Domain sensitivity in singular limits of compressible viscous fluids

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Seminaire Laurent Schwartz, Ecole Polytechnique, 5 April 2011

Domains with rapidly oscillating boundaries

$$\Omega_\varepsilon \subset R^3$$

- Ω_ε satisfy uniform $\delta-$cone condition ($\Rightarrow \Omega_\varepsilon \to \Omega$)
- $\partial\Omega_\varepsilon$ oscillate:

$$\lim_{r \to 0} \left(\liminf_{\varepsilon \to 0} \frac{1}{|\partial\Omega_\varepsilon \cap B_r(y)|_2} \int_{\partial\Omega_\varepsilon \cap B_r(y)} |(\mathbf{n} - \mathbf{n}_y) \cdot \mathbf{w}| \ \mathrm{dS}_x \right) > 0$$

$$\text{for any } |\mathbf{w}| = 1, \ \mathbf{w} \cdot \mathbf{n}_y = 0$$

$$\mathbf{w}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \ \mathbf{w}_\varepsilon \to \mathbf{w} \text{ weakly in } W^{1,2}(R^3; R^3)$$

$$\Rightarrow$$

$$\mathbf{w}|_{\partial\Omega} = 0$$

Low Mach number limit

Compressible Navier-Stokes system

$$\partial_t \varrho_\varepsilon + \mathrm{div}_x(\varrho \mathbf{u}_\varepsilon) = 0$$

$$\partial_t(\varrho_\varepsilon \mathbf{u}_\varepsilon) + \mathrm{div}_x(\varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon) + \boxed{\frac{1}{\varepsilon^2}\nabla_x p(\varrho_\varepsilon)} = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon)$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3}\mathrm{div}_x \mathbf{u}\mathbb{I}$$

Incompressible limit

$$\mathrm{div}_x \mathbf{U} = 0$$

$$\overline{\varrho}\left(\partial_t \mathbf{U} + \mathrm{div}_x(\mathbf{U} \otimes \mathbf{U})\right) + \nabla_x \Pi = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Boundary conditions

Complete slip

$$\Omega_\varepsilon \subset R^3, \ \mathbf{u}_\varepsilon \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \ [\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon)\mathbf{n}] \times \mathbf{n}|_{\partial\Omega} = 0$$

$$\Omega_\varepsilon = R^3 \setminus \mathcal{B}_\varepsilon \text{ exterior domains, } \boxed{\Omega_\varepsilon \to \Omega}$$

No-slip

$$\Omega \subset R^3, \ \mathbf{U}|_{\partial\Omega} = 0$$

Conditions for $|x| \to \infty$

$$\mathbf{u} \to 0 \text{ (or } \mathbf{U}_\infty), \ \varrho \to \overline{\varrho} \text{ as } |x| \to \infty,$$

Cost functional (Drag)

Drag

$$D_\varepsilon = \int_{\partial\Omega_\varepsilon} \left(\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) \cdot \mathbf{U}_\infty \cdot \mathbf{n} - \frac{1}{\varepsilon^2} p(\varrho_\varepsilon) \mathbf{U}_\infty \cdot \mathbf{n} \right) \, \mathrm{dS}_x.$$

$$D_{\tau_1,\tau_2} = \int_{\tau_1}^{\tau_2} D_\varepsilon(t) \, \mathrm{d}t$$

Alternative formula

$$\mathbf{u}_\varepsilon|_{\partial\Omega_\varepsilon} = \mathbf{U}_\varepsilon, \ \mathrm{div}_x \mathbf{u}_\varepsilon = 0, \ \mathbf{u}_\varepsilon \to 0 \text{ as } |x| \to \infty$$

$$\int_{\partial\Omega_\varepsilon} \left(\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) \cdot \mathbf{U}_\infty \cdot \mathbf{n} - \frac{1}{\varepsilon^2} p(\varrho_\varepsilon) \mathbf{U}_\infty \cdot \mathbf{n} \right) \, \mathrm{dS}_x$$

$$= \int_{\Omega_\varepsilon} \mathrm{div}_x \left(\mathbb{S} - \frac{1}{\varepsilon^2} p \mathbb{I} \right) \mathbf{u}_\infty + \int_{\Omega_\varepsilon} \mathbb{S} : \nabla_x \mathbf{u}_\infty$$

