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**Abstract**

We disprove a conjecture from [2] by showing the existence of a Fraïssé class $\mathcal{C}$ which does not admit a Katětov functor. On the other hand, we show that the automorphism group of the Fraïssé limit of $\mathcal{C}$ is universal, as it happens in the presence of a Katětov functor.

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**Keywords:** Katětov functor, Fraïssé limit.

## 1 Introduction

Let us recall some basic facts. A Fraïssé class $\mathcal{C}$ is a countable class of finitely generated structures for some countable language $\mathcal{L}$ together with embeddings which satisfy joint embedding property, amalgamation property, hereditary property. Let us denote by $\sigma\mathcal{C}$ the class of all countably generated structures that are colimits of countable chains from $\mathcal{C}$. Obviously $\mathcal{C} \subseteq \sigma\mathcal{C}$. Fraïssé theorem says that there exists a unique homogeneous structure $\mathrm{Flim}(\mathcal{C}) \in \sigma\mathcal{C}$, called the *Fraïssé limit* of $\mathcal{C}$, which is universal for $\sigma\mathcal{C}$. This means that every isomorphism between finitely generated substructures of $\mathrm{Flim}(\mathcal{C})$ extends to an automorphism of $\mathrm{Flim}(\mathcal{C})$, and every structure from $\mathcal{C}$ embeds into $\mathrm{Flim}(\mathcal{C})$. For more information see, e.g. [1].

In [2] the authors define a notion of a Katětov functor. Existence of such a functor leads to a uniform way of obtaining the Fraïssé limit and to prove, for example, that $\mathrm{Aut}(\mathrm{Flim}(\mathcal{C}))$ is universal for all $\mathrm{Aut}(X)$ where $X \in \sigma\mathcal{C}$. Let us recall the definition.

**Definition 1.** We say that $K\colon \mathcal{C} \to \sigma\mathcal{C}$ is a Katětov functor if

- $K$ is a covariant functor,
- there is a natural transformation $\{\eta_x\}_{x\in\mathcal{C}}$ between $id_{\mathcal{C}}$ and $K$,

$$\begin{array}{ccc} x & \xrightarrow{\eta_x} & K(x) \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle K(f)} \\ y & \xrightarrow[\eta_y]{} & K(y) \end{array}$$

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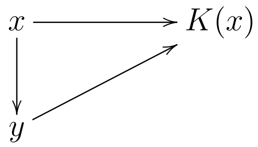
- every one point extension $y \in \mathcal{C}$ of $x \in \mathcal{C}$ is realized in $K(x)$ over $\eta_x(x)$.

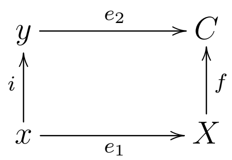



Every Katětov functor $K$ naturally extends to a functor $K\colon \sigma\mathcal{C} \to \sigma\mathcal{C}$ with similar properties, i.e., there is a natural transforamtion between $id_{\sigma\mathcal{C}}$ and $K$, every one point extension $y \in \mathcal{C}$ of $x \in \mathcal{C}$ where $x \subseteq X \in \sigma\mathcal{C}$ is realized in $K(X)$ over $\eta_X(x)$. Moreover, iterating the Katětov functor $\omega-$many times leads to another Katětov functor $K^\omega\colon \sigma\mathcal{C} \to \sigma\mathcal{C}$ such that for all $X \in \sigma\mathcal{C}$ it holds that $K(X) = \mathrm{Flim}(\mathcal{C})$. Easy observation (using functoriality) then gives us that $\mathrm{Aut}(X)$ embeds into $\mathrm{Aut}(\mathrm{Flim}(\mathcal{C}))$ and $\mathrm{Emb}(X, Y)$ embeds into $\mathrm{Emb}(\mathrm{Flim}(\mathcal{C}))$ i.e. $K^\omega\colon \sigma\mathcal{C} \to \{\mathrm{Flim}(\mathcal{C})\}$ is a faithful functor. See [2] for more information. It was an open problem whether there is a Fraïssé class without a Katětov functor. In this note we prove that the class of linearly ordered finite sets with colorings of pairs by countably many colors without monochromatic triangles is a Fraïssé class without a Katětov functor, however it admits a faithful functor as above.

