Surprising results in the theory of inviscid flows

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Outline

Setting of the problem

Let's start straight with the systems of equations describing the adiabatic flow of inviscid (in)compressible fluid.

The incompressible Euler system:

$$\left\{ \begin{array}{rcl} \operatorname{div}_x v &=& 0 \\ \partial_t v + \operatorname{div}_x (v \otimes v) + \nabla_x p &=& 0 \end{array} \right. \tag{1}$$

The compressible Euler system:

$$\left\{ \begin{array}{rcl} \partial_t \rho + \operatorname{div}_x (\rho v) &=& 0 \\ \partial_t (\rho v) + \operatorname{div}_x (\rho v \otimes v) + \nabla_x [p(\rho)] &=& 0 \end{array} \right. \tag{2}$$

Unknowns:

- $p(t,x) \setminus \rho(t,x)$ ... pressure \ density (scalar)
- $v(t,x)$ ... velocity (vector in $\mathbb{R}^n$)

In the compressible case, the pressure $p(\rho)$ is a given function.

Classical solutions

Let us study the Cauchy problem, i.e. prescribe initial data

$$v(0,x) = v^0(x) \text{ for } x \in \mathbb{R}^n. \quad (3)$$

Classical solution of the problem (1)-(3) is a couple of functions $(v,p) \in C^1([0,T) \times \mathbb{R}^n)$ such that they satisfy the equations pointwise for every $(t,x) \in [0,T) \times \mathbb{R}^n$.

This notion of solution is however not very useful. Why? Consider a very simple toy model.

Burger's equation

Look at the following partial differential equation in 1D:

$$\begin{cases} \partial_t u + \frac{1}{2}\partial_x(u^2) = 0 \\ u(\cdot,0) = u^0 . \end{cases} \tag{4}$$

It is easy to observe using the *method of characteristics* that

- the solution $u(t,x)$ is constant along the characteristic curves $x(t)$ in time-space
- the characteristic curves are in fact straight lines $x(t) = x_0 + u^0(x_0)t$

This is OK when $u^0(x)$ is increasing but not that OK when $u^0(x)$ is decreasing: in this case characteristics intersect and a shock is created.

Even for smooth initial data the solution creates discontinuities in finite time.

Weak solutions

It is natural to look for another concept of solutions which is able to capture phenomena like creation of shocks.

Similarly as functions are generalized to distributions, classical solutions are generalized to weak solutions. A function v is called a weak solution to the incompressible Euler system if

$$\int_0^T \int_{\mathbb{R}^n} \partial_t \phi \cdot v + \nabla_x \phi : v \otimes v \mathrm{d}x\mathrm{d}t + \int_{\mathbb{R}^n} \phi(0,x) \cdot v^0(x) \mathrm{d}x = 0 \quad (5)$$

for all test functions $\phi \in C_c^\infty([0,T) \times \mathbb{R}^n)$ with $\operatorname{div}_x \phi = 0$ and

$$\int_0^T \int_{\mathbb{R}^n} v \cdot \nabla_x \psi \mathrm{d}x\mathrm{d}t = 0 \quad (6)$$

for all test functions $\psi \in C_c^\infty([0,T) \times \mathbb{R}^n)$.

Note that the pressure dissapears completely in the weak formulation of the problem, in fact it can be determined up to a function depending only on time.

Weak solutions II

Weak solutions to equations of fluid mechanics are also more natural than classical solutions.

In fact, the derivation of the equations from the physical principles like conservation of mass, momentum and energy leads to the weak formulation.

The strong formulation (i.e. the partial differential equations like (1)) is then derived from the weak formulation assuming the functions are regular.

Function spaces

In the weak formulation no derivatives appear on the function v, it needs not to be differentiable. In order for the integrals appearing in the weak formulation to make sense, it is enough that the velocity field is in the Lebesgue space $L^2_{loc}((0, T) \times \mathbb{R}^n)$.