Stability issues

Energy inequality

$$\int_{\Omega_\varepsilon} \left(\frac{1}{2} \varrho_\varepsilon |\mathbf{u}_\varepsilon|^2 + \boxed{ \frac{1}{\varepsilon^2} E(\varrho_\varepsilon, \overline{\varrho}) } \right) (\tau, \cdot) \ \mathrm{d}x$$

$$+ \int_0^\tau \int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x \mathbf{u}_\varepsilon \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_{\Omega_\varepsilon} \left(\frac{1}{2} \varrho_{0,\varepsilon} |\mathbf{u}_{0,\varepsilon}|^2 + \boxed{ \frac{1}{\varepsilon^2} E(\varrho_{0,\varepsilon}, \overline{\varrho}) } \right) \ \mathrm{d}x$$

Relative entropy

$$E(\varrho, \overline{\varrho}) = P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho})$$

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \ \mathrm{d}z, \ \ P''(\varrho) = \frac{1}{\varrho} p'(\varrho)$$

Constitutive relations

$$p \in C^1[0,\infty) \cap C^2(0,\infty),\ p'(\varrho) > 0 \text{ for } \varrho > 0,\ \lim_{\varrho \to \infty} \frac{p'(\varrho)}{\varrho^{\gamma-1}} = p_\infty > 0$$

$$\gamma > \frac{3}{2}$$

$$\frac{1}{\varepsilon^2} E(\varrho, \overline{\varrho}) \approx \left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right)^2 \text{ for } \varrho \approx \overline{\varrho}$$

$$\frac{1}{\varepsilon^2} E(\varrho, \overline{\varrho}) \approx \frac{1}{\varepsilon^{2-\gamma}} \left(\frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right)^\gamma \text{ for } \varrho >> \overline{\varrho}$$

Ill-prepared initial data and uniform bounds

$$\varrho_\varepsilon(0,\cdot) - \overline{\varrho} = \varepsilon r_\varepsilon, \ \ r_\varepsilon \text{ bounded in } L^2 \cap L^\infty(\Omega_\varepsilon)$$

$$\mathbf{u}_\varepsilon(0,\cdot) = \mathbf{u}_{0,\varepsilon} \in L^2(\Omega_\varepsilon; R^3)$$

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon} \right\|_{L^2 + L^\gamma(\Omega_\varepsilon)} \leq c$$

$$\mathrm{ess} \sup_{t \in (0,T)} \left\| \sqrt{\varrho_\varepsilon} \mathbf{u}_\varepsilon \right\|_{L^2(\Omega_\varepsilon; R^3)} \leq c$$

Modified Korn's inequality:

$$\int_0^T \int_{\Omega_\varepsilon} |\nabla_x \mathbf{u}_\varepsilon|^2 \ \mathrm{d}x \ \mathrm{d}t \leq c$$

Family of domains

- $\Omega_\varepsilon \subset R^3$ is an exterior domain with C^2-boundary
- there is a $d > 0$ such that

$$R^3 \setminus \Omega_\varepsilon \subset B_d \equiv \{x \in R^3 \mid |x| < d\} \text{ for all } \varepsilon > 0;$$

- Ω_ε satisfy the uniform $\delta-$cone condition with $\delta > 0$ independent of ε;
- for each $x_0 \in \partial\Omega_\varepsilon$, there are two (open) balls $B_r[x_i] \equiv \{x; |x - x_i| < r\} \subset \Omega_\varepsilon$, $B_r[x_e] \subset R^3 \setminus \Omega_\varepsilon$ of radius $r > c_b \varepsilon^\beta$ such that

$$\overline{B_r[x_i]} \cap \overline{B_r[x_e]} = x_0,$$

with $c_b > 0$, $\boxed{0 \leq \beta < 1/4}$ independent of ε.

Helmholtz decomposition

$$\mathbf{u}_\varepsilon = \mathbf{H}_\varepsilon[\mathbf{u}_\varepsilon] + \mathbf{H}_\varepsilon^\perp[\mathbf{u}_\varepsilon], \ \mathbf{H}_\varepsilon^\perp[\mathbf{u}_\varepsilon] = \nabla_x \Phi_\varepsilon$$

$$\Delta \Phi_\varepsilon = \mathrm{div}_x \mathbf{u}_\varepsilon \text{ in } \Omega_\varepsilon, \ \nabla_x \Phi_\varepsilon \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0, \Phi_\varepsilon \to 0 \text{ as } |x| \to \infty$$

$$\mathbf{u}_\varepsilon \approx \varrho_\varepsilon \mathbf{u}_\varepsilon = \mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] + (\mathbf{H} - \mathbf{H}_\varepsilon)[\varrho_\varepsilon \mathbf{u}_\varepsilon] + \mathbf{H}_\varepsilon^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon]$$

