi,$$

$$\Phi(z) := \frac{1}{2}(f(z) - z\theta(z) - D)^2 + \tilde{G}(\theta(z)),$$
$$\Psi(z) := (f(z) - z\theta(z) - D)(g(\theta(z)) - g(V)).$$

From (2.7) and (1.22) it follows

$$\int_{-\infty}^{\infty} \big(\Phi(z)\frac{d}{dz}(z\eta(z)) + \Psi(z)\frac{d}{dz}\eta(z)\big)dz \leqslant 0 \tag{2.10}$$

and by Lemma II.4.16 the function $E(z) := \Psi(z) + z\Phi(z) - \int_0^z \Phi(\xi)d\xi$ is nondecreasing. We find $R > 0$ sufficiently large such that $\theta(z) = V$ for $|z| \geqslant R$. The functions Φ, Ψ are chosen in such a way that $\Phi(R) = \Phi(-R) = \Psi(R) = \Psi(-R) = 0, \Phi(z) \geqslant 0$ for all $z \in [-R, R]$.

The inequality $E(R) \geqslant E(-R)$ yields $\int_{-R}^{R} \Phi(\xi)d\xi \leqslant 0$, hence $\Phi(z) = 0$ for almost all $z \in\,]-R, R[$. This implies $\tilde{G}(\theta(z)) = 0$ for all z, hence $g(\theta(z)) = g(V)$ for all $z \in \mathbb{R}^1$ and (1.20)(ii),(iii) entail $\theta \equiv \text{const.} = V$ in $]-\infty, 0[\cup]0, \infty[$. □

The following example shows that the monotonicity assumption in Proposition 2.3 is substantial. Under the hypotheses of Proposition 1.7 we construct a nonconstant solution of (1.20) which satisfies the dissipation condition (2.5).

Example 2.4. We restrict ourselves for instance to the case B of the proof of Proposition 1.7. Assuming that $]c, d[\, = \,]-\infty, +\infty[$ we define the solution θ of (1.20) by formulas (1.34), (1.35). We now introduce a new function $\tilde{g}$ satisfying (1.8) such that θ is a solution of (1.20) and condition (2.5) holds with g replaced with $\tilde{g}$.

Let $\varphi_1 \in \mathcal{D}(]V+q, V[)$, $\varphi_2 \in \mathcal{D}(]V, V+s[)$, $\varphi_3 \in \mathcal{D}(]V+s, V+p[)$ be given auxiliary functions such that $\int_{V+q}^{V} \varphi_1(v)dv = 2$, $\int_{V}^{V+s} \varphi_2(v)dv = 3$, $\int_{V+s}^{V+p} \varphi_3(v)dv = 2$. For $K > 0$ and $v \in\,]a, b[$ put

$$g_K(v) := g(v) + K(\varphi_1(v) - \varphi_2(v) + \varphi_3(v)).$$

Then g_K satisfies (1.8) and θ is a solution of (1.20) with g replaced with g_K. By Remark 2.2 and inequality (2.9), condition (2.5) holds if and only if the following four conditions corresponding to jumps at the points z_1, z_2, z_3, z_4 are fulfilled:

$$0 \leqslant \frac{1}{2}p(g_K(V+p) + g_K(V)) - \int_V^{V+p} g_K(v)dv =$$
$$= \frac{1}{2}p(g(V+p) + g(V)) - \int_V^{V+p} g(v)dv + K,$$
$$0 \geqslant \frac{1}{2}(p-q)(g_K(V+p) + g_K(V+q)) - \int_{V+q}^{V+p} g_K(v)dv =$$
$$= \frac{1}{2}(p-q)(g(V+p) + g(V+q)) - \int_{V+q}^{V+p} g(v)dv - K,$$

$$0 \leqslant \frac{1}{2}(s-q)(g_K(V+s)+g_K(V+q)) - \int_{V+q}^{V+s} g_K(v)dv =$$

$$= \frac{1}{2}(s-q)(g(V+s)+g(V+q)) - \int_{V+q}^{V+s} g(v)dv + K,$$

$$0 \leqslant \frac{1}{2}s(g_K(V+s)+g_K(V)) - \int_{V}^{V+s} g_K(v)dv =$$

$$= \frac{1}{2}s(g(V+s)+g(V)) - \int_{V}^{V+s} g(v)dv + 3K.$$

The example is complete if we put $\tilde{g} := g_K$ for K sufficiently large.

Multiplicity of dissipative solutions

We now present another negative result showing that the dissipation condition (2.5) does not guarantee the uniqueness of solutions of (1.20) even in the "regular" case when g is increasing and smooth.

Proposition 2.5. *Let $g :]a,b[\to]c,d[$ be an increasing smooth function which has an inflection point $q_0 \in]a,b[$. Then there exist $V_+, V_- \in]a,b[$, $D_+, D_- \in \mathbb{R}^1$ such that problem (1.20) has infinitely many distinct solutions satisfying condition (2.5).*

P r o o f. We choose an interval $]q_0-k_1, q_0+k_2[\subset]a,b[$ such that one of the situations
(i) $g'' > 0$ in $]q_0-k_1, q_0[$, $g'' < 0$ in $]q_0, q_0+k_2[$,
(ii) $g'' < 0$ in $]q_0-k_1, q_0[$, $g'' > 0$ in $]q_0, q_0+k_2[$,
occurs. The construction will be different in each case (see Fig. 13)

(i) We fix some numbers $q_0 - k_1 < V_- < q_0 < p_0 < r_0 < V_+ < q_0 + k_2$ such that

$$\int_{V_-}^{r_0} g(v)dv < \frac{1}{2}(r_0 - V_-)(g(r_0)+g(V_-)), \tag{2.11}$$

$$\frac{g(r_0)-g(p_0)}{r_0-p_0} < \frac{g(r_0)-g(V_-)}{r_0-V_-} \tag{2.12}$$

and define $\theta(z)$ by the formula

$$\theta(z) := \begin{cases} V_- & \text{for} \quad z < z_1, \\ r & \text{for} \quad z \in]z_1, z_2[, \\ p & \text{for} \quad z \in]z_2, z_3[, \\ V_+ & \text{for} \quad z > z_3, \end{cases} \tag{2.13}$$

$$(2.14) \qquad z_1 := -\sqrt{\frac{g(r)-g(V_-)}{r-V_-}}, z_2 := -\sqrt{\frac{g(r)-g(p)}{r-p}}, z_3 := \sqrt{\frac{g(V_+)-g(p)}{V_+-p}}$$

for each (r,p) in a small neighborhood of (r_0,p_0) such that (2.11), (2.12) hold for (r,p). We put $D_- := 0$, $D_+ := h(p_0,r_0)$ with

$$h(p,r) := \sqrt{(g(r)-g(V_-))(r-V_-)} - \sqrt{(g(r)-g(p))(r-p)} - \sqrt{(g(V_+)-g(p))(V_+-p)}.$$

By Proposition 1.5, θ is a solution of (1.20) if and only if $h(p,r) = h(p_0,r_0)$. We obviously have $\frac{\partial}{\partial p}h(p,r) > 0$ and by the Implicit Function Theorem there exists a function $p(r)$ defined in a neighborhood of r_0 such that $h(p(r),r) = h(p_0,r_0)$ which determines a one-parametric family of solutions of (1.20) satisfying condition (2.5).

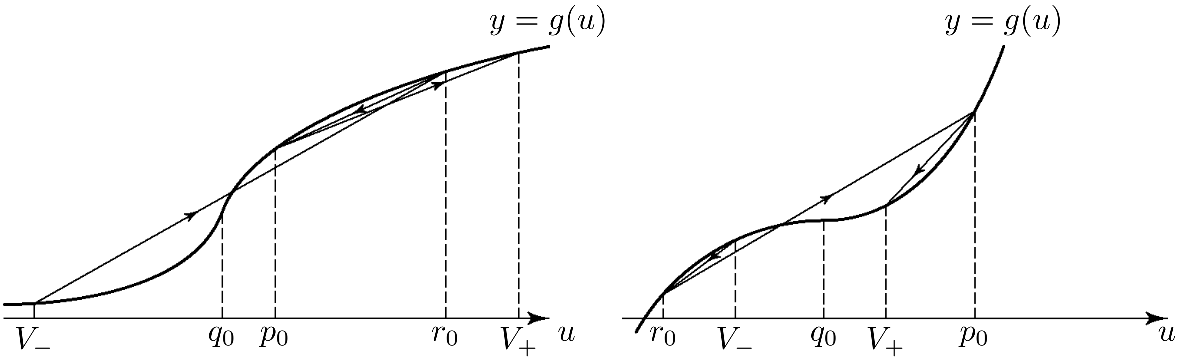


Fig. 13: *case (i)* *case (ii)*

(ii) Similarly as above, we fix some numbers $q_0 - k_1 < r_0 < V_- < q_0 < V_+ < p_0 < q_0 + k_2$ such that inequalities (2.12) and

$$(2.15) \qquad \int_{r_0}^{p_0} g(v)dv < \frac{1}{2}(p_0 - r_0)(g(p_0) + g(r_0))$$

hold. We put here $D_- := h(p_0,r_0)$, $D_+ := 0$. We easily check that the argument of (i) remains valid for the function θ defined by (2.13), (2.14). □

Remark 2.6. We can observe that the point $z = 0$ plays a particular role for solutions of (1.20). By Lemma 1.4, the function $z \mapsto g(\theta(z))$ is continuous at the point $z = 0$, hence the value

$$(2.16) \qquad Q := g(\theta(0))$$

is well defined for each solution θ of (1.20). Moreover, all possible discontinuities of θ across $z = 0$ are compatible with the dissipation condition (2.5). If equation (1.7) is interpreted as equation of motion, where $u, u_x, g(u_x)$ are the displacement, the strain and the stress, respectively, condition (2.16) represents a *boundary stress condition* at the point $x = 0$.

The idea now is to replace condition (1.20)(v) with the boundary condition (2.16) for an unknown value of $Q \in]c, d[$ which *is to be identified* in such a way that (1.20)(v) holds for given D_+, D_-. This procedure enables us to consider separately the cases $z < 0$ and $z > 0$. Introducing the functions

$$w_\pm(s) := \theta(\pm\sqrt{s}) \quad \text{for } s > 0 \tag{2.17}$$

we can easily rewrite the problem (1.20)(i)-(iv), (2.16) in the following way.

Proposition 2.7. *A function θ is a solution to (1.20)(i)-(iv), (2.16) if and only if each of the functions $w = w_+, w = w_-$ defined by formula (2.17) satisfies the conditions*

$$\begin{aligned} &\text{(i)} && w, g(w) \in L^\infty(0,\infty),\ w(s) \in]a,b[\ \textit{a.e.}, \\ &\text{(ii)} && \textit{the function} \quad s \mapsto sw(s) - g(w(s)) \quad \textit{is Lipschitz in }]0,+\infty[, \\ &\text{(iii)} && \frac{d}{ds}\big(sw(s) - g(w(s))\big) = w(s) \quad \textit{for a.e. } s > 0, \\ &\text{(iv)} && w(+\infty) = V,\ g(w(0)) = Q \end{aligned} \tag{2.18}$$

for $V = V_+, V = V_-$, respectively.

Moreover, the dissipation condition (2.5) is equivalent to the condition

$$\begin{aligned} &\text{The function } D(w) : s \mapsto G(w(s)) - w(s)g(w(s)) + \frac{s}{2}w^2(s) - \frac{1}{2}\int_0^s w^2(\sigma)d\sigma \\ &\text{is nondecreasing in }]0,+\infty[\end{aligned} \tag{2.19}$$

for each of the functions $w = w_+, w = w_-$.

The proof of Proposition 2.7 is elementary and we omit it here.

We conclude this section with the following complement to Proposition 2.5 which will be proved in the next section (see Remark 3.14).

Theorem 2.8. *Let $g :]a,b[\to]c,d[$ be a convex or concave function and let $V_+, V_- \in]a,b[$ be given. Then there exists an interval $]A,B[\subset \mathbb{R}^1$ such that*

(i) *problem (1.20) has a unique solution satisfying (2.5) provided $D_+ - D_- \in]A,B[$,*

(ii) *problem (1.20) has no solution satisfying (2.5) provided $D_+ - D_- \notin]A,B[$,*

(iii) *if $c = -\infty$ then $A = -\infty$ and if $d = +\infty$ then $B = +\infty$.*

IV.3 Minimal solutions

The aim of this section is to strengthen the dissipation condition (2.5) in order to ensure existence and uniqueness of solutions to the Riemann problem in the form (1.20) for an arbitrary nonlinearity g satisfying conditions (1.8).

The fact that the dissipation rate is nonnegative has been equivalently expressed for a solution w of system (2.18) by the condition that the function $D(w)$ in (2.19) is nondecreasing in $]0,\infty[$. Among all solutions to (2.18) we now select that one denoted by w^* which *maximizes the dissipation* in the sense that

(3.1) the total increment $D(w^*)(+\infty) - D(w^*)(0)$ of the dissipation function is maximal with respect to all solutions w of (2.18).

Condition (3.1) is meaningful if the set $g^{-1}(Q)$ contains a single point; otherwise the "initial" value $w(0) = V_0 \in g^{-1}(Q)$ can be arbitrarily chosen without affecting the dissipation condition (2.19). With the intention to eliminate the influence of the concrete choice of V_0 we formally compare only solutions with the same initial value V_0 and using the identity $D(w)(+\infty)-D(w)(0) = G(V)-G(V_0)-Vg(V)+V_0Q+\frac{1}{2}\int_0^\infty (V^2-w^2(s))\,ds$ we reformulate condition (3.1) in a more convenient way.

Definition 3.1. *Let $V \in]a,b[$ and $Q \in]c,d[$ be given. A solution w^* of (2.18) is called* minimal, *if the inequality*

$$\int_0^\infty \big(w^{*^2}(s) - w^2(s)\big)ds \leqslant 0 \tag{3.2}$$

holds for every solution w of (2.18).

This section is devoted to the proof of the following two statements.

Theorem 3.2. *For every $V \in]a,b[$ and $Q \in]c,d[$ there exists a unique minimal solution w^* of (2.18).*

Theorem 3.3. *Let $V_-, V_+ \in]a,b[$ be given. Then there exists an interval $]A,B[\subset \mathbb{R}^1$ with the following properties.*

(i) *For every $D \in]A,B[$ there exists a unique $Q \in]c,d[$ and a unique solution to (1.20) for $D_+ - D_- = D$ such that each of the functions w_+, w_- defined by formula (2.17) are minimal solutions of (2.18) with boundary conditions $g(w_+(0)) = g(w_-(0)) = Q$, $w_+(+\infty) = V_+$, $w_-(+\infty) = V_-$.*

(ii) *For $D \in \mathbb{R}^1 \setminus]A,B[$ no solution with the above property exists.*

(iii) *If $c = -\infty$, then $A = -\infty$ and if $d = +\infty$, then $B = +\infty$.*

The minimal solution will be found explicitly. The construction is based on the investigation of monotone solutions.

Monotone solutions

We start with an auxiliary lemma.

Lemma 3.4. *Let* $w : [s_1, s_2] \to \mathbb{R}^1$ *be a monotone function,* $w(s_1) = v_1, w(s_2) = v_2$ *and let its inverse* w^{-1} *be defined by the formula*

$$(3.3) \qquad w^{-1}(u) := \begin{cases} \sup S_-(u) & \text{for } u \in [v_2, v_1] \text{ if } w \text{ is nonincreasing,} \\ \sup S_+(u) & \text{for } u \in [v_1, v_2] \text{ if } w \text{ is nondecreasing,} \end{cases}$$

where $S_\pm(u) := \{s \in [s_1, s_2]; \pm w(s) \leqslant \pm u\}$. *Then we have*

$$(3.4) \qquad \text{(i)} \quad \int_{s_1}^{s_2} w(s)ds + \int_{v_1}^{v_2} w^{-1}(u)du = s_2 v_2 - s_1 v_1,$$

$$\text{(ii)} \quad \int_{s_1}^{s_2} w^2(s)ds + 2\int_{v_1}^{v_2} u w^{-1}(u)du = s_2 v_2^2 - s_1 v_1^2.$$

P r o o f. Both assertions follow from Fubini's theorem. We consider just the case of w nondecreasing (otherwise we pass from w to $-w$).

Let K be the rectangle $[s_1, s_2] \times [v_1, v_2]$. We define the maximal monotone graph $\Gamma_1 := \{(s, u) \in K; w(s-) \leqslant u \leqslant w(s+)\}$, where we put $w(s_1-) := w(s_1)$, $w(s_2+) := w(s_2)$, and the sets $A_1 := \{(s, u) \in K; v_1 \leqslant u < w(s-)\}$, $B_1 := \{(s, u) \in K; w(s+) < u \leqslant v_2\}$. The function w^{-1} is nondecreasing in $[v_1, v_2]$ and we have $B_1 = \{(s, u) \in K;\ s_1 \leqslant s < w^{-1}(u-)\}$, $K = \Gamma_1 \cup A_1 \cup B_1$, $A_1 \cap B_1 = \emptyset$, meas $\Gamma_1 = 0$, hence

$$(s_2 - s_1)(v_2 - v_1) = \int_{A_1} du\, ds + \int_{B_1} ds\, du = \int_{s_1}^{s_2} (w(s) - v_1)ds + \int_{v_1}^{v_2} (w^{-1}(u) - s_1)du$$

and (3.4)(i) follows easily.

To prove (3.4)(ii) we consider the cylinder in cylindrical coordinates

$$C := \{(r, \varphi, s); r \in [0, v_2 - v_1], \varphi \in [0, 2\pi], s \in [s_1, s_2]\},$$

define the sets $\Gamma_2 := \{(r, \varphi, s) \in C;\ (s, r + v_1) \in \Gamma_1\}$, $A_2 := \{(r, \varphi, s) \in C;\ (s, r + v_1) \in A_1\}$, $B_2 := \{(r, \varphi, s) \in C;\ (s, r + v_1) \in B_1\}$ and argue as above. $\square$

Formulas (3.4) enable us to identify monotone solutions of (2.18) with their trajectories in the phase plane. This will be done in the next three lemmas.

Lemma 3.5. *Let* $V \in]a, b[$ *and* $Q \in]c, d[$ *be given and let* w *be a solution of (2.18) which is monotone in* $]0, \infty[$, $w(0+) =: V_0 \in g^{-1}(Q)$, $w(+\infty) = V$. *Let* w^{-1} *be the inverse of* w *according to formula (3.3). For* $v \in \operatorname{Conv}\{V_0, V\}$ *put*

$$(3.5) \qquad g^*(v) := Q + \int_{V_0}^{v} w^{-1}(u)du.$$

Then $g(w(s)) = g^*(w(s))$ *for all* $s > 0$.