Function spaces II

The *energy space* however is not just $L^2_{loc}((0,T)\times\mathbb{R}^n)$ but $L^\infty(0,T,L^2(\mathbb{R}^n))$ for the following reason: Consider a classical solution v to the incompressible Euler system, multiply the momentum equation by v and integrate over $\mathbb{R}^n$.

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\mathbb{R}^n}|v|^2\,\mathrm{d}x = 0 \tag{7}$$

since

$$\operatorname{div}_x(v\otimes v)\cdot v = v_i v_{j,i} v_j = v_i\left(\frac{|v|^2}{2}\right)_{,i} \tag{8}$$

and integrating by parts this term vanish, the same holds also for the pressure term.

Thus we see that *classical solutions conserve the energy*.

Existence results

- In 2 space dimensions a classical results states that there exists a global strong solution for sufficiently smooth initial data.
- In 3D (and higher dimensions) only local in time existence (and uniqueness) of strong solutions is proved
- There is also a famous blow-up criterion of strong solutions in 3D (Beale, Kato, Majda): If T^* is a maximal time of existence of a strong solution, then

$$\limsup_{t \to T^*} \|\omega\|_{L^\infty} = +\infty, \tag{9}$$

  where $\omega = \nabla \times v$ is the vorticity of the fluid.
- Weak-strong uniqueness holds (strong solutions $\Leftrightarrow D(v) \in L^\infty$, weak solutions even measure-valued)

Difference between 2D and 3D

Difference between the 2D case and 3D can be ilustrated easily by taking the curl of the momentum equation. In the convective term we achieve using $\operatorname{div}_x v = 0$

$$\operatorname{curl}_x \operatorname{div}_x (v \otimes v) = \operatorname{curl}_x (v \cdot \nabla_x v) = v \cdot \nabla_x \omega - \omega \cdot \nabla_x v \tag{10}$$

and thus the vorticity equation is in general

$$\partial_t \omega + v \cdot \nabla_x \omega = \omega \cdot \nabla_x v. \tag{11}$$

However in 2D: $\omega = (0, 0, v_{2,1} - v_{1,2})$ and thus $\omega \cdot \nabla_x v \equiv 0$. Therefore the vorticity is transported freely by the flow.

Weak solutions are too weak

We know that strong solutions may not exist globally, however weak solutions have other types of bad behavior:

Theorem 1 (Scheffer, 1993)

There exist a weak solution $v \in L^2(\mathbb{R} \times \mathbb{R}^2)$ to the incompressible Euler system with compact support in space and time.

- In particular this shows nonuniqueness of weak solutions because $v = 0$ is also a solution.
- The proof of Scheffer is long and complicated, it is also not clear if his solution belongs to the energy space.
- In 1997, Shnirelman gave a simpler proof (working on periodic box instead of whole space) which is however also still quite complicated

Weak solutions are really too weak

The situation with weak solutions to the Euler equations is actually even worse than that:

Theorem 2 (De Lellis, Székelyhidi 2009)

There exist infinitely many compactly supported bounded ($L^\infty \cap L^2$) weak solutions to the incompressible Euler system in any space dimension greater than 1.

- These solutions are in particular all in the energy space $L^\infty(\mathbb{R}, L^2(\mathbb{R}^n))$
- Nonuniqueness for $v^0 = 0$.
- Proof is quite elegant and related to previous work in (at first sight) a different field of mathematics

Possible way out?

The questions now are:

- Is there some way to eliminate solutions which are obviously nonphysical?
- Can we benefit from some other - up to now not used - physical principle and achieve uniqueness of weak solutions?
- Or are the weak solutions to the incompressible Euler system simply a dead end and we should look for another notion of solution?

The answer could be: energy!
What happens when we prescribe moreover some (in)equality concerning the energy of the fluid?

Energy (in)equalities

- Weak energy inequality

$$\int_{\mathbb{R}^n} |v|^2 (x,t)\mathrm{d}x \leq \int_{\mathbb{R}^n} |v^0|^2 (x)\mathrm{d}x \qquad (12)$$

  for every $t > 0$.