- $\mathbf{H}[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to \overline{\varrho}\mathbf{U}$ a.a. pointwise
- $(\mathbf{H} - \mathbf{H}_\varepsilon)[\varrho_\varepsilon \mathbf{u}_\varepsilon] \to 0$ a.a. pointwise
- $\mathbf{H}^\perp[\varrho_\varepsilon \mathbf{u}_\varepsilon] \equiv \nabla_x \Phi_\varepsilon \to 0$ weakly (in time)

General strategy

$$\varrho_\varepsilon \mathbf{u}_\varepsilon = [\varrho_\varepsilon \mathbf{u}_\varepsilon]_{\rm ess} + [\varrho_\varepsilon \mathbf{u}_\varepsilon]_{\rm res}$$

$$[\varrho_\varepsilon \mathbf{u}_\varepsilon]_{\rm ess} \in L^\infty(0, T; L^2(\Omega_\varepsilon; R^3))$$

$$\left\| [\varrho_\varepsilon \mathbf{u}_\varepsilon]_{\rm res} \right\|_{L^\infty(0,T;L^r(\Omega_\varepsilon,R^3))} \leq \varepsilon^{1/\gamma} c, \ \ r = \frac{2\gamma}{\gamma + 1}$$

Helmholtz decomposition [Farwig, Kozono, and Sohr]

$$\left\| \mathbf{H}_\varepsilon [\mathbf{v}] \right\|_{(L^p \cap L^2)(\Omega_\varepsilon, R^3)} \leq \varepsilon^{-\beta \left(\frac{3}{2} - \frac{3}{p} \right)} c(p) \|\mathbf{v}\|_{(L^p \cap L^2)(\Omega_\varepsilon, R^3)}$$

$$2 \leq p < \infty$$

Acoustic equation

$$r_\varepsilon = \frac{\varrho_\varepsilon - \overline{\varrho}}{\varepsilon}, \ \mathbf{V}_\varepsilon = \varrho_\varepsilon \mathbf{u}_\varepsilon$$

$$\varepsilon \partial_t r_\varepsilon + \mathrm{div}_x \mathbf{V}_\varepsilon = 0$$

$$\varepsilon \partial_t \mathbf{V}_\varepsilon + p'(\overline{\varrho}) \nabla_x r_\varepsilon$$

$$= \varepsilon \mathrm{div}_x \left(\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \right)$$

$$- \frac{1}{\varepsilon} \nabla_x \left(p(\varrho_\varepsilon) - p(\overline{\varrho}) - p'(\overline{\varrho})(\varrho_\varepsilon - \overline{\varrho}) \right)$$

$$\mathbf{V}_\varepsilon \cdot \mathbf{n}|_{\partial \Omega_\varepsilon} = 0$$

Acoustic wave equation

$$\varepsilon \partial_t r_\varepsilon + \Delta \Phi_\varepsilon = 0$$

$$\varepsilon \partial_t \Phi_\varepsilon + p'(\overline{\varrho}) r_\varepsilon$$
$$= \varepsilon \Delta^{-1} \mathrm{div}_x \mathrm{div}_x \left(\mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) - \varrho_\varepsilon \mathbf{u}_\varepsilon \otimes \mathbf{u}_\varepsilon \right)$$
$$- \frac{1}{\varepsilon} \left(p(\varrho_\varepsilon) - p(\overline{\varrho}) - p'(\overline{\varrho})(\varrho_\varepsilon - \overline{\varrho}) \right)$$

$$\nabla_x \Phi_\varepsilon \cdot \mathbf{n}|_{\partial \Omega_\varepsilon} = 0$$

Abstract formulation

$$A_\varepsilon[v] = -\Delta v \text{ in } \Omega_\varepsilon,\ \nabla_x v \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0$$

A_ε a non-negative self-adjoint operator on $L^2(\Omega_\varepsilon)$

Abstract acoustic equation

$$\varepsilon \partial_t r_\varepsilon - A_\varepsilon[\Phi_\varepsilon] = 0$$

$$\varepsilon \partial_t \Phi_\varepsilon + r_\varepsilon = \varepsilon F(A_\varepsilon)[g_\varepsilon]$$

