

# Measure-valued solutions in fluid dynamics revisited

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# Compressible Navier-Stokes system

## Field equations

$$\partial_t \varrho + \operatorname{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \operatorname{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \operatorname{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

## Newton's rheological law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left( \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \operatorname{div}_x \mathbf{u} \mathbb{I} \right) + \eta \operatorname{div}_x \mathbf{u} \mathbb{I}, \quad \mu > 0, \quad \eta \geq 0$$

## No-slip boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0$$

# Dissipative solutions

## Energy (entropy) inequality

$$\frac{d}{dt} \int_{\Omega} \left( \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) dx + \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} dx \leq 0$$

$$P(\varrho) = \varrho \int_1^{\varrho} \frac{p(z)}{z^2} dz$$

## Known results

- **Local strong solution for any data and global weak solutions for small data.** Matsumura and Nishida [1983], Valli and Zajackowski [1986], among others
- **Global-in-time weak solutions.**  $p(\varrho) = \varrho^\gamma$ ,  $\gamma \geq 9/5$ ,  $N = 3$ ,  $\gamma \geq 3/2$ ,  $N = 2$  P.L. Lions [1998],  $\gamma > 3/2$ ,  $N = 3$ ,  $\gamma > 1$ ,  $N = 2$  EF, Novotný, Petzeltová [2000],  $\gamma = 1$ ,  $N = 2$  Plotnikov and Vaigant [2014]
- **Measure-valued solutions.** Neustupa [1993]

# Measure-valued solutions

## Parameterized (Young) measure

$$\nu_{t,x} \in L_{\text{weak}}^{\infty}((0, T) \times \Omega; \mathcal{P}([0, \infty) \times \mathbb{R}^N)), \quad N = 2, 3$$
$$\varrho(t, x) = \langle \nu_{t,x}; r \rangle, \quad \mathbf{u} = \langle \nu_{t,x}; \mathbf{v} \rangle \in L^2(0, T; W_0^{1,2}(\Omega; \mathbb{R}^N))$$

## Field equations revisited

$$\int_0^T \int_{\Omega} \langle \nu_{t,x}; r \rangle \partial_t \varphi + \langle \nu_{t,x}; \mathbf{r}\mathbf{v} \rangle \cdot \nabla_x \varphi \, dx \, dt = \langle R_1; \nabla_x \varphi \rangle$$

$$\int_0^T \int_{\Omega} \langle \nu_{t,x}; \mathbf{r}\mathbf{v} \rangle \cdot \partial_t \varphi + \langle \nu_{t,x}; \mathbf{r}\mathbf{v} \otimes \mathbf{v} \rangle \cdot \nabla_x \varphi + \langle \nu_{t,x}; p(r) \rangle \operatorname{div}_x \varphi \, dx \, dt$$
$$= \int_0^T \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \, dx \, dt + \langle R_2; \nabla_x \varphi \rangle$$

# Dissipativity

## Energy inequality

$$\int_{\Omega} \left\langle \nu_{\tau,x}; \left( \frac{1}{2} r |\mathbf{v}|^2 + P(r) \right) \right\rangle dx + \int_0^{\tau} \int_{\Omega} \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} dx dt + \mathcal{D}(\tau) \\ \leq \int_{\Omega} \left\langle \nu_0; \left( \frac{1}{2} r |\mathbf{v}|^2 + P(r) \right) \right\rangle dx$$

## Compatibility

$$|R_1[0, \tau] \times \bar{\Omega}| + |R_2[0, \tau] \times \bar{\Omega}| \leq \xi(\tau) \mathcal{D}(\tau), \quad \xi \in L^1(0, T)$$

$$\int_0^{\tau} \int_{\Omega} \langle \nu_{t,x}; |\mathbf{v} - \mathbf{u}|^2 \rangle dx dt \leq c_P \mathcal{D}(\tau)$$

# Do we need measure valued solutions?

## Limits of problems with higher order viscosities

Multipolar fluids with complex rheologies (Nečas - Šilhavý)

$$\begin{aligned} & \mathbb{T}(\mathbf{u}, \nabla_x \mathbf{u}, \nabla_x^2 \mathbf{u}, \dots) \\ &= \mathbb{S}(\nabla_x \mathbf{u}) + \delta \sum_{j=1}^{k-1} ((-1)^j \mu_j \Delta^j (\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}) + \lambda_j \Delta^j \operatorname{div}_x \mathbf{u} \mathbb{I}) \\ & \quad + \text{non-linear terms} \end{aligned}$$

Limit for  $\delta \rightarrow 0$

## Limits of numerical solutions

Numerical solutions resulting from Karlsen-Karper and other schemes

## Sub-critical parameters

$$p(\varrho) = a\varrho^\gamma, \quad \gamma < \gamma_{\text{critical}}$$

# Weak (mv) - strong uniqueness

**Theorem - EF, P.Gwiazda, A.Świerczewska-Gwiazda, E. Wiedemann [2015]**

A measure valued and a strong solution emanating from the same initial data coincide as long as the latter exists