- Strong energy inequality

$$\int_{\mathbb{R}^n} |v|^2 (x,t)\mathrm{d}x \leq \int_{\mathbb{R}^n} |v|^2 (x,s)\mathrm{d}x \qquad (13)$$

  for every $t > s \geq 0$.

- Local energy inequality

$$\partial_t \frac{|v|^2}{2} + \mathrm{div}_x \left(v \left(\frac{|v|^2}{2} + p \right) \right) \leq 0 \qquad (14)$$

  in the sense of distributions.

Weak solutions are really, really too weak

Theorem 3 (De Lellis, Székelyhidi 2010)

There exist bounded, compactly supported vector fields v^0 for which there are

(a) *infinitely many weak solutions satisfying both the strong and the local equalities*
(b) *infinitely many weak solutions satisfying the strong energy inequality but not equality*
(c) *infinitely many weak solutions satisfying the weak energy inequality but not the strong energy inequality*

In particular the message of this theorem is: The energy does not help us no matter what behavior we prescribe.

Wild initial data

The initial data in Theorem 3 cannot be smooth, otherwise this Theorem would collide with the classical local existence of smooth solutions - from the proof it is easily seen that nonuniqueness appears on any time interval $[0, \varepsilon)$.
A natural question therefore is: Are the initial data allowing for this wild behavior exceptions or are they generic? More bad news to come:

Theorem 4 (Székelyhidi, Wiedemann 2012)

The set of wild initial data v^0 for which the conclusions of Theorem 3 holds is dense in the set of L^2 solenoidal vector fields.

One really bad example

Up to now we know that wild initial data are generic, not some sort of exceptions. Of course they need to be irregular. But even the simpest irregular initial data you can think of is already bad enough:

Theorem 5 (Székelyhidi 2011)

The shear flow defined as

$$v^0(x) := \begin{cases} (-1,0) & \text{if } x_2 < 0 \\ (1,0) & \text{if } x_2 > 0, \end{cases} \tag{15}$$

is a wild initial data.

Other criteria

One could try to apply also other selection criteria which work for different types of differential equations in different situations. In the theory of hyperbolic conservation laws there are two other popular criteria:

- the vanishing viscosity limit
- the maximally dissipative solution

However, one can show that the vanishing viscosity limit solution for a shear flow initial data is the stationary solution - obviously conserving the energy. On the other hand some weak solutions with this initial data are dissipative due to Theorem 5. Therefore, even if these criteria singled out unique solutions (which is not clear for the maximally dissipative solution), they would be two different solutions.

Basic ideas

Let us consider the problem of constructing nontrivial compactly supported weak solutions and ilustrate the main ideas.
The most important idea of the whole construction of infinitely many solutions is the following:
Instead of solving directly the nonlinear differential equation, let's solve a linear differential equation with additional pointwise constraint.

Basic ideas II

Lemma 6

Suppose $v \in L^\infty(\mathbb{R}\times\mathbb{R}^n;\mathbb{R}^n)$, $u \in L^\infty(\mathbb{R}\times\mathbb{R}^n;\mathcal{S}^n_0)$ *and* $q \in L^\infty(\mathbb{R}\times\mathbb{R}^n)$ *solve the following linear system of PDEs in the sense of distributions*

$$\left\{\begin{array}{rcl} \mathrm{div}_x v &=& 0 \\ \partial_t v + \mathrm{div}_x u + \nabla_x q &=& 0. \end{array}\right. \tag{16}$$

If in addition

$$u = v\otimes v - \frac{1}{n}|v|^2\,\mathrm{Id} \qquad \textit{almost everywhere in } \mathbb{R}\times\mathbb{R}^n, \tag{17}$$

then v *and* $p = q - \frac{1}{n}|v|^2\,\mathrm{Id}$ *solve* (1) *in the sense of distributions. Conversely, if* v *and* p *solve* (1) *in the sense of distributions, then* v, $u = v\otimes v - \frac{1}{n}|v|^2\,\mathrm{Id}$ *and* $q = p + \frac{1}{n}|v|^2\,\mathrm{Id}$ *solve* (16), (17).

