Robust error bounds for finite element approximation of reaction-diffusion problems

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A posteriori error estimates

Error

$$e = u - u_h$$

Upper bound (reliability)

$$\|e\| \leq \eta$$

Local efficiency

$$C\eta_K \leq \|e\|_K$$

Reaction-diffusion problem

$$-\Delta u + \kappa^2 u = f \quad \text{in } \Omega \subset \mathbb{R}^d$$
$$\frac{\partial u}{\partial \mathbf{n}} = g_N \quad \text{on } \Gamma_\mathrm{N}$$
$$u = 0 \quad \text{on } \Gamma_\mathrm{D}$$

- $\kappa = \kappa(x) \geq 0$ piecewise constant
- Arbitrary dimension
- Mixed boundary conditions
- Robust upper bound on error
- Explicit bounds on trace constants

[M. Ainsworth, T. Vejchodský, Comput. Methods Appl. Mech. Engrg. 281, 2014, 184–199.]

Model problem

Weak formulation:

$$u \in V : \quad \mathcal{B}(u, v) = (f, v)_\Omega \quad \forall v \in V$$

Linear FEM on *d*-dimensional simplices:

$$u_h \in V_h : \quad \mathcal{B}(u_h, v_h) = (f, v_h)_\Omega \quad \forall v_h \in V_h$$

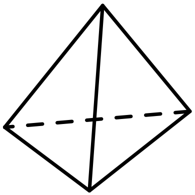
Notation:

$V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_\mathrm{D}\}$

$\mathcal{B}(u, v) = (\nabla u, \nabla v)_\Omega + (\kappa^2 u, v)_\Omega$

$(f, v)_\Omega = \int_\Omega fv \, \mathrm{d}x$

$V_h = \{v_h \in V : v_h|_K \in P^1(K), K \in \mathcal{T}_h\}$



Main result

Upper bound:

$$\|u - u_h\|^2 \leq \sum_{K \in \mathcal{T}_h} \left[\eta_K(\mathbf{y}) + \mathrm{osc}_K(f) + \mathrm{osc}_{\partial K \cap \Gamma_{\mathrm{N}}}(g_N)\right]^2$$

$$\forall \mathbf{y} \in \mathbf{H}(\mathrm{div}, \Omega)\text{:}\;\; \mathbf{y} \cdot \mathbf{n} = \Pi_\gamma^K g_N \text{ on all } \gamma \subset \partial K \cap \Gamma_{\mathrm{N}}$$
$$\mathrm{div}\,\mathbf{y} = -\Pi_K f \text{ on } K \text{ where } \kappa_K = 0$$

- $\eta_K^2(\mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_{0,K}^2 + \kappa_K^{-2}\|\Pi_K f - \kappa_K^2 u_h + \mathrm{div}\,\mathbf{y}\|_{0,K}^2$
- $\mathrm{osc}_K(f) = \min\left\{h_K/\pi, \kappa_K^{-1}\right\}\|f - \Pi_K f\|_{0,K}$
- $\mathrm{osc}_{\partial K \cap \Gamma_{\mathrm{N}}}(g_N) = \min\{C_{\mathrm{T}}, \overline{C}_{\mathrm{T}}\}\|g_N - \Pi_\gamma^K g_N\|_{0,\partial K \cap \Gamma_{\mathrm{N}}}$

- $\Pi_K f \in P^1(K)$: $\quad (f - \Pi_K f, \varphi)_K = 0 \quad \forall \varphi \in P^1(K)$
- $\Pi_\gamma^K g_N \in P^1(\gamma)$: $\quad (f - \Pi_\gamma^K g_N, \varphi)_\gamma = 0 \quad \forall \varphi \in P^1(\gamma)$

Main result

Local efficiency:

$$C\eta_K(\mathbf{y}) \le |||u - u_h|||_{\widetilde{K}} + \min\{h_K, \kappa_K^{-1}\}\|f - \Pi f\|_{0,\widetilde{K}}$$
$$+ \min\{h_K, \kappa_K^{-1}\}^{1/2}\|g_N - \Pi_\gamma^K g_N\|_{0,\partial K\cap\Gamma_{\mathrm{N}}} \text{ for a special } \mathbf{y}$$

- $\eta_K^2(\mathbf{y}) = \|\mathbf{y} - \nabla u_h\|_{0,K}^2 + \kappa_K^{-2}\|\Pi_K f - \kappa_K^2 u_h + \operatorname{div}\mathbf{y}\|_{0,K}^2$
- $\operatorname{osc}_K(f) = \min\left\{h_K/\pi, \kappa_K^{-1}\right\}\|f - \Pi_K f\|_{0,K}$
- $\operatorname{osc}_{\partial K\cap\Gamma_{\mathrm{N}}}(g_N) = \min\{C_{\mathrm{T}}, \overline{C}_{\mathrm{T}}\}\|g_N - \Pi_\gamma^K g_N\|_{0,\partial K\cap\Gamma_{\mathrm{N}}}$