# Relative energy (entropy)

## Relative energy functional

$$\begin{aligned} & \mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U})(\tau) \\ &= \int_{\Omega} \left\langle \nu_{\tau, x}; \frac{1}{2} s |\mathbf{v} - \mathbf{U}|^2 + P(s) - P'(r)(s - r) - P(r) \right\rangle dx \\ &= \int_{\Omega} \left\langle \nu_{\tau, x}; \frac{1}{2} s |\mathbf{v}|^2 + P(s) \right\rangle dx - \int_{\Omega} \langle \nu_{\tau, x}; s \mathbf{v} \rangle \cdot \mathbf{U} dx \\ & \quad + \int_{\Omega} \frac{1}{2} \langle \nu_{\tau, x}; s \rangle |\mathbf{U}|^2 dx \\ & \quad - \int_{\Omega} \langle \nu_{\tau, x}; s \rangle P'(r) dx + \int_{\Omega} p(r) dx \end{aligned}$$



# Relative energy (entropy) inequality

## Relative energy inequality

$$\begin{aligned} & \mathcal{E}(\varrho, \mathbf{u} \mid r, \mathbf{U}) + \int_0^\tau \mathbb{S}(\nabla_x \mathbf{u}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \, dx \, dt + \mathcal{D}(\tau) \\ & \leq \int_\Omega \left\langle \nu_{0,x}; \frac{1}{2} s |\mathbf{v} - \mathbf{U}_0|^2 + P(s) - P'(r_0)(s - r_0) - P(r_0) \right\rangle dx \\ & \quad + \int_0^\tau \mathcal{R}(\varrho, \mathbf{u} \mid r, \mathbf{U}) \, dt \end{aligned}$$

# Remainder

$$\begin{aligned} & \mathcal{R}(\varrho, \mathbf{u} \mid r, \mathbf{U}) \\ &= - \int_0^T \int_{\Omega} \langle \nu_{t,x}, \mathbf{sv} \rangle \cdot \partial_t \mathbf{U} \, dx \, dt \\ & - \int_0^T \int_{\Omega} [\langle \nu_{t,x}; \mathbf{sv} \otimes \mathbf{v} \rangle : \nabla_x \mathbf{U} + \langle \nu_{t,x}; p(s) \rangle \operatorname{div}_x \mathbf{U}] \, dx \, dt \\ & + \int_0^T \int_{\Omega} [\langle \nu_{t,x}; s \rangle \mathbf{U} \cdot \partial_t \mathbf{U} + \langle \nu_{t,x}; \mathbf{sv} \rangle \cdot \mathbf{U} \cdot \nabla_x \mathbf{U}] \, dx \, dt \\ & + \int_0^T \int_{\Omega} \left[ \left\langle \nu_{t,x}; \left(1 - \frac{s}{r}\right) \right\rangle p'(r) \partial_t r - \langle \nu_{t,x}; \mathbf{sv} \rangle \cdot \frac{p'(r)}{r} \nabla_x r \right] \, dx \, dt \\ & + \int_0^T \left\langle R_1; \frac{1}{2} \nabla_x (|\mathbf{U}|^2 - P'(r)) \right\rangle \, dt - \int_0^T \langle R_2; \nabla_x \mathbf{U} \rangle \, dt \end{aligned}$$

# Regularity

## Theorem - EF, P.Gwiazda, A. Świerczewska-Gwiazda, E. Wiedemann

Suppose that the initial data are smooth and satisfy the relevant compatibility conditions. Let  $\nu_{t,x}$  be a measure-valued solution to the compressible Navier-Stokes system with a dissipation defect  $\mathcal{D}$  such that

$$\text{supp } \nu_{t,x} \subset \left\{ (s, \mathbf{v}) \mid 0 \leq s \leq \bar{\varrho}, \mathbf{v} \in R^N \right\}$$

for a.a.  $(t, x) \in (0, T) \times \Omega$ .

Then  $\mathcal{D} = 0$  and

$$\nu_{t,x} = \delta_{\varrho(t,x), \mathbf{u}(t,x)}$$

where  $\varrho, \mathbf{u}$  is a smooth solution.

# Sketch of the proof

- The Navier-Stokes system admits a local-in-time smooth solution
- The measure-valued solution coincides with the smooth solution on its life-span
- The smooth solution density component remains bounded by  $\bar{\rho}$  as long as the solution exists
- Y. Sun, C. Wang, and Z. Zhang [2011]: The strong solution can be extended as long as the density component remains bounded

# Corrolarry

## **Convergence of numerical solutions**

Bounded numerical solutions emanating from smooth data that converge to a measure-valued solution converge, in fact, unconditionally to the unique strong solution