Basic ideas III

Obviously there is plenty of solutions to the linear system (16), so the crucial problem is to find those solutions which satisfy also (17).

Let us denote $y = (t, x) \in \mathbb{R}^{n+1}$ and

$$U = \begin{pmatrix} 0 & v \\ v^T & u + q\mathrm{Id} \end{pmatrix}. \tag{18}$$

Observe that the system (16) can be equivalently written simply as

$$\mathrm{div}_y\, U(y) = 0. \tag{19}$$

We now look for *plane wave solutions* of this linear equation, i.e. solution of the form

$$U(y) = Mh(y \cdot \xi) \tag{20}$$

with M being a constant state (matrix $(n+1) \times (n+1)$), $\xi \in \mathbb{R}^{n+1} \setminus \{0\}$ and $h : \mathbb{R} \to \mathbb{R}$.

Plane wave solutions

Now it is easy to observe that the set of all matrices M such that $U(y)$ defined as in (20) solves (19) for every h corresponds to

$$\Lambda = \left\{ (v, u, q) \in \mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R} : \det \begin{pmatrix} 0 & v \\ v^T & u + q\mathrm{Id} \end{pmatrix} = 0 \right\}. \tag{21}$$

This set is called a wave cone for the linear system (16).
The wave cone is actually very large, in fact for every (v, u) there exists $q \in \mathbb{R}$ such that $(v, u, q) \in \Lambda$.
Taking for example $h(x) = \cos x$, we see that such solution oscillates between states M and $-M$. The idea is to construct solutions of the linear equation (19) which would have similar property but will be compactly supported.

Localized plane waves

Indeed you can construct such localized plane waves, the price to pay is to introduce an error in the range of the "wave".

Lemma 7

Let $M = (v_0, u_0, q_0) \in \Lambda$ with $v_0 \neq 0$. Denote by σ the segment $[-M, M]$ in $\mathbb{R}^n \times \mathcal{S}_0^n \times \mathbb{R}$. Then for every $\varepsilon > 0$ there exists a smooth solution of (16) such that

- *the support of (v, u, q) is contained in $B_1(0) \subset \mathbb{R} \times \mathbb{R}^n$*
- *the range of (v, u, q) is contained in the ε–neighborhood of σ*
- $\int_{B_1(0)} |v(x,t)| \,\mathrm{d}x\,\mathrm{d}t \geq \alpha\,|v_0|$,

where α is a (dimensional) constant.

Actually, one can prove even more - no neighborhood in the pressure q is needed.

Geometric setup

We define

$$K = \left\{(v,u) \in \mathbb{R}^n \times \mathcal{S}_0^n : u = v \otimes v - \frac{1}{n}\mathrm{Id}, |v| = 1\right\} \tag{22}$$

and

$$\mathcal{U} = \left\{(v,u) \in \mathbb{R}^n \times \mathcal{S}_0^n : v \otimes v - u < \frac{1}{n}\mathrm{Id}\right\}. \tag{23}$$

Here the inequality between matrices $A < B$ means that $B - A$ is positive definite.

Note that finding solutions of the linear system (16) with values in K (a.e.) is due to Lemma 6 equivalent to finding solutions to the original Euler system (1).

It can be shown that $\mathcal{U}$ is the interior of the convex hull of K.

Subsolutions and solutions; abstract argument

Let us call *subsolutions* smooth solutions of the linear system (16) with values in $\mathcal{U}$, denote X_0 the space of all subsolutions and finally denote by X the closure of X_0 in the topology of L^∞ weak* convergence, i.e. the space of all (weak*) limits of sequences of subsolutions.

The suprising yet not very difficult fact is the following:

Lemma 8

The set of points of X which are solutions to the Euler equations is in a certain sense big, more precisely it is a residual set in the sense of Baire category. This means that the set of points of X which are not solutions is a set of first category.

