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Concept lattices and attribute implications from data with fuzzy attributes

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1 Introduction

1.1 Doctoral dissertation and this thesis

The submitted doctoral dissertation is devoted to foundations of formal concept analysis of data with fuzzy attributes, particularly to concept lattices, attribute implications, and related structures and topics. The dissertation consists of a collection of papers [A1]–[A21], accompanied by an annotation.

This thesis provides a summary of the dissertation and is organized as follows. Section 1 describes the content of the dissertation. Section 2 contains results on concept lattices and related structures. Section 3 deals with attribute implications. Section 4 overviews some further results and directions for future research. References are split into three parts. Part A (references [A1]–[A21]) contains publications of which the dissertation consists. Part B (references [B1]–[B16]) contains further publications of the author which are related to the topic of the thesis. Part C (references [C1]–[C49]) contains publications by other authors which are related to the dissertation.

1.2 Content of dissertation

Broadly speaking, the dissertation brings up results on a particular way of development of formal concept analysis (FCA) from the point of view of fuzzy logic. The aim of this section is to briefly present the basic ideas of FCA and outline the approach to FCA from the point of view of fuzzy logic which is adopted in the submitted dissertation.

Formal concept analysis Tables, i.e. two-dimensional arrays, represent perhaps the most popular way to describe data. Table rows usually correspond to objects of our interest, table columns correspond to some of their attributes, and table entries contain values of attributes on the respective objects. As an example, consider patients as objects and “name”, “weight”, “male”, “female”, etc. as attributes. Table rows and columns are usually labeled by objects' and attributes' names. A particular case arises when all the attributes are logical attributes (presence/absence attributes) like “male”, “headache”, “left-handed”, etc. A patient either is a male or not and, in general, either has a logical attribute or not. In this case, a table entry corresponding to object x and attribute y contains $\times$ or blank depending on whether object x has or does not have attribute y.

Many methods of various kinds have been and are being developed for the purpose to analyze tabular data, starting with classical statistical methods, and going through clustering and classification up to the most recent methods of data mining. The submitted dissertation is concerned with theoretical foundations

	y_1	y_2	y_3	$\cdots$
x_1	$\times$	$\times$	$\times$	
x_2	$\times$	$\times$		$\vdots$
x_3		$\times$	$\times$	
$\vdots$		$\cdots$		$\ddots$

	y_1	y_2	y_3	$\cdots$
x_1	1	1	0.7	
x_2	0.8	0.6	0.1	$\vdots$
x_3	0	0.9	0.9	
$\vdots$		$\cdots$		$\ddots$

Figure 1: Tables with logical attributes: crisp attributes (left), fuzzy attributes (right).

of one of these methods, namely, of FCA. Although some previous attempts exist, it is generally agreed and FCA started by Wille's seminal paper [C47]. Since then, FCA is being developed theoretically, algorithmically, and methodologically. Applications of FCA are basically in data analysis and knowledge discovery, both as a direct method delivering results to the user and a preprocessing method being used with other methods. Two monographs on FCA are [C17] (mainly mathematical foundations) and [C13] (mainly algorithms and applications). There are three international conferences devoted to FCA, namely ICFCA (Int. Conf. on Formal Concept Analysis), CLA (Concept Lattices and Their Applications), and ICCS (Int. Conf. on Conceptual Structures). In addition, further papers on FCA can be found in journals and proceedings of other conferences.

A table with logical attributes can be represented by a triplet $\langle X, Y, I\rangle$ where I is a binary relation between X and Y. Elements of X are called objects and correspond to table rows, elements of Y are called attributes and correspond to table columns, and for $x \in X$ and $y \in Y$, $\langle x, y\rangle \in I$ indicates that object x has attribute y while $\langle x, y\rangle \notin I$ indicates that x does not have y. For instance, Fig. 1.2 (left) depicts a table with logical attributes. The corresponding triplet $\langle X, Y, I\rangle$ is given by $X = \{x_1, x_2, x_3, x_4\}$, $Y = \{y_1, y_2, y_3\}$, and we have $\langle x_1, y_1\rangle \in I$, $\langle x_2, y_3\rangle \notin I$, etc. Since representing tables with logical attributes by triplets is common in FCA, we say just "table $\langle X, Y, I\rangle$" instead of "triplet $\langle X, Y, I\rangle$ representing a given table". FCA aims at obtaining two outputs out of a given table. The first one, called a concept lattice, is a partially ordered collection of particular clusters of objects and attributes. The second one consists of formulas, called attribute implications, describing particular attribute dependencies which are true in the table. The clusters, called formal concepts, are pairs $\langle A, B\rangle$ where $A \subseteq X$ is a set of objects and $B \subseteq Y$ is a set of attributes such that A is a set of all objects which have all attributes from B, and B is the set of all attributes which are common to all objects from A. For instance, $\langle\{x_1, x_2\}, \{y_1, y_2\}\rangle$ and $\langle\{x_1, x_2, x_3\}, \{y_2\}\rangle$ are examples of formal concepts of the (visible part of) left table in Fig. 1.2. An attribute implication is an expression $A \Rightarrow B$ with A and B

being sets of attributes. $A \Rightarrow B$ is true in table $\langle X, Y, I\rangle$ if each object having all attributes from A has all attributes from B as well. For instance, $\{y_3\} \Rightarrow \{y_2\}$ is true in the (visible part of) left table in Fig. 1.2, while $\{y_1, y_2\} \Rightarrow \{y_3\}$ is not (x_2 serves as a counterexample).

Formal concept analysis and fuzzy logic Contrary to classical (two-valued) logic, fuzzy logic uses intermediate truth degrees in addition to 0 (false) and 1 (true). Fuzzy logic thus allows to assign truth degrees like 0.8 to propositions like "Customer C is satisfied withe service s". In this example, assigning 0.8 to the above proposition means that customer C was quite satisfied but not completely. This way, fuzzy logic attempts to deal with fuzzy attributes (graded attributes) like "being tall", "being satisfied (with a given service)", etc. An example of a table with fuzzy attributes is presented in the right part of Fig. 1.2. A table entry corresponding to object x and attribute y contains a truth degree of "object x has attribute y". For instance, object x_1 has attribute y_1 to degree 1, x_2 has attribute y_1 to degree 0.8, x_2 has attribute y_3 to degree 0.1, etc. If objects are patients and y_1 is "intensive headache" than the table says that patient x_2 has a rather severe headache. Needless to say, dealing with fuzzy attributes by means of classical logic, i.e. using only 0 and 1, and "forcing" a user to decide whether or not a given customer was satisfied, is not appropriate. Using intermediate truth degrees in addition to 0 and 1 instead of 0 and 1 only has became known under the term fuzzy approach (graded approach).

With respect to the above outline, FCA in its basic setting deals with two-valued logical attributes (called also crisp attributes). The following question therefore arises:

> Is it feasible to extend the methods of FCA so as to have FCA applicable to tables with fuzzy attributes?

The submitted thesis attempts to provide a positive answer to this question.

Main contributions of the dissertation The following are the main contributions and features of the dissertation:

(1) We present a sound generalization of the mathematical foundations of FCA. This concerns mainly concept lattices and attribute implications, i.e. two main outputs of FCA, but also mathematical structures directly related to FCA like closure operators, closure systems, Galois connections, and complete lattices. We use complete residuated lattices as a general structure of truth degrees. The ordinary (i.e., "non-fuzzy") results on FCA turn out to be a particular case of our results when the complete residuated lattice is a two-element Boolean algebra.

(2) Although the computational aspects (design of efficient algorithms) are of secondary interest in the dissertation, we present algorithms with the same order of complexity as those known from the ordinary FCA (computation of fixed points of the fuzzy closure operators involved, computation of systems of pseudo-intents, computation of non-redundant bases of fuzzy attribute implications).

(3) Our approach is based on following closely fuzzy logic in narrow sense, see e.g. [C26]. Briefly speaking, our definitions result from considering appropriate formulas and evaluating these formulas according to the principles of fuzzy logic. Furthermore, when developing fuzzy attribute logic, i.e. a logical calculus for reasoning with rules $A \Rightarrow B$, we present both ordinary-style as well as Pavelka-style logics.

(3) We present various results (representation results, reduction results) on relationships between the new structures which result in our approach like fuzzy concept lattices, fuzzy Galois connections, fuzzy attribute implications, etc., and the ordinary structures, i.e. concept lattices, Galois connections, attribute implications, etc.

(4) We demonstrate that in a fuzzy setting, new phenomena arise. These phenomena are hidden in the ordinary setting but are interesting and important in a fuzzy setting. Two examples are presented in detail. First, factorization of concept lattices by similarity which allows us to consider simplified version of the original concept lattice, namely, its factor lattice. Second, usage of hedges (truth functions of connective "very true") to parameterize the underlying Galois connections. Hedges enable us to control the size of the resulting concept lattice. In addition to that, by setting hedges in an appropriate way, we obtain approaches proposed by other authors as a particular case of our approach.

(5) Results on fuzzy attribute implications show an interesting connection to an extension of Codd's relational model of data, a framework which underlies relational databases. The extension consists in considering similarity relations on domains in the relational model. The agenda around fuzzy attribute logic translated into the extension of Codd's model, plays the role of the agenda of functional dependencies in Codd's model. The extension by similarities and, in particular, functional dependencies in this extension have been studied by many authors. Our results offers a more elaborate approach: our concept of functional dependence is more general still quite natural, we deal with partial entailment of functional dependencies, etc.

(6) Some of our results, although developed in a fuzzy setting, are new even for

the ordinary setting. The method of reducing the size of concept lattices by closure operators is an example.

(7) FCA, in a sense, develops formally a theory of concepts proposed within so-called Port-Royal logic [C3] which was very influential and is still popular. The dissertation shows that Port-Royal theory of concepts, proposed originally informally in natural language and very probably bearing only bivalent (i.e. crisp) concepts in mind, may naturally be interpreted as a theory of fuzzy (graded) concepts.

(8) We outline some promising directions for future research. In addition to a further development of FCA in a fuzzy setting itself, we present two new promising directions and our preliminary results. First, relational factor analysis (the term was coined by us and is thus to be considered tentative). Second, the above mentioned extension of Codd's relational model of data.

(9) The dissertation presents a sound and comprehensive study of an area on the borderline of mathematics and computer science. As such, it can serve as an example of a proper development of a non-trivial area from the point of view of fuzzy approach.

1.3 Preliminaries

In what follows, we briefly outline basics of FCA and basics of fuzzy sets and fuzzy logic including references where one can find details. We proceed in a dry style, presenting just definitions, selected results, and short comments.

Formal concept analysis and related structures Let $\langle X, Y, I\rangle$ be a data table with crisp attributes, i.e. X and Y are finite sets (of objects and attributes) and $I \subseteq X \times Y$ is a binary relation between X and Y, see Section 1.2. $\langle X, Y, I\rangle$ is also called a formal context in FCA. Introduce operators ${}^{\uparrow} : 2^X \to 2^Y$ and ${}^{\downarrow} : 2^Y \to 2^X$ by putting for each $A \subseteq X$ and $B \subseteq Y$

$$\begin{aligned} A^{\uparrow} &= \{y \in Y \mid \text{for each } x \in A: \ \langle x, y\rangle \in I\}, \\ B^{\downarrow} &= \{x \in X \mid \text{for each } y \in B: \ \langle x, y\rangle \in I\}. \end{aligned}$$

A formal concept in $\langle X, Y, I\rangle$ is a pair $\langle A, B\rangle$ of $A \subseteq X$ and $B \subseteq Y$ such that $A^{\uparrow} = B$ and $B^{\downarrow} = A$. Put $\mathcal{B}(X, Y, I) = \{\langle A, B\rangle \mid A^{\uparrow} = B, B^{\downarrow} = A\}$, i.e., $\mathcal{B}(X, Y, I)$ is the set of all formal concepts in $\langle X, Y, I\rangle$. Introduce a partial order $\leq$ on $\mathcal{B}(X, Y, I)$ by $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ iff $A_1 \subseteq A_2$ (iff $B_2 \subseteq B_1$). The set $\mathcal{B}(X, Y, I)$ equipped by $\leq$ is called a concept lattice of $\langle X, Y, I\rangle$.

Note that $A^{\uparrow}$ is the set of all attributes shared by all objects from A; similarly for $B^{\downarrow}$. Therefore, $\langle A, B\rangle$ is a formal concept iff A is the set of all objects sharing all attributes from B and, *vice versa*, B is the set of all attributes shared by all objects from A. A and B are called an extent and an intent of $\langle A, B\rangle$; an extent (intent) is thought of as a collection of objects (attributes) to which the concept $\langle A, B\rangle$ applies. $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ means that $\langle A_2, B_2\rangle$ is more general than $\langle A_1, B_1\rangle$ since it applies to more objects (or, equivalently, applies to fewer attributes); $\leq$ therefore models the subconcept-superconcept hierarchy. Alternatively, formal concepts can be defined as maximal rectangles in table $\langle X, Y, I\rangle$ which are full of $\times$'s. The following assertion is called the Main theorem of concept lattices.

