INSTITUTE OF MATHEMATICS

THE CZECH ACADEMY OF SCIENCES

Generalized Garvan formulas

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Preprint No. 8-2016

PRAHA 2016

GENERALIZED GARVAN FORMULAS

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ABSTRACT. Using the elliptic version of the Fay's trisecant identities found as a result of computations of characters for a vertex operator superalgebra, we derive generalizations of the Garvan formula for powers of the modular discriminant in terms of deformed Weierstrass functions.

1. Introduction

Computations related to vertex algebras were always a source of new identities in number theory [K2]. While finding closed expressions for vertex operator algebra characters, we finally obtain relations among modular forms, fundamental kernels, and q-series.

Various ways to compute characters for vertex operator algebra modules lead to generation of modular forms as well as interesting identities [MTZ] for them in terms of elliptic functions. In particular, consideration of the partition function on the torus for the rank two free fermion vertex operator superalgebra allow us to provide a pure algebraic explanation of Jacobi triple product identity [K1]. At the same time, computation of higher correlation function on a genus one Riemann surface gives us [MTZ] an elliptic version of the Fay's trisecant identity [Fay] known from algebraic geometry. Various identities for powers of the η-function appear [K2] in frames of the affine Lie algebras.

Computation of characters for vertex operator (super)algebras [K1] brings about an algebraic way to regenerate known and find new identities for modular forms [Zh, DLM, MTZ, TZ]. For a module $\mathcal{V}$ of a vertex operator algebra V, we find closed formulas for characters of vertex operators $\mathcal{Y}$ on the torus with $q = e^{2\pi i\tau}$, $z_i \in \Sigma^{(g)}$, $v_i \in V$ [MTZ]:

$$Z^{(1)}_{\mathcal{V}}(v_1, z_1, \ldots, v_n, z_n; q) = \mathrm{STr}_{\mathcal{V}} \left(\mathcal{Y}(v_1, z_1) \ldots \mathcal{Y}(v_n, z_n)\; q^{L(0)-C/24} \right),$$

where $L(0)$ is the Virasoro algebra generator, and C is central charge. The formal parameter is associated to a complex parameter on the torus. Final expressions are given by determinants of matrices with elements being coefficients in the expansions of the regular parts of corresponding differentials (Bergman (bosons) A_a, $a = 1, 2$ or Szegő (fermions) Q kernels) [MTZ]. Modularity properties of n-point functions with respect to appropriate group follows explicitly from the vertex operator derivation.

1991 *Mathematics Subject Classification.* 17B69, 30F10, 32A25, 11F03.
Key words and phrases. Modular discriminant, Fay's trisecant identities, Modular forms.

In this paper we derive new generalization of the fundamental formulas for powers of the modular discriminant in terms of deformed versions [DLM] of the Weierstrass functions and Eisenstein series. In particular, we find that powers of the modular discriminant are expressed (up to theta-functions multiplier) via determinants of finite matrices containing combinations of deformed modular functions. In the proof we use the generalized form of the elliptic version of the Fay's trisecant identity for a vertex operator superalgebra.

2. Modular discriminant and Garvan formula

The modular discriminant is defined by

$$\Delta(\tau) = (2\pi)^{12}\eta(\tau)^{24},$$

where $\eta(\tau)$ is the Dedekind eta-function

$$\eta(\tau) = q^{1/24}\prod_{n=1}^{\infty}(1-q^n).$$

The fundamental classical formula for the modular discriminant is

$$\Delta(\tau) = \frac{1}{1728}\left(E_4^3(\tau) - E_6^2(\tau)\right) = \frac{1}{1728}\det\begin{pmatrix} \sqrt{7/3}E_4(\tau) & E_6(\tau) \\ E_6(\tau) & \sqrt{7/3}E_8(\tau)\end{pmatrix}.$$

The following formula was conjectured by F. Garvan

$$\Delta^2(\tau) = -\frac{691}{250\ (1728)^2}\ \det\begin{pmatrix} E_4(\tau) & E_6(\tau) & E_8(\tau) \\ E_6(\tau) & E_8(\tau) & E_{10}(\tau) \\ E_8(\tau) & E_{10}(\tau) & E_{12}(\tau)\end{pmatrix}, \tag{1}$$

which was then proved and generalized in [Mi]. In this paper we give a generalization for (1) for higher powers of the modular discriminant computed as a determinant of matrices containing deformed Weierstrass functions [DLM, MTZ].