- Set of first category is a countable union of nowhere dense sets
- Nowhere dense set: the interior of its closure is empty

Subsolutions and solutions; convex integration

- The abstract argument using the Baire category theory can be replaced by a more direct approach, where the desired solutions are obtained "directly" as limits of sequences of subsolutions
- The sequence starts with $(0, 0, 0)$ and next elements are achieved by adding (in a suitable way) highly oscillating localized plane waves at certain points
- The oscillations can be constructed in such a way that
    - the property of being a subsolution is satisfied for all elements of the sequence
    - the L^2 norm of the velocity component of the subsolutions increases (in a suitable way)
    - strong convergence in L^2 is obtained
- The limit of such sequence is the desired nontrivial compactly supported solution to the Euler system.

General criterion for existence of wild solutions

Lemma 9

Let $\Omega \subset \mathbb{R}^n$ *be an open set. Let* $\overline{e} \in C((0,T) \times \overline{\Omega}) \cap C([0,T], L^1(\Omega))$. *Assume there exists a smooth solution* (v_0, u_0, q_0) *of the linear system* (16) *on* $(0,T) \times \mathbb{R}^n$ *such that*

- $\operatorname{supp}(v_0(\cdot,t), u_0(\cdot,t)) \subset\subset \Omega$ *for all* $t \in (0,T)$
- $\frac{n}{2}\lambda_{max}(v_0(t,x) \otimes v_0(t,x) - u(t,x)) < \overline{e}(x,t)$ *for all* $(t,x) \in (0,T) \times \Omega$.

Then there exist infinitely many weak solutions to the Euler system (1) *in* $[0,T) \times \mathbb{R}^n$ *with pressure* $p = q_0 - \frac{|v|^2}{n}$ *such that*

- $v(x,t) = v_0(x,t)$ *for* $t = 0, T$ *a.e. in* $x \in \mathbb{R}^n$
- $\frac{|v(x,t)|^2}{2} = \overline{e}(x,t)$ *for every* $t \in (0,T)$ *a.e. in* $x \in \mathbb{R}^n$.

What does that mean?

Previous lemma is just a technical way to say the following:

- if you want to have infinitely many weak solutions starting from a certain initial data, it is enough to find a ("smooth") subsolution (not in the end points 0, T!) with the same initial data
- The energy of the fluid $|v|^2/2$ is in fact something like a free parameter, it can be prescribed!

This is very counter-intuitive as we are talking about system of $n+1$ partial differential equations for $n+1$ unknowns.

Now let's turn our attention to something little bit different.

Onsager's conjecture

We already know that classical (C^1) solutions to the Euler system conserve the energy whereas just bounded (L^∞) weak solutions may dissipate the energy. Natural question arises immediately:

Question 1

Is there a regularity threshold such that all solutions with better regularity conserve the energy and there exist dissipative solutions in every space with lower regularity?

Onsager's conjecture II

Such conjecture dates back to 1940s, when Lars Onsager (Nobel prize laureat in chemistry!) stated the following conjecture:

Conjecture 1 (Onsager's conjecture)

Every weak solution to the Euler system belonging to the class C^α for $\alpha > \frac{1}{3}$ is conservative. There exist dissipative solutions $v \in C^\alpha$ for all $\alpha < \frac{1}{3}$.

Recall that a function v is Hölder continuous with parameter α if

$$|v(x) - v(y)| \leq C\,|x - y|^\alpha \tag{24}$$

for all x, y in the domain of interest. The α-Hölder continuous functions lie on the following scale of function spaces

$$C^1 \subset W^{1,\infty} \text{ (Lipschitz) } \subset C^\alpha \subset C \subset L^\infty. \tag{25}$$

Onsager's conjecture III

There is actually very simple explanation why $\frac{1}{3}$ should be really the correct threshold. Let us recall, how we derived the energy conservation for classical solutions.