Theorem 1 ([C47]) (1) $\mathcal{B}(X, Y, I)$ *equipped with* $\leq$ *a complete lattice with infima and suprema given by*

$$\bigwedge_{j\in J}\langle A_j, B_j\rangle = \langle \bigcap_{j\in J} A_j, (\bigcup_{j\in J} B_j)^{\downarrow\uparrow}\rangle,\ \bigvee_{j\in J}\langle A_j, B_j\rangle = \langle (\bigcup_{j\in J} A_j)^{\uparrow\downarrow}, \bigcap_{j\in J} B_j\rangle.$$

(2) *Moreover, an arbitrary complete lattice* $\mathbf{V} = \langle V, \leq\rangle$ *is isomorphic to* $\mathcal{B}(X, Y, I)$ *iff there are mappings* $\gamma : X \to V$, $\mu : Y \to V$ *such that*

(i) $\gamma(X)$ *is* $\bigvee$*-dense in* V, $\mu(Y)$ *is* $\bigwedge$*-dense in* V;

(ii) $\gamma(x) \leq \mu(y)$ *iff* $\langle x, y\rangle \in I$. □

A subset K of a complete lattice $\mathbf{V}$ is called infimally (supremally) dense if each element of V is a infimum (supremum) of some elements of K.

Let $\langle U, \leq\rangle$ and $\langle V, \leq\rangle$ be complete lattices. A closure operator in $\langle U, \leq\rangle$ is a mapping $C : U \to U$ for which $u \leq C(u)$, $u \leq v$ implies $C(u) \leq C(v)$, $C(C(u)) = C(u)$, for each $u, v \in U$. A Galois connection between $\langle U, \leq\rangle$ and $\langle V, \leq\rangle$ is a pair of mappings $^\uparrow$ and $^\downarrow$ for which $u_1 \leq u_2$ implies $u_2^\uparrow \leq u_1^\uparrow$, $v_1 \leq v_2$ implies $v_2^\downarrow \leq v_1^\downarrow$, $u \leq u^{\uparrow\downarrow}$, $v \leq v^{\downarrow\uparrow}$, for each $u, u_i \in U$ and $v, v_i \in V$.

For further details we refer to [C17] and also to [C13].

Fuzzy sets and fuzzy logic We use complete residuated lattices as our structures of truth degrees. A complete residuated lattice is an algebra $\mathbf{L} = \langle L, \wedge, \vee, \otimes, \to, 0, 1\rangle$ such that $\langle L, \wedge, \vee, 0, 1\rangle$ is a complete lattice with 0 and 1 being the least and greatest element of L, respectively; $\langle L, \otimes, 1\rangle$ is a commutative monoid (i.e. $\otimes$ is commutative, associative, and $a \otimes 1 = 1 \otimes a = a$ for each $a \in L$); $\otimes$ and $\to$ satisfy $a \otimes b \leq c$ iff $a \leq b \to c$ (adjointness property) for each $a, b, c \in L$. Moreover, we use the following concept of a (truth-stressing) hedge (cf. [C26],[C27]). A hedge on a complete residuated lattice $\mathbf{L}$ is a mapping $^* : L \to L$ satisfying $1^* = 1$, $a^* \leq a$, $(a \to b)^* \leq a^* \to b^*$, $a^{**} = a^*$, for each $a, b \in L$. A biresiduum on $\mathbf{L}$ is a derived operation $\leftrightarrow$ defined by $a \leftrightarrow b = (a \to b) \wedge (b \to a)$.

Elements a of L are called truth degrees. $\otimes$ and $\to$ are (truth functions of) "fuzzy conjunction" and "fuzzy implication"; hedge * is a (truth function of) logical connective "very true"; $\leftrightarrow$ is a (truth function of) "fuzzy equivalence". A common choice of $\mathbf{L}$ is a structure with $L = [0,1]$ (real unit interval), $\wedge$ and $\vee$ being minimum and maximum, $\otimes$ being a left-continuous t-norm with the corresponding $\to$. Three most important pairs of adjoint operations on the unit interval are: Łukasiewicz: $a \otimes b = \max(a+b-1, 0)$, $a \to b = \min(1-a+b, 1)$; Gödel (minimum): $a \otimes b = \min(a,b)$, $a \to b = 1$ if $a \leq b$ and $a \to b = b$ if $a > b$; Goguen (product): $a \otimes b = a \cdot b$, $a \to b = 1$ if $a \leq b$ and $a \to b = b/a$ if $a > b$. Other examples are finite chains, e.g. $L = \{a_0 = 0, a_1, \ldots, a_n = 1\} \subseteq [0,1]$ ($a_0 < \cdots < a_n$) with $\otimes$ and $\to$ given by $a_k \otimes a_l = a_{\max(k+l-n,0)}$ and $a_k \to a_l = a_{\min(n-k+l,n)}$ (finite Łukasiewicz chain), or $\otimes$ and $\to$ being the restrictions of the above Gödel operations on $[0,1]$ to L. A special case is a two-element Boolean algebra which we will denote by $\mathbf{2}$.

An $\mathbf{L}$-set (fuzzy set) A in a universe U is a mapping $A\colon U \to L$, $A(u)$ being interpreted as "the degree to which u belongs to A". If $U = \{u_1, \ldots, u_n\}$ then A can be denoted by $A = \{{}^{a_1}/u_1, \ldots, {}^{a_n}/u_n\}$ meaning that $A(u_i)$ equals a_i; we write $\{u, {}^{0.5}/v\}$ instead of $\{{}^{1}/u, {}^{0.5}/v, {}^{0}/w\}$, etc. $\mathbf{L}^U$ (or L^U) denotes the collection of all $\mathbf{L}$-sets in U; basic operations with $\mathbf{L}$-sets are defined componentwise. An $\mathbf{L}$-set $A \in \mathbf{L}^U$ is called crisp if $A(u) \in \{0,1\}$ for each $u \in U$. Crisp $\mathbf{L}$-sets can be identified with ordinary sets. For a crisp A, we also write $u \in A$ for $A(u) = 1$ and $u \notin A$ for $A(u) = 0$. An $\mathbf{L}$-set $A \in \mathbf{L}^U$ is called empty (denoted by $\emptyset$) if $A(u) = 0$ for each $u \in U$. For $a \in L$ and $A \in \mathbf{L}^U$, an ordinary set ${}^aA = \{u \in U \mid A(u) \geq a\}$ is called an a-cut of A. Given $A, B \in \mathbf{L}^U$, we define a degree $S(A,B)$ to which A is contained in B and a degree $A \approx B$ to which A is equal to B by

$$S(A,B) \ = \ \textstyle\bigwedge_{u \in U}(A(u) \to B(u)), \qquad A \approx B \ = \ \textstyle\bigwedge_{u \in U}(A(u) \leftrightarrow B(u)).$$

In particular, we write $A \subseteq B$ iff $S(A,B) = 1$. A binary $\mathbf{L}$-relation $\approx$ on U is called an $\mathbf{L}$-equivalence if for any $u, v, w \in U$ we have $u \approx u = 1$ (reflexivity), $u \approx v = v \approx u$ (symmetry), $(u \approx v) \otimes (v \approx w) \leq (u \approx w)$ (transitivity); An $\mathbf{L}$-equality is an $\mathbf{L}$-equivalence satisfying $u = v$ whenever $u \approx v = 1$.

Throughout this thesis, we use the following convention. If we want to emphasize the structure $\mathbf{L}$ of truth degrees, we say "$\mathbf{L}$-set", "$\mathbf{L}$-Galois connection", etc., instead of "fuzzy set", "fuzzy Galois connection", etc., which we use if $\mathbf{L}$ is not important or clear from context.

For further details we refer to [C1], [C24], [C26], [C31], [C38].

2 Concept lattices and related structures

2.1 Concept lattices

Let $\mathbf{L}$ be a complete residuated lattice (our structure of truth degrees). If we say "a fuzzy set", "a fuzzy relation" and the like, we always mean a fuzzy set (fuzzy relation) with truth degrees taken from the support L of $\mathbf{L}$.

Data tables with fuzzy attributes A data table with fuzzy attributes, is a triplet $\langle X, Y, I\rangle$ where X and Y are sets, and $I : X \times Y \to L$ is a binary fuzzy relation between X and Y which takes values in the support L of $\mathbf{L}$. X and Y are usually assumed to be finite; elements of X and Y are called objects and attributes, respectively. A degree $I(x, y) \in L$ is interpreted as a degree to which object $x \in X$ has attribute $y \in Y$. The notion of a data table with fuzzy attributes is our formal counterpart to tables like the one in Fig. 1.2 (right) with an obvious correspondence: objects $x \in X$ and attributes $y \in Y$ correspond to table rows and columns, respectively; $I(x, y)$ is the table entry at the row corresponding to x and the column corresponding to y (the ordering of rows and columns does not play any role in our considerations).

Arrow operators, formal concepts, and concept lattices Each table $\langle X, Y, I\rangle$ with fuzzy attributes induces a pair of operators $^{\Uparrow} : \mathbf{L}^X \to \mathbf{L}^Y$ and $^{\Downarrow} : \mathbf{L}^Y \to \mathbf{L}^X$ defined by

$$A^{\Uparrow}(y) = \textstyle\bigwedge_{x\in X}(A(x) \to I(x, y)), \quad B^{\Downarrow}(x) = \textstyle\bigwedge_{y\in Y}(B(y) \to I(x, y)), \tag{1}$$

for each $A \in \mathbf{L}^X$ and $B \in \mathbf{L}^Y$, and $x \in X$ and $y \in Y$. A formal (fuzzy) concept of $\langle X, Y, I\rangle$ is a pair $\langle A, B\rangle$ of fuzzy sets $A \in \mathbf{L}^X$ and $B \in \mathbf{L}^Y$ satisfying $A^{\Uparrow} = B$ and $B^{\Downarrow} = A$. Put

$$\begin{aligned}
\mathcal{B}(X, Y, I) &= \{\langle A, B\rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^{\Uparrow} = B,\ B^{\Downarrow} = A\}, &(2)\\
\mathrm{Ext}(X, Y, I) &= \{A \in \mathbf{L}^X \mid \langle A, B\rangle \in \mathcal{B}(X, Y, I) \text{ for some } B\}, &(3)\\
\mathrm{Int}(X, Y, I) &= \{B \in \mathbf{L}^Y \mid \langle A, B\rangle \in \mathcal{B}(X, Y, I) \text{ for some } A\}. &(4)
\end{aligned}$$

That is, $\mathcal{B}(X, Y, I)$ is the set of all formal concepts in $\langle X, Y, I\rangle$. Introduce a partial order $\leq$ on $\mathcal{B}(X, Y, I)$ by

$$\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle \ \text{ iff } \ A_1 \subseteq A_2 \text{ (iff } B_2 \subseteq B_1). \tag{5}$$

The set $\mathcal{B}(X, Y, I)$ equipped by $\leq$ is called a (fuzzy) concept lattice of $\langle X, Y, I\rangle$.

Remark 1 (1) Using basic principles of fuzzy logic, one can see that $A^{\Uparrow}(y)$ is a truth degree of "for each object x: if x belongs to A then x has attribute

y". Therefore, $A^{\Uparrow}$ is a fuzzy set of all attributes shared by all objects from A. Analogously, $B^{\Downarrow}$ is a fuzzy set of all objects sharing all attributes from B.

(2) Therefore, $\langle A, B\rangle$ is a formal concept iff A is the fuzzy set of all objects sharing all attributes from B and, B is the fuzzy set of all attributes shared by all objects from A. Elements of $\text{Ext}(X, Y, I)$ are called extents; elements of $\text{Int}(X, Y, I)$ are called intents.

(3) An intuitive interpretation and terminology comes from Port-Royal approach to concepts [C3]. Under Port-Royal, a concept is understood as consisting of a collection A of objects to which it applies and a collection B of attributes to which it applies. Example: extent of concept DOG consists of all dogs, intent of DOG consists of all attributes common to dogs ("barks", "has a tail", etc.). Note that from the point of view of fuzzy approach it is quite natural that extents and intents of concepts are fuzzy sets. Namely, this allows to capture vaguely delineated concepts like LARGE DOG.

(4) Partial order $\leq$ is interpreted as a subconcept-superconcept hierarchy. Namely, $\langle A_1, B_1\rangle \leq \langle A_2, B_2\rangle$ means that $\langle A_2, B_2\rangle$ is more general than $\langle A_1, B_1\rangle$ since it applies to a larger collection of objects (or, equivalently, applies to a smaller collection of attributes). The structure of concept lattices will be investigated later. Among others, we will see that $\mathcal{B}(X, Y, I)$ equipped with $\leq$ is indeed a complete lattice.

(5) Later on, we will study modifications of $^{\Uparrow}$ and $^{\Downarrow}$. Nevertheless, we start with $^{\Uparrow}$ and $^{\Downarrow}$ since, as we will see later, they play the role of basic arrow operators.

(6) One can see that for $\mathbf{L} = \mathbf{2}$ (two-element Boolean algebra), the above notions coincide with the corresponding notions from ordinary FCA (provided we identify crisp fuzzy sets/relations with ordinary sets/relations).

(7) Historical remark: The first approach to concept lattices is [C8]. However, the authors did not use residuated implication and, as a result, did not go too far in their study. Note that later [C10], they proposed to use implications including residuated ones. Another study of FCA in a fuzzy setting is presented in Pollandt's [C42]. This study uses residuated implication and the arrow operators are defined as in (1). The author of the dissertation started to study FCA in a fuzzy setting around 1997, at that time unaware of [C42]. Prior to getting learning about [C42], there is some small overlap in the results of [C42] and the author's results. In general, the author's results go farther than those of [C42]. We will comment on this later on. There are several other definitions of the arrow operators which can be found in the literature. We will provide a brief comparison later.