P r o o f. By Lemma 3.4 and equation (2.18)(iii) we have for each $s > 0$

$$\int_{V_0}^{w(s)} w^{-1}(u)du = sw(s) - \int_0^s w(\sigma)d\sigma = g(w(s)) - Q,$$

hence $g^*(w(s)) = g(w(s))$ by definition of g^*. □

The function $y = g^*(u)$ describes the trajectory of the solution w along the strain-stress diagram $y = g(u)$ (see Fig. 14). From Lemma 3.5 we immediately derive two important properties, namely

(3.6) (i) g^* is convex and increasing in $[V_0, V]$ if w is nondecreasing and concave and increasing in $[V, V_0]$ if w is nonincreasing,

(ii) if $g^*(v) \neq g(v)$ for some $v \in \mathrm{Conv}\{V_0, V\}$, then g^* is affine in a neighborhood of v.

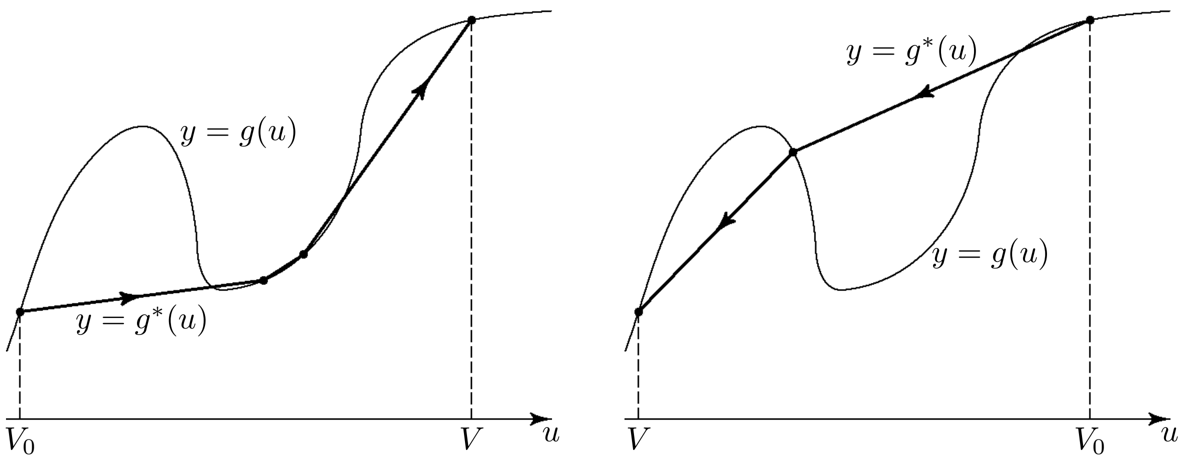


Fig. 14: *Trajectories of a nondecreasing and nonincreasing solution*

The proof of the converse of Lemma 3.5 is slightly more complicated.

Lemma 3.6.

(i) *Let $V_0, V \in]a,b[$ be given such that $V_0 < V, g(V_0) < g(V)$, and let $g^* : [V_0, V] \to]c,d[$ be a convex increasing function such that $g^*(V_0) = g(V_0)$, $g^*(V) = g(V)$ and implication (3.6)(ii) holds. Put $\bar{s} := g^{*'}(V-)$, $w^*(s) := \inf\{v \in [V_0, V]; g^{*'}(v) \geqslant s\}$ for $s \in]0, \bar{s}[$, $w^*(s) := V$ for $s \geqslant \bar{s}$. Then w^* is a nondecreasing solution of (2.18) with $Q = g(V_0)$, $w^*(0+) = V_0$ and its trajectory g^{**} defined according to Lemma 3.5 by the formula*

$$g^{**}(v) := g(V_0) + \int_{V_0}^{v} {w^*}^{-1}(u)du \quad \text{for } v \in [V_0, V] \tag{3.7}$$

coincides with g^*.

(ii) *Let* $V_0, V \in]a,b[$ *be given such that* $V_0 > V$, $g(V_0) > g(V)$, *and let* $g^* : [V, V_0] \to]c,d[$ *be a concave increasing function such that* $g^*(V_0) = g(V_0)$, $g^*(V) = g(V)$ *and implication (3.6)(ii) holds. Put* $\bar{s} := g^{*'}(V+)$, $w^*(s) := \sup\{v \in [V, V_0]; g^{*'}(v) \geqslant s\}$ *for* $s \in]0, \bar{s}[$, $w^*(s) := V$ *for* $s \geqslant \bar{s}$. *Then* w^* *is a nonincreasing solution of (2.18) with* $Q = g(V_0)$, $w^*(0+) = V_0$ *and its trajectory* g^{**} *defined by (3.7) coincides with* g^*.

P r o o f. It suffices to prove the statement (i). Part (ii) is then obtained by passing from $g(v)$ to $-g(-v)$.

The definition ensures that w^* is nondecreasing and

$$g^*(w^*(s)) - g^*(v) \leqslant s(w^*(s) - v) \quad \text{for all} \quad s > 0 \quad \text{and} \quad v \in [V_0, V], \tag{3.8}$$

hence

$$s_1(w^*(s_2) - w^*(s_1)) \leqslant g^*(w^*(s_2)) - g^*(w^*(s_1)) \leqslant s_2(w^*(s_2) - w^*(s_1))$$

for all $s_2 > s_1 > 0$. This yields

$$w^*(s_1)(s_2 - s_1) \leqslant s_2 w^*(s_2) - g^*(w^*(s_2)) - s_1 w^*(s_1) + g^*(w^*(s_1)) \leqslant w^*(s_2)(s_2 - s_1),$$

therefore the function $W^*(s) := s w^*(s) - g^*(w^*(s))$ is Lipschitz in $]0, \infty[, W^{*'}(s) = w^*(s)$ a.e.

To prove that w^* solves (2.18) it suffices to check that $g^*(w^*(s)) = g(w^*(s))$ for all $s > 0$. Assume on the contrary $g^*(w^*(s)) \neq g(w^*(s))$ for some $s > 0$. Then g^* is affine in a neighborhood of $w^*(s)$, say $g^{*'}(w^*(s) - \delta) = g^{*'}(w^*(s) + \delta) = s$, which contradicts the definition of w^*.

It remains to verify that $g^{**} = g^*$. In fact, we prove more, namely $g^{*'}(u+) = w^{*^{-1}}(u)$ for all $u \in]V_0, V[$. Indeed, for an arbitrary $s > w^{*^{-1}}(u)$ we have by (3.3) $u < w^*(s)$ and the definition of $w^*(s)$ entails $g^{*'}(u+) < s$, hence $g^{*'}(u+) \leqslant w^{*^{-1}}(u)$ for all $u \in]V_0, V[$. Conversely, for $s > g^{*'}(u+)$ there exists $\delta > 0$ such that $w^*(s) \geqslant u + \delta$, hence $s \geqslant w^{*^{-1}}(u)$. Consequently, $w^{*^{-1}}(u) = g^{*'}(u+)$ and Lemma 3.6 is proved. □

Lemma 3.4 enables us to express the value of the integral $\int_0^\infty \big(w_1^2(s) - w_2^2(s)\big)ds$ for two monotone solutions w_1, w_2 of (2.18) in terms of their convex (concave) trajectories g_1^*, g_2^*.

We first observe that integrating equation (2.18)(iii) we obtain

$$g(V) - Q = \int_0^\infty (V - w(s))ds \tag{3.9}$$

for each solution w of (2.18). If moreover we assume that w is monotone, then w is nondecreasing if $Q < g(V)$, nonincreasing if $Q > g(V)$ and constant if $Q = g(V)$.

Let now w_1, w_2 be two monotone solutions of (2.18) for given conditions $V \in]a,b[$ and $Q \in]c,d[$. We distinguish two cases.

A. $Q < g(V)$. Then both w_1 and w_2 are nondecreasing.

Assume for instance $w_1(0+) =: V_1 \leqslant V_2 := w_2(0+) < V$, $g(V_1) = g(V_2) = Q$. The convex trajectories g_1^*, g_2^* corresponding to w_1, w_2 are given by a formula analogous to (3.5) and satisfy $g_i^{*'}(u) = w_i^{-1}(u)$ for a.e. $u \in]V_i, V[$, $i = 1,2$. Identity (3.4)(ii) yields

$$\int_0^\infty (w_i^2(s) - V^2)ds + 2\int_{V_i}^V u g_i^{*'}(u)du = 0, \ i = 1,2,$$

and integrating by parts we obtain

$$\frac{1}{2}\int_0^\infty (w_1^2(s) - w_2^2(s))ds = \int_{V_1}^{V_2} (g_1^*(u) - Q)du + \int_{V_2}^V (g_1^*(u) - g_2^*(u))du. \tag{3.10}$$

B. $Q > g(V)$. Then both w_1 and w_2 are nonincreasing.

Assume $w_1(0+) =: V_1 \geqslant V_2 := w_2(0+) > V$, $g(V_1) = g(V_2) = Q$. For the corresponding concave trajectories g_1^*, g_2^* we have analogously as above

$$\frac{1}{2}\int_0^\infty (w_1^2(s) - w_2^2(s))ds = \int_{V_2}^{V_1} (Q - g_1^*(u))du + \int_V^{V_2} (g_2^*(u) - g_1^*(u))du. \tag{3.11}$$

We see that the minimization problem (3.2) in the class of monotone solutions consists in finding the minimal convex trajectory in the case A and the maximal concave trajectory in the case B. This suggests the following definition (cf. Fig. 15)

Definition 3.7. *Let $V \in]a,b[$ and $Q \in]c,d[$ be given. Put*

$$V_Q := \begin{cases} \max\left(g^{-1}(Q) \cap]a,V]\right) & \text{if} \quad Q \leqslant g(V), \\ \min\left(g^{-1}(Q) \cap [V,b[\right) & \text{if} \quad Q > g(V), \end{cases}$$

$\Omega(Q,V) := \mathrm{Conv}\left\{(u,y) \in \left(\mathrm{Conv}\{V_Q, V\}\right) \times]c,d[;\ y = g(u)\right\}$. *Then the function g^* defined for $u \in \mathrm{Conv}\{V_Q, V\}$ by the formula*

$$g^*(u) := \begin{cases} \min\{y \in]c,d[;\ (u,y) \in \Omega(Q,V)\} & \text{if} \quad Q < g(V), \\ \max\{y \in]c,d[;\ (u,y) \in \Omega(Q,V)\} & \text{if} \quad Q > g(V), \\ g(u) & \text{if} \quad Q = g(V), \end{cases} \tag{3.12}$$

is called the minimal trajectory *from Q to V.*

We immediately see that the minimal trajectory satisfies the hypotheses of Lemma 3.6. From identity (3.11) we easily conclude that the solution w^* of (2.18) associated to g^* by Lemma 3.6 is minimal with respect to all monotone solutions. We now prove that it is minimal in the sense of Definition 3.1.

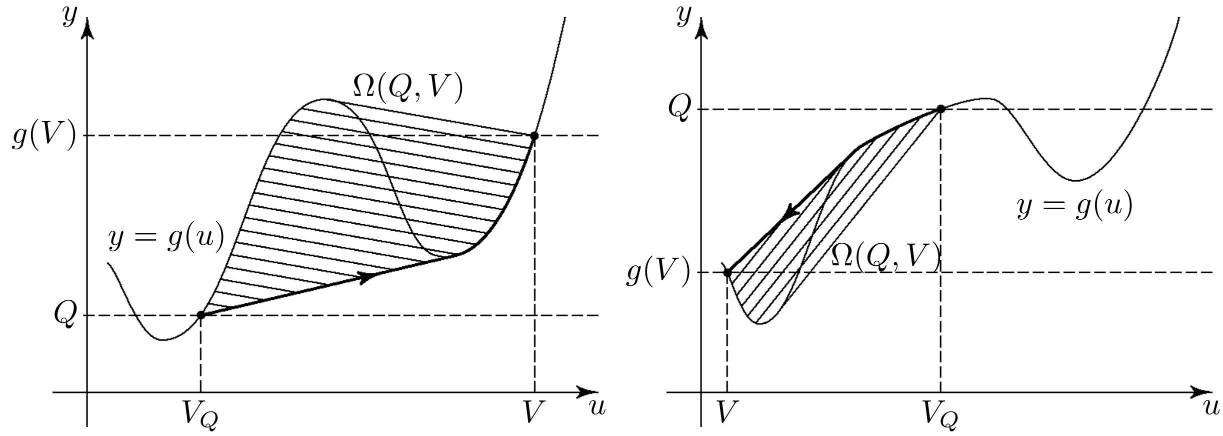


Fig. 15: *Minimal convex trajectory* *Maximal concave trajectory*

Existence and uniqueness of minimal solutions

Theorem 3.2 will be proved in the following form.

Proposition 3.8. *Let* $V \in]a, b[$ *and* $Q \in]c, d[$ *be given and let* g^* *be the minimal trajectory from* Q *to* V*. Let* w^* *be the solution associated to* g^* *by Lemma 3.6 in the case* $Q \neq g(V)$, $w^* \equiv V$ *if* $Q = g(V)$*. Then for every solution* $w \neq w^*$ *of (2.18) we have*

$$\int_0^\infty \big(w^{*^2}(s) - w^2(s)\big) ds < 0. \tag{3.13}$$

This fact is less obvious. Its original proof in Krejčí, Straškraba (1993) is relatively complicated. We present here a simple and elegant proof which is due to Lovicar (1994). It consists of two steps (Lemmas 3.9 - 3.10).

We first observe that the case $Q = g(V)$ follows trivially from identity (3.9) which entails for every solution w of (2.18)

$$\begin{aligned}\int_0^\infty (w^2(s) - V^2) ds &= \int_0^\infty (w(s) - V)^2 ds + 2V \int_0^\infty (w(s) - V) ds \\ &= \int_0^\infty (w(s) - V)^2 ds > 0.\end{aligned}$$

On the other hand, passing from w^* to $-w^*$ and from $g(v)$ to $-g(-v)$ we see that the cases $Q > g(V)$ and $Q < g(V)$ are symmetrical. For the sake of definiteness we assume in the sequel $Q < g(V)$.

Let us suppose now that there exists a solution $w \neq w^*$ of (2.18). We introduce the functions

$$\begin{cases} W^*(s) := s w^*(s) - g(w^*(s)) \\ W(s) := s w(s) - g(w(s)) \end{cases} \quad \text{for} \quad s > 0. \tag{3.14}$$

Both W and W^* are Lipschitz, $W' = w$, ${W^*}' = w^*$ a.e., W^* is convex and there exists $L > 0$ such that $W(s) = W^*(s) = sV - g(V)$ for $s \geqslant L$, $W(0+) = W^*(0+) = -Q$. We define the sets

$$\left\{\begin{aligned} M_0 &:= \{s > 0;\, W^*(s) = W(s)\},\\ M_+ &:= \{s > 0;\, W^*(s) > W(s)\},\\ M_- &:= \{s > 0;\, W^*(s) < W(s)\}.\end{aligned}\right. \tag{3.15}$$

We have $[L, +\infty[\subset M_0$, hence both M_+ and M_- are open bounded sets. They have the form $M_+ = \bigcup_{k=1}^{\infty}]\alpha_k^+, \beta_k^+[$, $M_- = \bigcup_{k=1}^{\infty}]\alpha_k^-, \beta_k^-[$, with $\alpha_k^\pm, \beta_k^\pm \in M_0$, provided we include the case $\alpha_k^\pm = 0$.

For almost all $s \in M_0$ we have $w^*(s) = w(s)$, hence

$$\int_{M_0} \big({w^*}^2(s) - w^2(s)\big)\,ds = 0. \tag{3.16}$$

Lemma 3.9. *For all $k \in \mathbb{N}$ we have*

$$\int_{\alpha_k^+}^{\beta_k^+} \big({w^*}^2(s) - w^2(s)\big)\,ds < 0. \tag{3.17}$$

P r o o f. We have $W^*(s) > W(s)$ for all $s \in]\alpha_k^+, \beta_k^+[$, $W^*(\alpha_k^+) = W(\alpha_k^+)$, $W^*(\beta_k^+) = W(\beta_k^+)$, hence

$$\int_{\alpha_k^+}^{\beta_k^+} r(s)\big(w(s) - w^*(s)\big)\,ds \geqslant 0$$

for each bounded nondecreasing function $r :]\alpha_k^+, \beta_k^+[\to \mathbb{R}^1$. Indeed, this follows trivially from the integration by parts provided r is smooth. In the general case we approximate r by a pointwise convergent sequence $r_n \to r$ of smooth nondecreasing functions and pass to the limit.

This yields

$$\begin{aligned} 0 < \int_{\alpha_k^+}^{\beta_k^+} \big(w^*(s) - w(s)\big)^2 ds &= \int_{\alpha_k^+}^{\beta_k^+} \big(w^2(s) - {w^*}^2(s)\big)\,ds - 2\int_{\alpha_k^+}^{\beta_k^+} w^*(s)\big(w(s) - w^*(s)\big)\,ds \\ &\leqslant \int_{\alpha_k^+}^{\beta_k^+} \big(w^2(s) - {w^*}^2(s)\big)\,ds \end{aligned}$$

and Lemma 3.9 is proved. □

Lemma 3.10. *For all* $k \in \mathbb{N}$ *we have*

$$\int_{\alpha_k^-}^{\beta_k^-} \big(w^{*^2}(s) - w^2(s)\big)ds < 0. \tag{3.18}$$

P r o o f. Lemmas 3.6, 3.5 and inequality (3.8) yield

$$sw^*(s) - g(w^*(s)) \geqslant sv - g^*(v) \geqslant sv - g(v) \tag{3.19}$$

for all $s > 0$ and $v \in [V_Q, V]$. On the other hand, for $s \in]\alpha_k^-, \beta_k^-[$ we have by hypothesis $sw^*(s) - g(w^*(s)) < sw(s) - g(w(s))$, hence $w(s) \notin [V_Q, V]$ for $s \in]\alpha_k^-, \beta_k^-[$.

Put $A_+ := \{s \in]\alpha_k^-, \beta_k^-[;\ w(s) > V\}$, $A_- := \{s \in]\alpha_k^-, \beta_k^-[;\ w(s) < V_Q\}$. We have $]\alpha_k^-, \beta_k^-[= A_+ \cup A_-$ and

$$\int_{A_-} \big(w^2(s) - w^{*^2}(s)\big)ds > (V_Q + V)\int_{A_-} \big(w(s) - w^*(s)\big)ds,$$
$$\int_{A_+} \big(w^2(s) - w^{*^2}(s)\big)ds > (V_Q + V)\int_{A_+} \big(w(s) - w^*(s)\big)ds,$$

therefore

$$\int_{\alpha_k^-}^{\beta_k^-} \big(w^2(s) - w^{*^2}(s)\big)ds > (V_Q + V)\big(W(\beta_k^-) - W^*(\beta_k^-) - W(\alpha_k^-) + W^*(\alpha_k^-)\big) = 0,$$

and inequality (3.18) is proved. □

To finish the proof of Proposition 3.8 which in turn implies Theorem 3.2, it suffices to combine Lemmas 3.9, 3.10 and identity (3.16).

The natural question whether the minimal solution of (2.18) satisfies the dissipation condition (2.19) can easily be answered.