We multiplied the momentum equation by the velocity v and integrated over the domain. The pressure term dissapeared due to the incompressibility condition

$$\int_\Omega \nabla_x p \cdot v \mathrm{d}x = -\int_\Omega p \mathrm{div}_x v \mathrm{d}x = 0 \tag{26}$$

and the problematic term was

$$\Pi = \int_\Omega \mathrm{div}_x (v \otimes v) \cdot v \mathrm{d}x = \int_\Omega ((v \cdot \nabla_x) v) \cdot v \mathrm{d}x \tag{27}$$

Onsager's conjecture IV

$$\Pi = \int_\Omega \mathrm{div}_x(v \otimes v) \cdot v \mathrm{d}x = \int_\Omega ((v \cdot \nabla_x)v) \cdot v \mathrm{d}x$$

In the absence of some smoothness of v we cannot justify integration by parts to conclude $\Pi = 0$. However, absolutely formally, let's look at ∇_x as a multiplication operator. Then

$$\Pi \sim \int_\Omega (|\nabla_x|^{1/3} v)^3 \mathrm{d}x. \tag{28}$$

It appears that if v has Hölder continuity $\frac{1}{3}$ we can at least make sense of the flux Π and any better regularity would be sufficient to justify integration by parts.

Onsager's conjecture V

Interesting fact is that Onsager's own justification of the threshold $\alpha = \frac{1}{3}$ was completely different and unrelated to the one presented above. Onsager was motivated by the works of Kolmogorov on turbulence.

Consider the Navier-Stokes equations

$$\left\{ \begin{array}{rcl} \mathrm{div}_x v & = & 0 \\ \partial_t v + \mathrm{div}_x (v \otimes v) + \nabla_x p & = & \nu \Delta_x v \end{array} \right. \tag{29}$$

The energy equality is obtained in a similar way as in the case of Euler equations:

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^n} |v|^2 \,\mathrm{d}x = -\nu \int_{\mathbb{R}^n} |\nabla_x v|^2 \,\mathrm{d}x. \tag{30}$$

Anomalous dissipation

Kolomogorov studyied the behaviour of the quantity

$$\varepsilon^\nu = \left\langle \nu \left| \nabla_x v^\nu \right|^2 \right\rangle \tag{31}$$

in the inviscid limit $\nu \to 0$. Here the notation $< \ldots >$ denotes some kind of averaging process (space-time averaging, ensemble averaging) and the quantity ε^ν is the mean energy dissipation rate. The theory of Kolmogorov is based on the following fact, which is observed experimentally as well as numerically:

$$\varepsilon^\nu \to \varepsilon \neq 0 \qquad \text{as} \quad \nu \to 0. \tag{32}$$

Even though the limiting equations are energy preserving (for smooth solutions) in the vanishing viscosity limit, dissipation of energy is observed. Thus the notion of *anomalous dissipation* is used for this phenomenon.

Anomalous dissipation II

The anomalous dissipation is explained in the turbulence theory by the *energy cascade*: due to the presence of the nonlinear term $v \cdot \nabla_x v$ the energy propagates from large scales to small scales. Actually, the energy spectrum is predicted to have (in 3D) a form

$$E(\kappa) = \frac{1}{2}\frac{d}{d\kappa}\left\langle |v_{<\kappa}|^2 \right\rangle \sim \varepsilon^{\frac{2}{3}}\kappa^{-\frac{5}{3}}, \qquad (33)$$

here $v_{<\kappa}$ denotes the filtered velocity field containing only frequencies below κ.

This was the starting point for Onsager to construct his conjecture.

Onsager's conjecture: mathematics

From the point of view of mathematics, Onsager's conjecture is *almost* proved, however most of the work has been done very recently.