Alternatively, formal concepts can be defined as maximal rectangles contained in $\langle X, Y, I\rangle$. Call a rectangle any pair $\langle A, B\rangle \in \mathbf{L}^X \times \mathbf{L}^Y$. Put $\langle A_1, B_1\rangle \sqsubseteq \langle A_2, B_2\rangle$ iff for each $x \in X$ and $y \in Y$ we have $A_1(x) \leq A_2(x)$ and $B_1(y) \leq B_2(y)$ ($\langle A_1, B_1\rangle$ is a subrectangle of $\langle A_2, B_2\rangle$). We say that $\langle A, B\rangle$ is contained in I iff for each $x \in X$ and $y \in Y$ we have $A(x) \otimes B(y) \leq I(x, y)$. Then we have

Theorem 2 ([B5]) *$\langle A, B\rangle$ is a formal concept of $\langle X, Y, I\rangle$ iff $\langle A, B\rangle$ is maximal (w.r.t. $\sqsubseteq$) rectangle contained in I.*

Remark 2 Theorem 2 provides a useful way of looking at formal concepts. In crisp case (table contains $\times$'s and blanks), Theorem 2 says that formal concepts are maximal rectangles in the table which are full of $\times$'s.

2.2 Fuzzy Galois connections and closure operators

We now turn to selected results on Galois connections and closure operators in a fuzzy setting which are the basic structures related to the arrow operators $^{\Uparrow}$ and $^{\Downarrow}$. These results are taken from [A1], [A3], [A5], [A6], to which we refer for details (further results, comments, examples, etc.).

Fuzzy Galois connections

Throughout this section, K denotes a $\leq$-filter in $\mathbf{L}$, i.e. $K \subseteq L$ satisfies that if $a \in K$ and $a \leq b$ then $b \in K$. Sometimes, K is assumed to be a filter in $\mathbf{L}$, i.e. a $\leq$-filter satisfying $a \otimes b \in K$ whenever $a, b \in K$. An $\mathbf{L}_K$-Galois connection between non-empty sets X and Y is a pair $\langle ^{\Uparrow}, ^{\Downarrow}\rangle$ of mappings $^{\Uparrow} : \mathbf{L}^X \to \mathbf{L}^Y$, $^{\Downarrow} : \mathbf{L}^Y \to \mathbf{L}^X$, satisfying

$$S(A_1, A_2) \leq S(A_2^{\Uparrow}, A_1^{\Uparrow}) \quad \text{whenever } S(A_1, A_2) \in K, \tag{6}$$

$$S(B_1, B_2) \leq S(B_2^{\Downarrow}, B_1^{\Downarrow}) \quad \text{whenever } S(B_1, B_2) \in K, \tag{7}$$

$$A \subseteq A^{\Uparrow\Downarrow}, \tag{8}$$

$$B \subseteq B^{\Downarrow\Uparrow}, \tag{9}$$

for every $A, A_1, A_2 \in \mathbf{L}^X$, $B, B_1, B_2 \in \mathbf{L}^Y$.

Remark 3 (1) We usually omit the term "between X nd Y" and say just $\mathbf{L}_K$-Galois connection. For $\mathbf{L} = \mathbf{2}$ (ordinary case), we obtain the usual notion of a Galois connection between sets.

(2) K controls the meaning of the antitony conditions (6) and (7). Two important cases are $K = L$ and $K = \{1\}$. For instance, (6) becomes "$S(A_1, A_2) \leq S(A_2^{\Uparrow}, A_1^{\Uparrow})$" for $K = L$, and it becomes "if $A_1 \subseteq A_2$ then $A_2^{\Uparrow} \subseteq A_1^{\Uparrow}$" for $K = \{1\}$. Clearly, for $K_1 \subseteq K_2$, each $\mathbf{L}_{K_2}$-Galois connection is also an $\mathbf{L}_{K_1}$-Galois connection.

(3) (6)–(9) can be simplified [A3]: $\langle ^{\Uparrow}, ^{\Downarrow}\rangle$ is an $\mathbf{L}_K$-Galois connection iff $S(A, B) \in K$ or $S(B, A) \in K$ implies $S(A, B^{\Downarrow}) = S(B, A^{\Uparrow})$.

(4) $\mathbf{L}_K$-Galois connections obey several useful properties which we omit here due to lack of space.

Axiomatic characterization of arrow operators The arrow operators defined by (1) can be characterized axiomatically. Namely, they turn out to be just $\mathbf{L}_L$-Galois connections:

Theorem 3 ([A1]) *For a binary* $\mathbf{L}$*-relation* I *between* X *and* Y *denote by* $\langle {}^{\Uparrow_I}, {}^{\Downarrow_I} \rangle$ *the mappings defined by* (1)*. For an* $\mathbf{L}_L$*-Galois connection* $\langle {}^{\Uparrow}, {}^{\Downarrow} \rangle$ *between* X *and* Y *denote* $I_{\langle \Uparrow, \Downarrow \rangle}$ *a binary* $\mathbf{L}$*-relation between* X *and* Y *defined by*

$$I_{\langle \Uparrow, \Downarrow \rangle}(x,y) \;=\; \{{}^1/x\}^{\Uparrow}(y) \;=\; \{{}^1/y\}^{\Downarrow}(x).$$

Then $\langle {}^{\Uparrow_I}, {}^{\Downarrow_I} \rangle$ *is an* $\mathbf{L}_L$*-Galois connection and* $I \mapsto \langle {}^{\Uparrow_I}, {}^{\Downarrow_I} \rangle$ *and* $\langle {}^{\Uparrow}, {}^{\Downarrow} \rangle \mapsto I_{\langle \Uparrow, \Downarrow \rangle}$ *define a bijective correspondence between binary* $\mathbf{L}$*-relations and* $\mathbf{L}_L$*-Galois connections between* X *and* Y*.*

Remark 4 Theorem 3 generalizes a classical result by Ore [C39].

Representation by ordinary Galois connections: case 1 A natural question regarding the relationship of ordinary and fuzzy concept lattices is the following: Isn't there some simple relationship between the arrow operators ${}^{\Uparrow_I}$ and ${}^{\Downarrow_I}$ induced by a fuzzy relation I on the one hand, and the ordinary arrow operators ${}^{\Uparrow_{{}^aI}}$ and ${}^{\Downarrow_{{}^aI}}$ induced by a-cuts ${}^aI$ of I? For instance, isn't it the case that ${}^a(A^{\Uparrow_I}) = ({}^aA)^{\Uparrow_{{}^aI}}$, i.e. that $A^{\Uparrow_I}$ can be computed cut-by-cut using ${}^{\Uparrow_{{}^aI}}$'s? If yes, this would imply some simple relationships between $\mathcal{B}(X,Y,I)$ and $\mathcal{B}(X,Y,{}^aI)$. It turns out that the answer to the above question is negative. Nevertheless, there is a relationship between fuzzy Galois connections and ordinary Galois connections which we present here. It consists in establishing a bijective correspondence between $\mathbf{L}_L$-Galois connections and particular systems of ordinary Galois connections.

A system $\{\langle {}^{\Uparrow_a}, {}^{\Downarrow_a} \rangle \mid a \in L\}$ of ordinary Galois connections between X and Y is called $\mathbf{L}$-nested if (1) for each $a, b \in L$, $a \leq b$, $A \subseteq X$, $B \subseteq Y$, we have $A^{\Uparrow_a} \supseteq A^{\Uparrow_b}$, $B^{\Downarrow_a} \supseteq B^{\Downarrow_b}$, (2) for each $x \in X$, $y \in Y$, the set $\{a \in L \mid y \in \{x\}^{\Uparrow_a}\}$ has a greatest element. The we have:

Theorem 4 ([A1],[B1]) *For an* $\mathbf{L}_L$*-Galois connection* $\langle {}^{\Uparrow}, {}^{\Downarrow} \rangle$ *denote* $\mathcal{C}_{\langle \Uparrow, \Downarrow \rangle} = \{\langle {}^{\Uparrow_a}, {}^{\Downarrow_a} \rangle \mid a \in L\}$ *where* ${}^{\Uparrow_a} : 2^X \to 2^Y$ *and* ${}^{\Downarrow_a} : 2^Y \to 2^X$ *are defined by* $A^{\Uparrow_a} = {}^a(A^{\Uparrow})$ *and* $B^{\Downarrow_a} = {}^a(B^{\Downarrow})$ *for* $A \in 2^X$, $B \in 2^Y$*. For an* $\mathbf{L}$*-nested system* $\mathcal{C} = \{\langle {}^{\Uparrow_a}, {}^{\Downarrow_a} \rangle \mid a \in L\}$ *of ordinary Galois connections denote* $\langle {}^{\Uparrow_\mathcal{C}}, {}^{\Downarrow_\mathcal{C}} \rangle$ *the pair of* ${}^{\Uparrow_\mathcal{C}} : L^X \to L^Y$ *and* ${}^{\Downarrow_\mathcal{C}} : L^Y \to L^X$ *defined for* $A \in L^X$, $B \in L^Y$ *by*

$$A^{\Uparrow_\mathcal{C}}(y) = \bigvee\{a \mid y \in \textstyle\bigcap_{b \in L} ({}^bA)^{\Uparrow_{a \otimes b}}\}, \quad B^{\Downarrow_\mathcal{C}}(x) = \bigvee\{a \mid x \in \textstyle\bigcap_{b \in L} ({}^bB)^{\Uparrow_{a \otimes b}}\}.$$

Then

(1) $\mathcal{C}_{\langle \Uparrow, \Downarrow \rangle}$ *is a nested system of* $\mathbf{L}$*-Galois connections,*

(2) $\langle {}^{\Uparrow_\mathcal{C}}, {}^{\Downarrow_\mathcal{C}} \rangle$ *is an* $\mathbf{L}$*-Galois connection,*
(3) $\langle {}^{\Uparrow}, {}^{\Downarrow} \rangle \mapsto \mathcal{C}_{\langle \Uparrow, \Downarrow \rangle}$ *and* $\mathcal{C} \mapsto \langle {}^{\Uparrow_\mathcal{C}}, {}^{\Downarrow_\mathcal{C}} \rangle$ *define bijective correspondence between* $\mathbf{L}_L$*-Galois connections and* $\mathbf{L}$*-nested systems of ordinary Galois connections.*

Remark 5 (1) Note that Theorem 4 can be obtained as a consequence of results on cut-like semantics for fuzzy logic as presented in [B1]. A particular (and trivial) case of the cut-like semantics is a result on representation of fuzzy sets by their a-cuts.

(2) Theorem 4 can be used to get insight to some approaches to FCA in a fuzzy setting which are based on decomposing $\langle X, Y, I \rangle$ into the cuts $\langle X, Y, {}^aI \rangle$, see [B23].

Representation by ordinary Galois connections: case 2 We now present another representation of fuzzy Galois connections by ordinary Galois connections. It consists in establishing a bijective correspondence between $\mathbf{L}_{\{1\}}$-Galois connections between X and Y and particular ordinary Galois connections between $X \times L$ and $Y \times L$. This representation is useful for establishing a relationship between fuzzy and ordinary concept lattices.

For $A \in \mathbf{L}^U$ let $\lfloor A \rfloor \subseteq U \times L$ be defined by $\lfloor A \rfloor = \{\langle u, a \rangle \mid a \leq A(u)\}$. Thus, $\lfloor A \rfloor$ is the "area below the membership function A" in $U \times L$. For $A \subseteq U \times L$ let $\lceil A \rceil \in \mathbf{L}^U$ be defined by $\lceil A \rceil(u) = \bigvee\{a \mid \langle u, a \rangle \in A\}$. Thus, $\lceil A \rceil$ is a fuzzy set in U resulting as an "upper envelope of A". Call an ordinary Galois connection $\langle {}^\wedge, {}^\vee \rangle$ between $X \times L$ and $Y \times L$ is called commutative w.r.t. $\lfloor \lceil\ \rceil \rfloor$ if for each $A \subseteq X \times L$, $B \subseteq Y \times L$ we have

$$\lfloor \lceil A \rceil \rfloor^\wedge = \lfloor \lceil A^\wedge \rceil \rfloor \quad \text{and} \quad \lfloor \lceil B \rceil \rfloor^\vee = \lfloor \lceil B^\vee \rceil \rfloor. \tag{10}$$

For a pair $\langle {}^\wedge, {}^\vee \rangle$ of mappings ${}^\wedge : X \times L \to Y \times L$, ${}^\vee : Y \times L \to X \times L$ introduce a pair of mappings ${}^{\Uparrow_{\langle \wedge, \vee \rangle}} : L^X \to L^Y$, ${}^{\Downarrow_{\langle \wedge, \vee \rangle}} : L^Y \to L^X$ by

$$A^{\Uparrow_{\langle \wedge, \vee \rangle}} = \lceil \lfloor A \rfloor^\wedge \rceil \quad \text{and} \quad B^{\Downarrow_{\langle \wedge, \vee \rangle}} = \lceil \lfloor B \rfloor^\vee \rceil \tag{11}$$

for $A \in L^X$, $B \in L^Y$. For a pair $\langle {}^\Uparrow, {}^\Downarrow \rangle$ of mappings ${}^\Uparrow : L^X \to L^Y$, ${}^\Downarrow : L^Y \to L^X$ define a pair of mappings ${}^{\wedge_{\langle \Uparrow, \Downarrow \rangle}} : X \times L \to Y \times L$, ${}^{\vee_{\langle \Uparrow, \Downarrow \rangle}} : Y \times L \to X \times L$ by

$$A^{\wedge_{\langle \Uparrow, \Downarrow \rangle}} = \lfloor \lceil A \rceil^\Uparrow \rfloor \quad \text{and} \quad B^{\vee_{\langle \Uparrow, \Downarrow \rangle}} = \lfloor \lceil B \rceil^\Downarrow \rfloor \tag{12}$$

for $A \subseteq X \times L$, $B \subseteq Y \times L$. Then we have:

Theorem 5 ([A4]) *Let* $\langle {}^\Uparrow, {}^\Downarrow \rangle$ *be an* $\mathbf{L}_{\{1\}}$*-Galois connection between* X *and* Y *and* $\langle {}^\wedge, {}^\vee \rangle$ *be a ordinary Galois connection between* $X \times L$ *and* $Y \times L$ *which is commutative w.r.t.* $\lfloor \lceil\ \rceil \rfloor$*. Then*

(1) $\langle {}^{\wedge_{\langle \Uparrow, \Downarrow \rangle}}, {}^{\vee_{\langle \Uparrow, \Downarrow \rangle}} \rangle$ *is an ordinary Galois connection between* $X \times L$ *and* $Y \times L$ *which is commutative w.r.t.* $\lfloor \lceil\ \rceil \rfloor$*;*

(2) $\langle {}^{\Uparrow_{\langle \wedge, \vee \rangle}}, {}^{\Downarrow_{\langle \wedge, \vee \rangle}} \rangle$ *is an* $\mathbf{L}_{\{1\}}$*-Galois connection between* X *and* Y*;*

(3) *Sending* $\langle {}^{\Uparrow}, {}^{\Downarrow} \rangle$ *to* $\langle {}^{\wedge_{\langle \Uparrow, \Downarrow \rangle}}, {}^{\vee_{\langle \Uparrow, \Downarrow \rangle}} \rangle$ *and* $\langle {}^{\wedge}, {}^{\vee} \rangle$ *to* $\langle {}^{\Uparrow_{\langle \wedge, \vee \rangle}}, {}^{\Downarrow_{\langle \wedge, \vee \rangle}} \rangle$ *defines a bijective correspondence between* $\mathbf{L}_{\{1\}}$*-Galois connections between* X *and* Y *and commutative Galois connections between* $X \times L$ *and* $Y \times L$*.*

This observation has some important consequences for the relationship between fuzzy concept lattices and ordinary concept lattices. We now present selected results. Under the above notation, denote $\mathcal{B}(X, Y, \langle {}^{\Uparrow}, {}^{\Downarrow} \rangle) = \{\langle A, B \rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^{\Uparrow} = B, B^{\Downarrow} = A\}$ and $\mathcal{B}(X \times L, Y \times L, \langle {}^{\wedge}, {}^{\vee} \rangle) = \{\langle A, B \rangle \in 2^{X \times L} \times 2^{Y \times L} \mid A^{\wedge} = B, B^{\vee} = A\}$, i.e. the sets of fixpoints of the respective Galois connections. Note that if $\langle {}^{\Uparrow}, {}^{\Downarrow} \rangle$ are the arrow operators induced by $\langle X, Y, I \rangle$, then $\mathcal{B}(X, Y, \langle {}^{\Uparrow}, {}^{\Downarrow} \rangle)$ is just the $\mathbf{L}$-concept lattice $\mathcal{B}(X, Y, I)$. Then, using

Lemma 1 ([A4]) *For any* $\mathbf{L}_K$*-Galois connection* $\langle {}^{\Uparrow}, {}^{\Downarrow} \rangle$*, if* $\langle {}^{\wedge}, {}^{\vee} \rangle = \langle {}^{\wedge_{\langle \Uparrow, \Downarrow \rangle}}, {}^{\vee_{\langle \Uparrow, \Downarrow \rangle}} \rangle$ *as in Theorem 5, then* $\mathcal{B}\left(X, Y, \langle {}^{\Uparrow}, {}^{\Downarrow} \rangle\right)$ *and* $\mathcal{B}\left(X \times L, Y \times L, \langle {}^{\wedge}, {}^{\vee} \rangle\right)$ *are isomorphic lattices. Moreover,* $\mathcal{B}\left(X \times L, Y \times L, \langle {}^{\wedge}, {}^{\vee} \rangle\right) = \mathcal{B}\left(X \times L, Y \times L, I^{\times}\right)$ *where* $I^{\times} \subseteq (X \times L) \times (Y \times L)$ *is defined by* $\langle \langle x, a \rangle, \langle y, b \rangle \rangle \in I^{\times}$ *iff* $b \leq \{ {}^{a}/x \}^{\Uparrow}(y)$*.*

one can prove

Theorem 6 ([A4]) *Any* $\mathbf{L}$*-concept lattice* $\mathcal{B}(X, Y, I)$ *is isomorphic to the ordinary concept lattice* $\mathcal{B}\left(X \times L, Y \times L, I^{\times}\right)$ *where* $\langle \langle x, a \rangle, \langle y, b \rangle \rangle \in I^{\times}$ *iff* $a \otimes b \leq I(x, y)$*. An isomorphism is given by sending* $\langle A, B \rangle \in \mathcal{B}(X, Y, I)$ *to* $\langle \lfloor A \rfloor, \lfloor B \rfloor \rangle \in \mathcal{B}\left(X \times L, Y \times L, I^{\times}\right)$*.*

An almost direct consequence of Lemma 1 and Theorem 6 we get a theorem characterizing the lattice of fixed points of $\mathbf{L}_{\{1\}}$-Galois connections [A4] (Theorem 3.4) a particular case of which is the following theorem.

Theorem 7 ([A4]) *Let* $\langle X, Y, I \rangle$ *be a data table with fuzzy attributes.* (1) *Then* $\mathcal{B}(X, Y, I)$ *is a complete lattice w.r.t.* $\leq$ *where the suprema and infima are given by*

$$\bigwedge_{j \in J} \langle A_j, B_j \rangle = \langle \bigcap_{j \in J} A_j, (\bigcup_{j \in J} B_j)^{\Downarrow\Uparrow} \rangle, \bigvee_{j \in J} \langle A_j, B_j \rangle = \langle (\bigcup_{j \in J} A_j)^{\Uparrow\Downarrow}, \bigcap_{j \in J} B_j \rangle .$$

(2) *Moreover, an arbitrary complete lattice* $\mathbf{V} = \langle V, \leq \rangle$ *is isomorphic to* $\mathcal{B}(X, Y, I)$ *iff there are mappings* $\gamma : X \times L \to V$*,* $\mu : Y \times L \to V$ *such that*

(i) $\gamma(X, L)$ *is supremally dense in* V*,* $\mu(Y, L)$ *is infimally dense in* V*;*

(ii) $\gamma(x, a) \leq \mu(y, b)$ *iff* $a \otimes b \leq I(x, y)$*.*

Note that Theorem 6 is a "reduction theorem" which, in principle, enables us to reduce several problems concerning fuzzy concept lattices (e.g., computing a fuzzy concept lattice) to the corresponding problems of ordinary concept lattices. We will go back to this issue later on. Theorem 7 plays a role of a Main theorem for concept lattices in a fuzzy setting. Note that Theorem 1, i.e. the Main theorem for ordinary concept lattices, is a particular case of Theorem 7. As we will see in Section 2.3, Theorem 7 is a version of the main theorem for concept lattices which concerns crisp order on $\mathcal{B}(X, Y, I)$. The other version, concerning fuzzy order on $\mathcal{B}(X, Y, I)$, will be presented in Section 2.3 where we will also see an alternative way to prove Theorem 7 (directly, not via reduction to the ordinary case).

Fuzzy closure operators

Fuzzy closure operators are important structures widely studied in fuzzy set theory, see e.g. [B1], [C21]. They are closely related to FCA in a fuzzy setting, but play a role in other areas as well, analogously as in case of ordinary closure operators. Let K be a filter in $\mathbf{L}$ (in some cases, $\leq$-filter suffices). An $\mathbf{L}_K$-closure operator in a non-empty set X is a mapping $\mathrm{C} : L^X \to L^X$ satisfying

$$A \subseteq \mathrm{C}(A), \tag{13}$$

$$S(A_1, A_2) \leq S(\mathrm{C}(A_1), \mathrm{C}(A_2)) \quad \text{whenever } S(A_1, A_2) \in K, \tag{14}$$

$$\mathrm{C}(A) = \mathrm{C}(\mathrm{C}(A)) \tag{15}$$

for every $A, A_1, A_2 \in L^X$.

Remark 6 As in case of $\mathbf{L}_K$-Galois connections, K influences the meaning of the monotony condition (14). Two important cases are $K = L$ and $K = \{1\}$ for which (14) becomes "$S(A_1, A_2) \leq S(\mathrm{C}(A_1), \mathrm{C}(A_2))$" and "if $A_1 \subseteq A_2$ then $\mathrm{C}(A_1) \subseteq \mathrm{C}(A_2)$". Note that most of the literature on fuzzy closure operators deals with $K = \{1\}$ only.

Results of the dissertation related to fuzzy closure operators are contained mainly in [A3], [A5], [A6]. In what follows, we present selected results of these papers.

The first result concerns a characterization of systems of fixpoints of $\mathbf{L}_K$ closure operators. Recall that it is well known from an ordinary case that a system $\mathcal{S}$ of subsets of X is a system of fixpoints of some closure operator on X iff it is closed under arbitrary intersections. In our setting we have

Theorem 8 ([A3]) *A system $\mathcal{S} \subseteq \mathbf{L}^X$ is a system of fixpoints of some $\mathbf{L}_K$-closure operator C in X, i.e. $\mathcal{S} = \{A \in \mathbf{L}^X \mid A = \mathrm{C}(A)\}$, iff for each $a \in K$ and $A \in \mathcal{S}$ we have $a \to A \in \mathcal{S}$ and for any $A_i \in \mathcal{S}$ $(i \in I)$ we have $\bigcap_{i \in I} A_i \in \mathcal{S}$.*

Remark 7 (1) Note that $a \to A$ (a-shift of A) is defined by $(a \to A)(x) = a \to A(x)$. That is, systems of fixpoints are just systems closed under a-shifts for $a \in K$ and closed under arbitrary intersections.

(2) [A3] contains further characterizations of systems of fixpoints of fuzzy closure operators and describes explicitly the operators bijective mappings between $\mathbf{L}_K$-closure operators and systems of their fixpoints. For interior operators, the corresponding results are in [A10] (note that due to lack of the law of double negation, we cannot get these results dually from the results on closure operators as in the ordinary case).

(3) [A5] contains further results on $\mathbf{L}_K$-closure operators, namely: fuzzy closure operators induced by binary fuzzy relations; representation of $\mathbf{L}_{\{1\}}$-closure operators in X by ordinary closure operators in $X \times L$; operators of consequence and some furhter results.

Fuzzy closure operators induced by similarity An important case of fuzzy closure operators studied in literature comes from the formula

$$\mathrm{C}_{\approx}(A)(y) = \bigvee_{y \in X}(A(x) \otimes (x \approx y))$$

where $\approx$ is an $\mathbf{L}$-equivalence on X (called also similarity in fuzzy set literature), see Section 1.3. $\mathrm{C}_{\approx}$ is a mapping of $\mathbf{L}^X$ into itself. $\mathrm{C}_{\approx}(A)$ is usually called an extensional hull of A. Fuzzy sets A which are closed under $\mathrm{C}_{\approx}$ are sometimes called compatible with $\approx$ (alternative characteziration: $A(x) \otimes (x \approx y) \leq A(y)$). It turns out that $\mathrm{C}_{\approx}$ is a particular fuzzy closure operator (fuzzy closure operator induced by similarity) the characterization of which is given by the following theorem:

Theorem 9 ([A6]) (1) *Let $\approx$ be an $\mathbf{L}$-equivalence on X. Then $\mathrm{C}_{\approx}$ is an $\mathbf{L}_L$-closure operator on X satisfying, moreover,*

$$\begin{aligned}
\mathrm{C}_{\approx}(\textstyle\bigcup_{i \in I} A_i) &= \textstyle\bigcup_{i \in I} \mathrm{C}_{\approx}(A_i), && (16)\\
\mathrm{C}_{\approx}(\{{}^{a}/x\}) &= a \otimes \mathrm{C}_{\approx}(\{{}^{a}/x\}), && (17)\\
\mathrm{C}_{\approx}(\{{}^{1}/x\})(y) &= \mathrm{C}_{\approx}(\{{}^{1}/y\})(x), && (18)
\end{aligned}$$

for any $A_i \in \mathbf{L}^X$ $(i \in I)$, $x, y \in X$, $a \in L$.
(2) *Let* C *be an $\mathbf{L}_L$-closure operator on X satisfying* (16)–(18). *Then putting $x \approx_{\mathrm{C}} y = \mathrm{C}(\{{}^{1}/x\})(y)$, $\approx_{\mathrm{C}}$ is an $\mathbf{L}$-equivalence on X.*
(3) *The mappings sending $\approx$ to $\mathrm{C}_{\approx}$ and* C *to $\approx_{\mathrm{C}}$ provide a bijection between $\mathbf{L}$-equivalences and $\mathbf{L}_L$-closure operators in X satisfying* (16)–(18).

Remark 8 [A6] contains some further results: characterization of systems of fixpoints of of fuzzy closure operators $\mathrm{C}_{\approx}$ (they are just systems $\mathcal{S}$ of fixpoints of $\mathbf{L}_L$-closure operators which are, moreover, closed under arbitrary unions and satisfy $A \to a \in \mathcal{S}$ a $a \otimes A \in \mathcal{S}$ for any $a \in L$ and $A \in \mathcal{S}$); results on relationship of similarity-based closure and metric closure.

Fuzzy closure operators and Galois connections In this section, we present selected results on relationships between fuzzy closure operators and fuzzy Galois connections. We have seen that the arrow operators $^{\Uparrow}$ and $^{\Downarrow}$ induced by a table with fuzzy attributes form an $\mathbf{L}_L$-Galois connection. The following result is an excerpt of results from [A3] which describe a bijective correspondence between $\mathbf{L}_K$-Galois connections and pairs of $\mathbf{L}_K$-closure operators with dually isomorphic systems of fixpoints.