$F(A)$ may become singular for $A \to 0+$, $A \to \infty$

Variation-of-constants formula

$$\Phi_\varepsilon(t) = \frac{1}{2}\exp\left(\mathrm{i}\sqrt{A_\varepsilon}\frac{t}{\varepsilon}\right)\left[\Phi_{0,\varepsilon} + \mathrm{i}\frac{1}{\sqrt{A_\varepsilon}}[r_{0,\varepsilon}]\right]$$

$$+\frac{1}{2}\exp\left(-\mathrm{i}\sqrt{A_\varepsilon}\frac{t}{\varepsilon}\right)\left[\Phi_{0,\varepsilon} - \mathrm{i}\frac{1}{\sqrt{A_\varepsilon}}[r_{0,\varepsilon}]\right]$$

$$\frac{1}{2}\int_0^t \left(\exp\left(\mathrm{i}\sqrt{A_\varepsilon}\frac{t-s}{\varepsilon}\right) + \exp\left(-\mathrm{i}\sqrt{A_\varepsilon}\frac{t-s}{\varepsilon}\right)\right) F(A_\varepsilon)[g_\varepsilon(s)]\,\mathrm{d}s,$$

$\{\Phi_{0,\varepsilon}\}_{\varepsilon>0},\ \{r_{0,\varepsilon}\}_{\varepsilon>0},\ \{g_\varepsilon\}_{\varepsilon>0}$ determined by the data

- Bounds on g_ε
- Dispersive estimates on the local decay of $\exp(\sqrt{A_\varepsilon}t/\varepsilon)$

Bounds on the forcing terms - example

$$\left| \int_{\Omega_\varepsilon} \mathbb{S}(\nabla_x \mathbf{u}_\varepsilon) : \nabla_x^2 \Delta_{N,\varepsilon}^{-1} \varphi \ \mathrm{d}x \right| \leq \|\mathbb{S}\|_{L^2} \|\nabla_x^2 \Delta_{N,\varepsilon}^{-1} \varphi\|_{L^2}$$

Elliptic estimates

$$\|\nabla_x^2 v\|_{L^p(\Omega_\varepsilon; R^{3\times 3})} \leq c(p) \left(\|\Delta_x v\|_{L^p(\Omega_\varepsilon)} + \frac{1}{\varepsilon^{2\beta}} \|v\|_{L^p(\Omega_\varepsilon)} \right)$$

$$1 < p < \infty$$

$$\|\nabla_x^2 \Delta_{N,\varepsilon}^{-1} \varphi\|_{L^2} \leq c \left(\|\varphi\|_{L^2} + \varepsilon^{-2\beta} \|\Delta_{N,\varepsilon}^{-1}[\varphi]\|_{L^2} \right)$$

$$\Delta_{N,\varepsilon}^{-1} \mathrm{div}_x \mathrm{div}_x \mathbb{S} = \varepsilon^{-2\beta} F(A_\varepsilon) g_\varepsilon, \ F(z) = 1 + \frac{1}{z}, \ g_\varepsilon \in L^2(0, T; L^2(\Omega_\varepsilon)).$$

Dispersive estimates

$$\left\langle \exp\left(\mathrm{i}\sqrt{A_\varepsilon}\frac{t}{\varepsilon}\right) G(A_\varepsilon)[\psi], \varphi \right\rangle$$

$$= \int_0^\infty \exp\left(\mathrm{i}\sqrt{y}\frac{t}{\varepsilon}\right) G(y)\tilde{\psi}(y)\mathrm{d}\mu_{\varepsilon,\varphi}(y)$$

$\mu_{\varepsilon,\varphi}$ − spectral measure associated to φ

$$\|\tilde{\psi}\|_{L^2_\mu[0,\infty)} \leq \|\psi\|_{L^2(\Omega_\varepsilon)}$$

$$\int_0^T \left| \left\langle \exp\left(\mathrm{i}\sqrt{A_\varepsilon}\frac{t}{\varepsilon}\right) G(A_\varepsilon)[\psi], \varphi \right\rangle \right|^2 \, \mathrm{d}t$$

$$= \int_0^T \int_0^\infty \int_0^\infty \exp\left(\mathrm{i}\left(\sqrt{y} - \sqrt{x}\right)\frac{t}{\varepsilon}\right)$$

$$G(y)G(x)\tilde{\psi}(y)\overline{\tilde{\psi}(x)} \, \mathrm{d}\mu_{\varepsilon,\varphi}(y) \, \mathrm{d}\mu_{\varepsilon,\varphi}(x)$$