- The part that $\alpha-$Hölder solutions conserve the energy for $\alpha > \frac{1}{3}$ was proved in 1994 by Constantin, E and Titi, building heavily on the work of Eyink (1994) (so much that the paper of Constantin, E and Titi has just 3 pages including abstract, acknowledgment and references!)
- Continuous solutions to Euler equations that dissipate energy were first constructed by De Lellis and Székelyhidi in the work that followed their L^∞ theory in 2012 (published 2013)
- Then several results in this direction came quite quickly

Onsager's conjecture: mathematics II

- $\alpha < \frac{1}{10}$: De Lellis, Székelyhidi, 2013 (published 2014)
- $\alpha < \frac{1}{5}$: Isett (Ph.D. student of Klainerman in Princeton), 2013
- $\alpha < \frac{1}{5}$: Buckmaster, De Lellis, Székelyhidi, 2013 (much shorter proof than the one by Isett)
- $\alpha < \frac{1}{3}$, $L^1(0, T, C^\alpha)$: Buckmaster, De Lellis, Székelyhidi, 2014

Note that the Onsager conjectured the space $L^\infty(0, T, C^\alpha)$ with any $\alpha < \frac{1}{3}$ to contain dissipative solutions of Euler, so the conjecture is still not proved completely.

Relation to geometry

The crucial ideas of the method of De Lellis and Székelyhidi are not completely new, they have been used for different problems quite a long time before. The starting point of the whole story is the following theorem.

Theorem 10 (Nash, Kuiper)

Let (M^n, g) be n-dimensional smooth compact manifold, let $m \geq n + 1$ and let

$$u : M^n \hookrightarrow \mathbb{R}^m \tag{34}$$

be a short embedding. Then u can be uniformly approximated by C^1 isometric embeddings.

Note that *short* map is a map which shrink distances, in other words

$$\text{length}(u \circ \gamma) \leq \text{length}(\gamma) \tag{35}$$

Nash-Kuiper theorem explained

How to visualize the Nash-Kuiper theorem? Consider a unit sphere $S^2 \subset \mathbb{R}^3$ and as a short embedding u the map which shrinks it to a sphere of radius ε, $u : S^2 \to \varepsilon S^2$.

The theorem implies that in a C^0 neighborhood of this map there exist C^1 isometric embeddings, i.e. in an arbitrarily small neighborhood of the shrinked sphere there are C^1 *isometric* images of S^2. In other words you can wrinkle your unit sphere (in a C^1 way) to preserve lengths of curves and put it inside a 3D ball of radius ε.

Without wanting the C^1 property this would be quite easy, imagine just crumbling of paper. However this is not really C^1 but only Lipschitz.

Nash-Kuiper theorem explained II

It is known that no such isometry can be constructed in C^2: There is a unique (standard) C^2 isometric embedding of S^2 into $\mathbb{R}^3$ (up to a rigid motion).

The theorem was first proved by Nash in 1954 in the case $m \geq n+2$ and later improved by Kuiper to the case $m = n+1$.

To see at least some mathematics, let us rewrite the theorem in its local version and in the Nash case.

Theorem 11 (Nash, local)

Let $m \geq n+2$, $\Omega \subset \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}^m$ be smooth, strictly short map, i.e. such that $\nabla u^T \nabla u = (\partial_i u \cdot \partial_j u)_{i,j=1,\dots,n} < g$ in $\overline{\Omega}$. For any $\varepsilon > 0$ there exists $\tilde{u} \in C^1(\overline{\Omega}, \mathbb{R}^m)$ such that $\|u - \tilde{u}\|_{C^0(\Omega)} < \varepsilon$ and

$$\nabla \tilde{u}^T \nabla \tilde{u} = g \quad \text{in } \overline{\Omega} \tag{36}$$

Analogy between Euler and Nash-Kuiper

The analogy between this problem and Euler equations is the following: We can rewrite the problem of finding $\tilde{u}$ satisfying

$$\nabla\tilde{u}^T\nabla\tilde{u} = g$$

as a problem of finding a matrix A satisfying a linear constraint

$$\operatorname{curl} A = 0$$

and a pointwise nonlinear relation

$$A^T A = g.$$

Then short maps can be considered as subsolutions to the problem of isometric embedding.