Theorem 10 ([A3]) *Let $\langle{}^{\Uparrow},{}^{\Downarrow}\rangle$ be an $\mathbf{L}_L$-Galois connection between X and Y, C be an $\mathbf{L}_L$-closure operator on X. Then*

(1) $\mathrm{C}_{\langle\Uparrow,\Downarrow\rangle}: \mathbf{L}^X \to \mathbf{L}^X$ *defined by* $\mathrm{C}_{\langle\Uparrow,\Downarrow\rangle}(A) = A^{\Uparrow\Downarrow}$ *is an $\mathbf{L}_L$-closure operator on X;*

(2) *for $Y = \{A \in \mathbf{L}^X \mid A = \mathrm{C}(A)\}$, operators $^{\Uparrow_\mathrm{C}}$ and $^{\Downarrow_\mathrm{C}}$ defined by*

$$A^{\Uparrow_\mathrm{C}}(A') = S(A, A'), \quad B^{\Downarrow_\mathrm{C}}(x) = \textstyle\bigwedge_{A\in Y} B(A) \to A(x)$$

form an $\mathbf{L}_L$-Galois connection between X and Y;

(3) $\mathrm{C} = \mathrm{C}_{\langle\Uparrow_\mathrm{C},\Downarrow_\mathrm{C}\rangle}$.

Therefore, given $\langle X, Y, I\rangle$, both $^{\Uparrow\Downarrow}$ and $^{\Downarrow\Uparrow}$ are $\mathbf{L}_L$-closure operators.

Computing a concept lattice Since (easy to see)

$$\mathcal{B}(X,Y,I) = \{\langle A, A^{\Uparrow}\rangle \mid A \in \mathrm{Ext}(X,Y,I)\} \text{ and } \mathrm{Ext}(X,Y,I) = \mathrm{fix}(^{\Uparrow\Downarrow}),$$

where $\mathrm{fix}(^{\Uparrow\Downarrow})$ is the set of all fixpoints of $^{\Uparrow\Downarrow}$, in order to compute $\mathcal{B}(X,Y,I)$, it is sufficient if we are able to compute $\mathrm{fix}(\mathrm{C})$ for a given fuzzy closure operator C. Computing systems of fixpoints of fuzzy closure operators appears several times in FCA (we will see some cases later). For this purpose, we now briefly present and algorithm which is an extension of Ganter's NextClosure algorithm [C17] to our setting, for details see [A8]. The algorithm outputs all fixed points of C in a lexicographic order defined below.

Suppose $X = \{1, 2, \ldots, n\}$; $L = \{0 = a_1 < a_2 < \cdots < a_k = 1\}$ (the assumption that L is linearly ordered is in fact not essential). For $i, r \in \{1, \ldots, n\}$, $j, s \in \{1, \ldots, k\}$ we put

$$(i, j) \leq (r, s) \quad \text{iff} \quad i < r \quad \text{or} \quad i = r, a_j \geq a_s.$$

For $A \in \mathbf{L}^X$, $(i, j) \in X \times \{1, \ldots, k\}$, put

$$A \oplus (i, j) := \mathrm{C}((A \cap \{1, 2, \ldots, i-1\}) \cup \{\, {}^{a_j}/i\}).$$

Furthermore, for $A, B \in \mathbf{L}^X$, define

$$A <_{(i,j)} B \quad \text{iff} \quad A \cap \{1, \dots, i-1\} = B \cap \{1, \dots, i-1\} \text{ and } A(i) < B(i) = a_j,$$
$$A < B \quad \text{iff} \quad A <_{(i,j)} B \quad \text{for some } (i, j).$$

$<$ is a lexicographic order on $\mathbf{L}^X$ and we have:

Theorem 11 ([A8]) *The least fixed point A^+ which is greater (w.r.t. $<$) than a given $A \in \mathbf{L}^X$ is given by*

$$A^+ = A \oplus (i, j)$$

where (i, j) is the greatest one with $A <_{(i,j)} A \oplus (i, j)$.

The algorithm for computing fix($^{\Uparrow\Downarrow}$) starts with C($\emptyset$) (the least fixpoint of C) and using Theorem 11 generates all other fixpoints up to X in a lexicographic order $<$, see [A8]. Note that due to Theorem 6, $\mathcal{B}(X, Y, I)$ can, in principle, be computed using algorithms for ordinary concept lattices.

2.3 Main theorem on concept lattices

From certain point of view, Theorem 7 is not satisfactory. It concerns an ordinary partial order $\leq$ on $\mathcal{B}(X, Y, I)$, while $\mathcal{B}(X, Y, I)$ can naturally be considered as equipped with a fuzzy partial order $\preceq$ and a fuzzy equality $\approx$ defined by

$$(\langle A_1, B_1 \rangle \preceq \langle A_2, B_2 \rangle) = \bigwedge_{x \in X}(A_1(x) \to A_2(x)) = \bigwedge_{y \in Y}(B_2(y) \to B_1(y)),$$
$$(\langle A_1, B_1 \rangle \approx \langle A_2, B_2 \rangle) = \bigwedge_{x \in X}(A_1(x) \leftrightarrow A_2(x)) = \bigwedge_{y \in Y}(B_1(y) \leftrightarrow B_2(y)). \quad (19)$$

Moreover, in the ordinary case, $\mathcal{B}(X, Y, I)$ equipped with $\leq$ is isomorphic to $\mathcal{B}(\mathcal{B}(X, Y, I), \mathcal{B}(X, Y, I), \leq)$ and if $\langle V, \leq \rangle$ is a partially ordered set then $\mathcal{B}(V, V, \leq)$ is the Dedekind-MacNeille completion of $\langle V, \leq \rangle$ [C17]. Therefore, it is interesting to ask whether we can have analogous results and notions (like that of a complete lattice) in a fuzzy setting as well. This problem was studied in [A7]. Without going into details, we now summarize the main results.

An **L**-ordered set is a pair $\langle \langle V, \approx \rangle \preceq \rangle$ where $\approx$ is an **L**-equality on V (see Section 1.3) and $\preceq$ is an **L**-order on $\langle V, \approx \rangle$, i.e. $\preceq$ is reflexive, transitive (see Section 1.3), and satisfies $(u \preceq v) \wedge (v \preceq u) \leq (u \approx v)$ (antisymmetry). Then, one can introduce the notions of infimum, supremum, infimal and supremal density, etc., in an **L**-ordered set and obtain the following theorem which, from the above point of view is "the proper" version of the Main theorem of concept lattices in a fuzzy setting:

Theorem 12 ([A7]) (1) *$\langle \langle \mathcal{B}(X, Y, I), \approx \rangle, \preceq \rangle$ is completely lattice* **L**-*ordered set in which infima and suprema are described as in* [A7].

(2) *Moreover, a completely lattice* $\mathbf{L}$*-ordered set* $\mathbf{V} = \langle\langle V, \approx\rangle, \preceq\rangle$ *is isomorphic to* $\langle\langle \mathcal{B}(X,Y,I), \approx\rangle, \preceq\rangle$ *iff there are mappings* $\gamma : X \times L \to V$, $\mu : Y \times L \to V$*, such that* $\gamma(X \times L)$ *is supremally dense in* $\mathbf{V}$*,* $\mu(Y \times L)$ *is infimally dense in* $\mathbf{V}$*, and* $((a \otimes b) \to I(x,y)) = (\gamma(x,a) \preceq \mu(y,b))$ *for all* $x \in X$*,* $y \in Y$*,* $a, b \in L$*. In particular,* $\mathbf{V}$ *is isomorphic to* $\mathcal{B}(V, V, \preceq)$*.*

Remark 9 (1) The ordinary Main theorem of concept lattices is a particular case of Theorem 12 for $\mathbf{L} = \mathbf{2}$. Moreover, inspecting the proof of Theorem 12 gives us a direct proof of Theorem 7.

(2) If $\langle\langle V, \approx\rangle, \sqsubseteq\rangle$ is an $\mathbf{L}$-ordered set, $\langle\langle \mathcal{B}(V, V, \sqsubseteq), \approx\rangle, \preceq\rangle$ behaves the same way as the Dedekind-MacNeille completion in the ordinary case, see [A7].

(3) An interesting property was shown in [B11]: a complete lattice $\mathbf{L}$-order $\preceq$ is uniquely given by its 1-cut ${}^1{\preceq}$.

2.4 Factorization by similarity

Factor lattice by similarity In [A2], we investigated similarity relations in concept lattices and in FCA. For illustration, we now focus on factorization by similarity. Fuzzy equivalence $\approx$ defined by (19) can be interpreted as a similarity on $\mathcal{B}(X,Y,I)$. Since $\mathcal{B}(X,Y,I)$ might be large, it is natural to ask whether one can "put sufficiently similar formal concepts together" and consider a simplified version of $\mathcal{B}(X,Y,I)$ in which one identifies the "sufficiently similar" formal concepts. These ideas, studied in [A2] and then in [A10], lead to a construction of a factor lattice $\mathcal{B}(X,Y,I)/{}^a{\approx}$ of $\mathcal{B}(X,Y,I)$ which is driven by a parameter $a \in L$ supplied by a user. A brief description follows.

For a given parameter $a \in L$ (similarity threshold, supplied by a user), consider the a-cut ${}^a{\approx}$. In general, ${}^a{\approx}$ is a tolerance (i.e., reflexive and symmetric) relation on $\mathcal{B}(X,Y,I)$ containing pairs of formal concepts which are pairwise similar in degree at least a. Note that, in general, algebras can be factorized using congruence relations, i.e. compatible equivalences. Surprisingly, Czédli [C14] and later Wille [C17] showed that in case of complete lattices, factorization is possible even with compatible tolerance relations. As can be shown, ${}^a{\approx}$ is compatible with infima and suprema in $\mathcal{B}(X,Y,I)$ [A2] and, thus, we can define a factor lattice $\mathcal{B}(X,Y,I)/{}^a{\approx}$:

(1) elements of $\mathcal{B}(X,Y,I)/{}^a{\approx}$ are blocks of ${}^a{\approx}$, i.e. maximal sets $B \subseteq \mathcal{B}(X,Y,I)$ of concepts s. t. any two concepts from B are similar in degree at least a;

(2) each block B is, in fact, an interval in $\mathcal{B}(X,Y,I)$, i.e. $B = [c_1, c_2] = \{d \in \mathcal{B}(X,Y,I) \mid c_1 \le d \le c_2\}$ for some $c_1, c_2 \in \mathcal{B}(X,Y,I)$;

(3) putting $[c_1, c_2] \preceq [d_1, d_2]$ iff $c_1 \le d_1$ (iff $c_2 \le d_2$) we get:

Theorem 13 ([A2]) $\mathcal{B}(X,Y,I)/{}^a{\approx}$ *equipped with* $\preceq$ *is a complete lattice, the so-called factor lattice of* $\mathcal{B}(X,Y,I)$ *by similarity* $\approx$ *and threshold* a*.*

Elements of $\mathcal{B}(X,Y,I)/{}^a\!\approx$ can be seen as similarity-based granules of formal concepts from $\mathcal{B}(X,Y,I)$. $\mathcal{B}(X,Y,I)/{}^a\!\approx$ thus provides a granular view on (the possibly large) $\mathcal{B}(X,Y,I)$. If ${}^a\!\approx$ is transitive then it is a congruence relation on $\mathcal{B}(X,Y,I)$ and $\mathcal{B}(X,Y,I)/{}^a\!\approx$ is the usual factor lattice modulo a congruence.

Fast factorization by similarity In order to compute $\mathcal{B}(X,Y,I)/{}^a\!\approx$ using its definition one has (1) to compute the whole concept lattice $\mathcal{B}(X,Y,I)$ and then (2) to compute ${}^a\!\approx$-blocks on $\mathcal{B}(X,Y,I)$, which can be quite demanding. A question is if $\mathcal{B}(X,Y,I)/{}^a\!\approx$ can be computed directly from $\langle X,Y,I\rangle$ and a, i.e. without computing the possibly large $\mathcal{B}(X,Y,I)$. A positive answer was presented in [A10]. A brief description follows.

The method is based on the fact that each element of $\mathcal{B}(X,Y,I)/{}^a\!\approx$ is in fact an interval in $\mathcal{B}(X,Y,I)/{}^a\!\approx$, i.e. is of the form $[\langle C,D\rangle,\langle A,B\rangle]$. Furthermore, it can be shown that $\langle C,D\rangle$ is uniquely given by $\langle A,B\rangle$ and, since $B=A^{\Uparrow}$, by A. In order to generate $\mathcal{B}(X,Y,I)/{}^a\!\approx$, it is thus enough if we know how to generate the set

$$\mathrm{ESB}(a)=\{A\in \mathbf{L}^X \mid \langle A,B\rangle\in\mathcal{B}(X,Y,I) \text{ and } [\ldots,\langle A,B\rangle]\in\mathcal{B}(X,Y,I)/{}^a\!\approx\}$$

of all extents of suprema of ${}^a\!\approx$-blocks. It turns out that $\mathrm{ESB}(a)$ is just the set of fixpoints of a suitable fuzzy closure operator:

Theorem 14 ([A10]) *For any $\langle X,Y,I\rangle$ and a threshold $a\in L$, a mapping C_a sending a fuzzy set A in X to a fuzzy set $a\rightarrow(a\otimes A)^{\Uparrow\Downarrow}$ in X is a fuzzy closure operator in X for which $\mathrm{fix}(C_a)=\mathrm{ESB}(a)$.*

Computing $\mathrm{fix}(C_a)$ can be accomplished using the above algorithm. As demonstrated in [A10], the procedure just described leads to a significant speed-up compared to the "naive" method consisting in computing first $\mathcal{B}(X,Y,I)$ and then computing the ${}^a\!\approx$-blocks.

