Measure-valued solutions to the compressible Euler system revisited

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Compressible Euler system

Compressible Euler system

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0$$

Energy (entropy) inequality

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) \mathbf{u} + p(\varrho)\mathbf{u} \right] \leq 0$$

$$P(\varrho) = \varrho \int_1^{\varrho} \frac{p(z)}{z^2} \, \mathrm{d}z$$

Measure-valued solutions

Compressible Euler system

$$\partial_t \overline{\varrho} + \mathrm{div}_x (\overline{\varrho \mathbf{u}}) = 0$$

$$\partial_t (\overline{\varrho \mathbf{u}}) + \mathrm{div}_x (\overline{\varrho \mathbf{u} \otimes \mathbf{u}}) + \nabla_x \overline{p(\varrho)} = 0$$

Parameterized measure

$$\nu_{t,x} \in L^\infty_{\mathrm{weak}}((0,T) \times \Omega; \mathcal{P}([0,\infty) \times R^N), \ N = 2, 3$$

$$\overline{b(\varrho, \mathbf{u})}(t,x) = \langle \nu_{t,x}; b(s, \mathbf{v}) \rangle \ \text{ for a.a. } (t,x)$$

Young measure

$$\overline{b(\varrho, \mathbf{u})} = \ \text{weak limit of } b(\varrho_\varepsilon, \mathbf{u}_\varepsilon)$$

Do we need measure valued solutions?

Existence

Measure-valued solutions may be the only global in time solutions available for the compressible Euler system, cf. DiPerna and Majda [1987]. In general false, there are “many” weak solutions, see DeLellis, Székelyhidi and collaborators [2012]

Oscillatory data

Measure-valued solutions describe the behavior of systems with oscillatory (measure-valued) data.

Measure-valued solutions are the right ones (?)

Measure-valued solutions are the physically relevant ones obtained in the artificial viscosity approximations, cf. numerical experiments Mishra [2013–2015]

Measure-valued solutions vs. weak solutions

Basic question

Can every measure-valued solution to the compressible Euler equations be approximated by a sequence of weak solutions?

Incompressible Euler system

Székelyhidi, Wiedemann [2012]

- Any measure-valued solutions of the incompressible Euler system is generated by a sequence of weak solutions
- If the initial data for a measure-valued solution are represented by an L^2-function, then the generating sequence can be chosen in such a way that the initial energies are close and the energy inequality satisfied
- There is a dense set of initial data (in L^2) for which the incompressible Euler system admits infinitely many solutions satisfying the energy inequality

Subsolutions

New variables

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x \left(\varrho \mathbf{u} \otimes \mathbf{u} - \frac{1}{3} \varrho |\mathbf{u}|^2 \right) + \nabla_x \left(p(\varrho) + \frac{1}{3} \varrho |\mathbf{u}|^2 \right) = 0$$

Measure valued subsolution

$$\partial_t \varrho + \mathrm{div}_x \mathbf{m} = 0$$

$$\partial_t \mathbf{m} + \mathrm{div}_x \mathbb{U} + \nabla_x q = 0$$

Young measure

$$\mu_{t,x} \in L^\infty_{\mathrm{weak}}\Big((0,T) \times \Omega; \mathcal{P}([0,\infty); R^3, \mathbb{R}^{3\times 3}_{0,\mathrm{sym}}, R) \Big)$$

Lifting

Measure-valued solution of the Euler system

$$\langle \nu; G(\varrho, \mathbf{u}) \rangle$$

Measure-valued subsolution

$$\langle \mu; f(\varrho, \mathbf{m}, \mathbb{U}, q) \rangle$$

$$= \left\langle \nu; f\left(\varrho, \varrho\mathbf{u}, \varrho\mathbf{u}\otimes\mathbf{u} - \frac{1}{3}\varrho|\mathbf{u}|^2\mathbb{I}, p(\varrho) + \frac{1}{3}\varrho|\mathbf{u}|^2\right) \right\rangle$$