3. Generalized Garvan formulas

3.1. **Triple Jacobi identity.**

Computations of the twisted partition function $Z_V^{(1)}\left[\begin{array}{c} f \\ g\end{array}\right]$ for free fermion vertex operator superalgebra leads to two alternative expressions (see, e.g., [MTZ, K1]) as expansion over a basis:

$$Z_V^{(1)}\left[\begin{array}{c} f \\ g\end{array}\right](\tau) = q^{\kappa^2/2-1/24}\prod_{l\geq 1}(1-\theta^{-1}q^{l-\frac{1}{2}-\kappa})(1-\theta q^{l-\frac{1}{2}+\kappa}), \tag{2}$$

and

$$Z_V^{(1)}\left[\begin{array}{c} f \\ g\end{array}\right](\tau) = \frac{e^{2\pi i(\alpha+1/2)(\beta+1/2)}}{\eta(\tau)}\vartheta^{(1)}\left[\begin{array}{c} -\beta+\frac{1}{2} \\ \alpha+\frac{1}{2}\end{array}\right](0,\tau),$$

in terms of the torus theta series with characteristics:

$$\vartheta^{(1)}\left[\begin{array}{c} a \\ b\end{array}\right](z,\tau) = \sum_{n\in\mathbb{Z}}\exp\left[i\pi(n+a)^2\tau + (n+a)(z+2\pi i b)\right].$$

In (2) we define $f = e^{2\pi i\alpha a(0)}$, $g = e^{2\pi i\beta a(0)}$, with some parameters α, $\beta \in \mathbb{R}$, and zero mode $a(0)$ of a Heisenberg subalgebra in the rank two free fermionic vertex operator superalgebra [MTZ]. We also define $\phi = e^{-2\pi i\beta}$ and $\theta = e^{-2\pi i\alpha}$. Note that $Z_V^{(1)} \left[\begin{array}{c} f \\ g \end{array} \right] (\tau) = 0$ for $(\theta, \phi) = (1, 1)$, i.e., $(\alpha, \beta) \equiv (0, 0) \pmod{\mathbb{Z}}$. Comparing to (2) we obtain Jacobi triple product formula [K1] which can be rewritten in the form:

$$\eta(\tau) = q^{-\kappa^2/2+1/24} e^{2\pi i(\alpha+1/2)(\beta+1/2)}\ \vartheta^{(1)} \left[\begin{array}{c} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{array} \right] (0, \tau)\ \det(R),$$

where the determinant

$$\det(R) = \left(\prod_{l \geq 1} (1 - \theta^{-1} q^{l - \frac{1}{2} - \kappa})(1 - \theta q^{l - \frac{1}{2} + \kappa}) \right)^{-1},$$

corresponds to sphere self-sewing to form a torus [TZ]. Thus we get the identity for the first power of the η-function.

Let us introduce the notation:

$$\Theta^{(1)}_{r,s,(m_i,n_i)}(x, y, \tau) \equiv \frac{\prod\limits_{1 \leq i \leq r, 1 \leq j \leq s} \vartheta^{(1)} \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] (x_i - y_j, \tau)^{m_i n_j}}{\prod\limits_{1 \leq i < k \leq r} \vartheta^{(1)} \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] (x_i - x_k, \tau)^{m_i m_k} \prod\limits_{1 \leq j < l \leq s} \vartheta^{(1)} \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] (y_j - y_l, \tau)^{n_j n_l}}.$$

In [Mi] Garvan formula for powers of the modular discriminant was obtained. Similarly, for any power of the modular discriminant we obtain a formula, generalizing the Garvan's formulas.