2.5 Concept lattices with hedges

The approach In [B19], we studied so-called crisply generated fuzzy concepts and related concept lattices, i.e. formal fuzzy concepts $\langle A,B\rangle\in\mathcal{B}(X,Y,I)$ such that $A=D^{\Downarrow}$ and $B=D^{\Downarrow\Uparrow}$ for some crisp set $D\subseteq Y$ of attributes. Crisply generated concepts can be identified with crisp sets of attributes and are usually considered as "the natural" concepts by users. In addition, the number of crisply generated concepts is usually significantly smaller than the number of all formal concepts, which is another advantage. Later on [A11], we introduced a parameterized approach to fuzzy concept lattices using so-called hedges, see Section 1.3. The resulting concept lattices play an interesting role. A brief description follows.

Let $*_X$ and $*_Y$ be hedges. Consider the following modification of arrow operators induced by $\langle X, Y, I\rangle$:

$$A^{\uparrow}(y) = \bigwedge_{x\in X}\big(A(x)^{*_X} \to I(x,y)\big), \quad B^{\downarrow}(x) = \bigwedge_{y\in Y}\big(B(y)^{*_Y} \to I(x,y)\big).$$

Hedges $*_X$ and $*_Y$ play the role of parameters. Note that the verbal description of $^{\uparrow}$ and $^{\downarrow}$ is almost the same as that of $^{\Uparrow}$ and $^{\Downarrow}$. For instance, $A^{\uparrow}(y)$ is a truth degree of "for each $x \in X$: if it is very true that x belongs to A then x has attribute y", etc. For $\mathbf{L} = \mathbf{2}$ (crisp case), both $\langle ^{\Uparrow}, ^{\Downarrow}\rangle$ and $\langle ^{\uparrow}, ^{\downarrow}\rangle$ coincide with the ordinary operators. Hence, with hedges, the meaning remains the same and we deal with a sound generalization of the ordinary case. A fuzzy concept lattice with hedges is then the set

$$\mathcal{B}(X^{*_X}, Y^{*_Y}, I) = \{\langle A, B\rangle \in \mathbf{L}^X \times \mathbf{L}^Y \mid A^{\uparrow} = B,\ B^{\downarrow} = A\}.$$

$\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$, equipped with a partial order $\leq$ defined by (5) is a complete lattice. The following is the Main theorem for concept lattices with hedges ($\mathrm{fix}(*) = \{a \in L \mid a^* = a\}$ denotes the fixpoints of $*$):

Theorem 15 ([A11]) (1) $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ *is under* $\leq$ *a complete lattice where the infima and suprema are given by*

$$\bigwedge_{j\in J}\langle A_j, B_j\rangle = \langle(\bigcap_{j\in J} A_j)^{\uparrow\downarrow}, (\bigcup_{j\in J} B_j^{*_Y})^{\downarrow\uparrow}\rangle, \quad \bigvee_{j\in J}\langle A_j, B_j\rangle = \langle(\bigcup_{j\in J} A_j^{*_X})^{\uparrow\downarrow}, (\bigcap_{j\in J} B_j)^{\downarrow\uparrow}\rangle.$$

(2) *Moreover, an arbitrary complete lattice* $\mathbf{K} = \langle K, \leq\rangle$ *is isomorphic to* $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ *iff there are mappings* $\gamma : X \times \mathrm{fix}(*_X) \to K$, $\mu : Y \times \mathrm{fix}(*_Y) \to K$ *such that*

(i) $\gamma(X \times \mathrm{fix}(*_X))$ *is* $\bigvee$*-dense in* K, $\mu(Y \times \mathrm{fix}(*_Y))$ *is* $\bigwedge$*-dense in* V*;*

(ii) $\gamma(x, a) \leq \mu(y, b)$ *iff* $a \otimes b \leq I(x, y)$*.*

Further results The following are selected results on concept lattices with hedges:

(1) Mutual relationships of concept lattices with hedges for different choices of hedges (stronger hedges lead to smaller concept lattices), [A11], [B25].
(2) Galois connections closure operators for the case with hedges; they play a similar role as fuzzy Galois connections and closure operators in the basic approach without hedges, see [A14], [A14].
(3) Reduction theorem analogous to Theorem 6, see [A11].
(4) $\mathcal{B}(X^{*_X}, Y, I)$ (case in which $*_Y$ is identity) plays an important role for attribute implications, see Section 3.

Other approaches to concept lattices Concept lattices with hedges provide a common generalization for several approaches to FCA in a fuzzy setting which appeared in the literature (see [B23] for details on comparison):

(1) If both $*_X$ and $*_Y$ are identities, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is just the fuzzy concept lattice (without hedges), see (2).

(2) If both $*_X$ and $*_Y$ are globalizations, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is isomorphic (and almost equal) to the ordinary concept lattice $\mathcal{B}(X, Y, {}^1I)$.

(3) If $*_X$ is identity and $*_Y$ is globalization, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ coincides with the crisply generated fuzzy concept lattice [B19]. In addition, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is isomorphic (and almost identical) to a what is called a fuzzy concept lattice in [C48]. If $*_X$ is globalization and $*_Y$ is identity, $\mathcal{B}(X^{*_X}, Y^{*_Y}, I)$ is isomorphic (and almost identical) to a "one-sided fuzzy concept lattice" of [C32].

(4) Recently introduced fuzzy concept lattices with thresholds [C16] are, again, isomorphic to concept lattices with hedges, see [B26].

2.6 Constrained concept lattices

In its basic setting, FCA (both in ordinary and fuzzy setting) works with a table $\langle X, Y, I\rangle$ is the only input data. It is, however, often the case that a user has some additional information along with the input $\langle X, Y, I\rangle$. For instance, the additional information $\mathcal{C}$ may concern the importance of attributes. $\mathcal{C}$ can then be used as a constraint in such a way that only those formal concepts which satisfy the constraint $\mathcal{C}$ are considered relevant. That is, instead of the whole $\mathcal{B}(X, Y, I)$, we are interested in

$$\mathcal{B}_{\mathcal{C}}(X, Y, I) = \{\langle A, B\rangle \in \mathcal{B}(X, Y, I) \mid \langle A, B\rangle \text{ satisfies constraint } \mathcal{C}\}.$$

In [B12], [B17], [B18], we studied some particular cases of constraints. It turned out that several seemingly different constraints are particular cases "constraints by (fuzzy) closure operators" which were introduced in [A20]. We now briefly describe the idea and some examples of these constraints. Note that the idea of constrained concept lattices provides a new method not only in a fuzzy setting but also in the ordinary setting. In our approach, a constraint is represented by a fuzzy closure operator C in the set Y of attributes (or, dually, in X). Given C, a constrained concept lattice is defined by

$$\mathcal{B}_{\mathrm{C}}(X, Y, I) = \{\langle A, B\rangle \in \mathcal{B}(X, Y, I) \mid B = \mathrm{C}(B)\}.$$

That is, a formal concept $\langle A, B\rangle$ satisfies a user's constraint (is interesting) iff B is a fixed point of C. Constrained lattices are, indeed, complete lattices:

Theorem 16 ([A20]) *Then* $\mathcal{B}_{\mathrm{C}}(X,Y,I)$ *equipped with* $\leq$ *defined by* (5) *is a complete lattice which is a* $\bigvee$*-sublattice of* $\mathcal{B}(X,Y,I)$*.*

Furthermore, the following gives a way to compute $\mathcal{B}_{\mathcal{C}}(X,Y,I)$ for finite $\mathbf{L}$: For any $B \in \mathbf{L}^Y$ define fuzzy sets B_i and $\mathcal{C}(B)$ by

$$B_i = \begin{cases} B & \text{if } i = 0, \\ \mathrm{C}({B_{i-1}}^{\downarrow\uparrow}) & \text{if } i \geq 1. \end{cases} \qquad \mathcal{C}(B) = \bigcup_{i=1}^{\infty} B_i. \tag{20}$$

Theorem 17 ([A20]) $\mathcal{C}$ *is a fuzzy closure operator such that* $\mathrm{fix}(\mathcal{C}) = \{B \in \mathbf{L}^Y \mid \langle B^{\Downarrow}, B\rangle \in \mathcal{B}_{\mathrm{C}}(X,Y,I)\}$*.*

Therefore, $\mathcal{B}_{\mathrm{C}}(X,Y,I)$ can easily be restored from the fixpoints $\mathrm{fix}(\mathcal{C})$ of $\mathcal{C}$ and the fixpoints of $\mathcal{C}$ can be computed by the algorithm presented above.

We now present selected examples of constraining fuzzy closure operators. The operators will be represented by their sets of fixpoints.

(1) INCL(Z) where $Z \in \mathbf{L}^Y$: $\mathrm{fix}(\mathrm{INCL}(Z)) = \{B \in \mathbf{L}^Y \mid Z \subseteq B\}$,
i.e. B is considered interesting iff B contains a prescribed collection Z of attributes.

(2) CARDLEQ(n) where $n \in \mathbb{N}$: $\mathrm{fix}(\mathrm{CARDLEQ}(n)) = \{B \in \mathbf{L}^Y \mid |B| \leq n\} \cup \{Y\}$,
where $|\cdots|$ is a suitably defined cardinality. Thus, B is considered interesting iff B contains at most n attributes (or $B = Y$).

(3) SUPP(n) where $n \in \mathbb{N}$: $\mathrm{fix}(\mathrm{SUPP}(n)) = \{B \in \mathbf{L}^Y \mid |B^{\Downarrow}| \geq n\} \cup \{Y\}$,
where $|\cdots|$ is a suitably defined cardinality. Thus, B is considered interesting iff the support of B (in terms of mining association rules, i.e. the number of elements sharing all attributes from B) is at least n. It is interesting to note that in crisp case ($\mathbf{L} = \mathbf{2}$), $\langle A, B\rangle \in \mathcal{B}_{\mathrm{SUPP}(n)}(X,Y,I)$ iff B is a so-called closed frequent itemset. Closed frequent itemsets are used for mining non-redundant associaton rules, see e.g. [C49].

(4) FACTOR(a) where $a \in L$: $[\mathrm{FACTOR}(a)](A) = a \to (a \otimes A)^{\Uparrow\Downarrow}$.
This shows that factorization by similarity described in Section 2.4 can be considered a particular case of constraining by fuzzy closure operators. Namely, $\mathcal{B}_{\mathrm{FACTOR}(a)}(X,Y,I)$ is isomorphic to the factor lattice $\mathcal{B}(X,Y,I)/{}^a{\approx}$.

Further examples (e.g. further constraints concerning presence/absence of attributes, constraints imposed by required attribute dependencies, “conjunctions” of constraints) can be found in [A20] and also in [B27].

3 Attribute implications

Attribute implications (AIs) are formulas/expressions $A \Rightarrow B$ describing particular attribute dependencies. In addition to FCA, AIs are known in several other areas. In data mining, AIs are called association rules, see e.g. [C50] but also [C28]. In relational databases, AIs are called functional dependencies, see e.g. [C37]. In this part of the thesis we present selected results of the dissertation which concern attribute implications in a fuzzy setting. Section 3.1 provides basic notions. Section 3.2 deals with semantic issues like semantic consequence, non-redundant bases, etc. Section 3.3 presents two kinds of logics for reasoning with attribute dependencies with their completeness theorems. Section 3.4 deals with computational aspects. In Section 3.5, we provide a database semantics for AIs and deal with functional dependencies in a fuzzy setting.

3.1 Attribute implications, validity, theories and models

Fuzzy attribute implications Suppose Y is a finite set (of attributes). A fuzzy attribute implication over Y (FAI) is an expression $A \Rightarrow B$, where $A, B \in \mathbf{L}^Y$ (A and B are fuzzy sets of attributes). FAIs are our basic formulas. We want to interpret them in data tables $\langle X, Y, I\rangle$ with fuzzy attributes. The intended meaning of $A \Rightarrow B$ being true in $\langle X, Y, I\rangle$ is, basically: "for each row $x \in X$: if x has all attributes from A then X has all attributes from B". We proceed in a general way using a hedge $*$ (see later for comments).

Validity Let thus $M \in \mathbf{L}^X$ be a fuzzy set of attributes (e.g. of some object, i.e. a row in $\langle X, Y, I\rangle$). Define a degree $||A \Rightarrow B||_M$ to which $A \Rightarrow B$ is true in M by

$$||A \Rightarrow B||_M = S(A, M)^* \to S(B, M), \tag{21}$$

where $S(\cdots)$ is a degree of subsethood, see Section 1.3. For a system $\mathcal{M}$ of $\mathbf{L}$-sets in Y, define a degree $||A \Rightarrow B||_{\mathcal{M}}$ to which $A \Rightarrow B$ is true in (each M from) $\mathcal{M}$ by

$$||A \Rightarrow B||_{\mathcal{M}} \;=\; \textstyle\bigwedge_{M \in \mathcal{M}} ||A \Rightarrow B||_M. \tag{22}$$

Finally, a data table $\langle X, Y, I\rangle$ with fuzzy attributes, define a degree $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ to which $A \Rightarrow B$ is true in $\langle X, Y, I\rangle$ by

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} \;=\; ||A \Rightarrow B||_{\{I_x \,|\, x \in X\}}, \tag{23}$$

where $I_x \in \mathbf{L}^Y$ is defined by $I_x(y) = I(x, y)$, i.e. I_x is a fuzzy set of attributes of object x (row corresponding to x in the table).

Remark 10 (1) Since $*$ is a truth function of "very true", if M is a fuzzy set of attributes of object x, $||A \Rightarrow B||_M$ is a truth degree of "if it is very true that x has all attributes from A then x has all attributes from B". Therefore, the above definitions give us the desired interpretation of FAIs.