Compressible Euler system

The compressible Euler system reads as follows:

$$\left\{\begin{array}{rcl} \partial_t \rho + \operatorname{div}_x(\rho v) & = & 0 \\ \partial_t(\rho v) + \operatorname{div}_x(\rho v \otimes v) + \nabla_x[p(\rho)] & = & 0 \end{array}\right. \tag{37}$$

The pressure $p(\rho)$ is a given function.

Compressible Euler system II

- It is a hyperbolic system of conservation laws
- The theory of hyperbolic conservation laws is far frome being completely understood
- Solutions develop singularities in finite time even for smooth initial data (Burger's equation)
- Admissibility comes into play due to the entropy inequality ("selector" of physical solutions in case of existence of many solutions)
- There are satisfactory results in the case of scalar conservation laws (in 1D as well as in multi-D), there is a lot of entropies: $\Rightarrow$ Kruzkov, 1970: Well–posedness theory in BV.
- There are also satisfactory results in the case of systems of conservation laws in 1D: Lax, Glimm, Bressan, Bianchini, ...

Compressible Euler system III

Back to our case, the compressible Euler system:

- In more than 1D there is only one (entropy, entropy flux) pair, which is

$$\left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2}, \left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2} + p(\rho)\right) v\right)$$

  with the internal energy $\varepsilon(\rho)$ given through

$$p(\rho) = \rho^2\varepsilon'(\rho).$$

- Local existence of strong (and therefore admissible) solutions is proved
- On the other hand global existence of weak solutions in general (it is a system in multi D!) is still an open problem, there are only partial results
- The weak–strong uniqueness property holds for this system

Admissible solutions

Reasonable admissibility criterion for weak solutions therefore seems to be the (mathematical) entropy or (physical) energy inequality

$$\partial_t(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2}) + \mathrm{div}_x \left(\left(\rho\varepsilon(\rho) + \frac{\rho|v|^2}{2} + p(\rho)\right) v\right) \leq 0$$

(in the sense of distributions).

But is it really? The answer is again - not quite.

Ill-posedness results

Using the method of De Lellis and Székelyhidi in a clever way the following results were obtained.

- De Lellis, Székelyhidi (2010): There exists initial data $(\rho^0, v^0) \in L^\infty$ for which there exist infinitely many admissible weak solutions
- Chiodaroli (2012): For every $\rho^0 \in C^1$ there exists $v^0 \in L^\infty$ such that there exist infinitely many admissible weak solutions (on short time interval $(0, T^*)$)
- Feireisl (2013): Same as above on long time intervals provided $\left|\nabla_x \rho^0\right| < \varepsilon$
- Chiodaroli, De Lellis, K. (2013): There exists Lipschitz initial data (ρ^0, v^0) for which there exist infinitely many admissible weak solutions

Finally we have that even for nice initial data there is nonuniqueness of admissible weak solutions.

Riemann problem

Denote $x = (x_1, x_2) \in \mathbb{R}^2$ and consider the special initial data

$$(\rho^0(x), v^0(x)) := \begin{cases} (\rho_-, v_-) & \text{if } x_2 < 0 \\ (\rho_+, v_+) & \text{if } x_2 > 0, \end{cases} \tag{38}$$

where $\rho_\pm, v_\pm$ are constants.
In particular the initial data are "1D" and there is a classical theory about self-similar solutions to the Riemann problem in 1D (they are unique in the class of BV functions).
In the case of system (37), the initial singularity can resolve to at most 3 structures (rarefaction wave, admissible shock or contact discontinuity) connected by constant states.

Results for Riemann problem

Analysis of the Riemann problem yielded up to now the following results. If the initial data are such that the 1D self-similar solutions consists

- only of rarefaction waves, then this solution is unique in the class of all bounded admissible weak solutions (Feireisl, K., 2014)
- of two shocks, then there exists infinitely many admissible weak solutions (Chiodaroli, K., 2014). Moreover some of these solutions dissipate more total energy than the self-similar one
- one rarefaction wave and one shock, nothing is known in general, but there is example of such initial data, for which there is nonuniqueness (Chiodaroli, De Lellis, K. (2013))

Thank you

Thank you for your attention.