Proposition 1. *For $(\theta, \phi) \neq (1, 1)$ one has*

$$\Delta^n(\tau) = -\frac{1}{(2\pi)^{12}}\ \frac{\vartheta^{(1)} \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] (0, \tau)\ \Theta^{(1)}_{8n,8n,(1,1)}(x, y, \tau).}{\vartheta^{(1)} \left[\begin{array}{c} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{array} \right] \left(\sum\limits_{i=1}^{8n} (x_i - y_i), \tau \right)} \det\left(\mathbf{P}_{8n}(\theta, \phi) \right). \tag{3}$$

For $(\theta, \phi) = (1, 1)$,

$$\Delta^n(\tau) = \frac{i}{(2\pi)^{12}}\ \frac{\Theta^{(1)}_{8n+1,8n+1,(1,1)}(x, y, \tau)}{\vartheta^{(1)} \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] \left(\sum\limits_{i=1}^{8n+1} (x_i - y_i), \tau \right)} \det\left(\mathbf{Q}_{8n+1} \right). \tag{4}$$

Remark. As it was proved in [MTZ]

$$P_1 \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (z, \tau) = \frac{1}{z} - \sum_{n \geq 1} \frac{1}{n} E_n \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (\tau) z^{n-1},$$

we can also express (3) and (4) in terms of deformed Eisenstain series by substitution the last formula to (7) leading to a quite involved formula which we do not give here.

Proof. Recall the notion of the genus one prime form $K^{(1)}(z,\tau)$ [Mu, Fay]. It can be expressed via the formula

$$K^{(1)}(z,\tau) = -\frac{i}{\eta^3(\tau)}\, \vartheta^{(1)} \left[\begin{array}{c} \frac{1}{2} \\ \frac{1}{2} \end{array} \right] (z,\tau). \tag{5}$$

In [MTZ] the elliptic function version of the Fay's generalized trisecant identity [Fay] was derived. For $(\theta,\phi) \neq (1,1)$ one has

$$\det\left(\mathbf{P}_n(\theta,\phi)\right) = \frac{\vartheta^{(1)} \left[\begin{array}{c} -\beta+\frac{1}{2} \\ \alpha+\frac{1}{2} \end{array} \right] \left(\sum\limits_{i=1}^{n} (x_i - y_i), \tau \right)}{\vartheta^{(1)} \left[\begin{array}{c} -\beta+\frac{1}{2} \\ \alpha+\frac{1}{2} \end{array} \right] (0,\tau)} \cdot \frac{\prod\limits_{1\le i<j\le n} K^{(1)}(x_i - x_j,\tau) K^{(1)}(y_i - y_j,\tau)}{\prod\limits_{1\le i,j\le n} K^{(1)}(x_i - y_j,\tau)}, \tag{6}$$

where $\mathbf{P}_n(\theta,\phi)$ is the $n \times n$ matrix:

$$\mathbf{P}_n(\theta,\phi) = \left(P_1 \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (x_i - y_j,\tau) \right), \quad (1 \le i,j \le n). \tag{7}$$

The deformed Weierstrass function in (7) was defined in [DLM, MTZ]

$$P_1 \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (z,\tau) = -\sum_{n\in\mathbb{Z}+\lambda}{}^{'} \frac{q_z^n}{1-\theta^{-1}q^n}, \tag{8}$$

for $q = e^{2\pi i\tau}$, and where $\sum'$ means we omit $n = 0$ if $(\theta,\phi) = (1,1)$. For $(\theta,\phi) = (1,1)$, one has

$$\det\left(\mathbf{Q}_n\right) = -\frac{K^{(1)}\left(\sum\limits_{i=1}^{n} (x_i - y_i), \tau \right) \prod\limits_{1\le i<j\le n} K^{(1)}(x_i - x_j,\tau) K^{(1)}(y_i - y_j,\tau)}{\prod\limits_{1\le i,j\le n} K^{(1)}(x_i - y_j,\tau)}, \tag{9}$$

where $\mathbf{Q}_n$ is the $(n+1)\times(n+1)$ matrix:

$$\mathbf{Q}_n = \left(\begin{array}{cccc} P_1(x_1 - y_1,\tau) & \ldots & P_1(x_1 - y_n,\tau) & 1 \\ \vdots & \ddots & & \vdots \\ P_1(x_n - y_1,\tau) & & P_1(x_n - y_n,\tau) & 1 \\ 1 & \ldots & 1 & 0 \end{array} \right).$$

Using (5), we express the prime forms to rewrite the identities (3) and (4) to get formulas for powers of η-function.