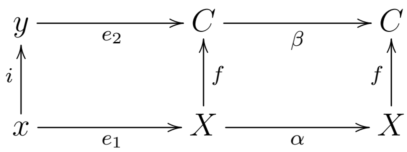
## 2 The construction

Let us fix some Fraïssé class $\mathcal{C}$ and denote its closure on colimits of countable chains by $\sigma\mathcal{C}$. When we have a Katětov functor $K\colon \sigma\mathcal{C} \to \sigma\mathcal{C}$ we will always assume that $K(X)$ is an extension of $X$ i.e. $\eta_X$ is inclusion. Let us denote a one point extension $y \in \mathcal{C}$ of $x \in \mathcal{C}$ by $y = \langle x, t\rangle$ (meaning that $t$ is a single generator). Similarly, $\langle X, t\rangle$ will denote a one point extension of $X \in \sigma\mathcal{C}$, namely, a structure in $\sigma\mathcal{C}$ generated by $X \cup \{t\}$.

**Definition 2.** Let $x, y \in \mathcal{C}$ such that $x \subseteq y$ and $X, C \in \sigma\mathcal{C}$. Consider the following commutative diagram.




Then we say that $C$ is a *homogeneous extension* of $X$ over $x \subseteq y$ if for all $\alpha \in \mathrm{Aut}(X)$ such that $\alpha \restriction x = id_x$, there is $\beta \in \mathrm{Aut}(C)$ such that $\beta \restriction y = id_y$ and the following diagram commutes.




The most important case is when $C$ is generated by $X \cup y$ and $y$ is generated by one element over $x$. In that case we in fact deal with extending one point extension $\langle x, t\rangle$ to a homogeneous one point extension $\langle X, t\rangle$.

**Proposition 1.** *Assume that there is a Katětov functor $K$ for $\mathcal{C}$. Then for all $x \in \mathcal{C}$, all one point extensions $\langle x, t\rangle \in \mathcal{C}$ and all embeddings $e : x \to \mathrm{Flim}(\mathcal{C})$ there exists a homogeneous one point extension $\langle \mathrm{Flim}(\mathcal{C}), t\rangle$ of $\mathrm{Flim}(\mathcal{C})$ over $x \subseteq \langle x, t\rangle$.*

*Proof.* Assume we have the following commutative diagram, where $\alpha$ is an automorphism.

$$\begin{array}{ccc} x & \xrightarrow{e} & \mathrm{Flim}(\mathcal{C}) \\ \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle \alpha} \\ x & \xrightarrow[e]{} & \mathrm{Flim}(\mathcal{C}) \end{array}$$

This diagram can be moved by our Katětov functor $K$ to the following commutative diagram

$$\begin{array}{ccc} K(x) & \xrightarrow{K(e)} & K(\mathrm{Flim}(\mathcal{C})) \\ \uparrow{\scriptstyle id} & & \uparrow{\scriptstyle K(\alpha)} \\ K(x) & \xrightarrow[K(e)]{} & K(\mathrm{Flim}(\mathcal{C})) \end{array}$$

Where $K(\alpha)$ is again an automorphism, and these two diagrams are connected by the natural transformation $\eta$. Then $\langle x, t\rangle$ can be embedded to $K(x)$ by the definition of $K$, so we may assume that this embedding is inclusion and put $e(t) := K(e)(t)$. By commutativity we see that $K(\alpha)(e(t)) = e(t)$. So $K(\alpha)$ and $K(\alpha)^{-1}$ are invariant on $\langle \mathrm{Flim}(\mathcal{C}), e(t)\rangle$ which is exactly what we needed to prove. $\square$

Another consequence of the existence of a Katětov functor is a nontrivial pair of embeddings $e\colon \mathrm{Flim}(\mathcal{C}) \to \mathrm{Flim}(\mathcal{C})$ and $E\colon \mathrm{Aut}(\mathrm{Flim}(\mathcal{C})) \to \mathrm{Aut}(\mathrm{Flim}(\mathcal{C}))$ such that the following diagram

$$\begin{array}{ccc} \mathrm{Flim}(\mathcal{C}) & \xrightarrow{e} & \mathrm{Flim}(\mathcal{C}) \\ \uparrow{\scriptstyle \alpha} & & \uparrow{\scriptstyle E(\alpha)} \\ \mathrm{Flim}(\mathcal{C}) & \xrightarrow[e]{} & \mathrm{Flim}(\mathcal{C}) \end{array}$$

commutes for all $\alpha \in \mathrm{Aut}(\mathrm{Flim}(\mathcal{C}))$. Assume that we have such a pair $(e, E)$, fix $x \subseteq \mathrm{Flim}(\mathcal{C})$ and $t \in \mathrm{Flim}(\mathcal{C}) \setminus e[\mathrm{Flim}(\mathcal{C})]$. Define

$$\mathcal{G} := \{\alpha \in \mathrm{Aut}(\mathrm{Flim}(\mathcal{C})) : E(\alpha) \restriction \langle x, t\rangle = id_{\langle x,t\rangle}\}.$$