Proposition 3.11. *For every* $V \in]a, b[$ *and* $Q \in]c, d[$ *the minimal solution* w^* *of (2.18) fulfils the dissipation condition (2.19).*

P r o o f. By Lemmas 3.4, 3.6 we have for all $s > 0$

$$\frac{1}{2}\int_0^s w^{*^2}(\sigma)d\sigma = \frac{1}{2}sw^*(s) - \int_{V_Q}^{w^*(s)} uw^{*^{-1}}(u)du$$
$$= \frac{1}{2}sw^*(s) - w^*(s)g^*(w^*(s)) + V_Q g^*(V_Q) + \int_{V_Q}^{w^*(s)} g^*(u)du.$$

The function $D(w^*)$ in (2.19) has therefore the form

$$D(w^*)(s) = \int_{V_Q}^{w^*(s)} \big(g(u) - g^*(u)\big)du + G(V_Q) - QV_Q.$$

For $Q < g(V)$ we have $g(u) \geqslant g^*(u)$ for all $u \in [V_Q, V]$ and w^* is nondecreasing, for $Q > g(V)$ we have $g(u) \leqslant g^*(u)$ for all $u \in [V, V_Q]$ and w^* is nonincreasing, for $Q = g(V)$ the solution w^* is constant, hence in all cases condition (2.19) holds. □

Existence and uniqueness in the Riemann problem

Let us state now two easy lemmas which enable us to prove Theorem 3.3.

Lemma 3.12. *Let* $Q_1, Q_2 \in]c, d[$ *and* $V \in]a, b[$ *be given such that* $Q_1 < Q_2 < g(V)$. *According to Definition 3.7 put* $V_i := V_{Q_i}$ *and* $g_i^*(u) := \min\{y \in]c, d[; (u, y) \in \Omega(Q_i, V)\}$ *for* $u \in [V_i, V]$, $i = 1, 2$. *Then* $g_1^*(u) \leqslant g_2^*(u)$ *for all* $u \in [V_2, V]$ *and* $g_1^{*'}(u) \geqslant g_2^{*'}(u)$ *for a.e.* $u \in]V_2, V[$.

P r o o f. We obviously have $\Omega(Q_2, V) \subset \Omega(Q_1, V)$ and $V > V_2 > V_1$, hence $g_1^* \leqslant g_2^*$ in $[V_2, V]$. Let us assume now $g_1^{*'}(u) < g_2^{*'}(u)$ for some Lebesgue point $u \in]V_2, V[$ of both $g_1^{*'}$ and $g_2^{*'}$. Then $g_1^*(u) < g_2^*(u) \leqslant g(u)$, hence g_1^* is affine in a neighborhood of u. Put $\bar u := \min\{v \in]u, V[; g(v) = g_1^*(v)\}$. The points $(\bar u, g(\bar u))$ and $(u, g_2^*(u))$ belong to $\Omega(Q_2, V)$, hence for all $\alpha \in]0, 1[$ we have $g_2^*(\alpha\bar u + (1-\alpha)u) \leqslant \alpha g(\bar u) + (1-\alpha) g_2^*(u)$, or equivalently

$$\frac{g_2^*(u + \alpha(\bar u - u)) - g_2^*(u)}{\alpha(\bar u - u)} \leqslant \frac{g(\bar u) - g_2^*(u)}{\bar u - u}.$$

Passing to the limit as $\alpha \to 0+$ we obtain

$$g_2^{*'}(u) \leqslant \frac{g(\bar u) - g_2^*(u)}{\bar u - u} < \frac{g(\bar u) - g_1^*(u)}{\bar u - u} = g_1^{*'}(u).$$

which is a contradiction. □

Lemma 3.13. *Let* $V \in]a, b[$ *and* $c < Q_1 < Q_2 < d$ *be given. Let* w_1^*, w_2^* *be the minimal solutions of (2.18) for* $Q = Q_1, Q = Q_2$, *respectively. Then* $w_1^*(s) \leqslant w_2^*(s)$ *for all* $s > 0$.

P r o o f. The cases $Q_1 \leqslant g(V) < Q_2$ or $Q_1 < g(V) \leqslant Q_2$ are obvious. We may therefore assume $Q_1 < Q_2 < g(V)$ (the opposite situation $g(V) < Q_1 < Q_2$ is again covered by the usual transformation $g(v) \mapsto -g(-v)$). By Lemma 3.6 we have for all $s > 0$

$$w_i^*(s) = \inf\{u \in [V_i, V]; g_i^{*'}(u) \geqslant s\},\ i = 1, 2,$$

where V_i, g_i^* are as in Lemma 3.12. For all $s > 0$ and $u \in [V_2, V]$ such that $u < w_1^*(s)$ we have by Lemma 3.12 $g_2^{*'}(u+) \leqslant g_1^{*'}(u+) < s$. This entails $u < w_2^*(s)$, hence $w_1^*(s) \leqslant w_2^*(s)$. □

P r o o f of Theorem 3.3. For an arbitrary $Q \in]c, d[$ we denote by w_+^Q, w_-^Q the minimal solutions of (2.18) with boundary conditions $g(w_+^Q(0)) = g(w_-^Q(0)) = Q, w_+^Q(+\infty) = V_+, w_-^Q(+\infty) = V_-$, and put

$$\theta_Q(z) := \begin{cases} w_+^Q(z^2) & \text{for} \quad z \geqslant 0, \\ w_-^Q(z^2) & \text{for} \quad z < 0. \end{cases}$$

By Proposition 2.7, $\theta_Q(z)$ solves (1.20)(i) - (iv) for all $Q \in]c, d[$. To handle condition (1.20)(v) we introduce the function

$$\varphi(Q) := \int_{-\infty}^{\infty} \big(\theta_Q(z) - P_0(z)\big)dz \tag{3.20}$$

with the intention to put

$$A := \varphi(c+), \quad B := \varphi(d-). \tag{3.21}$$

The proof of Theorem 3.3 will be complete as soon as we prove that the function φ defined by (3.20) is continuous and increasing and implications (iii) hold.

The fact that φ is increasing follows immediately from Lemma 3.13. To prove the continuity, we fix an arbitrary compact interval $[c', d'] \subset]c, d[$ such that $g(V_+)$, $g(V_-) \in]c', d'[$. Put $a' := \min\{v \in]a, b[; g(v) \geqslant c'\}$, $b' := \max\{v \in]a, b[; g(v) \leqslant d'\}$, $L' := \sup\big\{\big|\frac{g(u)-g(v)}{u-v}\big|; a' \leqslant v < u \leqslant b'\big\} < +\infty$.

From Lemma 3.6 we infer that $\theta_Q(z) = P_0(z)$ for all $|z| > \sqrt{L'}$ and all $Q \in [c', d']$. Integrating equation (1.20)(iii) $\int_0^{\sqrt{L'}} dz$ and $\int_{-\sqrt{L'}}^0 dz$ we obtain for all $Q \in [c', d']$

$$\begin{cases} L'V_+ - g(V_+) + Q = 2\int_0^{\sqrt{L'}} z\theta_Q(z)dz, \\ -L'V_- + g(V_-) - Q = 2\int_{-\sqrt{L'}}^0 z\theta_Q(z)dz, \end{cases} \tag{3.22}$$

hence

$$\int_{-\sqrt{L'}}^{\sqrt{L'}} |z|\big(\theta_{Q_1}(z) - \theta_{Q_2}(z)\big)dz = \int_{-\infty}^{\infty} |z|\big(\theta_{Q_1}(z) - \theta_{Q_2}(z)\big)dz = Q_1 - Q_2 \tag{3.23}$$

for all $Q_1, Q_2 \in [c', d']$, $Q_1 > Q_2$. Note that by Lemma 3.13 we have $\theta_{Q_1}(z) \geqslant \theta_{Q_2}(z)$ for a.e. $z \in \mathbb{R}^1$. Using the estimates

$$\begin{aligned} \varphi(Q_1) - \varphi(Q_2) &= \int_{-\infty}^{\infty} \big(\theta_{Q_1}(z) - \theta_{Q_2}(z)\big)dz \\ &\leqslant \int_{-\sqrt{Q_1-Q_2}}^{\sqrt{Q_1-Q_2}} \big(\theta_{Q_1}(z) - \theta_{Q_2}(z)\big)dz + \frac{1}{\sqrt{Q_1 - Q_2}} \int_{-\infty}^{\infty} |z|\big(\theta_{Q_1}(z) - \theta_{Q_2}(z)\big)dz \\ &\leqslant (2(b' - a') + 1)\sqrt{Q_1 - Q_2} \end{aligned}$$

we conclude that φ is locally $\frac{1}{2}$-Hölder continuous in $]c,d[$.

Parts (i), (ii) of Theorem 3.3 now follow from (3.20). It remains to prove one of the implications (iii), the other one is analogous. Assume for instance $d=+\infty$, $V_+ \geqslant V_-$, and put

$$L := \sup\Big\{\frac{g(v)-g(V_+)}{v-V_+};\ v\in\,]V_+,b[\Big\} > 0.$$

We distinguish two cases.

A. $L<+\infty$.

Then for every $Q>g(V_+)$ the slope of the minimal trajectory (3.12) from Q to V_+ does not exceed the value of L, and therefore $\theta_Q(z)=V_+$ for $z>\sqrt{L}$. Using formula (3.22) for $L'=L$ we obtain

$$(3.24)\quad \varphi(Q) \geqslant \int_0^{\sqrt{L}} (\theta_Q(z)-V_+)dz \geqslant \frac{1}{\sqrt{L}}\int_0^{\sqrt{L}} z(\theta_Q(z)-V_+)dz \geqslant \frac{1}{2\sqrt{L}}(Q-g(V_+)).$$

B. $L=+\infty$.

Put $\hat{L} := \limsup_{v\to V_+} \frac{g(v)-g(V_+)}{v-V_+} < +\infty$. For $\lambda>\hat{L}$ we define

$$V_\lambda := \min\Big\{v\in\,]V_+,b[;\ \frac{g(v)-g(V_+)}{v-V_+}=\lambda\Big\},\ Q_\lambda := g(V_\lambda).$$

The minimal trajectory g^* from Q_λ to V_+ is then affine, namely $g^*(u)=g(V_+)+\lambda(u-V_+)$ for $u\in[V_+,V_\lambda]$. This yields

$$\theta_{Q_\lambda}(z) = \begin{cases} V_+ & \text{for} \quad z>\sqrt{\lambda}. \\ V_\lambda & \text{for} \quad z\in\,]0,\sqrt{\lambda}[, \\ w_-^{Q_\lambda}(z^2)\geqslant V_- & \text{for} \quad z<0, \end{cases}$$

therefore

$$(3.25)\qquad \varphi(Q_\lambda) \geqslant \int_0^{\sqrt{\lambda}} (\theta_{Q_\lambda}(z)-V_+)dz = \sqrt{(Q_\lambda-g(V_+))(V_\lambda-V_+)}.$$

In both cases (3.24), (3.25) we obtain $\varphi(Q)\to+\infty$ as $Q\to+\infty$. Theorem 3.3 is proved. □

Remark 3.14. Theorem 3.3 enables us now to prove Theorem 2.8 from the preceding section. In fact, it suffices to prove that for a convex constitutive function g the dissipation condition (2.19) and the minimality criterion (3.2) for solutions of (2.18) coincide. The case of g concave is then obtained in a standard way. We prove the following theorem.

Theorem 3.15. *Let g be convex and let $V \in]a,b[$, $Q \in]c,d[$ be given. Let w be a solution of (2.18) satisfying the dissipation condition (2.19) and let w^* be the minimal solution of (2.18). Then $w = w^*$ a.e.*

In the proof we make use of an auxiliary lemma. Notice that a convex function satisfying (1.8) is increasing, hence every solution w of (2.18) can be continuously extended to $s = 0$.

Lemma 3.16. *Let the hypotheses of Theorem 3.15 hold. Assume that there exist Lebesgue points s_1, s_2 of w such that $0 \leqslant s_1 < s_2$ and $w(s_1) =: v_1 < v_2 := w(s_2)$. Then $s_2 \geqslant g'(v_2-)$, $s_1 \leqslant g'(v_1+)$, $w(s) = \inf\{u \in [v_1, v_2]; g'(u) \geqslant s\}$ for a.e. $s \in [s_1, g'(v_2-)[$, $w(s) = v_2$ for $s \in]g'(v_2-), s_2]$.*

P r o o f of Lemma 3.16. The function $w_0 : [0,\infty[\to]a,b[$ defined as $w_0(s) := w(s)$ for $s \in]s_1, s_2[, w_0(s) := v_1$ for $s \in [0, s_1], w_0(s) := v_2$ for $s \in [s_2, +\infty[$ solves (2.18) with $V = v_2, Q = g(v_1)$. The minimal convex trajectory g_0^* from $g(v_1)$ to v_2 coincides with g and the corresponding minimal solution w_0^* is given by the formula (cf. Lemma 3.6) $w_0^*(s) = \inf\{u \in [v_1, v_2], g'(u) \geqslant s\}$ for $s \in [0, g'(v_2-)[$, $w_0^*(s) = v_2$ for $s > g'(v_2-)$, ${w_0^*}^{-1}(u) = g'(u)$ for a.e. $u \in]v_1, v_2[$.

By (2.19) we have $D(w)(s_2) \geqslant D(w)(s_1)$, hence

$$\frac{1}{2}\int_{s_1}^{s_2} w^2(s)ds \leqslant -\int_{v_1}^{v_2} ug'(u)du + \frac{1}{2}s_2v_2^2 - \frac{1}{2}s_1v_1^2.$$

Lemma 3.4 yields $-\int_{v_1}^{v_2} ug'(u)du = \frac{1}{2}\int_0^\infty \big({w_0^*}^2(s) - v_2^2\big)ds$, therefore $\frac{1}{2}\int_0^\infty \big(w_0^2(s) - {w_0^*}^2(s)\big)ds \leqslant 0$.

By Proposition 3.8 we conclude $w_0 = w_0^*$ a.e. and Lemma 3.16 follows. □

P r o o f of Theorem 3.15. The assertion is an immediate consequence of Lemma 3.16 if $Q < g(V)$ (we simply put $s_1 = 0$ and let s_2 tend to $+\infty$). The case $Q \geqslant g(V)$ is slightly more complicated. In fact, it suffices to prove that w is nonincreasing in $[0, +\infty[$, since the only concave trajectory from Q to V in this case is the minimal one which is affine.

Let us suppose on the contrary that there exist Lebesgue points s_1, s_2 of w such that $0 \leqslant s_1 < s_2$ and $w(s_1) := v_1 < v_2 := w(s_2)$. We distinguish 2 cases.

A. $g(v_1) < Q$.

Put $\bar{v} := \operatorname{supess}\{w(s); s \in [0, s_1]\}$. Then $\bar{v} \geqslant g^{-1}(Q) > v_1$ and there exists a sequence $\{\sigma_n\} \subset [0, s_1]$ of Lebesgue points of w such that $\sigma_n \to \bar{s} < s_1, w(\sigma_n) \to \bar{v}$. Passing to the limit as $n \to \infty$ in the identity $\int_{\sigma_n}^{s_1} w(s)ds = s_1v_1 - g(v_1) - \sigma_n w(\sigma_n) + g(w(\sigma_n))$ we obtain $0 > \int_{\bar{s}}^{s_1}(w(s) - \bar{v})ds = g(\bar{v}) - g(v_1) - s_1(\bar{v} - v_1)$, hence $s_1 > \frac{g(\bar{v}) - g(v_1)}{\bar{v} - v_1} \geqslant g'(v_1+)$, which is in contradiction with Lemma 3.16.

B. $g(v_1) \geqslant Q$.

Analogously as above put $\underline{v} := \inf \operatorname{ess}\{w(s);\, s \in [s_2, +\infty[\}$. We have $\underline{v} \leqslant V < v_2$ and $w(s) = V$ for sufficiently large, therefore there exists a convergent sequence $\{\sigma_n\} \subset [s_2, +\infty[$ of Lebesgue points of w such that $\sigma_n \to \underline{s} > s_2, w(\sigma_n) \to \underline{v}$ as $n \to \infty$. Passing to the limit in the identity $\int_{s_2}^{\sigma_n} w(s)ds = \sigma_n w(\sigma_n) - s_2 v_2 - g(w(\sigma_n)) + g(v_2)$ yields $0 < \int_{s_2}^{\underline{s}} (w(s) - \underline{v})ds = g(v_2) - g(\underline{v}) - s_2(v_2 - \underline{v})$ hence $s_2 < \frac{g(v_2) - g(\underline{v})}{v_2 - \underline{v}} \leqslant g'(v_2-)$ which again contradicts Lemma 3.16. □

At the end of this section we show an interesting example.

Example 3.17. Let g be the function $g(u) := e^u - 1$ for $u \in \mathbb{R}^1$, $V_+ = V_- := 0$. Then the hypotheses of Theorem 2.8 are fulfilled with $a = -\infty$, $b = d = +\infty$, $c = -1$ and we can explicitly compute the values of A and B. In fact, we have $B = +\infty$ by Theorem 3.3.

The minimal solution w^* of (2.18) with $V = 0$ is given by the formula

$$w^*(s) = \begin{cases} \log(1+Q) & \text{for} \quad 0 \leqslant s < \frac{Q}{\log(1+Q)} \\ 0 & \text{for} \quad s > \frac{Q}{\log(1+Q)} \end{cases} \qquad \text{if} \quad Q > 0,$$

$$w^*(s) = \begin{cases} \log(1+Q) & \text{for} \quad 0 \leqslant s \leqslant 1+Q \\ \log s & \text{for} \quad 1+Q < s < 1 \\ 0 & \text{for} \quad s \geqslant 1 \end{cases} \qquad \text{if} \quad Q \in]-1, 0].$$

The function φ defined by (3.20) has the form

$$\varphi(Q) = 2\int_0^\infty w^*(z^2)dz = \begin{cases} 2\sqrt{Q\log(1+Q)} & \text{for} \quad Q > 0, \\ 4\big(\sqrt{1+Q} - 1\big) & \text{for} \quad Q \in]-1, 0]. \end{cases}$$

By (3.21) we have $A = -4$. Since g is convex, we infer from Theorems 3.15, 3.3 that problem (1.20) with $D_+ - D_- \leqslant -4$ has no solution satisfying the dissipation condition (2.5). On the other hand, putting for an arbitrary $u < 0$.

$$\hat{\theta}_u(z) := \begin{cases} u & \text{for} \quad |z| < \sqrt{\frac{g(u)}{u}} \\ 0 & \text{for} \quad |z| > \sqrt{\frac{g(u)}{u}} \end{cases} \tag{3.26}$$

we see that $\hat{\theta}_u$ solves problem (1.20) with $V_+ = V_- = 0$, $D_+ - D_- = \int_{-\infty}^{+\infty} \hat{\theta}_u(z)dz = -2\sqrt{ug(u)}$, hence for arbitrary $D_+ < D_-$ there exists a solution of (1.20) of the form (3.26). In other words, *problem (1.20) with* $D_+ - D_- \leqslant -4$ *admits only solutions which violate the dissipation condition*!

IV.4 Entropy conditions

In this section we study the relationship of the minimal solution from Definition 3.1 to entropy conditions arising from various physical or geometrical considerations. We already proved Theorem 2.8 and Propositions 2.5, 3.11 which state that the minimality always implies the dissipation condition (2.5) and that these two conditions are equivalent (modulo some smoothness) if and only if g is convex or concave.