$\square$

4. Higher formulas

Using the full version of the Fay's generalized trisecant identity [MTZ] for integers m_i, $n_j \geq 0$, satisfying

$$\sum_{i=1}^{r} m_i = \sum_{j=1}^{s} n_j,$$

we derive the following

Lemma 1. *For $(\theta, \phi) \neq (1,1)$, $\zeta \in \mathbb{N}$,*

$$\Delta^{\zeta}(\tau) = \frac{(-i)^{\Phi/24}}{(2\pi)^{12}} \frac{\vartheta^{(1)} \left[\begin{array}{c} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{array} \right] (0,\tau)\, \Theta^{(1)}_{r,s,(m,n)}(x,y,\tau)}{\vartheta^{(1)} \left[\begin{array}{c} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{array} \right] \left(\sum\limits_{i=1}^{r} m_i x_i - \sum\limits_{j=1}^{s} n_j y_j, \tau \right)} \det(\mathbf{M}_{r,s}), \tag{10}$$

where

$$\Phi = \sum_{1 \leq i \leq r, 1 \leq k \leq s} m_i n_j - \sum_{1 \leq i < k \leq r} m_i m_k - \sum_{1 \leq j < l \leq s} n_j n_l.$$

Proof. In [MTZ] generalizations of the formulas (6), (9) were derived. For $(\theta, \phi) \neq (1,1)$ we have

$$\begin{aligned} \det(\mathbf{M}_{r,s}) \quad &= \quad \frac{\vartheta^{(1)} \left[\begin{array}{c} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{array} \right] \left(\sum\limits_{i=1}^{r} m_i x_i - \sum\limits_{j=1}^{s} n_j y_j, \tau \right)}{\vartheta^{(1)} \left[\begin{array}{c} -\beta + \frac{1}{2} \\ \alpha + \frac{1}{2} \end{array} \right] (0,\tau)} \\ &\quad \cdot \frac{\prod\limits_{1 \leq i < k \leq r} K^{(1)}(x_i - x_k, \tau)^{m_i m_k} \prod\limits_{1 \leq j < l \leq s} K^{(1)}(y_j - y_l, \tau)^{n_j n_l}}{\prod\limits_{1 \leq i \leq r, 1 \leq j \leq s} K^{(1)}(x_i - y_j, \tau)^{m_i n_j}}, \end{aligned} \tag{11}$$

where $\mathbf{M}_{r,s}$ is the block matrix

$$\mathbf{M}_{r,s} = \begin{pmatrix} \mathbf{D}^{(11)} & \dots & \mathbf{D}^{(1s)} \\ \vdots & \ddots & \vdots \\ \mathbf{D}^{(r1)} & \dots & \mathbf{D}^{(rs)} \end{pmatrix},$$

with $\mathbf{D}^{(ab)}$ the $m_a \times n_b$ matrix

$$\mathbf{D}^{(ab)}(i,j) = D \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (i, j, x_a - y_b, \tau), \quad (1 \leq i \leq m_a, 1 \leq j \leq n_b),$$

for $1 \leq a \leq r$ and $1 \leq b \leq s$. The ingredients of $\mathbf{D}^{(ab)}$ are given by the analytic expansions [MTZ]:

$$P_1 \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (z + z_1 - z_2, \tau) \quad = \quad \sum_{k,l \geq 1} D \left[\begin{array}{c} \theta \\ \phi \end{array} \right] (k,l,z) z_1^{k-1} z_2^{l-1},$$

where for $k, l \geq 1$,

$$D\left[\begin{array}{c}\theta\\ \phi\end{array}\right](k,l,z,\tau) \quad = \quad (-1)^{k+1}\binom{k+l-2}{k-1} P_{k+l-1}\left[\begin{array}{c}\theta\\ \phi\end{array}\right](\tau,z).$$

Using (11) we obtain (10).

$\square$

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