Convergence via Kato's theorem

$$\int_0^\infty \exp\left(- \frac{|\sqrt{x} - \sqrt{y}|^2}{\varepsilon^2} \frac{T^2}{4} \right) \mathrm{d}\mu_{\varepsilon,\varphi}(y)$$

$$= \sum_{n=0}^\infty \int_{\varepsilon n \leq |\sqrt{y} - \sqrt{x}| < \varepsilon(n+1)} \exp\left(- \frac{|\sqrt{x} - \sqrt{y}|^2}{\varepsilon^2} \frac{T^2}{4} \right) \mathrm{d}\mu_{\varepsilon,\varphi}(y)$$

$$\leq \sup_{n \geq 0} \int_{\varepsilon n \leq |\sqrt{y} - \sqrt{x}| < \varepsilon(n+1)} 1 \mathrm{d}\mu_{\varepsilon,\varphi}(y) \sum_{n=0}^\infty \exp\left(- \frac{n^2 T^2}{4} \right)$$

for $x \in \mathrm{supp}[G]$

Stone's formula:

$$\mu_{\varepsilon,\varphi}(a,b)$$

$$= \lim_{\delta \to 0+} \lim_{\eta \to 0+} \int_{a+\delta}^{b-\delta} \left\langle \left(\frac{1}{A_\varepsilon - \lambda - \mathrm{i}\eta} - \frac{1}{A_\varepsilon - \lambda + \mathrm{i}\eta} \right) \varphi, \varphi \right\rangle \, \mathrm{d}\lambda$$

Limiting absorption principle:

$$\left[\begin{array}{c} \textit{Operators} \\ \\ \mathcal{V} \circ (A_\varepsilon - \lambda \pm \mathrm{i}\eta)^{-1} \circ \mathcal{V} : L^2(\Omega) \to L^2(\Omega), \\ \\ \mathcal{V}[v] = (1 + |x|^2)^{-s/2},\ s > 1 \\ \\ \textit{are bounded uniformly for } \lambda \in [a, b],\ 0 < a < b,\ \eta > 0, \end{array} \right]$$

Conclusion via Kato's result

Operators A_ε satisfy Limiting Absorption Principle

$$\Longrightarrow$$

$$\mu_{\varepsilon,\varphi}[I] \leq c_\delta |I| \text{ for any interval } I \subset [\delta, 1/\delta],\ \delta > 0$$

$$\Longrightarrow$$

$$\int_0^T \left| \left\langle \exp\left(\mathrm{i}\sqrt{A_\varepsilon}\frac{t}{\varepsilon} \right) G(A_\varepsilon)[\psi], \varphi \right\rangle \right|^2 \ \mathrm{d}t \leq \varepsilon c(G,\varphi) \|\psi\|^2_{L^2(\Omega_\varepsilon)}$$

Stone's formula $\Rightarrow$

$$\mu_{\varepsilon,\varphi}(a,b) = \int_a^b \left\langle \left(w^-_{\varepsilon,\lambda} - w^+_{\varepsilon,\lambda} \right), \varphi \right\rangle_{L^2(\Omega_\varepsilon)} \ \mathrm{d}\lambda, \ 0 < a < b,$$

where $w^\pm_{\varepsilon,\lambda}$ solve

$$\Delta w^\pm_{\varepsilon,\lambda} + \lambda w^\pm_{\varepsilon,\lambda} = \varphi \text{ in } \Omega_\varepsilon, \ \nabla_x w^\pm_{\varepsilon,\lambda} \cdot \mathbf{n}|_{\partial\Omega_\varepsilon} = 0,$$

Sommerfeld radiation condition

$$\lim_{r\to\infty} r \left(\partial_r \pm \mathrm{i}\sqrt{\lambda} \right) w^\pm_{\varepsilon,\lambda} = 0, \ r = |x|$$

Reduction to bounded domain case:

$$\mathrm{supp}[\varphi] \subset \{|x| \leq R\}$$

$$w^{\pm}_{\varepsilon,\lambda_\varepsilon}(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^m Y_l^m(\theta,\phi) \text{ for } |x| = 2R$$

(r,θ,ϕ) polar coordinates

$$w^{\pm}_{\varepsilon,\lambda_\varepsilon}(x) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_l^m Y_l^m(\theta,\phi) \frac{h_l^{(1)}(\pm\sqrt{\lambda}r)}{h_l^{(1)}(\pm\sqrt{\lambda}2R)} \text{ for all } x \in R^3 \setminus \overline{B}_{2R},$$

Y_l^m spherical harmonics of order l

$h_l^{(1)}$ spherical Bessel functions