Here we recall four more or less classical selection rules for the relevant solution. Three of them, namely the Lax entropy condition, Liu's shock admissibility criterion and the vanishing viscosity criterion are compatible with the minimal solution, while Dafermos' maximal entropy rate condition is not in general.

Lax entropy condition

The Lax (1957) shock admissibility condition for systems of conservation laws can be defined only under some regularity assumption. Following Aumann (1969) we introduce here the space $\mathcal{R}(\alpha,\beta)$ of *regulated functions* as the space of all functions $f :]\alpha,\beta[\to \mathbb{R}^1$ such that there exist finite limits $f(\alpha+), f(\beta-)$ and $f(r+), f(r-)$ for all $r \in]\alpha,\beta[$, and this space is endowed with the sup-norm $|\cdot|_\infty$.

It is clear that regulated functions are bounded and have at most countable many discontinuities. More information about the spaces $\mathcal{R}(\alpha,\beta)$ can be found e.g. in Fraňková (1991) or Tvrdý (1989).

Definition 4.1. *Let us assume that the derivative* $g'(u) = \frac{dg}{du}$ *belongs to* $\mathcal{R}(a,b)$. *A weak solution* θ *of system (1.20) is said to satisfy the* Lax entropy condition *at a point* $z \in \mathbb{R}^1$ *if the one-sided limits* $\theta(z+) \neq \theta(z-)$ *exist and*

(i) $\quad zg'(\theta(z-)+) \geqslant z^3 \geqslant zg'(\theta(z+)-) \quad$ *if* $\quad \theta(z-) < \theta(z+)$,

(ii) $\quad zg'(\theta(z-)-) \geqslant z^3 \geqslant zg'(\theta(z+)+) \quad$ *if* $\quad \theta(z-) > \theta(z+)$.

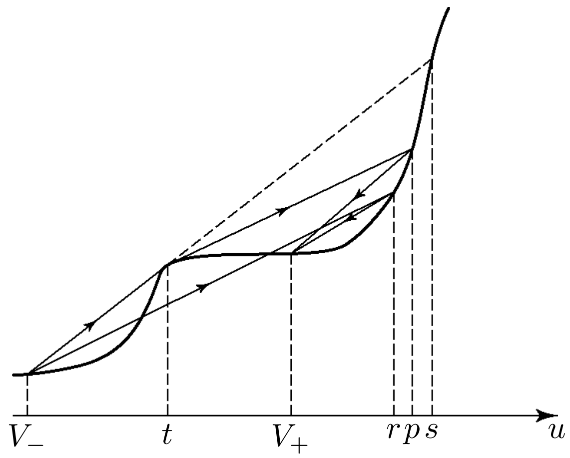
The fact that the minimal solution follows the minimal trajectory along g implies immediately the following result.

Proposition 4.2. *Let the hypotheses of Theorem 3.3 be satisfied and let* g' *belong to* $\mathcal{R}(a,b)$. *For* $D_+ - D_- \in]A,B[$ *denote by* θ *the solution of problem* (1.20) *defined in Theorem 3.3. Then* θ *satisfies the Lax entropy condition at each point of discontinuity* $z \in \mathbb{R}^1$.

Notice that the Lax entropy condition does not follow from the dissipation condition (2.5) if g has inflection points: the solutions defined by formula (2.13) in the proof of Proposition 2.6 obviously violate the Lax entropy condition. On the other hand, it is easy to infer from inequality (2.9) that the Lax entropy condition does not necessarily imply the dissipation condition if g is allowed to have more than one inflection point. Below in Example 4.3 we show that even the combination of both Lax condition and dissipation condition does not guarantee the uniqueness of solutions of the Riemann problem. It suffices to work with smooth increasing constitutive functions g having at least two inflection points.

Example 4.3. Let $g :]a,b[\to]c,d[$ be an increasing smooth function and let there exist numbers $a < V_- < q < V_+ < s < b$ such that
(i) $g'' > 0$ in $]V_-, q[\cup]V_+, s[$, $g'' < 0$ in $]q, V_+[$,
(ii) there exists $t \in]q, V_+[$ such that $\frac{g(t)-g(V_-)}{t-V_-} = \frac{g(s)-g(V_-)}{s-V_-} = \max\left\{\frac{g(u)-g(V_-)}{u-V_-}; u \in]V_-, s]\right\}$ (see Fig. 16)

Fig. 16



We fix some $r \in]V_+, s[$ such that $\int_{V_-}^{r} g(u)du < \frac{1}{2}(r - V_-)(g(r) + g(V_-))$ and put

$$\theta(z) := \begin{cases} V_- & \text{for} \quad z < -\sqrt{\frac{g(r)-g(V_-)}{r-V_-}}, \\ r & \text{for} \quad z \in \left] -\sqrt{\frac{g(r)-g(V_-)}{r-V_-}}, \sqrt{\frac{g(r)-g(V_+)}{r-V_+}} \right[, \\ V_+ & \text{for} \quad z > \sqrt{\frac{g(r)-g(V_+)}{r-V_+}}. \end{cases} \tag{4.1}$$

Then θ is a solution of (1.20) with

$$D_+ - D_- = \sqrt{(g(r) - g(V_-))(r - V_-)} + \sqrt{(g(r) - g(V_+))(r - V_+)}. \tag{4.2}$$

For $p \in [r, s[$ we further define

$$\theta_p(z) := \begin{cases} w_p^*(z^2) & \text{for} \quad z < 0, \\ p & \text{for} \quad z \in \left[0, \sqrt{\frac{g(p)-g(V_+)}{p-V_+}}\right[, \\ V_+ & \text{for} \quad z > \sqrt{\frac{g(p)-g(V_+)}{p-V_+}}, \end{cases} \tag{4.3}$$

where w_p^* is the minimal solution of (2.18) with $V = V_-, Q = g(p)$ and we check that the value of p can be chosen in such a way that θ_p satisfies (1.20) with $D_+ - D_-$ given by (4.2). Using Lemmas 3.4, 3.6 we obtain

$$\int_{-\infty}^{\infty} \big(\theta_p(z) - P_0(z)\big) dz = \int_0^{\infty} \big(w_p^*(z^2) - V_-\big) dz + \sqrt{\big(g(p) - g(V_+)\big)(p - V_+)} =$$
$$= \int_{V_-}^{p} \sqrt{g_p^{*\prime}(u)}\, du + \sqrt{\big(g(p) - g(V_+)\big)(p - V_+)},$$

where g_p^* is the minimal (concave) trajectory from $g(p)$ to V_-. Put

$$h(p) := \int_{V_-}^{p} \sqrt{g_p^{*\prime}(u)}\, du + \sqrt{\big(g(p) - g(V_+)\big)(p - V_+)} -$$
$$- \sqrt{\big(g(r) - g(V_-)\big)(r - V_-)} - \sqrt{\big(g(r) - g(V_+)\big)(r - V_+)}.$$

We claim that $p \in [r, s[$ can be chosen in such a way that $h(p) = 0$. Indeed, we have

$$h(s) = \sqrt{\big(g(s) - g(V_-)\big)(s - V_-)} + \sqrt{\big(g(s) - g(V_+)\big)(s - V_+)} -$$
$$- \sqrt{\big(g(r) - g(V_-)\big)(r - V_-)} - \sqrt{\big(g(r) - g(V_+)\big)(r - V_+)} > 0,$$

and Hölder's inequality yields

$$\int_{V_-}^{r} \sqrt{g_r^{*\prime}(u)} du \leqslant \sqrt{\big(g(r) - g(V_-)\big)(r - V_-)},$$

hence $h(r) \leqslant 0$. The function h is continuous in $[r, s[$, hence $h(p) = 0$ for some $p \in [r, s[$. We thus dispose of two solutions θ, θ_p of problem (1.20) with $D_+ - D_-$ given by formula (4.2). Both θ and θ_p satisfy the Lax condition and the dissipation condition. To check that $\theta \neq \theta_p$ we notice that $g_p^*(t) = g(t)$, hence $\frac{g_p^*(p) - g_p^*(t)}{p-t} = \frac{g(p)-g(t)}{p-t} < \frac{g(s)-g(t)}{s-t} = \frac{g(t)-g(V_-)}{t-V_-} = \frac{g_p^*(t)-g_p^*(V_-)}{t-V_-}$. This implies that g_p^* is not affine, consequently the solutions θ, θ_p are distinct.

Liu's shock admissibility criterion

Definition 4.4. (Liu (1981)) *A solution θ to problem (1.20) is said to satisfy* Liu's shock admissibility criterion *at a point $z \in \mathbb{R}^1$ if the limits $\theta(z+) \neq \theta(z-)$ exist and*

$$(4.4) \quad z\left(\frac{g(u) - g(\theta(z-))}{u - \theta(z-)} - \frac{g(\theta(z+)) - g(\theta(z-))}{\theta(z+) - \theta(z-)}\right) \geqslant 0 \quad \forall u \in \operatorname{Conv}\big\{\theta(z-), \theta(z+)\big\}.$$

It is obvious that the minimal solution defined in Theorem 3.3 satisfies Liu's criterion at each point of discontinuity. The converse is true in the class of regulated functions.

Proposition 4.5. *Let the problem (1.20) admit a solution* $\theta \in \mathcal{R}(-\infty, \infty)$ *such that condition (4.4) holds at each point* $z \in \mathbb{R}^1$ *of discontinuity of* θ*. Then* θ *is minimal in the sense of Theorem 3.3.*

Proof. Let us first assume for instance that θ is nondecreasing in $]0,\infty[$. Let $w_+(s) := \theta(\sqrt{s})$ be the corresponding solution of (2.18) and let g^* be its trajectory according to Lemma 3.5. If for some $u \in]\theta(0+), V_+[$ we have $g(u) \neq g^*(u)$, then by Lemma 3.5 there exists $s > 0$ such that $u \in]w(s-), w(s+)[$ and $g^*(u) = g(w(s-)) + (u - w(s-))\frac{g(w(s+))-g(w(s-))}{w(s+)-w(s-)}$. Condition (4.4) then entails $g(u) \geqslant g^*(u)$, hence g^* is the minimal trajectory. The same argument works for θ nonincreasing and for the interval $]-\infty, 0[$.

On the other hand, condition (4.4) excludes nonmonotonicities of θ in $]-\infty,0[$ and $]0,\infty[$. This can be seen again by considering just the interval $]0,\infty[$ only. Let us assume for instance that there exist $z_3 > z_1 > 0$ and $z_2 \in [z_1, z_3]$ such that the values $v_1 := \theta(z_1-)$, $v_3 := \theta(z_3+)$, $v_2 := \inf\{\theta(z); z \in [z_1, z_3]\}$ satisfy $v_2 < v_1 < v_3$, $v_2 = \theta(z_2+)$ or $v_2 = \theta(z_2-)$, $\theta(z) \in [v_2, v_1]$ for $z \in [z_1, z_2]$, $\theta(z) \in [v_2, v_3]$ for $z \in [z_2, z_3]$ (the other possibilities are analogous).

It is more convenient to work with the solution w of (2.18) defined by the formula $w(s) := \theta(\sqrt{s})$ for $s > 0$. Put $s_i := z_i^2$ for $i = 1,2,3$, $A := \{s \in]s_2, s_3[; w(s+) = v_1$ or $w(s-) = v_1\}$. For $A \neq \emptyset$ put $s_A := \inf A$. Integrating equation (2.18)(iii) we obtain

$$s_2(v_2 - v_1) - g(v_2) + g(v_1) = \int_{s_1}^{s_2} (w(s) - v_1)ds \leqslant 0, \tag{4.5}$$

$$s_2(v_1 - v_2) - g(v_1) + g(v_2) = \int_{s_2}^{s_A} (w(s) - v_1)ds. \tag{4.6}$$

Put $\bar{s} := \inf\{s \in [s_2, s_3]; w(s+) > v_1\}$. We have either $\bar{s} = s_2$ or $\bar{s} > s_2$. In the latter case it follows from (4.5), (4.6) that $[s_2, \bar{s}] \cap A = \emptyset$, hence in both cases we obtain $w(\bar{s}-) < v_1 < w(\bar{s}+)$. Put $\bar{v} := w(\bar{s}-) \in [v_2, v_1[$. Hypothesis (4.4) and Lemma 1.4 then yield

$$\frac{g(v_1) - g(\bar{v})}{v_1 - \bar{v}} \geqslant \bar{s}, \tag{4.7}$$

consequently

$$\int_{s_1}^{\bar{s}} (w(s) - v_1)ds = \bar{s}(\bar{v} - v_1) - g(\bar{v}) + g(v_1) \geqslant 0. \tag{4.8}$$

By construction, we have $\int_{s_1}^{\bar{s}} (w(s) - v_1)ds < 0$, which is a contradiction. Proposition 4.5 is proved. ☐

Dafermos maximal entropy rate criterion

The idea of Dafermos (1973) is similar to that which leads to the minimality condition (3.1), namely to maximize the dissipation of energy. Its advantage is that it can easily be formulated for arbitrary (not necessarily self-similar) solutions to equation (1.1).

Definition 4.6. *A solution u of problem (1.9) - (1.11) is said to satisfy the* Dafermos maximal entropy criterion, *if for every solution $\tilde{u}$ to (1.9) - (1.11) we have*

$$\frac{d}{dt}\int_{-\infty}^{\infty}\Big(\frac{1}{2}u_t^2 + G(u_x)\Big) - \Big(\frac{1}{2}\tilde{u}_t^2 + G(\tilde{u}_x)\Big)dx \leqslant 0 \tag{4.9}$$

in the sense of distributions.

In the class of self-similar solutions we can rewrite condition (4.9) in a simple way.

Proposition 4.7. *For $r \in]a,b[$ and $z \in \mathbb{R}^1$ put $\mathcal{G}(z,r) := G(r) - rg(r) + \frac{3}{2}z^2r^2$. Then a self-similar solution u of (1.9) - (1.11) satisfies condition (4.9) with respect to all self-similar solutions $\tilde{u}$ of (1.9)-(1.11) if and only if*

$$\int_{-\infty}^{\infty}\big(\mathcal{G}(z,\theta(z)) - \mathcal{G}(z,\tilde{\theta}(z))\big)dz \leqslant 0 \tag{4.10}$$

for all solutions $\tilde{\theta}$ to (1.20), where θ is the solution to (1.20) associated to u according to Proposition 1.2.

The proof of Proposition 4.7 is a simple exercise analogous to the proof of Proposition 1.2 based on the integration-by-parts formula

$$\int_{-\infty}^{\infty}(f^2 - \tilde{f}^2)dz = \int_{-\infty}^{\infty}2z(\theta f - \tilde{\theta}\tilde{f})dz = -\int_{-\infty}^{\infty}\big[\theta(z^2\theta - g(\theta)) - \tilde{\theta}(z^2\tilde{\theta} - g(\tilde{\theta}))\big]dz$$

and we leave it to the reader.

The comparison of the maximum principles (3.2) and (4.10) is not easy in general. We can better understand their meaning when looking at piecewise constant solutions of the form (1.26). Let us denote by $A(\theta_i,\theta_{i+1}) := G(\theta_{i+1}) - G(\theta_i) - \frac{1}{2}(\theta_{i+1}-\theta_i)(g(\theta_{i+1}) + g(\theta_i))$ the signed area between the graph of the constitutive function g and the segment connecting the points $(\theta_i, g(\theta_i))$ and $(\theta_{i+1}, g(\theta_{i+1}))$ which represents the trajectory of the shock at the point $z = z_i$. While condition (4.10) consists in maximizing the sum

$$\sum_{z_i\in\mathbb{R}^1} z_i A(\theta_i,\theta_{i+1}), \tag{4.11}$$

condition (3.2) requires to maximize separately the expressions

$$\sum_{z_i>0} A(\theta_i, \theta_{i+1}), \qquad -\sum_{z_i<0} A(\theta_i, \theta_{i+1}) \tag{4.12}$$

with an unknown intermediate condition $g(\theta(0)) = Q$ which is to be identified.

The construction in Example 2.4 shows that these conditions are not equivalent. Condition (4.12) naturally selects the constant solution. On the other hand, the expression (4.11) vanishes for the constant solution and is positive for the nonconstant one. Paradoxically, the constant solution does not maximize the entropy rate in Dafermos' sense.

Vanishing viscosity

It has been observed in various situations that the fact of neglecting small dissipation effects may lead to a loss of well-posedness of the problem (one example of this kind has recently been studied by Lovicar, Straškraba, Krejčí (1993).

Here, the nonlinear elastic-stress constitutive law $\sigma = g(\varepsilon)$ can be considered as the limit case of the parallel viscoelastic law $\sigma = g(\varepsilon) + \eta\dot{\varepsilon}$ (see Sect. I.1 on rheological models) as the viscosity coefficient η tends to 0. In the case that the solutions u^η of the corresponding equation of motion

$$u^\eta_{tt} - g(u^\eta_x)_x - \eta u^\eta_{xxt} = 0 \tag{4.13}$$

converge in some sense to a solution of equation (1.7), it is natural to declare that this limit is the relevant solution of equation (1.7). In other words, the selection rule is imposed by the limit process.

With respect to self-similar solutions, it is more convenient to replace (4.13) with the equation

$$u^\eta_{tt} - g_{\eta,K}(u^\eta_x)_x - \frac{\eta}{2} t u^\eta_{xxt} = 0, \tag{4.14}$$

where $g_{\eta,K} \in C^1(]a,b[)$ is a regularization of the function g that we briefly describe here.

Let g satisfy condition (1.8). For a fixed compact interval $K \subset]a,b[$ and a number $\eta > 0$ put

$$g_{\eta,K}(u) := e^{\frac{1}{\eta}(u_K - u)} g(u_K) + \int_{u_K}^{u} \frac{1}{\eta} e^{\frac{1}{\eta}(v-u)} g(v)\, dv \quad \text{for } u \in]a,b[, \tag{4.15}$$

where $u_K := \min K$. The identity

$$\eta g'_{\eta,K}(u) = g(u) - g_{\eta,K}(u) \quad \forall u \in]a,b[\tag{4.16}$$

has the following immediate consequences (the proof is left to the reader).

Lemma 4.8. *Let* $K \subset]a,b[$ *be a compact interval. Put* $L_K := \sup\left\{\left|\frac{g(u)-g(v)}{u-v}\right|;\ u,v \in K, u \neq v\right\}$. *Then for every* $\eta > 0$ *the function* $g_{\eta,K}$ *is continuously differentiable in* $]a,b[$ *and for every* $u \in K$ *we have*

(i) $|g(u) - g_{\eta,K}(u)| \leqslant \eta L_K$,

(ii) $|g'_{\eta,K}(u)| \leqslant L_K$.

In terms of self-similar solutions, approximating equation (1.7) by (4.14) corresponds to the approximation of problem (2.18) by the equation

$$\eta\big(sw'_\eta(s)\big)' = w_\eta(s) - \big(sw_\eta(s) - g_{\eta,K}(w_\eta(s))\big)' \tag{4.17}$$

for a suitable choice of boundary conditions and of the compact interval K. This can be done in the following way.