(2) In fact, $*$ controls the semantics of FAIs. Two boundary cases of $*$ give us basic different ways to the meaning of FAIs: For $*$ being identity and globalization, $||A \Rightarrow B||_M = 1$ ($A \Rightarrow B$ is fully true) means

$$S(A, M) \leq S(B, M), \qquad \text{and} \quad \text{“if } A \subseteq M \text{ then } B \subseteq M\text{''},$$

respectively.

(3) For $\mathbf{L} = \mathbf{2}$, FAIs coincide with ordinary AIs and the above semantics coincides with the ordinary one.

(4) Degrees $A(y)$ and $B(y)$ can be seen as thresholds. This is best seen when $*$ is globalization. Then, $||A \Rightarrow B||_{\langle X,Y,I\rangle} = 1$ means that "for each object $x \in X$: if for each attribute $y \in Y$, x has y to degree greater than or equal to (a threshold) $A(y)$, then for each $y \in Y$, x has y to degree at least $B(y)$". In general, $||A \Rightarrow B||_{\langle X,Y,I\rangle}$ is a truth degree of the latter proposition. That is, having A and B fuzzy sets allows a rich expressibility of relationships between attributes.

Theories and models Each fuzzy set T of FAIs will be called a theory. A degree $T(A \Rightarrow B)$ is interpreted as a degree to which $A \Rightarrow B$ is prescribed (justified) by T (see also [C21], [C26], [C38]). As a particular case, sets of FAIs are theories. For a theory T of FAIs, a set $\text{Mod}(T)$ of all models of T is defined by

$$\text{Mod}(T) = \{M \in \mathbf{L}^Y \,|\, \text{for each } A, B \in \mathbf{L}^Y : T(A \Rightarrow B) \leq ||A \Rightarrow B||_M\}.$$

That is, M is a model of T, i.e. $M \in \text{Mod}(T)$, means that for each $A \Rightarrow B$, a degree to which $A \Rightarrow B$ holds in M is higher than or at least equal to a degree $T(A \Rightarrow B)$ prescribed by T. Models of theories T have an interesting property. Note first that an $\mathbf{L}^*$-system is a system of fixpoints of an $\mathbf{L}^*$-closure operator, i.e. an operator C satisfying (13), (15), and $S(A, B)^* \leq S(C(A), C(B))$.

Theorem 18 ([A19]) *A system $\mathcal{S} \subseteq \mathbf{L}^Y$ is system of all models of some theory T iff $\mathcal{S}$ is an $\mathbf{L}^*$-closure system.*

Further results on models of FAIs can be found in [A19].

Relationship to concept lattices with hedges In the ordinary case, several issues in AIs are related to concept lattices. In our setting, FAIs correspond to particular concept lattice with hedges. Namely, consider arrow operators, cf. Section 2.5, defined by

$$A^{\uparrow}(y) = \bigwedge_{x \in X} \big(A(x)^* \to I(x,y)\big), \quad B^{\downarrow}(x) = \bigwedge_{y \in Y} \big(B(y) \to I(x,y)\big),$$

the corresponding concept lattice $\mathcal{B}(X^*, Y, I)$, and the corresponding set

$$\mathrm{Int}(X^*, Y, I) = \{B \mid \langle B^{\downarrow}, B\rangle \in \mathcal{B}(X^*, Y, I)\}$$

of intents. The following is an excerpt of a theorem from [A12] illustrating some basic relationships (we will see more relationships later).

Theorem 19 ([A12]) *For a data table $\langle X, Y, I\rangle$ with fuzzy attributes,*

$$||A \Rightarrow B||_{\langle X,Y,I\rangle} = ||A \Rightarrow B||_{\mathrm{Int}(X^*,Y,I)} = S(B, A^{\downarrow\uparrow}).$$

3.2 Semantic entailment and non-redundant bases

We now turn our attention to the notions of semantic entailment, completeness in data tables, non-redundant basis, etc.

Entailment and completeness in data A degree $||A \Rightarrow B||_T$ to which $A \Rightarrow B$ semantically follows from a fuzzy set T of FAIs is defined by

$$||A \Rightarrow B||_T = ||A \Rightarrow B||_{\mathrm{Mod}(T)}, \tag{24}$$

i.e., $||A \Rightarrow B||_T$ can be seen as a degree to which $A \Rightarrow B$ is true in each model of T. From now on in this section, we will assume that T is an ordinary set of fuzzy attribute implications. A set T of attribute implications is called complete (in $\langle X, Y, I\rangle$) if

$$||A \Rightarrow B||_T = ||A \Rightarrow B||_{\langle X,Y,I\rangle}$$

for each FAI $A \Rightarrow B$, i.e., a degree to which $A \Rightarrow B$ is true in $\langle X, Y, I\rangle$ equals the degree to which $A \Rightarrow B$ follows from T. If T is complete and no proper subset of T is complete, then T is called a non-redundant basis (of $\langle X, Y, I\rangle$).

The following observation is interesting. Call T 1-complete in $\langle X, Y, I\rangle$ provided $||A \Rightarrow B||_T = 1$ iff $||A \Rightarrow B||_{\langle X,Y,I\rangle} = 1$ for each $A \Rightarrow B$. Clearly, if T is complete then it is also 1-complete. Surprisingly, we have also

Theorem 20 ([B20]) *T is 1-complete in $\langle X, Y, I\rangle$ iff T is complete in $\langle X, Y, I\rangle$.*

The following assertion shows that the models of a complete set of fuzzy attribute implications are exactly the intents of the corresponding concept lattice.

Theorem 21 ([A9]) *T is complete in $\langle X, Y, I\rangle$ iff* $\mathrm{Mod}(T) = \mathrm{Int}(X^*, Y, I)$.

Guigues-Duquenne bases We now focus on the so-called Guigues-Duquenne basis, i.e. a non-redundant basis based on the notion of a pseudo-intent which was introduced in the ordinary setting by Guigues and Duquenne [C17], [C25]. As we will see, the situation is somewhat different from what we know from the ordinary case. We start by the notion of a system of pseudo-intents.

Given $\langle X, Y, I\rangle$, $\mathcal{P} \subseteq \mathbf{L}^Y$ (system of fuzzy sets of attributes) is called a system of pseudo-intents of $\langle X, Y, I\rangle$ if for each $P \in \mathbf{L}^Y$ we have:

$$P \in \mathcal{P} \quad \text{iff} \quad P \neq P^{\downarrow\uparrow} \text{ and } ||Q \Rightarrow Q^{\downarrow\uparrow}||_P = 1 \text{ for each } Q \in \mathcal{P} \text{ with } Q \neq P.$$

It is easily seen that if $*$ is globalization, the above condition simplifies to

$$P \in \mathcal{P} \quad \text{iff} \quad P \neq P^{\downarrow\uparrow} \text{ and } Q^{\downarrow\uparrow} \subseteq P \text{ for each } Q \in \mathcal{P} \text{ with } Q \subset P.$$

In addition, in case of finite $\mathbf{L}$, for each data table with finite set of attributes there is exactly one system of pseudo-intents which can be described recursively the same way as in the ordinary case [C17], [C25]:

Theorem 22 ([A13]) *Let* $\mathbf{L}$ *be finite,* $*$ *be globalization. For each* $\langle X, Y, I\rangle$ *there is a unique system of pseudo-intents* $\mathcal{P}$ *of* $\langle X, Y, I\rangle$ *and*

$$\mathcal{P} = \{P \in \mathbf{L}^Y \mid P \neq P^{\downarrow\uparrow} \text{ and } Q^{\downarrow\uparrow} \subseteq P \text{ holds for each } Q \in \mathcal{P} \text{ such that } Q \subset P\}.$$

Neither the uniqueness of $\mathcal{P}$ nor the existence of $\mathcal{P}$ can be guaranteed in general, see [A13]. For $\mathbf{L} = \mathbf{2}$, the system of pseudointents described by Theorem 22 coincides with the ordinary one. The next theorem shows the role of systems of pseudointents.

Theorem 23 ([A13]) *Let* $\mathcal{P}$ *be a system of pseudointents of* $\langle X, Y, I\rangle$*. Then* $T = \{P \Rightarrow P^{\downarrow\uparrow} \mid P \in \mathcal{P}\}$ *is a non-redundant basis of* $\langle X, Y, I\rangle$ *(so-called Guigues-Duquenne basis).*

Non-redundancy of T does not ensure that T is minimal in terms of its size. The following theorem shows a generalization of a well-known result saying that Guigues-Duquenne basis is minimal in terms of its size.

Theorem 24 ([A13]) *Let* $\mathbf{L}$ *be finite,* $*$ *be globalization,* T *be the Guigues-Duquenne basis of* $\langle X, Y, I\rangle$*. If* T' *is complete in* $\langle X, Y, I\rangle$ *then* $|T| \leq |T'|$.

For hedges other than globalization we can have several systems of pseudointents. The systems of pseudointents may have different numbers of elements, see [A13].

Remark 11 (1) The first study on FAIs is S. Pollandt's [C42]. Pollandt uses the same notion of a FAI, i.e. $A \Rightarrow B$ where A, B are fuzzy sets, and obtains several results. Pollandt's notion of validity is a special case of ours, namely the one for * being identity. On the other hand, the notion of a pseudo-intent in

[C42] corresponds to * being globalization. That is why Pollandt did not get a proper generalization of results leading to Guigues-Duquenne basis.

(2) [B22] and [A12] contain some reduction theorems concerning relationships of FAIs in $\langle X, Y, I \rangle$ vs. ordinary AIs in some tables with binary attributes obtained from $\langle X, Y, I \rangle$.

3.3 Fuzzy attribute logic

In this section we present two kinds of logics for reasoning with FAIs including their completeness theorems. The logics are inspired by so-called Armstong axioms [C2], well known from the theory of database systems [C37]. Throughout this section, we assume that $\mathbf{L}$ is a finite residuated lattice (for infinite case, see [B33]).

Ordinary-style fuzzy attribute logic The logic has the following deduction rules:

$$\text{(Ax)} \ \frac{}{A \cup B \Rightarrow A}, \qquad \text{(Cut)} \ \frac{A \Rightarrow B,\ B \cup C \Rightarrow D}{A \cup C \Rightarrow D}, \qquad \text{(Mul)} \ \frac{A \Rightarrow B}{c^* \otimes A \Rightarrow c^* \otimes B},$$

for each $A, B, C, D \in \mathbf{L}^Y$, and $c \in L$. Note that the first system of rules was introduced in [B20]. The present one, introduced in [A15], has the following advantage: With A, B, C, D being ordinary sets, (Ax) and (Cut) are well-known deduction rules from the ordinary case for which it is known that they are complete (w.r.t. both database semantics an the semantics given by tables with binary attributes). (Mul) is a new rule in a fuzzy setting (rule of multiplication). Therefore, the above system results by taking ordinary rules (and replacing sets by fuzzy sets in these rules) and adding (Mul) as a single "fuzzy rule". It can be easily seen that if we take any system of rules which is complete in the ordinary case and replace ordinary sets by fuzzy sets in these rules, then adding (Mul), we get a system of deduction rules which is equivalent the the above rules (Ax)–(Mul).

in a usual way, we can now introduce: a FAI $A \Rightarrow B$ is provable from a set T of FAIs (denoted by $T \vdash A \Rightarrow B$) iff there is a proof of $A \Rightarrow B$, i.e. a sequence $\varphi_1, \ldots, \varphi_n$ of FAIs such that φ_n is $A \Rightarrow B$ and for each φ_i, either $\varphi_i \in T$ or φ_i is inferred (in one step) from some of the preceding formulas using some of deduction rules (Ax)–(Mul). Writing $T \models A \Rightarrow B$ instead of $||A \Rightarrow B||_T = 1$ (i.e., $A \Rightarrow B$ semantically follows from T in degree 1), we can get the ordinary completeness:

Theorem 25 ([A15]) *For any set T of FAIs and a FAI $A \Rightarrow B$ we have*

$$T \models A \Rightarrow B \quad \textit{iff} \quad T \vdash A \Rightarrow B.$$

Pavelka-style fuzzy attribute logic The above completeness theorem does not capture degrees of entailment. We now present a so-called Pavelka-style logic [C21], [C26], [C38], [C40], and refer to [A21] for details.