Abstract setting

Differential constraints

$$\mathcal{A}\mathbf{Z} = \sum_{i=1}^{N} \mathbb{A}^i \frac{\partial \mathbf{Z}}{\partial y_i} = 0$$

$$A(\mathbf{w}) = \sum_{i=1}^{N} w_i \mathbb{A}^i$$

Constant rank

$$\mathrm{rank} A(\mathbf{w}) = r \text{ for all } \mathbf{w} \in S^{N-1}$$

$\mathcal{A}-$quasiconvexity

Definition

A function $F : R^D \to R$ *is called* $\mathcal{A}-$*quasiconvex if*

$$F(\mathbf{Z}) \leq \int_{\mathbb{T}^N} F\left(\mathbf{Z} + \mathbf{w}(x)\right) \, \mathrm{d}x$$

for all $\mathbf{Z} \in R^D$ *and all* $\mathbf{w} : \mathbb{T}^N \to R^D$ *such that*

$$\mathbf{w} \in C^\infty(\mathbb{T}^N; R^D), \ \mathcal{A}\mathbf{w} = 0, \ \int_{\mathbb{T}^N} \mathbf{w} \, \mathrm{d}x = 0.$$

A result by Fonseca and Mueller

Theorem

Let $\mathcal{A}$ have the constant rank property, $1 \leq p < \infty$, and ν_y a weakly measurable family of probability measures on R^D, $y \in \Omega$. Then there is a sequence of $p-$equiintegrable functions $\{\mathbf{Z}_n\} \subset L^p(\Omega; R^D)$, $\mathcal{A}\mathbf{Z}_n = 0$ generating the Young measure ν_y if and only if:

- $\mathbf{Z}(y) = \langle \nu_y, \mathbf{Z} \rangle \in L^p(\Omega; R^D)$ *satisfies* $\mathcal{A}\mathbf{Z} = 0$;
- $\int_\Omega \langle \nu_y; |\mathbf{Z}|^p \rangle \, \mathrm{d}y < \infty$;
- $$F\left(\langle \nu_y; \mathbf{Z} \rangle\right) \leq \langle \nu_y; F(\mathbf{Z}) \rangle \ \text{ for a.a. } y \in \Omega$$

  for any $\mathcal{A}$-quasiconvex F satisfying the growth restriction

  $$|F(\mathbf{Z})| \leq C(1 + |\mathbf{Z}|^p).$$

Wave cone

Wave cone associated to $\mathcal{A}$

The wave cone of the operator $\mathcal{A}$ is the set of all $\overline{\mathbf{Z}} \in R^D \setminus \{0\}$ for which there is $\xi \in R^N \setminus \{0\}$ such that

$$\mathbf{Z}(y) = h(y \cdot \xi)\overline{\mathbf{Z}}$$

satisfies $\mathcal{A}\mathbf{Z} = 0$ for any h.
Equivalently, $\overline{\mathbf{Z}} \in \Lambda$ if and only if $\overline{\mathbf{Z}} \neq 0$ and there exists $\xi \in R^N \setminus \{0\}$ such that $A(\xi)\overline{\mathbf{Z}} = 0$.

$\mathcal{A}-$free rigidity

Theorem

Let

$$\|\mathbf{Z}_n\|_{L^p(\Omega;R^D)} \leq c,\ \mathcal{A}\mathbf{Z}_n = 0 \text{ in } \mathcal{D}'(\Omega)$$

generate a compactly supported Young measure ν_y*,*

$$\mathrm{supp}[\nu_y] \subset \left\{\lambda\overline{Z}_1 + (1-\lambda)\overline{Z}_2;\ \lambda \in [0,1]\right\}$$

$$\overline{Z}_1 \neq \overline{Z}_2,\ \overline{Z}_2 - \overline{Z}_1 \notin \Lambda.$$

Then

$$\mathbf{Z}_n \to \mathbf{Z}_\infty \text{ in } L^p(\Omega; R^D),$$

where $\mathbf{Z}_\infty$ *is a constant,*

$$\mathbf{Z}_\infty = \lambda_\infty\overline{Z}_1 + (1-\lambda_\infty)\overline{Z}_2.$$

Negative result

Theorem E. Chiodaroli, E.F., O. Kreml, E.Wiedemann [2015]

There exists a measure-valued solution of the compressible Euler system which is not generated by any sequence of L^p-bounded weak solutions (for any choice of $p > 1$).

Strategy

- Show that the linearized differential operator generated by the compressible Euler system (subsolutions) enjoys the constant rank property
- Apply the result by Fonseca and Mueller
- Find constant states $\mathbf{Z}_1$, $\mathbf{Z}_2$ such that $\mathbf{Z}_1 - \mathbf{Z}_2 \notin \Lambda$ but

  $$\frac{1}{2}\delta_{\mathbf{Z}_1} + \frac{1}{2}\delta_{\mathbf{Z}_2}$$

  is a measure-valued subsolution obtained from a measure-valued solution of the compressible Euler system by lifting
- Apply the abstract result