Let Q, V be given data in (2.18) and let us define V_Q as in Definition 3.7. We can assume for the sake of definiteness that $Q < g(V)$ leaving the other cases to the reader.

We fix an open bounded interval J, $[V_Q, V] \subset J \subset \bar{J} \subset]a,b[$ and put $K := \bar{J}$. For an arbitrary $\beta > L_K$ we prescribe boundary conditions

$$w_\eta(\eta) = V_Q, \quad w_\eta(\beta) = V. \tag{4.18}$$

We first verify that problem (4.17), (4.18) cannot have multiple solutions.

Lemma 4.9. *Let* $0 < \delta < s_1 < s_2$ *be given and let* $w, \tilde{w}$ *be two solutions of (4.17) in the interval* $]s_1 - \delta, s_2 + \delta[$. *Assume* $w(s_i) = \tilde{w}(s_i)$ *for* $i = 1, 2$. *Then* $w(s) = \tilde{w}(s)$ *for all* $s \in]s_1 - \delta, s_2 + \delta[$.

P r o o f. If the set $B := \{s \in [s_1, s_2]; w(s) = \tilde{w}(s)\}$ is infinite, then it contains a convergent sequence and its limit point $\bar{s}$ satisfies $w(\bar{s}) = \tilde{w}(\bar{s})$, $w'(\bar{s}) = \tilde{w}'(\bar{s})$. The general theory of ordinary differential equations then yields $w \equiv \tilde{w}$.

Assume that B is finite. We choose two consecutive points $\sigma_1, \sigma_2 \in B$, so that for instance $w(\sigma_i) = \tilde{w}(\sigma_i)$ for $i = 1, 2$, $w(s) > \tilde{w}(s)$ for $s \in]\sigma_1, \sigma_2[$. Integrating $\int_{\sigma_1}^{\sigma_2} ds$ the identity

$$\big(\eta s(w' - \tilde{w}')\big)' = (w - \tilde{w}) - \big(s(w - \tilde{w}) - g(w) + g(\tilde{w})\big)'$$

we obtain

$$\eta\big[\sigma_2\big(w'(\sigma_2) - \tilde{w}'(\sigma_2)\big) - \sigma_1\big(w'(\sigma_1) - \tilde{w}'(\sigma_1)\big)\big] = \int_{\sigma_1}^{\sigma_2} (w - \tilde{w})ds > 0,$$

hence either $w'(\sigma_2) > \tilde{w}'(\sigma_2)$ or $w'(\sigma_1) < \tilde{w}'(\sigma_1)$, which is a contradiction. □

For a fixed $\eta > 0$ we have the following existence result.

Theorem 4.10. *Problem (4.17), (4.18) has a unique classical solution* w_η. *Moreover, there exists* $\eta_0 > 0$ *such that for* $\eta < \eta_0$ *the solution* w_η *can be extended to an interval* $]\alpha_\eta, +\infty[$ *for some* $\alpha_\eta \in]0, \eta[$, *it is twice continuously differentiable and increasing in its domain of definition.*

P r o o f. We define recursively for $s \in [\eta, \beta]$ a sequence $\{w^{(n)}(s); n \in \mathbb{N} \cup \{0\}\}$ by the formula

$$w^{(0)}(s) := V_Q + (V - V_Q)\frac{s-\eta}{\beta-\eta},$$

$$w^{(n)}(s) := V_Q + c^{(n-1)} \int_\eta^s \frac{1}{\tau} e^{\frac{1}{\eta}\int_\eta^\tau \left(\frac{g'_{\eta,K}(w^{(n-1)}(\sigma))}{\sigma} - 1\right)d\sigma} d\tau,$$

where

$$c^{(n-1)} := (V - V_Q)\left[\int_\eta^\beta \frac{1}{\tau} e^{\frac{1}{\eta}\int_\eta^\tau \left(\frac{g'_{\eta,K}(w^{(n-1)}(\sigma))}{\sigma} - 1\right)d\sigma} d\tau\right]^{-1}.$$

We immediately see that $\{w^{(n)}\} \subset C^2([\eta, \beta])$ is a sequence of increasing functions satisfying boundary conditions (4.18) and that there exists a constant M_η independent of n such that $0 < c^{(n)} \leqslant M_\eta$, $\left|w^{(n)'}(s)\right| \leqslant M_\eta$ for all $s \in]\eta, \beta[$.

From the Arzelà-Ascoli theorem V.2.1 it follows that there exist convergent subsequences of $\{c^{(n)}\}$ and $\{w^{(n)}\}$ (still indexed by n) such that the limits $c_\eta := \lim_{n\to\infty} c^{(n)}$, $w_\eta := \lim_{n\to\infty} w^{(n)}$ satisfy

$$w_\eta(s) = V_Q + c_\eta \int_\eta^s \frac{1}{\tau} e^{\frac{1}{\eta}\int_\eta^\tau \left(\frac{g'_{\eta K}(w_\eta(\sigma))}{\sigma} - 1\right)d\sigma} d\tau, \tag{4.19}$$

hence w_η is a solution of (4.17), (4.18).

The function w_η can be extended to a maximal solution of (4.17) $w_\eta :]\alpha_\eta, \beta_\eta[\to]a, b[$ for some $\alpha_\eta < \eta$, $\beta_\eta > \beta$. Identity (4.19) remains valid for $s \in]\alpha_\eta, \beta_\eta[$, hence w_η is twice continuously differentiable and increasing in its maximal domain of definition. Lemma 4.9 then entails that this solution is unique.

It remains to prove that $\beta_\eta = +\infty$ for η sufficiently small. Put

$$\gamma_\eta := \sup\{s \in]\alpha_\eta, \beta_\eta[; w_\eta(s) \in K\}, \quad \delta := \frac{1}{4}(\beta - L_K).$$

We have $\gamma_\eta > \beta$ and the identity

$$\big(s w'_\eta(s)\big)' = \frac{1}{\eta}\Big(\frac{g'_{\eta,K}(w_\eta(s))}{s} - 1\Big)\big(s w'_\eta(s)\big) \tag{4.20}$$

combined with Lemma 4.8 (ii) entails for $s\in]L_K+\delta,\gamma_\eta[$

$$(4.21)\qquad \big(sw_\eta'(s)\big)'\leqslant \frac{-\delta}{\eta(L_K+\delta)}sw_\eta'(s).$$

Putting $p:=\frac{\delta}{L_K+\delta}>0$ we rewrite (4.21) in the form $\big(e^{\frac{p}{\eta}s}sw_\eta'(s)\big)'\leqslant 0$, hence also $\big(e^{\frac{p}{\eta}s}w_\eta'(s)\big)'\leqslant 0$. This yields for $s\in]L_K+2\delta,\gamma_\eta[$

$$\delta e^{\frac{p}{\eta}s}w_\eta'(s)\leqslant\int_{L_K+\delta}^{L_K+2\delta}e^{\frac{p}{\eta}t}w_\eta'(t)\,dt\leqslant e^{\frac{p}{\eta}(L_k+2\delta)}(V-V_Q),$$

hence

$$(4.22)\qquad w_\eta(s)\leqslant w_\eta(L_K+3\delta)+\frac{\eta(V-V_Q)}{\delta p}e^{-\frac{\delta p}{\eta}}\quad\text{for}\quad s\in]L_K+3\delta,\gamma_\eta[.$$

For $\eta>0$ sufficiently small, say $\eta<\eta_0$, we thus have $w_\eta(s)\in K$ for all $s\in]\alpha_\eta,\beta_\eta[$, hence $\beta_\eta=+\infty$. This completes the proof of Theorem 4.10. $\square$

We now pass to the limit as $\eta\to 0+$. The following Theorem says that the solution obtained by the vanishing viscosity selection rule coincides with the minimal solution defined in Sect. IV.3.

Theorem 4.11. *Let $Q\in]c,d[$ and $V\in]a,b[$ be given and let w_η be the solution of (4.17), (4.18) for $\eta\in]0,\eta_0[$. Let w^* be the minimal solution of (2.18). Then $w_\eta(s)\to w^*(s)$ as $\eta\to 0+$ for a.e. $s>0$.*

P r o o f. For $\eta<\eta_0$ we define auxiliary functions

$$(4.23)\qquad \hat w_\eta(s):=\begin{cases} w_\eta(s), & s\in[\eta,+\infty[,\\ V_Q, & s\in[0,\eta[.\end{cases}$$

It suffices to assume $Q<g(V)$ ($Q>g(V)$ is analogous and $Q=g(V)$ is trivial). By (4.22), the system $\{w_\eta;\eta<\eta_0\}$ converges uniformly to the constant V on $[\beta-\delta,+\infty[$ as $\eta\to 0+$. On $[0,\beta],\{\hat w_\eta;\eta>0\}$ is an equibounded system of continuous nondecreasing functions, and from Helly's Selection Principle (Kolmogorov, Fomin (1970)) we deduce the existence of a nondecreasing function $\overline{w}:[0,\beta]\to[V_Q,V]$ and of a sequence $\eta_k\to 0+$ as $k\to\infty$ such that

$$(4.24)\qquad \hat w_{\eta_k}(s)\to\overline{w}(s)\quad\forall s\in[0,\beta]\quad\text{as}\quad k\to\infty.$$

Let $\varphi \in \mathcal{D}(0,\infty)$ be arbitrarily chosen. For k sufficiently large we have

$$\int_0^\infty \big[\big(s\hat{w}_{\eta_k}(s) - g_{\eta_k,K}(\hat{w}_{\eta_k}(s))\big)\varphi'(s) + \hat{w}_{\eta_k}(s)\varphi(s)\big]ds = \\ = \eta_k \int_0^\infty \hat{w}_{\eta_k}(s)\big(\varphi'(s) + s\varphi''(s)\big)ds$$

and passing to the limit as $k \to \infty$ we obtain

$$\int_0^\infty \big[\big(s\overline{w}(s) - g(\overline{w}(s))\big)\varphi'(s) + \overline{w}(s)\varphi(s)\big]ds = 0.$$

Consequently, $\overline{w}$ is a nondecreasing solution of (2.18) with $\overline{w}(s) = V$ for $s \geqslant \beta$ and $\overline{w}(0+) = \bar{V} \in [V_Q, V]$.

For each $\eta > 0$ and $s > \eta$ we have

$$\eta^2 w_\eta'(\eta) = \eta s w_\eta'(s) + s w_\eta(s) - \eta V_Q - g_{\eta,K}(w_\eta(s)) + g_{\eta,K}(V_Q) - \int_\eta^s w_\eta(\sigma)d\sigma \tag{4.25}$$

and integrating the last identity $\int_\eta^\beta ds$ we obtain

$$(\beta-\eta)\eta^2 w_\eta'(\eta) = \eta\big[\beta V - \eta V_Q - \int_\eta^\beta w_\eta(s)ds\big] + \\ + \int_0^\beta \big(s\hat{w}_\eta(s) - g_{\eta,K}(\hat{w}_\eta(s)) + g_{\eta,K}(V_Q) - \int_0^s \hat{w}_\eta(\sigma)d\sigma\big)ds.$$

For $\eta = \eta_k$ we pass to the limit as $k \to \infty$. This yields

$$\beta \lim_{k\to\infty} \eta_k^2 w_{\eta_k}'(\eta_k) = \int_0^\beta \Big(s\overline{w}(s) - g(\overline{w}(s)) + g(V_Q) - \int_0^s \overline{w}(\sigma)d\sigma\Big)ds \\ = \int_0^\beta \big(g(V_Q) - g(\bar{V})\big)ds = \beta\big(g(V_Q) - g(\bar{V})\big) \leqslant 0.$$

We conclude

$$\bar{V} = V_Q, \quad \lim_{k\to\infty} \eta_k^2 w_{\eta_k}'(\eta_k) = 0. \tag{4.26}$$

According to Lemma 3.5, we define the convex trajectory g^* of the solution $\overline{w}$ by the formula

$$g^*(u) := Q + \int_{V_Q}^u \overline{w}^{-1}(v)dv$$

analogous to (3.5). We are done if we prove

$$g(u) \geqslant g^*(u) \quad \forall u \in [V_Q, V]. \tag{4.27}$$

Indeed, then g^* is the minimal trajectory from Q to V and by Lemma 3.6, $\overline{w}$ is the minimal solution of (2.18). The limit function $\overline{w}$ is then independent of the choice of the sequence $\{\eta_k\}$, so the assertion of Theorem 4.11 holds.

To prove (4.27), we choose an arbitrary $u \in]V_Q, V[$ and find $s > 0$ such that $u \in \big[\overline{w}(s-), \overline{w}(s+)\big]$. Following Lemma 3.5 we have $g^*(\overline{w}(s\pm)) = g(\overline{w}(s\pm))$, hence it remains to consider the case

$$\overline{w}(s-) < u < \overline{w}(s+). \tag{4.28}$$

Let $\{s_k\}$ be the sequence such that $w_{\eta_k}(s_k) = u$ for all $k \in \mathbb{N}$ and let us assume that a subsequence (denoted again by s_k) converges to some $\bar{s} \neq s$. For $\bar{s} > s$ and $\sigma \in]s, \bar{s}[$ we have $\overline{w}(s+) \leqslant \overline{w}(\sigma) = \lim\limits_{k\to\infty} w_{\eta_k}(\sigma) \leqslant u$, which is a contradiction. The case $\bar{s} < s$ is analogous, so $s_k \to s$ as $k \to \infty$.

Put $\Delta := g(u) - g^*(u)$. Lemma 3.4 entails

$$\begin{aligned}\Delta &= g(u) - g(V_0) - su + \int_0^s \overline{w}(\sigma)d\sigma \\ &= g_{\eta_k,K}\big(w_{\eta_k}(s_k)\big) - g_{\eta_k,K}(V_0) - s_k w_{\eta_k}(s_k) + \eta_k V_0 + \int_{\eta_k}^{s_k} w_{\eta_k}(\sigma)d\sigma + I_k,\end{aligned}$$

where

$$\begin{aligned}I_k := \big(g(u) - g_{\eta_k,K}(u)\big) - \big(g(V_0) - g_{\eta_k,K}(V_0)\big) + \\ + (s_k - s)u + \int_0^s \big(\overline{w}(\sigma) - \hat{w}_{\eta_k}(\sigma)\big)d\sigma - \int_s^{s_k} w_{\eta_k}(\sigma)d\sigma.\end{aligned}$$

We have $\lim\limits_{k\to\infty} I_k = 0$ and identity (4.25) yields $\Delta = \eta_k s_k w'_{\eta_k}(s_k) - \eta_k^2 w'_{\eta_k}(\eta_k) + I_k$.

From (4.26) we conclude

$$\Delta = \lim_{k\to\infty} \eta_k s_k w'_{\eta_k}(s_k) \geqslant 0$$

which is nothing but inequality (4.27). Theorem 4.11 is proved. □

V. Appendix: Function spaces

The calculus of functions of one real variable with values in a Banach space has originally been developed as an auxiliary tool for the semigroup theory, see for instance Hille, Phillips (1957), Yosida (1965) or Brézis (1973). Special results that we need here either do not exist at all or, as the opposite extreme, exist only in a form which is too general for our purposes. This is also the case of embedding theorems for anisotropic Sobolev spaces that we use in Chapter III for solving partial differential equations with hysteresis.

About 20 pages are thus devoted here to a survey, where new results incorporated into a simplified general theory constitute an exposition that the reader will hopefully find elementary and consistent.

V.1 Integration of vector-valued functions

In this section we recall basic notions of the Bochner integral and of the theory of functions of bounded variation that are directly needed in the preceding chapters. One of the main goals is to give a self-contained proof of Theorem 1.15 on the relationship between the strong and weak convergences of integrable functions which seems to be new and plays a important role in the study of vector hysteresis operators in Sect. I.3.

For the reader's convenience, we include those proofs which are simple enough and do not require special knowledge of other branches of analysis.

Definition 1.1. *Let X be a real Banach space endowed with norm $|\cdot|_X$ and let $[a,b] \subset \mathbb{R}^1$ be a compact interval. A function $u : [a,b] \to X$ is called*

(i) simple, *if there exists a partition $[a,b] = \bigcup_{k=1}^{N} E_k$ of the interval $[a,b]$ into a finite union of pairwise disjoint Lebesgue measurable sets $\{E_k;\ k = 1,\dots,N\}$ and a sequence $\{x_k;\ k = 1,\dots,N\} \subset X$ such that for almost all $t \in [a,b]$ we have*

$$u(t) = \sum_{k=1}^{N} x_k\, \chi_{E_k}(t), \tag{1.1}$$

where χ_{E_k} is the characteristic function *of the set E_k,*

$$\chi_{E_k}(t) = \begin{cases} 0 & \text{if} \quad t \notin E_k, \\ 1 & \text{if} \quad t \in E_k; \end{cases}$$

(ii) strongly measurable, *if there exists a sequence* $\{u_n;\, n \in \mathbb{N}\}$ *of simple functions such that* $\lim_{n\to\infty} |u_n(t) - u(t)|_X = 0$ *for a.e.* $t \in [a,b]$.

It is easy to see that for a strongly measurable function $u : [a,b] \to X$ the scalar-valued function $t \mapsto |u(t)|_X$ is Lebesgue measurable.

In order to fix the terminology we first list basic properties of Lebesgue measurable and integrable functions.

Theorem 1.2. (Egoroff) *Let* $\{f_n; n \in \mathbb{N}\} : [a,b] \to [0,\infty[$ *be a sequence of Lebesgue measurable functions. Then the following conditions are equivalent:*

(i) $\lim_{n\to\infty} f_n(t) = 0$ *for almost all* $t \in [a,b]$;

(ii) *for every* $\delta > 0$ *there exists a measurable set* $M_\delta \subset [a,b]$, $\text{meas}(M_\delta) < \delta$ *such that* $\lim_{n\to\infty} \sup\{f_n(t); t \in [a,b] \setminus M_\delta\} = 0$, *where* meas *denotes the Lebesgue measure.*

An elementary proof of Egoroff's Theorem can be found in Yosida (1965). The following two statements deal with Lebesgue integrable functions. Proposition 1.3 is a straightforward consequence of the additivity of the Lebesque integral and Proposition 1.4 follows from Egoroff's Theorem and Proposition 1.3.

Proposition 1.3. (Absolute continuity of the integral). *For each Lebesgue integrable function* $f : [a,b] \to \mathbb{R}^1$ *we have*

$$\forall \varepsilon > 0 \quad \exists \delta > 0 \quad \forall A \subset [a,b] : \quad \text{meas}(A) < \delta \Rightarrow \int_A |f(t)| dt < \varepsilon. \tag{1.2}$$

Proposition 1.4. (Fatou's Lemma). *Let* $\{f_k;\, k \in \mathbb{N} \cup \{0\}\} : [a,b] \to [0,\infty[$ *be a sequence of integrable functions,* $f_0(t) = \lim_{k\to\infty} f_k(t)$ *for a.e.* $t \in [a,b]$. *Then* $\int_a^b f_0(t)dt \leqslant \liminf_{k\to\infty} \int_a^b f_k(t)dt$.