Our logic uses the following deduction rules:

$$\text{(Ax)}\ \frac{}{\langle A\cup B \Rightarrow A, 1\rangle}, \qquad \text{(Cut)}\ \frac{\langle A\Rightarrow B, a\rangle, \langle B\cup C\Rightarrow D, b\rangle}{\langle A\cup C\Rightarrow D, a^*\otimes b\rangle},$$

$$\text{(Mul)}\ \frac{\langle A\Rightarrow B, a\rangle}{\langle c^*\otimes A\Rightarrow c^*\otimes B, a\rangle}, \qquad \text{(Sh)}\ \frac{\langle A\Rightarrow B, a\rangle}{\langle A\Rightarrow C, S(C, a\otimes B)\rangle},$$

for each $A, B, C, D \in \mathbf{L}^Y$, and $a, b, c \in L$; $S(\cdots)$ denotes a subsethood degree, see Section 1.3. Note that, in fact, (Sh) is a parameterized rule; we have one rule (Sh_C) for each C. Note that, e.g., (Cut) can be read as follows: having inferred a FAI $A \Rightarrow B$ in degree (at least) $a \in L$, and a FAI $B \cup C \Rightarrow D$ in degree at least b, we can infer $A \cup C \Rightarrow D$ in degree $a^* \otimes b$. As usual in Pavelka-style logic, a proof of $\langle A \Rightarrow B, a\rangle$ is a sequence of pairs $\langle \varphi_1, a_1\rangle, \ldots, \langle \varphi_n, a_n\rangle$ (φ_i a FAI, $a_i \in L$) such that $\langle A \Rightarrow B, a\rangle = \langle \varphi_n, a_n\rangle$ and for each $i = 1, \ldots, n$ we have $a_i = T(\varphi_i)$ or $\langle \varphi_i, a_i\rangle$ is obtained by some rule (Ax)–(Sh) from some $\langle \varphi_j, a_j\rangle$'s ($j < i$). A degree $|A \Rightarrow B|_T$ of provability of a FAI $A \Rightarrow B$ from T is defined by

$$|A \Rightarrow B|_T = \bigvee\{a \mid \ \ldots, \langle A \Rightarrow B, a\rangle \text{ is a proof from } T\}.$$

Then we have the following Pavelka-style completeness:

Theorem 26 ([A21]) *For each fuzzy set T of FAIs and a FAI $A \Rightarrow B$ we have*

$$||A \Rightarrow B||_T \ = \ |A \Rightarrow B|_T.$$

Reducing Pavelka-style completeness to ordinary completeness It is interesting to note that due to some special properties, we can get Pavelka-style completeness using a "technical trick". Our approach is conceptually the same as the way Hájek proved completeness of Rational Pavelka logic in [C26].

For a fuzzy set T of FAIs and for $A \Rightarrow B$ we define a degree $|A \Rightarrow B|_T \in L$ to which $A \Rightarrow B$ is provable from T (there is a clash with the above definition but it will turn out that the definitions coincide) by

$$|A \Rightarrow B|_T = \bigvee\{c \in L \mid \mathrm{c}(T) \vdash A \Rightarrow c \otimes B\},$$

where $\mathrm{c}(T)$ is an ordinary set of FAIs defined by

$$\mathrm{c}(T) = \{A \Rightarrow T(A \Rightarrow B) \otimes B \mid A, B \in \mathbf{L}^Y \text{ and } T(A \Rightarrow B) \otimes B \neq \emptyset\}.$$

Then we have (a consequence of Theorem 25 and some further facts):

Theorem 27 ([A15]) *For each fuzzy set T of FAIs and a FAI $A \Rightarrow B$ we have*

$$||A \Rightarrow B||_T = |A \Rightarrow B|_T.$$

3.4 Computation of non-redundant bases

This section presents selected results related to computation of non-redundant bases. Throughout this section, we assume that $\mathbf{L}$ is finite.

* being globalization If * is globalization, there is a unique system $\mathcal{P}$ of pseudointents for $\langle X, Y, I\rangle$, see Theorem 22. An algorithm for computing $\mathcal{P}$, extending Ganter's algorithm for computing ordinary pseudointents [C17], can be obtained as follows [A9]: For $Z \in \mathbf{L}^Y$ put

$$\begin{aligned} Z^{T^*} &= Z \cup \bigcup\{B \otimes S(A, Z)^* \mid A \Rightarrow B \in T \text{ and } A \neq Z\}, \\ Z^{T^*_0} &= Z, \\ Z^{T^*_n} &= (Z^{T^*_{n-1}})^{T^*}, \quad \text{for } n \geq 1, \end{aligned}$$

and define an operator cl_{T^*} on $\mathbf{L}$-sets in Y by

$$cl_{T^*}(Z) = \textstyle\bigcup_{n=0}^{\infty} Z^{T^*_n}.$$

Theorem 28 ([A9]) *cl_{T^*} is a fuzzy closure operator, and*

$$\{cl_{T^*}(Z) \mid Z \in \mathbf{L}^Y\} = \mathcal{P} \cup \mathrm{Int}(X^*, Y, I).$$

Therefore, pseudo-intents can be obtained using Theorem 28 and the above algorithm for computing fixpoints of fuzzy closure operators.

Arbitrary * If * is an arbitrary hedge, systems of pseudo-intents for $\langle X, Y, I\rangle$ can be computed using algorithms for generating maximal independent sets in graphs. Namely, systems of pseudo-intents can be identified with particular maximal independent sets, for details see [A18]: For $\langle X, Y, I\rangle$ define a set V of fuzzy sets of attributes by

$$V = \{P \in \mathbf{L}^Y \mid P \neq P^{\downarrow\uparrow}\}. \tag{25}$$

If $V \neq \emptyset$, define a binary relation E on V by

$$E = \{\langle P, Q\rangle \in V \mid P \neq Q \text{ and } ||Q \Rightarrow Q^{\downarrow\uparrow}||_P \neq 1\}. \tag{26}$$

Consider the graph $\mathbf{G} = \langle V, E \cup E^{-1}\rangle$. For any $Q \in V$ and $\mathcal{P} \subseteq V$ define the following subsets of V: $\mathrm{Pred}(Q) = \{P \in V \mid \langle P, Q\rangle \in E\}$, and $\mathrm{Pred}(\mathcal{P}) = \bigcup_{Q \in \mathcal{P}} \mathrm{Pred}(Q)$.

Theorem 29 ([A18]) *Let $\mathbf{L}$ be finite, * be any hedge, $\langle X, Y, I\rangle$ be a data table with fuzzy attributes, $\mathcal{P} \subseteq \mathbf{L}^Y$, V and E be defined by* (25) *and* (26)*, respectively. Then the following statements are equivalent.*

(i) $\mathcal{P}$ *is a system of pseudo-intents;*

(ii) $V - \mathcal{P} = \text{Pred}(\mathcal{P})$*;*

(iii) $\mathcal{P}$ *is a maximal independent set in* $\mathbf{G}$ *such that* $V - \mathcal{P} = \text{Pred}(\mathcal{P})$*.*

Theorem 29 gives a way to compute systems of pseudo-intents. One needs to find all maximal independent sets in $\mathbf{G}$ (algorithms exist for this problem, e.g. [C30]) and check which of them satisfy the additional condition $V - \mathcal{P} = \text{Pred}(\mathcal{P})$. Further details can be found in [A18].

Further way to get non-redundant bases [A19] contains another way to obtain non-redundant bases for general *: First, one computes a set T of FAIs which is complete for a given $\langle X, Y, I\rangle$ (in a way similar to computing pseudo-intents when * is globalization). Second, one removes FAIs from T until it becomes non-redundant. This is based on checking whether a FAI $A \Rightarrow B$ follows in degree 1 from a set T of FAIs which can be done by checking whether B is contained in the least model M of $T - \{A \Rightarrow B\}$ which contains A. M can be computed as a closure under a particular fuzzy closure operator, see [A19].

3.5 Functional dependencies in tables over domains with similarity relations

In this section, we briefly describe a "database interpretation" of FAIs. It turns out that this interpretation has the same notion of semantic entailment. As a result, the logics presented in Section 3.3 give us completeness theorem for the database interpretation. We refer to [A16] and [A17] for details. Following common usage, we also call a FAI $A \Rightarrow B$ a (fuzzy) functional dependence (FFD) in this section.

A data table over domains with similarity relations is a tuple $\mathcal{D} = \langle X, Y, \{\langle D_y, \approx_y\rangle \mid y \in Y\}, T\rangle$ where

- X is a non-empty set (of objects, table items),
- Y is a non-empty finite set (of attributes),
- for each $y \in Y$, D_y is a non-empty set (of values of attribute y) and $\approx_y$ is a binary fuzzy relation which is reflexive and symmetric (we call it a similarity),
- T is a mapping assigning to each $x \in X$ and $y \in Y$ a value $T(x, y) \in D_y$ (value of attribute y on object x, denoted also $x[y]$).

Remark 12 Consider $L = \{0, 1\}$ (ordinary case). If each $\approx_y$ is an equality (i.e. $a \approx_y b = 1$ iff $a = b$), then $\mathcal{D}$ can be identified with what is called a relation on relation scheme Y with domains D_y ($y \in Y$) [C37], i.e. one of the basic concepts of Codd's relational model of data.

$\mathcal{D}$ can be seen as a table with rows and columns corresponding to $x \in X$ and $y \in Y$, respectively, and with table entries containing values $T(x, y) \in D_y$. Moreover, each domain D_y is equipped with an additional information about similarity of elements from D_y. We now introduce a condition for a functional dependence $A \Rightarrow B$ to be true in $\mathcal{D}$ which says basically the following: "for any two objects $x_1, x_2 \in X$: if x_1 and x_2 have similar values on attributes from A then x_1 and x_2 have similar values on attributes from B". Define first for a given $\mathcal{D}$, objects $x_1, x_2 \in X$, and a fuzzy set $C \in \mathbf{L}^Y$ of attributes a degree $x_1(C) \approx x_2(C)$ to which x_1 and x_2 have similar values on attributes from C (agree on attributes from C) by

$$x_1(C) \approx x_2(C) \;=\; \textstyle\bigwedge_{y \in Y} \big(C(y) \to (x_1[y] \approx_y x_2[y])\big).$$

That is, $x_1(C) \approx x_2(C)$ is truth degree of "for each attribute $y \in Y$: if y belongs to C then the value $x_1[y]$ of x_1 on y is similar to the value $x_2[y]$ of x_2 on y", which can be seen as a degree to which x_1 and x_2 have similar values on attributes from C. Then, a degree $||A \Rightarrow B||_{\mathcal{D}}$ to which $A \Rightarrow B$ is true in $\mathcal{D}$ is defined by

$$||A \Rightarrow B||_{\mathcal{D}} = \textstyle\bigwedge_{x_1, x_2 \in X} \big((x_1(A) \approx x_2(A))^* \to (x_1(B) \approx x_2(B))\big).$$

Remark 13 (1) $\mathbf{L} = \mathbf{2}$, the above definition gives the well-known notion of a functional dependence being true in a relation over relation scheme Y.

(2) $A(y) \in L$ and $B(y) \in L$ can be seen as thresholds, as in case of FAIs, cf. Remark 10.

We now have two semantics for FAIs: one given by data tables with fuzzy attributes, the second one given by tables over domains with similarities. As it will turn out, both of them have the same notion of semantic entailment. For a fuzzy set T of FFD, the set $\mathrm{Mod}^{\mathrm{FD}}(T)$ of all models of T is defined by $\mathrm{Mod}^{\mathrm{FD}}(T) = \{\mathcal{D} \,|\, \text{for each } A, B \in \mathbf{L}^Y : T(A \Rightarrow B) \leq ||A \Rightarrow B||_{\mathcal{D}}\}$, where $\mathcal{D}$ stands for an arbitrary data table over domains with similarities. A degree $||A \Rightarrow B||_T^{\mathrm{FD}} \in L$ to which $A \Rightarrow B$ semantically follows from a fuzzy set T of FFDs is defined by $||A \Rightarrow B||_T^{\mathrm{FD}} = \bigwedge_{\mathcal{D} \in \mathrm{Mod}^{\mathrm{FD}}(T)} ||A \Rightarrow B||_{\mathcal{D}}$. Denoting now $||A \Rightarrow B||_T$, see (24), by $||A \Rightarrow B||_T^{\mathrm{AI}}$, one can prove the following theorem.

Theorem 30 ([A16]) *For each fuzzy set T of FAIs and any FAI $A \Rightarrow B$ we have*

$$||A \Rightarrow B||_T^{\mathrm{FD}} = ||A \Rightarrow B||_T^{\mathrm{AI}}.$$

Remark 14 Various notions of FFDs have been studied. Our approach seems to be quite general and our results go beyond the results which can be found in the literature. See [B36] for a comparison. Note that [B30] extends the tables over domains with similarities by ranks assigned to table rows. This enables us to consider a table as an answer to a similarity-based query.

4 Further directions

This section presents a brief overview of some of further topics including topics for future research.

Formal concept analysis Algorithms for FCA: Development of efficient algorithms was not our main focus. Up to now, we only focused on presenting computational feasibility of FCA of data with fuzzy attributes. A detailed design and study of algorithms for FCA is a topic for future research. Other approachesto FCA: Recently, there have been proposed several new approaches to FCA of data with fuzzy attributes, see e.g. [C20], [C16], [C34]. The mutual relationships between these approaches as well as their theoretical and computational tractability needs to be explored. Further studies of structures behind FCA: We presented some results on the mathematical structures behind FCA of data with fuzzy attributes. These structures are of intest in fuzzy set theory per se (e.g., fuzzy closure operators and systems, fuzzy order). A further study of these structures is an interesting problem.

Relational factor analysis In relational factor analysis (RFA, the term was coined by us and is thus to be considered tentative), the aim is to decompose a given objects×attributes fuzzy relation into a relational product of an objects×factors fuzzy relation and a factors×attributes fuzzy relation with the number of factors possibly less than the number of attributes. RFA generalizes Boolean factor analysis (BFA), on which there are many papers in the literature, and is an example of a non-linear factor analysis. In [A31], we demonstrated that concept lattices are of crucial importance for RFA. Namely, one can take formal concepts as factors in the decomposition. This is an optimal approach, see [A31] for BFA, and analogous results hold true for RFA in general. Note that taking formal concept as factors was proposed by Keprt and Snášel.

Extension of Codd's relational model of data Although there are many papers on the extension of Codd's relational model of data in the literature, this topic is not properly developed. We already mentioned functional dependencies in this extension. A proper use of fuzzy logic is expected to lead to further development of the extension. Note that according to [C1], foundations of uncertainty and imprecision in databases are considered an important research topic. For instance, in [B30], we showed that the top_k query, on which there has been an intensive research recently, see e.g. [C36], is a relational query in the extension by similarities and ranks. It seems that the extension of Codd's model provides appropriate foundations for the ongoing research on ranked queries in the database community.

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