- $\Pi_K f \in P^1(K): \quad (f - \Pi_K f, \varphi)_K = 0 \quad \forall\varphi\in P^1(K)$
- $\Pi_\gamma^K g_N \in P^1(\gamma): \quad (f - \Pi_\gamma^K g_N, \varphi)_\gamma = 0 \quad \forall\varphi\in P^1(\gamma)$

Flux reconstruction

- Compute robust inter-element fluxes g_K
  - $g_K \approx \nabla u \cdot \mathbf{n}_K$ on ∂K

  [Ainsworth, Babuška, 1999], [Ainsworth, Vejchodský, 2011]
- Flux #1: for all elements K with $\kappa_K \rho_K \leq 1$ construct $\mathbf{y}_K^{(1)}$:
  - $\mathbf{y}_K^{(1)} \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K$
  - $\Pi_K f - \kappa_K^2 u_h + \operatorname{div} \mathbf{y}_K^{(1)} = 0$
- Flux #2: for all elements K with $\kappa_K \rho_K > 1$ construct $\mathbf{y}_K^{(2)}$:
  - $\mathbf{y}_K^{(2)} \cdot \mathbf{n}_K = g_K \quad \text{on } \partial K$
  - Correct asymptotic behavior w.r.t. h and κ_K
- $\mathbf{y}|_K = \begin{cases} \mathbf{y}_K^{(1)} & \text{if } \kappa_K \rho_K \leq 1, \\ \mathbf{y}_K^{(2)} & \text{if } \kappa_K \rho_K > 1, \end{cases}$

Flux reconstruction #1

Definition:

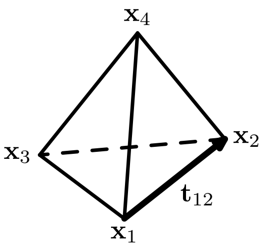
$$\mathbf{y}_K^{(1)} = \nabla u_h + \mathbf{y}_K^L + \mathbf{y}_K^Q$$

$$\mathbf{y}_K^L = -\sum_{n=1}^{d+1} \lambda_n \sum_{\substack{m=1\\ m\neq n}}^{d+1} R_{|\gamma_m}(\mathbf{x}_n)\, |\nabla \lambda_m|\, \mathbf{t}_{nm}$$

$$\mathbf{y}_K^Q = \frac{1}{d+1} \sum_{n=1}^{d+1} \sum_{\substack{m=2\\ m>n}}^{d+1} \lambda_m \lambda_n \mathbf{t}_{mn} \mathbf{t}_{mn}^T \nabla r(\bar{\mathbf{x}}_K)$$

Notation:

$R = g_K - \nabla u_h \cdot \mathbf{n}_K$
$r = \Pi_K f - \kappa_K^2 u_h$
λ_n ... barycentric coords in K
$\mathbf{t}_{mn} = \mathbf{x}_m - \mathbf{x}_n$ ... edge vector
$\bar{\mathbf{x}}_K$ ... barycentre of K



Flux reconstruction #2

Definition:

$$\mathbf{y}_K^{(2)} = \nabla u_h + \mathbf{y}_K^O$$

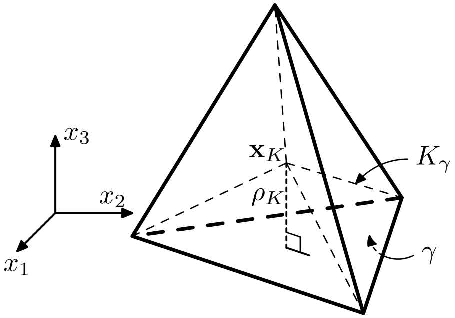
$$\mathbf{y}_K^O|_{K_\gamma} = \frac{1}{\rho_K} \mathrm{e}^{-\kappa_K x_d} (\mathbf{x} - \mathbf{x}_K) R(x_1, \dots, x_{d-1}) \quad \text{for all } \gamma \subset \partial K$$

Notation:

$R = g_K - \nabla u_h \cdot \mathbf{n}_K$

$\mathbf{x}_K$ ... incentre of K

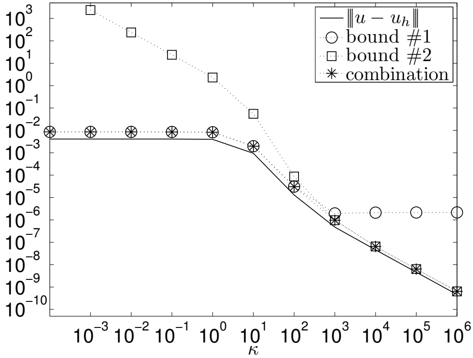
ρ_K ... inradius of K



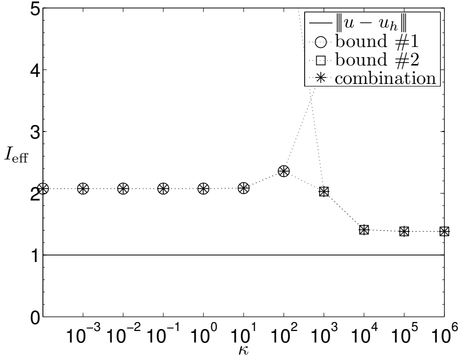
Example: $\kappa = \text{const.}$

$$\Omega = (-1,1)^3,\ \Gamma_D = \partial\Omega$$

Error estimators



Effectivity indices



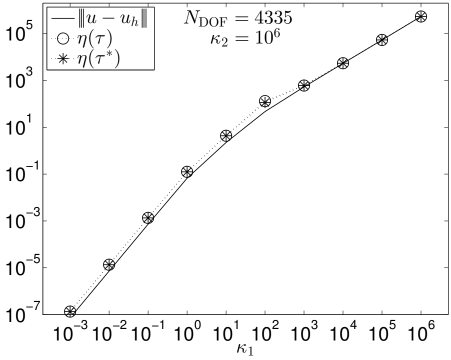
$N_{\text{DOF}} = 29791 \qquad h = 0.03125$

$$I_{\text{eff}} = \frac{\eta}{\|u - u_h\|}$$

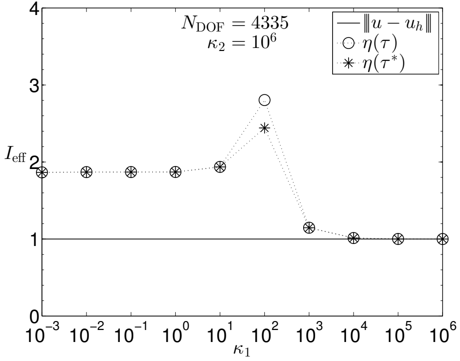
Example: piecewise constant κ

$$\Omega = (-1,1)^3,\ \Gamma_D = \partial\Omega, \qquad \kappa = \begin{cases} \kappa_1 & \text{if } x < 0 \\ \kappa_2 & \text{if } x \geq 0 \end{cases}$$

Error estimators



Effectivity indices



$N_{DOF} = 4335 \qquad h = 0.216$

$$I_{eff} = \frac{\eta}{\|u - u_h\|}$$

Trace constants

Neumann oscillations:

$\operatorname{osc}_{\partial K \cap \Gamma_\mathrm{N}}(g_N) = \min\{C_\mathrm{T}, \overline{C}_\mathrm{T}\} \|g_N - \Pi_\gamma^K g_N\|_{0,\partial K \cap \Gamma_\mathrm{N}}$

Trace theorem: For all $v \in H^1(K)$ we have

(1) $\|v\|_{0,\gamma} \leq C_\mathrm{T} \|\!| v |\!\|_K$ for $\kappa_K > 0$,

(2) $\|v - \overline{v}_\gamma\|_{0,\gamma} \leq \overline{C}_\mathrm{T} \|\!| v |\!\|_K$ where $\overline{v}_\gamma = \frac{1}{|\gamma|} \int_\gamma v \, \mathrm{d}x$.

where

- $C_\mathrm{T}^2 = \frac{|\gamma|}{d|K|} \frac{1}{\kappa_K} \sqrt{(2h_K)^2 + (d/\kappa_K)^2}$
- $\overline{C}_\mathrm{T}^2 = \frac{|\gamma|}{d|K|} \min\{h_K/\pi, \kappa_K^{-1}\} \left(2h_K + d \min\{h_K/\pi, \kappa_K^{-1}\}\right)$

Conclusions

Highlights

- Guaranteed upper bound on error (reliability)
- Robust for all values of κ_K (efficiency)
- Fast flux reconstruction (easily parallelizable)
- A bound on the total error

Generality

- Linear reaction-diffusion problems
- Piecewise constant κ
- Arbitrary dimension
- Mixed boundary conditions

Thank you for your attention

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