We shall study more in detail the relationship between the pointwise convergence almost everywhere and convergence of integrals. Let us recall that a set S of integrable functions $f : [a,b] \to \mathbb{R}^1$ is called *equiintegrable*, if relation (1.2) holds for δ independent of the choice $f \in S$.

Proposition 1.5. *Let* $\{f_k;\, k \in \mathbb{N}\}$ *be an equiintegrable sequence such that* $\lim_{k\to\infty} f_k(t) = 0$ *for* a.e. $t \in [a,b]$. *Then* $\lim_{k\to\infty} \int_a^b f_k(t)dt = 0$.

We omit the proof which is very easy (one can use for instance Egoroff's Theorem and property (1.2)).

Let us come back to vector-valued strongly measurable functions. We first show some kind of countable structure in the convergence almost everywhere.

Proposition 1.6. (Diagonalization Principle). *Let* $\{u_n;\, n \in \mathbb{N}\}$ *and* $\{v_n^k;\, k, n \in \mathbb{N}\}$ *be two sequences of strongly measurable functions* $[a,b] \to X$ *and let* $u : [a,b] \to X$ *be a function such that*

$$\lim_{n\to\infty} |u_n(t) - u(t)|_X = 0, \quad \text{a.e.},$$
$$\lim_{k\to\infty} |v_n^k(t) - u_n(t)|_X = 0 \quad \text{a.e.} \quad \forall n \in \mathbb{N}.$$

Then there exists a sequence $\{k_n;\, n \in \mathbb{N}\} \subset \mathbb{N}$ *such that* $\lim\limits_{n\to\infty} |v_n^{k_n}(t) - u(t)|_X = 0$ *a.e..*

One immediately realizes that the statement is false if the convergence almost everywhere is replaced with convergence at each point $t \in [a,b]$; it suffices to choose u in the second Baire's class and v_n^k continuous.

P r o o f of Proposition 1.6. By Egoroff's Theorem, for each $n \in \mathbb{N}$ there exists a set $M_n \subset [a,b]$, $\operatorname{meas}(M_n) < 2^{-n}$ and a number $k_n \in \mathbb{N}$ such that for every $j \geqslant k_n$ and $t \in [a,b] \backslash M_n$ we have $|v_n^j(t) - u_n(t)|_X < \frac{1}{n}$. Put $M := \bigcap\limits_{l=1}^{\infty} \bigcup\limits_{n=l}^{\infty} M_n$. Then $\operatorname{meas}(M) = 0$ and for each $t \in [a,b] \backslash M$ we have for n sufficiently large $|v_n^{k_n}(t) - u(t)|_X \leqslant \frac{1}{n} + |u_n(t) - u(t)|_X$ and the assertion follows. $\square$

We leave to the reader the detailed proof of the next three consequences of Egoroff's Theorem and of the Diagonalization Principle.

Corollary 1.7. *Let* $\{u_n;\, n \in \mathbb{N}\} : [a,b] \to X$ *be a sequence of strongly measurable functions such that* $\lim\limits_{n\to\infty} |u_n(t) - u(t)|_X = 0$ *a.e. Then* $u : [a,b] \to X$ *is strongly measurable.*

Corollary 1.8. *A function* $u : [a,b] \to X$ *is strongly measurable if and only if there exists a sequence* $\{u_n;\, n \in \mathbb{N}\}$ *of continuous functions* $[a,b] \to X$ *such that* $\lim\limits_{n\to\infty} |u_n(t) - u(t)|_X = 0$ *a.e.*

Corollary 1.9. (Lusin's Theorem). *A function* $u : [a,b] \to X$ *is strongly measurable if and only if for every* $\delta > 0$ *there exist a closed set* $F_\delta \subset [a,b]$ *and a* continuous *function* $w : [a,b] \to X$ *such that* $\operatorname{meas}([a,b] \setminus F_\delta) < \delta$, $u(t) = w(t)$ *for all* $t \in F_\delta$ *and* $\sup\limits_{[a,b]} |w(t)|_X \leqslant \sup\limits_{[a,b]} |u(t)|_X$.

Bochner integral

We now introduce the Bochner integral in a standard way following Yosida (1965) and Hille, Phillips (1957).

Definition 1.10. *For a simple function* $u : [a, b] \to X$ *of the form (1.1) we define its* Bochner integral *over a measurable set* $A \subset [a, b]$ *by the formula.*

$$\int_A u(t)dt := \sum_{k=1}^{N} x_k \text{ meas}(E_k \cap A) \in X. \tag{1.3}$$

An arbitrary function $u : [a, b] \to X$ *is said to be* Bochner integrable *in* $[a, b]$ *if there exists a sequence* $\{u_n; n \in \mathbb{N}\}$ *of simple functions* $[a, b] \to X$ *such that* $\lim_{n\to\infty} \int_a^b |u_n(t) - u(t)|_X dt = 0$ *and we define its Bochner integral over a measurable set* $A \subset [a, b]$ *as*

$$\int_A u(t)dt := \lim_{n\to\infty} \int_A u_n(t)dt \in X. \tag{1.4}$$

Notice that the sequence $x_n := \int_A u_n(t)dt$ in Definition 1.10 is fundamental in X and its limit (1.4) is independent of the choice of the sequence $\{u_n\}$. The definition immediately implies

$$\left|\int_A u(t)dt\right|_X \leqslant \int_A |u(t)|_X dt < \infty \tag{1.5}$$

for each Bochner integrable function u and measurable set $A \subset [a, b]$.

Bochner's Theorem 1.11 below gives an elegant characterization of Bochner integrable functions.

Theorem 1.11. (Bochner's Theorem). *A function* $u : [a, b] \to X$ *is Bochner integrable if and only if it is strongly measurable and* $\int_a^b |u(t)|_X dt < \infty$.

We show here a simple proof which is based on the following Lemma.

Lemma 1.12. *Let* $\{u_n; n \in \mathbb{N}\}$ *be a sequence of Bochner integrable functions* $[a, b] \to X$ *such that*

$$\forall \varepsilon > 0 \quad \exists n_\varepsilon \in \mathbb{N} \quad \forall m, \ell \geqslant n_\varepsilon \, : \, \int_a^b |u_m(t) - u_\ell(t)|_X dt < \varepsilon.$$

Then there exists a strongly measurable function $u : [a, b] \to X$ *and a subsequence* $\{u_{n_k}\} \subset \{u_n\}$ *such that*

$$\lim_{k\to\infty} |u_{n_k}(t) - u(t)|_X = 0 \quad \text{for a.e.} \quad t \in [a, b]. \tag{1.6}$$

P r o o f of Lemma 1.12. We choose n_k in such a way that the implication $m, \ell \geqslant n_k \Rightarrow \int_a^b |u_m(t) - u_\ell(t)|_X dt < 2^{-2k}$ holds. Put $M_k := \{t \in [a,b];\ |u_{n_k}(t) - u_{n_{k+1}}(t)|_X \geqslant 2^{-k}\}$, $M := \bigcap_{\ell=1}^{\infty} \bigcup_{k=\ell}^{\infty} M_k$. Then $\mathrm{meas}(M_k) < 2^{-k}$, hence $\mathrm{meas}(M) = 0$ and for each $t \in [a,b] \setminus M$ the limit $u(t) := \lim_{k\to\infty} u_{n_k}(t)$ exists.

To prove that u is strongly measurable it suffices to prove that every Bochner integrable function $v : [a,b] \to X$ is strongly measurable. So, let $\{w_n\}$ be a sequence of simple functions such that $\lim_{n\to\infty} \int_a^b |w_n(t) - v(t)|_X dt = 0$. The above argument shows that there exists a function $w : [a,b] \to X$ and a subsequence $\{w_{n_k}\} \subset \{w_n\}$ such that $\lim_{k\to\infty} |w_{n_k}(t) - w(t)|_X = 0$ a.e. The function w is strongly measurable by definition and from Egoroff's Theorem we conclude that $v(t) = w(t)$ a.e. $\square$

P r o o f of Theorem 1.11. The "only if" part follows from Lemma 1.12 and inequality (1.5). To prove the converse we choose an arbitrary strongly measurable function $u : [a,b] \to X$ with $\int_a^b |u(t)|_X dt < \infty$ and an arbitrary sequence $\{w_k\}$ of simple functions such that $\lim_{k\to\infty} |w_k(t) - u(t)|_X = 0$ a.e. For every $n \in \mathbb{N}$ we apply Egoroff's theorem to find a set $M_n \subset [a,b]$, $\mathrm{meas}(M_n) < \frac{1}{n}$ and an index k_n such that $|w_{k_n}(t) - u(t)|_X < \frac{1}{n}$ for all $t \in [a,b] \setminus M_n$. Putting

$$\hat{w}_n(t) := \begin{cases} w_{k_n}(t) & \text{for} \quad t \in [a,b] \setminus M_n \\ 0 & \text{for} \quad t \in M_n \end{cases}$$

we obtain $\int_a^b |\hat{w}_n(t) - u(t)|_X dt \leqslant \frac{b-a}{n} + \int_{M_n} |u(t)|_X dt$ and it suffices to use Proposition 1.3. $\square$

We define in a standard way in the class of strongly measurable functions an equivalence relation $u \sim v \Leftrightarrow u(t) = v(t)$ a.e. Identifying in an obvious sense functions with their equivalence classes we can define the normed linear spaces

(1.7) (i) $L^1(a,b;X)$ of Bochner integrable functions $u : [a,b] \to X$ endowed with norm $|u|_1 := \int_a^b |u(t)|_X dt$,

(ii) $L^p(a,b;X)$ for $1 < p < \infty$ of functions $u \in L^1(a,b;X)$ such that $|u|_p := \left(\int_a^b |u(t)|_X^p dt\right)^{1/p} < \infty$, endowed with norm $|\cdot|_p$,

(iii) $L^\infty(a,b;X)$ of a.e. bounded strongly measurable functions $u : [a,b] \to X$ endowed with norm $|u|_\infty := \inf\{\sup\{|u(t)|_X; t \in [a,b] \setminus M\}; M \subset [a,b], \mathrm{meas}(M) = 0\}$,

(iv) $C([a,b];X)$ of continuous functions $u : [a,b] \to X$ endowed with norm $|\cdot|_\infty$.

The fact that $|\cdot|_p$ is a norm is well known (Adams (1975)). It is not difficult to infer from Lemma 1.12 and Propositions 1.4, 1.5 that $L^p(a,b,X)$ are Banach spaces for $p \in [1,\infty[$. The completeness of $L^\infty(a,b;X)$ and $C([a,b];X)$ is obvious, indeed.

Let us mention the following classical results.

Proposition 1.13. (Lebesgue Dominated Convergence Theorem). *Let* $\{v_n;\, n \in \mathbb{N} \cup \{0\}\} \subset L^p(a,b;X), \{g_n;\, n \in \mathbb{N} \cup \{0\}\} \subset L^p(a,b;\mathbb{R}^1)$ *be given sequences for some* $p \in [1,\infty[$. *Let us assume*

(i) $\lim\limits_{n\to\infty} \int_a^b |g_n(t) - g_0(t)|^p dt = 0,$

(ii) $\lim\limits_{n\to\infty} |v_n(t) - v_0(t)|_X = 0 \quad$ *a.e.,*

(iii) $|v_n(t)|_X \leqslant g_n(t) \quad$ *a.e. for all* $\quad n \in \mathbb{N} \cup \{0\}$.

Then $\lim\limits_{n\to\infty} |v_n - v_0|_p = 0.$

Proof. Put $f_n(t) = |v_n(t) - v_0(t)|_X^p$ for all $t \in [a,b]$. We have $0 \leqslant |f_n(t)| \leqslant (g_n(t) + g_0(t))^p \leqslant 2^{p-1}\big(|g_n(t) - g_0(t)|^p + 2^p|g_0(t)|^p\big)$, hence $\{f_n\}$ is an equiintegrable sequence and we use Proposition 1.5. □

Proposition 1.14. (Mean Continuity Theorem). *For every* $p \in [1,\infty[$ *and* $u \in L^p(a,b;X)$ *we have*

$$\lim_{\delta\to 0+} \int_a^{b-\delta} \big|u(t) - u(t+\delta)\big|_X^p dt = 0. \tag{1.8}$$

Proof. Let $\varepsilon > 0$ be given. For $n \in \mathbb{N}$ put $E_n := \{t \in [a,b]; |u(t)|_X \geqslant n\}$. We find $\eta > 0$ such that $\int_A |u(t)|_X^p < \varepsilon^p$ for each set $A \subset [a,b]$ with meas$(A) < \eta$, and $n_0 \in \mathbb{N}$ such that meas$(E_{n_0}) < \eta$. Put $u_0(t) := u(t)(1 - \chi_{E_{n_0}}(t))$ for $t \in [a,b]$. Then $|u_0|_\infty \leqslant n_0$ and by Lusin's Theorem (Corollary 1.9) there exists a set M, meas$(M) < \left(\frac{\varepsilon}{n_0}\right)^p$ and a function $v \in C([a,b];X)$ such that $|v|_\infty \leqslant n_0$ and $v(t) = u_0(t)$ for all $t \in [a,b] \setminus M$.

We fix $\delta_0 > 0$ such that $\sup\{|v(t) - v(s)|_X^p;\, t,s \in [a,b], |t-s| < \delta_0\} < \frac{\varepsilon^p}{b-a}$. The triangle inequality in L^p for $\delta \in]0,\delta_0[$ then yields

$$\Big(\int_a^{b-\delta} |u_0(t) - u_0(t+\delta)|_X^p dt\Big)^{1/p} \leqslant \Big(\int_a^{b-\delta} |v(t) - v(t+\delta)|_X^p dt\Big)^{1/p} + \\ + 2\Big(\int_M |u_0(t) - v(t)|_X^p dt\Big)^{1/p} \leqslant \varepsilon + 4\varepsilon,$$

therefore

$$\Big(\int_a^{b-\delta} |u(t) - u(t+\delta)|_X^p dt\Big)^{1/p} < 7\varepsilon \quad \text{for} \quad \delta < \delta_0,$$

and (1.8) is proved. □

We now state the main result of this section.

Theorem 1.15. *Let* X *be a Hilbert space endowed with a scalar product* $\langle\cdot,\cdot\rangle$. *Let* $\{v_n; n \in \mathbb{N}\cup\{0\}\} \subset L^1(a,b;X), \{g_n; n \in \mathbb{N}\cup\{0\}\} \subset L^1(a,b;\mathbb{R}^1)$ *be given sequences such that*

(i) $\lim\limits_{n\to\infty} \int_a^b \langle v_n(t),\varphi(t)\rangle dt = \int_a^b \langle v_0(t),\varphi(t)\rangle dt \quad \forall \varphi \in C([a,b];X)$,

(ii) $\lim\limits_{n\to\infty} \int_a^b |g_n(t) - g_0(t)| dt = 0$,

(iii) $|v_n(t)|_X \leqslant g_n(t)$ *a.e.* $\forall n \in \mathbb{N}$,

(iv) $|v_0(t)|_X = g_0(t)$ *a.e.*

Then $\lim\limits_{n\to\infty} |v_n - v_0|_1 = 0$.

Notice that Theorem 1.15 does not follow from Proposition 1.13, since *we do not assume the pointwise convergence* here.

Proof of Theorem 1.15. We first prove that property (i) is satisfied for every $\varphi \in L^\infty(a,b;X)$. For a fixed $\varphi \in L^\infty(a,b;X)$ and $\delta > 0$ we use Lusin's Theorem to find a function $\psi \in C([a,b];X)$ and a set $M_\delta \subset [a,b]$ such that $\text{meas}(M_\delta) < \delta$ and $\psi(t) = \varphi(t)$ for all $t \in [a,b]\setminus M_\delta, |\psi|_\infty \leqslant |\varphi|_\infty$. We then have

$$\Big|\int_a^b \langle v_n(t) - v_0(t), \varphi(t)\rangle dt\Big| \leqslant \Big|\int_a^b \langle v_n(t) - v_0(t), \psi(t)\rangle\Big| + \\ + 2\|\varphi\|_\infty\Big(\int_a^b |g_n(t) - g_0(t)| dt + 2\int_{M_\delta} g_0(t) dt\Big),$$

and Proposition 1.3 entails

$$\lim_{n\to\infty}\int_a^b \langle v_n(t),\varphi(t)\rangle dt = \int_a^b \langle v_0(t),\varphi(t)\rangle dt \quad \forall \varphi \in L^\infty(a,b;X). \tag{1.9}$$

Let us note that the transition from (i) to (1.9) is related to the Dunford-Pettis Theorem, see Edwards (1965). To prove Theorem 1.15 we put for $t \in [a,b]$

$$\varphi(t) := \begin{cases} 0 & \text{if } v_0(t) = 0, \\ \frac{v_0(t)}{g_0(t)} & \text{if } v_0(t) \neq 0. \end{cases}$$

Then $\varphi \in L^\infty(a,b;X)$ and the inequality

$$|v_n(t) - v_0(t)|_X^2 \leqslant g_n^2(t) - 2\langle v_n(t), v_0(t)\rangle + g_0^2(t) = \\ = |g_n(t) - g_0(t)|^2 + 2g_0(t)\big[g_n(t) - g_0(t) + \langle v_0(t),\varphi(t)\rangle - \langle v_n(t),\varphi(t)\rangle\big]$$

holds for a.e. $t \in [a,b]$. By Hölder's inequality we have

$$\int_a^b |v_n(t) - v_0(t)|_X dt \leqslant \int_a^b |g_n(t) - g_0(t)|\, dt +$$
$$+ \Big(\int_a^b 2g_0(t)dt\Big)^{1/2} \Big(\int_a^b \big[g_n(t) - g_0(t) + \langle v_0(t), \varphi(t)\rangle - \langle v_n(t), \varphi(t)\rangle\big] dt\Big)^{1/2}$$

and the assertion follows from (1.9). □

Functions of bounded variation

Definition 1.16. *A partition* $S := \{\{t_0, \dots, t_N\};\ a = t_0 < t_1 < \dots < t_N = b\}$ *of the interval* $[a,b]$ *is said to be* δ-fine *for* $\delta > 0$, *if* $\max\{|t_i - t_{i-1}|;\ i = 1, \dots, N\} \leqslant \delta$. *We denote by* $\Delta_\delta(a,b)$ *the set of* δ*-fine partitions of the interval* $[a,b]$, $\Delta_0(a,b) := \bigcup_{\delta>0} \Delta_\delta(a,b)$.

Definition 1.17. *Let* $S = \{t_0, \dots, t_N\} \in \Delta_0(a,b)$ *and a function* $u : [a,b] \to X$ *be given. We define the* S-variation $\mathcal{V}_S(u)$ *of* u *and the total variation* $\operatorname*{Var}_{[a,b]} u$ *of* u *in* $[a,b]$ *by the formulae*

$$\mathop{\mathcal{V}}_S(u) := \sum_{i=1}^N |u(t_i) - u(t_{i-1})|_X,$$
$$\operatorname*{Var}_{[a,b]} u := \sup\{\mathop{\mathcal{V}}_S(u);\ S \in \Delta_0(a,b)\}.$$

We denote by $BV(a,b;X) := \{u : [a,b] \to X;\ \operatorname*{Var}_{[a,b]} u < \infty\}$ *the set of all functions of bounded total variation.*

The definition entails that every function $u \in BV(a,b;X)$ is bounded, the one-sided limits $u(t+)\,(u(t-))$ exist for all $t \in [a,b[$ $(t \in]a,b]$, respectively) and the set $\big\{t \in [a,b];\ u(t+) \neq u(t)$ or $u(t-) \neq u(t)\big\}$ of discontinuity points is at most countable.

An important example of functions of bounded variation are the *step functions*

$$\xi(t) := \sum_{j=1}^N x_j \chi_{]t_{j-1},t_j[}(t) + \sum_{j=0}^N y_j \chi_{\{t_j\}}(t) \tag{1.10}$$

as a special case of (1.1), where $S := \{t_0, \dots, t_N\} \subset \Delta_0(a,b)$ is a given partition and $\{x_j\}, \{y_j\} \subset X$ are given sequences.

The following statement shows that functions of bounded variation are strongly measurable and that $BV(a,b;X)$ endowed with the norm

$$|u|_{BV} := \sup\{|u(t)|_X;\, t \in [a,b]\} + \operatorname*{Var}_{[a,b]} u \tag{1.11}$$

is a Banach space.

Proposition 1.18.

(i) *For every* $u \in BV(a,b;X)$ *there exists a sequence* $\{\xi_n;\, n \in \mathbb{N}\}$ *of step functions such that* $\lim_{n\to\infty} \sup_{[a,b]} |u(t) - \xi_n(t)|_X = 0$, $\operatorname*{Var}_{[a,b]} \xi_n \leqslant \operatorname*{Var}_{[a,b]} u$.

(ii) *Let* $\{u_n; n \in \mathbb{N}\} \subset BV(a,b;X)$ *be a sequence and let* $u : [a,b] \to X$ *be a function such that* $\lim_{n\to\infty} |u_n(t) - u(t)|_X = 0$ *for all* $t \in [a,b]$. *Then* $\operatorname*{Var}_{[a,b]} u \leqslant \liminf_{n\to\infty} \operatorname*{Var}_{[a,b]} u_n$.

P r o o f. (i) The function $V(t) := \operatorname*{Var}_{[a,t]} u$ is nondecreasing in $[a,b]$. For $n \in \mathbb{N}$ put $N(n) := \max(\mathbb{N} \cap [0, nV(b)])$ and $t_j^n := \sup\{t \in [a,b];\, V(t) \leqslant \frac{j}{n}\}$ for $j = 1, \ldots, N(n)$, $t_{N(n)+1}^n := b$, $t_0^n := a$. The assertion holds for $\xi_n(t_j^n) := u(t_j^n)$, $\xi_n(t) := u\big(\frac{1}{2}(t_j^n + t_{j+1}^n)\big)$ for $t \in]t_j^n, t_{j+1}^n[$, $j = 0, \ldots, N(n)$, $\xi_n(b) := u(b)$, with the convention $]t_j^n, t_{j+1}^n[= \emptyset$ if $t_j^n = t_{j+1}^n$.

Part (ii) follows immediately from Definition 1.17. □

As a consequence of Proposition 1.18 we see that step functions form a dense subset of $BV(a,b;X)$ with respect to the so-called *strict metric* defined by the formula $d_s(u,v) = \sup\{|u(t) - v(t)|_X;\, t \in [a,b]\} + |\operatorname*{Var}_{[a,b]} u - \operatorname*{Var}_{[a,b]} v|$.

Let us pass to another important concept.

Definition 1.19. *A function* $u : [a,b] \to X$ *is called* absolutely continuous, *if for every* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that the implication*

$$\sum_{k=1}^{n} (b_k - a_k) < \delta \Rightarrow \sum_{k=1}^{n} \big|u(b_k) - u(a_k)\big|_X < \varepsilon$$

holds for every sequence of intervals $]a_k, b_k[\subset [a,b]$ *such that* $]a_k, b_k[\cap]a_j, b_j[= \emptyset$ *for* $k \neq j$.

Exercise 1.20. Prove the implication: u is absolutely continuous in $[a,b] \Rightarrow u \in C([a,b];X) \cap BV(a,b;X)$.

The following result is taken from Brézis (1973) (Proposition A.2). We cite it without proof.

Proposition 1.21. *Let* u *be absolutely continuous and put* $V(t) := \operatorname*{Var}_{[a,t]} u$ *for* $t \in [a,b]$. *Then* $V : [a,b] \to [0,\infty[$ *is nondecreasing, absolutely continuous and*

$$\dot V(t) = \lim_{h\to 0} \Big|\frac{1}{h}(u(t+h)-u(t))\Big|_X \quad \textit{for} \quad \text{a.e. } t \in]a,b[. \tag{1.12}$$

In general, the problem of differentiability of absolutely continuous vector-valued functions is nontrivial (see Brézis (1973)). For our purposes it is sufficient to consider a simpler case, namely

$$X \quad \textit{is a separable Hilbert space.} \tag{1.13}$$

Proposition 1.22. *Let (1.13) hold. Then for every absolutely continuous function* $u : [a,b] \to X$ *there exists an element* $\dot u \in L^1(a,b;X)$ *such that*

(i) $\dot u(t) = \lim_{h\to 0} \frac{1}{h}(u(t+h)-u(t))$ *a.e.,*

(ii) $u(t) - u(s) = \int_s^t \dot u(\tau)d\tau$ *for all* $a \leqslant s < t \leqslant b$.

P r o o f. Let $\langle \cdot,\cdot \rangle$ be a scalar product in X, let $\{e_k; k\in\mathbb{N}\}$ be an orthonormal basis in X and let $u : [a,b] \to X$ be an absolutely continuous function. Then the real-valued functions $v_k(t) := \langle u(t), e_k\rangle$ are absolutely continuous. There exists a set $M \subset [a,b]$ of measure zero such that for all $t \in]a,b[\setminus M$ identity (1.12) holds and the derivative $\dot v_k(t) = \frac{dv_k}{dt}(t)$ exists for all $k \in \mathbb{N}$.

Let $X_f \subset X$ be the space of finite linear combinations $x = \sum_{k=1}^N a_k e_k$, $\{a_1,\dots,a_N\} \subset \mathbb{R}^1$. For each $t \in]a,b[\setminus M$ we define a linear functional $\Phi_t : X_f \to \mathbb{R}^1$ by the formula

$$\Phi_t(x) := \sum_{k=1}^N a_k \dot v_k(t) = \lim_{h\to 0} \Big\langle x, \frac{1}{h}(u(t+h)-u(t))\Big\rangle \quad \text{for } x \in X_f,$$

and (1.12) entails

$$|\Phi_t(x)| \leqslant |x|_X \dot V(t) \tag{1.14}$$

for all $x \in X_f$ and $t \in]a,b[\setminus M$.

From the density of X_f in X we infer that the closure of Φ_t (still denoted by Φ_t) is a bounded linear functional on X and can be represented by an element $\dot u(t) \in X$ in the form

$$\Phi_t(x) = \langle x, \dot u(t)\rangle \quad \forall x \in X, \quad \forall t \in]a,b[\setminus M. \tag{1.15}$$

We have in particular $\dot v_k(t) = \langle e_k, \dot u(t)\rangle$ for all t and k and

$$\frac{1}{h}(u(t+h)-u(t)) \to \dot u(t) \quad \text{weakly in} \quad X \quad \text{as} \quad h \to 0. \tag{1.16}$$

From (1.12), (1.16) we obtain

$$|\dot{u}(t)|_X \leqslant \dot{V}(t) \qquad \forall t \in]a,b[\backslash M. \tag{1.17}$$

On the other hand, putting $u^{(N)}(t) := \sum_{k=1}^{N} v_k(t)e_k$ we have for all $t \in]a,b[\backslash M$

$$\lim_{N\to\infty} \left|\dot{u}^{(N)}(t) - \dot{u}(t)\right|_X = 0,$$
$$\lim_{h\to 0} \left|\dot{u}^{(N)}(t) - \frac{1}{h}(u^{(N)}(t+h) - u^{(N)}(t)\right|_X = 0, \quad \forall N \in \mathbb{N},$$

and the Diagonalization Principle (Proposition 1.6) entails that $\dot{u} : [a,b] \to X$ is strongly measurable. Using Bochner's Theorem and inequality (1.17) we check that $\dot{u} \in L^1(a,b;X)$.

For $t \in [a,b]$ put $v(t) := u(a) + \int_a^t \dot{u}(\tau)d\tau$. We have $\langle x, v(t)\rangle = \langle x, u(t)\rangle$ for all $x \in X_f$ and $t \in [a,b]$, hence (ii) holds. This implies in particular $V(t) - V(s) \leqslant \int_s^t |\dot{u}(\tau)|_X d\tau$, consequently $\dot{V}(t) \leqslant |\dot{u}(t)|_X$ a.e. It follows from (1.17), (1.12) that the convergence in (1.16) is strong and Proposition 1.22 is proved. □

Similarly as in the scalar-valued case we denote by $W^{1,1}(a,b;X)$ the space of absolutely continuous functions with values in a Hilbert space X and by $W^{1,p}(a,b;X)$ for $p \in]1,\infty]$ the space of all functions $u \in W^{1,1}(a,b;X)$ such that $\dot{u} \in L^p(a,b;X)$. The spaces $W^{1,p}$ are Banach spaces endowed with the norm $|u|_{1,p} := |u|_p + |\dot{u}|_p$.

Stieltjes integral

Let X be a separable Hilbert space with a scalar product $\langle \cdot,\cdot \rangle$. For arbitrary functions $u \in C([a,b];X)$ and $\xi \in BV(a,b;X)$ and for an arbitrary partition $S = \{t_0,\ldots,t_N\} \in \Delta_\delta(a,b)$ we define the Riemann-Stieltjes sum

$$I_S(u,\xi) := \sum_{k=1}^{N} \langle u(t_k), \xi(t_k) - \xi(t_{k-1})\rangle \tag{1.18}$$

with the intention to pass to the limit as $\delta \to 0$.

We denote by $\mu_u(\delta)$ the *modulus of continuity* of a function $u \in C([a,b];X)$, i.e.

$$\mu_u(\delta) := \sup\{|u(t) - u(s)|_X ; |t-s| \leqslant \delta\}. \tag{1.19}$$

Lemma 1.23. *Let $u \in C([a,b];X)$ and $\xi \in BV(a,b;X)$ be given. Then for every $\varepsilon > 0$ there exists $\delta > 0$ such that for arbitrary partitions $S, S' \in \Delta_\delta(a,b)$ we have*

$$|I_S(u,\xi) - I_{S'}(u,\xi)| < \varepsilon.$$

P r o o f. Let $\varepsilon > 0$ be given. We find $\delta > 0$ such that $\mu_u(\delta) \operatorname*{Var}_{[a,b]} \xi < \varepsilon$. For $S, S' \in \Delta_\delta(a,b)$, $S = \{t_0, \dots, t_N\}$, $S' = \{s_0, \dots, s_K\}$ we define $\hat{S} := S \cup S' = \{\tau_0, \dots \tau_L\} \in \Delta_\delta(a,b)$. For each $k = 0, \dots, L$ there exist $j_k \in \{1, \dots, N\}, i_k \in \{1, \dots, K\}$ such that $\tau_k \in]t_{j_{k-1}}, t_{j_k}] \cap]s_{i_{k-1}}, s_{i_k}]$ and we have

$$|I_S(u,\xi) - I_{S'}(u,\xi)| = \Big| \sum_{k=1}^{L} \big\langle u(t_{j_k}) - u(s_{i_k}), \xi(\tau_k) - \xi(\tau_{k-1}) \big\rangle \Big| \leqslant \mu_u(\delta) \operatorname*{Var}_{[a,b]} \xi < \varepsilon.$$

Lemma 1.23 is proved. □

Lemma 1.23 shows that the limit $\lim\limits_{\delta \to 0+} I_S(u,\xi)$ exists and is independent of the choice of $S \in \Delta_\delta(a,b)$. This limit is called the *Riemann-Stieltjes integral* and denoted by $\int_a^b \big\langle u(t), d\xi(t) \big\rangle$.

It is easy to see that the Riemann-Stieltjes integral is linear with respect to both u and ξ and that the estimate

$$\Big| \int_a^b \big\langle u(t), d\xi(t) \big\rangle \Big| \leqslant |u|_\infty \operatorname*{Var}_{[a,b]} \xi \tag{1.20}$$

holds for all $u \in C([a,b];X)$ and $\xi \in BV(a,b;X)$.

Exercise 1.24. Prove that for every $u \in C([a,b];X) \cap BV(a,b;X)$ we have

$$\int_a^b \big\langle u(t), du(t) \big\rangle = \frac{1}{2}(|u(b)|_X^2 - |u(a)|_X^2)! \tag{1.21}$$

H i n t. Use the identity $\big\langle u(t_k), u(t_k) - u(t_{k-1}) \big\rangle = \frac{1}{2}(|u(t_k) - u(t_{k-1})|_X^2 + |u(t_k)|_X^2 - |u(t_{k-1})|_X^2)$.

An immediate consequence of identity (1.21) is the integration-by-parts formula

$$\int_a^b \big\langle u(t), d\xi(t) \big\rangle + \int_a^b \big\langle \xi(t), du(t) \big\rangle = \big\langle u(b), \xi(b) \big\rangle - \big\langle u(a), \xi(a) \big\rangle \tag{1.22}$$

for every $u, \xi \in C([a,b];X) \cap BV(a,b;X)$.

The relation between Riemann-Stieltjes and Lebesgue integrals can be expressed in the following way.

Lemma 1.25. *For all $u \in C([a,b];X)$ and $\xi \in W^{1,1}(a,b;X)$ we have*

$$\int_a^b \big\langle u(t), d\xi(t) \big\rangle = \int_a^b \big\langle u(t), \dot{\xi}(t) \big\rangle dt. \tag{1.23}$$

P r o o f. For an arbitrary partition $S = \{t_0, \dots, t_N\} \in \Delta_\delta(a,b)$ we have by Proposition 1.22(ii) $\big| \int_a^b \langle u(t), \dot\xi(t)\rangle dt - I_S(u,\xi)\big| = \big| \sum_{k=1}^N \int_{t_{k-1}}^{t_k} \langle u(t) - u(t_k), \dot\xi(t)\rangle dt\big| \leqslant \mu_u(\delta) \int_a^b |\dot\xi(t)|_X dt$, so (1.23) holds. □

We can derive useful integration formulas in the case that ξ is a step function of the form (1.10). For an arbitrary $u \in C([a,b];X)$ we then have

$$\int_a^b \big\langle u(t), d\xi(t)\big\rangle = \sum_{j=1}^{N-1} \big\langle u(t_j), x_{j+1} - x_j\big\rangle + \big\langle u(a), x_1 - y_0\big\rangle + \big\langle u(b), y_N - x_N\big\rangle. \tag{1.24}$$

If moreover $u \in W^{1,1}(a,b;X)$, then

$$\int_a^b \big\langle u(t), d\xi(t)\big\rangle = \big\langle u(b), y_N\big\rangle - \big\langle u(a), y_0\big\rangle - \int_a^b \big\langle \xi(t), \dot u(t)\big\rangle dt. \tag{1.25}$$

Notice that the integrals (1.24), (1.25) are independent of the values of $y_1, \dots, y_{N-1}$!

The Riemann-Stieltjes integral depends continuously on the functions u and ξ in the following sense.

Theorem 1.26. *Let $\{u_n; n \in \mathbb{N}\} \subset C([a,b];X), \{\xi_n; n \in \mathbb{N}\} \subset BV(a,b;X)$ be given sequences and $u, \xi : [a,b] \to X$ be given functions such that*

(i) $\lim_{n\to\infty} |u_n - u|_\infty = 0,$

(ii) $\lim_{n\to\infty} |\xi_n(t) - \xi(t)|_X = 0$ *for all* $t \in [a,b]$,

(iii) $\operatorname*{Var}_{[a,b]} \xi_n(t) \leqslant c$, *where* $c > 0$ *is a constant independent of* n.

Then $\lim_{n\to\infty} \int_a^b \big\langle u_n(t), d\xi_n(t)\big\rangle = \int_a^b \big\langle u(t), d\xi(t)\big\rangle$.

Notice that the integral $\int_a^b \big\langle u(t), d\xi(t)\big\rangle$ is meaningful by Proposition 1.18. The proof of Theorem 1.26 relies on three Lemmas.

Lemma 1.27. *For every $u \in W^{1,1}(a,b;X)$ and every sequence $\{\xi_n; n \in \mathbb{N}\}$ of step functions such that $|\xi_n|_\infty \leqslant c$ and $\lim_{n\to\infty} \xi_n(t) = 0$ for all $t \in [a,b]$ we have*

$$\lim_{n\to\infty} \int_a^b \big\langle u(t), d\xi_n(t)\big\rangle = 0. \tag{1.26}$$

P r o o f. It suffices to use formula (1.25) and Proposition 1.13 for $X = \mathbb{R}^1$ and $p = 1$. □

Lemma 1.28. *For every* $u \in W^{1,1}(a,b;X)$ *and every sequence* $\{\xi_n; n \in \mathbb{N}\} \subset BV(a,b;X)$ *such that* $|\xi_n|_\infty \leqslant c$ *and* $\lim_{n\to\infty} \xi_n(t) = 0$ *for all* $t \in [a,b]$ *identity (1.26) holds.*

P r o o f. For every $n \in \mathbb{N}$ we find a step function $\hat{\xi}_n$ such that $|\xi_n - \hat{\xi}_n|_\infty < \frac{1}{n}$. For every partition $S = \{t_0, \ldots, t_N\} \subset \Delta_\delta(a,b)$ we have

$$\sum_{k=1}^{N} \big\langle u(t_k), \xi_n(t_k) - \xi_n(t_{k-1})\big\rangle = \sum_{k=1}^{N} \big\langle u(t_k), \hat{\xi}_n(t_k) - \hat{\xi}_n(t_{k-1})\big\rangle + \big\langle u(b), \xi_n(b) - \hat{\xi}_n(b)\big\rangle -$$

$$- \big\langle u(a), \xi_n(a) - \hat{\xi}_n(a)\big\rangle - \sum_{k=0}^{N-1} \big\langle u(t_{k+1}) - u(t_k), \xi_n(t_k) - \hat{\xi}_n(t_k)\big\rangle.$$

Passing to the limit as $\delta \to 0+$ we obtain

$$\Big| \int_a^b \big\langle u(t), d\xi_n(t)\big\rangle \Big| \leqslant \Big| \int_a^b \big\langle u(t), d\hat{\xi}_n(t)\big\rangle \Big| + \frac{1}{n}(2|u|_\infty + |\dot{u}|_1)$$

and the assertion follows from Lemma 1.27. □

Lemma 1.29. *For every* $u \in C([a,b];X)$ *and every sequence* $\{\xi_n; n \in \mathbb{N}\} \subset BV(a,b;X)$ *such that* $\operatorname*{Var}_{[a,b]} \xi_n \leqslant c$ *and* $\lim_{n\to\infty} \xi_n(t) = 0$ *for all* $t \in [a,b]$ *identity (1.26) holds.*

P r o o f. Let $\varepsilon > 0$ be given. We choose $\hat{u} \in W^{1,1}(a,b;X)$ such that $|u - \hat{u}|_\infty < \frac{\varepsilon}{2c}$ (for instance $\hat{u}$ piecewise linear). By Lemma 1.28 there exists n_0 such that for $n \geqslant n_0$ we have $\int_a^b \big\langle \hat{u}(t), d\xi_n(t)\big\rangle < \frac{\varepsilon}{2}$. From inequality (1.20) we obtain

$$\Big| \int_a^b \big\langle u(t), d\xi_n(t)\big\rangle \Big| \leqslant \Big| \int_a^b \big\langle u(t) - \hat{u}(t), d\xi_n(t)\big\rangle \Big| + \Big| \int_a^b \big\langle \hat{u}(t), d\xi_n(t)\big\rangle \Big| < \varepsilon,$$

hence (1.26) holds. □

We now can finish the proof of Theorem 1.26.

P r o o f of Theorem 1.26. Using the inequality

$$\Big| \int_a^b \big\langle u_n(t), d\xi_n(t)\big\rangle - \int_a^b \big\langle u(t), d\xi(t)\big\rangle \Big| \leqslant$$

$$\leqslant \Big| \int_a^b \big\langle u_n(t) - u(t), d\xi_n(t)\big\rangle \Big| + \Big| \int_a^b \big\langle u(t), d(\xi_n - \xi)(t)\big\rangle \Big|$$

we obtain the assertion from inequality (1.20) and Lemma 1.29. □

To conclude this section, we prove another important theorem.

Theorem 1.30. *For every* $\xi \in C([a,b];X) \cap BV(a,b;X)$ *put*

$$M(\xi) := \sup\Big\{ \int_a^b \big\langle u(t), d\xi(t)\big\rangle;\ u \in C([a,b];X), |u|_\infty \leqslant 1\Big\}.$$

Then $M(\xi) = \operatorname*{Var}_{[a,b]} \xi$.

Indeed, Theorem 1.30 does not hold for arbitrary $\xi \in BV(a,b;X)$. Easy counterexamples can be found in the class of step functions according to formula (1.24).

P r o o f of Theorem 1.30. Let $a = t_0 < t_1 < \ldots < t_N = b$ be an arbitrary partition. For $k = 1, \ldots, N$ put

$$v_k := \begin{cases} 0 & \text{if} \quad \xi(t_k) = \xi(t_{k-1}), \\ \frac{\xi(t_k) - \xi(t_{k-1})}{|\xi(t_k) - \xi(t_{k-1})|_X} & \text{if} \quad \xi(t_k) \neq \xi(t_{k-1}). \end{cases}$$

For $0 < \varepsilon < \frac{1}{2}\min\{t_k - t_{k-1}; k = 1, \ldots, N\}$ we define a function $u \in C([a,b];X)$ by the formula

$$u(t) := \begin{cases} v_1\ , & t \in [a, t_1 - \varepsilon[, \\ v_N, & t \in\ ]t_{N-1} + \varepsilon, b], \\ v_k\ , & t \in\ ]t_{k-1} + \varepsilon, t_k - \varepsilon[, \quad k = 2, \ldots, N-1, \\ \text{linear in } [t_k - \varepsilon, t_k + \varepsilon], & k = 1, \ldots, N-1. \end{cases}$$

Using formulas (1.22), (1.23) we obtain

$$\int_a^b \big\langle u(t), d\xi(t)\big\rangle = \sum_{k=1}^N \big|\xi(t_k) - \xi(t_{k-1})\big|_X + \\ + \frac{1}{2\varepsilon}\sum_{k=1}^{N-1} \int_{t_k-\varepsilon}^{t_k+\varepsilon} \big\langle v_{k+1} - v_k, \xi(t_k) - \xi(t)\big\rangle dt. \tag{1.27}$$

We obviously have $|u|_\infty \leqslant 1$, hence (1.27) yields

$$\sum_{k=1}^N \big|\xi(t_k) - \xi(t_{k-1})\big|_X \leqslant M(\xi) + \mu_\xi(\varepsilon) \sum_{k=1}^{N-1} \big|v_{k+1} - v_k\big|_X,$$

where μ_ξ is the modulus of continuity (1.19). Letting $\varepsilon \to 0+$ we obtain $\sum_{k=1}^N |\xi(t_k) - \xi(t_{k-1})|_X \leqslant M(\xi)$, hence $\operatorname*{Var}_{[a,b]} \xi \leqslant M(\xi)$. From (1.20) it follows $M(\xi) \leqslant \operatorname*{Var}_{[a,b]} \xi$ and the proof is complete. □

Remark 1.31. Theorem 1.26 and Proposition 1.18(i) immediately imply that formula (1.25) holds for arbitrary $\xi \in BV(a,b;X)$ and $u \in W^{1,1}(a,b;X)$ with $\xi(a) = y_0$, $\xi(b) = y_N$.

V.2 Embedding theorems

Classical monographs Adams (1975), Kufner, John, Fučík (1977), Besov, Il'in, Nikol'skii (1975) on Sobolev spaces and their embeddings deal mainly with isotropic spaces, where the derivatives with respect to different variables belong to the same L^p-space. The anisotropy in Besov, Il'in, Nikol'skii (1975) concerns merely the L^p-space itself. However, hysteresis operators which occur in partial differential equations produce in a natural way functions which behave differently with respect to the time and space variables. For the readers's convenience we prove by classical methods of Adams (1975) or Kufner, John, Fučík (1977) the simple Theorem 2.4 below which is extensively used in Chap. III and does not immediately follow from well-known embedding formulas.

Let us first recall the following classical result the proof of which is elementary and can be found e.g. in Yosida (1965).

Theorem 2.1 (Arzelà - Ascoli). *Let X, Y be compact metric spaces endowed with metrics d_X, d_Y, respectively. Let $C(X;Y)$ be the space of continuous functions $f : X \to Y$ endowed with the metric $d_c(f_1, f_2) := \max\{d_Y\big(f_1(x), f_2(x)\big); x \in X\}$. Then a subset $A \subset C(X, Y)$ is relatively compact if and only if it is* equicontinuous, *i.e.*

(2.1) $$\forall \varepsilon > 0\ \exists \delta > 0\ \forall f \in A\ \forall x_1, x_2 \in X\ : d_X(x_1, x_2) < \delta \Rightarrow d_Y\big(f(x_1), f(x_2)\big) < \varepsilon.$$

Definition 2.2. *Let X, Y be Banach spaces endowed with norms $|\cdot|_X, |\cdot|_Y$, respectively. We say that*

(i) *Y is* embedded *in X and denote $Y \hookrightarrow X$ if $Y \subset X$ and*

(2.2) $$\exists c > 0\ \forall y \in Y : |y|_X \leqslant c|y|_Y;$$

(ii) *Y is* compactly embedded *in X and denote $Y \hookrightarrow\hookrightarrow X$ if $Y \hookrightarrow X$ and every bounded set in Y is relatively compact in X.*

For the sake of completeness we mention Sobolev Embedding Theorems for isotropic spaces in the following classical form (see any of the monographs cited above).

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^N$ be a bounded open domain with a Lipschitzian boundary. Then for $N = 1$ we have*

(2.3) $$\text{(i)}\quad W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C(\bar{\Omega}) \quad \textit{for} \quad 1 < p \leqslant \infty,$$
$$\text{(ii)}\quad W^{1,1}(\Omega) \hookrightarrow C(\bar{\Omega}),\ W^{1,1}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \quad \textit{for} \quad 1 \leqslant q < \infty.$$

For $N > 1$ *and* $1 \leqslant p \leqslant \infty$ *put* $s := \frac{1}{p} - \frac{1}{N}$ *with the convention* $\frac{1}{\infty} = 0$. *Then*

$$\text{(2.4)} \quad \text{(i)} \quad s < 0 \Rightarrow W^{1,p}(\Omega) \hookrightarrow\hookrightarrow C(\bar{\Omega}),$$

$$\text{(ii)} \quad s = 0 \Rightarrow W^{1,p}(\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \quad \textit{for} \quad 1 \leqslant q < \infty,$$

$$\text{(iii)} \quad s > 0 \Rightarrow W^{1,p}(\Omega) \hookrightarrow L^{1/s}(\Omega), W^{1,p}((\Omega) \hookrightarrow\hookrightarrow L^q(\Omega) \quad \textit{for} \quad 1 \leqslant q < \frac{1}{s}.$$

In Chap. III we deal with anisotropic function spaces of the type $L^p\big(0,T;L^q(\Omega)\big)$, $L^q\big(\Omega;L^p(0,T)\big), L^q\big(\Omega;C([0,T])\big)$ for $1 \leqslant p,q \leqslant \infty$, where $\Omega \subset \mathbb{R}^N$ is a regular open bounded set and $]0,T[$ is a time interval. According to the general theory of Besov, Il'in, Nikol'skii (1975), these spaces are Banach spaces endowed with norms of the form (1.7). Another kind of anisotropy is related to functions $u \in L^1(\Omega\times]0,T[)$ whose generalized partial derivatives $\partial_0 u := \frac{\partial u}{\partial t}$, $\partial_i u := \frac{\partial u}{\partial x_i}$, $i = 1,\dots,N$ belong to the spaces $L^{p_i}\big(0,T;L^{q_i}(\Omega)\big)$ with $1 \leqslant p_i, q_i \leqslant \infty$, $i = 0,\dots,N$ endowed with norms $|\cdot|_{(p_i,q_i)}$ given by (1.7). We denote such a space by $W^{1,\mathbf{p}}(0,T;\Omega)$, where $\mathbf{p}$ is the multiindex $\{(p_0,q_0),\dots,(p_N,q_N)\}$ and easily check that it is reflexive if $1 < p_i < \infty, 1 < q_i < \infty$ for all $i = 0,\dots,N$. We similarly treat the spaces with $\partial_i u \in L^{q_i}\big(\Omega;L^{p_i}(0,T)\big)$.

We do not give an exhaustive list of embedding formulas for all possible combinations of multiindices. Instead, we present a detailed proof of one typical anisotropic embedding theorem which is used several times in Chap. III.

Theorem 2.4. *Let* $[a,b] \subset \mathbb{R}^1$ *be a compact interval and let* $T > 0$ *and a multiindex* $\mathbf{p} = \big((p_0,q_0),(p_1,q_1)\big)$ *be given,* $1 \leqslant p_0,q_0,p_1,q_1 \leqslant \infty$. *Put*

$$\alpha := 1 - \tfrac{1}{q_1} + \tfrac{1}{q_0},\ \beta := 1 - \tfrac{1}{p_0} + \tfrac{1}{p_1},\ \kappa := \big(1 - \tfrac{1}{p_0}\big)\big(1 - \tfrac{1}{q_1}\big) - \tfrac{1}{p_1 q_0}$$

with the convention $\frac{1}{\infty} = 0$ *and assume* $\kappa > 0$. *Then for every* $u \in W^{1,\mathbf{p}}(0,T;]a,b[)$ *and every* $(x,t),(y,s) \in]a,b[\times]0,T[$ *such that*

$$\max\{|t-s|^{1/\alpha}, |x-y|^{1/\beta}\} \leqslant \min\{T^{1/\alpha}, (b-a)^{1/\beta}\} \tag{2.5}$$

we have

$$|u(x,t) - u(y,s)| \leqslant \frac{2}{\kappa}(\alpha|u_t|_{(p_0,q_0)} + \beta|u_x|_{(p_1,q_1)}) \max\{|t-s|^{\frac{\kappa}{\alpha}}, |x-y|^{\frac{\kappa}{\beta}}\}. \tag{2.6}$$

P r o o f. We follow the strategy of Adams (1975) or Kufner, John, Fučík (1977). Assume first

$$1 \leqslant p_0,q_0,p_1,q_1 < \infty. \tag{2.7}$$

Then the space $C^1([a,b]\times[0,T])$ is dense in $W^{1,\mathbf{p}}(0,T;]a,b[)$ and it suffices to assume $u\in C^1([a,b]\times[0,T])$.

Let $(x,t),(y,s)$ be two distinct points of $]a,b[\times]0,T[$. Put

$$\eta := \max\{|t-s|^{\frac{1}{\alpha}}, |x-y|^{\frac{1}{\beta}}\}. \tag{2.8}$$

Let $Q := [x_1,x_2]\times[t_1,t_2]\subset]a,b[\times]0,T[$ be a rectangle such that

$$\begin{aligned} &\text{(i)} && (x,t),(y,s)\in\partial Q,\\ &\text{(ii)} && t_2-t_1=\eta^\alpha,\ x_2-x_1=\eta^\beta. \end{aligned} \tag{2.9}$$

We choose arbitrarily $(\xi,\tau)\in Q$ and for $\sigma\in[0,1]$ put $\varphi(\sigma):=u\big(x+\sigma^\beta(\xi-x), t+\sigma^\alpha(\tau-t)\big)$. We have

$$\begin{aligned} u(\xi,\tau)-u(x,t) = \int_0^1 &\big[\beta\sigma^{\beta-1}(\xi-x)u_x\big(x+\sigma^\beta(\xi-x),t+\sigma^\alpha(\tau-t)\big)+\\ &+\alpha\sigma^{\alpha-1}(\tau-t)u_t\big(x+\sigma^\beta(\xi-x),t+\sigma^\alpha(\tau-t)\big)\big]\,d\sigma. \end{aligned} \tag{2.10}$$

and integrating identity (2.10) with respect to (ξ,τ) we obtain

$$\begin{aligned} &\Big|\iint_Q u(\xi,\tau)d\xi\,d\tau-\eta^{\alpha+\beta}u(x,t)\Big|\leqslant\\ &\leqslant \beta\eta^\beta\int_0^1\sigma^{\beta-1}\iint_Q\big|u_x\big(x+\sigma^\beta(\xi-x),t+\sigma^\alpha(\tau-t)\big)\big|d\xi\,d\tau\,d\sigma+\\ &+\alpha\eta^\alpha\int_0^1\sigma^{\alpha-1}\iint_Q\big|u_t\big(x+\sigma^\beta(\xi-x),t+\sigma^\alpha(\tau-t)\big)\big|d\xi\,d\tau\,d\sigma. \end{aligned} \tag{2.11}$$

The substitution $\xi\mapsto\varphi=x+\sigma^\beta(\xi-x)$, $\tau\mapsto\psi=t+\sigma^\alpha(\tau-t)$ and Hölder's inequality yield

$$\begin{aligned} &\iint_Q\big|u_x\big(x+\sigma^\beta(\xi-x),t+\sigma^\alpha(\tau-t)\big)\big|d\xi\,d\tau=\\ &=\sigma^{-\alpha-\beta}\int_{t+\sigma^\alpha(t_1-t)}^{t+\sigma^\alpha(t_2-t)}\int_{x+\sigma^\beta(x_1-x)}^{x+\sigma^\beta(x_2-x)}\big|u_x(\varphi,\psi)\big|d\varphi\,d\psi\\ &\leqslant \big|u_x\big|_{(p_1,q_1)}\sigma^{-\frac{\alpha}{p_1}-\frac{\beta}{q_1}}\,\eta^{\alpha(1-\frac{1}{p_1})+\beta(1-\frac{1}{q_1})} \end{aligned} \tag{2.12}$$

and similarly

$$\begin{aligned} &\iint_Q\big|u_t\big(x+\sigma^\beta(\xi-x),t+\sigma^\alpha(\tau-t)\big)\big|d\xi\,d\tau\leqslant\\ &\leqslant\big|u_t\big|_{(p_0,q_0)}\sigma^{-\frac{\alpha}{p_0}-\frac{\beta}{q_0}}\,\eta^{\alpha(1-\frac{1}{p_0})+\beta(1-\frac{1}{q_0})}. \end{aligned} \tag{2.13}$$

We thus obtain from (2.11)

$$(2.14) \qquad \Big| \iint_Q u(\xi,\tau) d\xi\, d\tau - \eta^{\alpha+\beta} u(x,t) \Big| \leqslant$$

$$\leqslant \eta^{\alpha+\beta+\kappa} \big(\beta |u_x|_{(p_1,q_1)} + \alpha |u_t|_{(p_0,q_0)}\big) \int_0^1 \sigma^{\kappa-1} d\sigma$$

The same inequality (2.14) holds for $\Big| \iint_Q u(\xi,\tau) d\xi\, d\tau - \eta^{\alpha+\beta} u(y,s) \Big|$ and we conclude

$$(2.15) \qquad |u(x,t) - u(y,s)| \leqslant \frac{2}{\kappa} \eta^{\kappa} \big(\beta |u_x|_{(p_1,q_1)} + \alpha |u_t|_{(p_0,q_0)}\big)$$

and (2.6) follows.

Let now $u \in W^{1,\mathbf{P}}(0,T;]a,b[)$ be arbitrary and let $\{u^{(n)}; n \in \mathbb{N}\} \subset C^1([a,b] \times [0,T])$ be a sequence such that $u^{(n)} \to u$ in $W^{1,\mathbf{P}}(0,T;]a,b[)$. By inequality (2.6) for $u^{(n)}$ and Arzelà - Ascoli Theorem 2.1 the sequence $\{u^{(n)}\}$ converges uniformly to u and we can pass to the limit in (2.6).

To complete the proof we have to remove assumption (2.7). Let p_0, q_0, p_1, q_1 be arbitrary. We construct sequences $\{p_0^{(n)}, q_0^{(n)}, p_1^{(n)}, q_1^{(n)};\ n \in \mathbb{N}\}$ satisfying assumption (2.7) such that $\alpha = 1 - \frac{1}{q_1^{(n)}} + \frac{1}{q_0^{(n)}}$, $\beta = 1 - \frac{1}{p_0^{(n)}} + \frac{1}{p_1^{(n)}}$, $\kappa_n := \big(1 - \frac{1}{p_0^{(n)}}\big)\big(1 - \frac{1}{q_1^{(n)}}\big) - \frac{1}{p_1^{(n)} q_0^{(n)}} > 0$, $p_i^{(n)} \nearrow p_i$, $q_i^{(n)} \nearrow q_i$ as $n \to \infty$, $i = 0,1$. Every $u \in W^{1,\mathbf{P}}(0,T;]a,b[)$ satisfies $|u(x,t) - u(y,s)| \leqslant \frac{2}{\kappa_n} \big(\alpha |u_t|_{(p_0^{(n)},q_0^{(n)})} + \beta |u_x|_{(p_1^{(n)},q_1^{(n)})}\big) \max\big\{|t-s|^{\frac{\kappa_n}{\alpha}}, |x-y|^{\frac{\kappa_n}{\beta}}\big\}$ and passing to the limit we obtain the assertion. $\square$

Combining Arzelà - Ascoli Theorem 2.1 and Theorem 2.4 we obtain the following embedding result.

Corollary 2.5. *Under the hypotheses of Theorem 2.4 the compact embedding* $W^{1,\mathbf{P}}(0,T;]a,b[) \hookrightarrow\hookrightarrow C([a,b] \times [0,T])$ *holds.*

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