This is an open subgroup of $\mathrm{Aut}(\mathrm{Flim}(\mathcal{C}))$ because it is an analytic subgroup (i.e. it has the Baire property) and it has countable index. So there is $x \subseteq x' \subseteq \mathrm{Flim}(\mathcal{C})$ such that $\mathcal{H} := \{\alpha\colon \alpha \restriction x' = id_{x'}\} \leq \mathcal{G}$ i.e. $\langle \mathrm{Flim}(\mathcal{C}), t\rangle$ is a one point homogeneous extension of $\mathrm{Flim}(\mathcal{C})$ over $\langle x', t\rangle$. Hence we may say that for a nontrivial pair of embeddings $(e, E)$ and every one point extension $\langle x, t\rangle$ such that $x \subseteq \mathrm{Flim}(\mathcal{C})$ and $t$ is realized in $\mathrm{Flim}(\mathcal{C})$ over $e[x]$ outside $e[\mathrm{Flim}(\mathcal{C})]$ there are $x' \supseteq x$ and a homogeneous one point extension $\langle \mathrm{Flim}(\mathcal{C}), t\rangle$ of $\mathrm{Flim}(\mathcal{C})$ over $\langle x', t\rangle$. In particular, if $(e, E)$ is nontrivial than there are nontrivial homogeneous one point extensions.

As it was mentioned in the introduction, iterating $\omega$ many times a fixed Katětov functor $K$ leads to the functor $K^\omega$ such that $K^\omega(X) = \mathrm{Flim}(\mathcal{C})$ for every $X \in \sigma\mathcal{C}$. In particular, $K^\omega\colon \sigma\mathcal{C} \to \{\mathrm{Flim}(\mathcal{C})\}$ is a faithful functor.

**Theorem 1.** *There exists a Fraïssé class $\mathcal{C}$ without a Katětov functor, yet with a faithful functor from $\sigma\mathcal{C}$ to $\{\mathrm{Flim}(\mathcal{C})\}$.*

*Proof.* Let $Q$ be a countable set. Let us define the class $\mathcal{C}$. An element of $\mathcal{C}$ is a finite set $x$ with a linear order and with a function $c_x : [x]^2 \to Q$ such that there are no monochromatic triangles. We will denote the coloring function $c_x$ always by $c$, omitting the subscript $x$. It can be easily seen that $\mathcal{C}$ is a Fraïssé class.

Let us first prove that there are no homogeneous one point extensions of $\operatorname{Flim}(\mathcal{C})$. Let us fix $x \subseteq \operatorname{Flim}(\mathcal{C})$ and any one point extension $\langle x, t\rangle \in \mathcal{C}$. Let us assume that there is a homogeneous one point extension $\langle \operatorname{Flim}(\mathcal{C}), t\rangle$ of $\operatorname{Flim}(\mathcal{C})$ over $\langle x, t\rangle$. Let us pick any $t_1 \in \operatorname{Flim}(\mathcal{C})$ which realizes the same type over $x$ as $t$. Let $q = c((t, t_1))$. Using the saturation of $\operatorname{Flim}(\mathcal{C})$ we find an element $t_2 \in \operatorname{Flim}(\mathcal{C})$ such that $c((t_2, t_1)) = q$ and the mapping $\alpha\colon \langle x, t_1\rangle \to \langle x, t_2\rangle$, defined by conditions $\alpha \restriction x = id_x$ and $\alpha(t_1) = t_2$, is an isomorphism. Take any automorphism $\alpha_0\colon \operatorname{Flim}(\mathcal{C}) \to \operatorname{Flim}(\mathcal{C})$ which extends $\alpha$. From the definition of a homogeneous extension, $\alpha_0$ extends to $\langle \operatorname{Flim}(\mathcal{C}), t\rangle$ such that $\alpha_0(t) = t$. Then we have that $\{t, t_1, t_2\}$ is a monochromatic triangle, which is a contradiction.

From the arguments above it follows that not only there is no Katětov functor for $\mathcal{C}$ but there is no nontrivial pair of embeddings $(e, E)$ for $\operatorname{Flim}(\mathcal{C})$ as well.

In order to prove that $\operatorname{Emb}(\operatorname{Flim}(\mathcal{C}))$ is universal for $\sigma\mathcal{C}$, let us define a suitable sequence of Fraïssé classes $\{\mathcal{C}_i\}_{i\leq\omega}$. First let us fix a sequence of countable sets $\{Q_i\}_{i\leq\omega}$ such that $|Q_{i+1} \setminus Q_i| = \omega$ and $Q_\omega = \bigcup_{i<\omega} Q_i$. The definition of $\mathcal{C}_i$ is similar as of $\mathcal{C}$, the only difference is that the set of colors is $Q_i$. We have that $\sigma\mathcal{C}_i \subseteq \sigma\mathcal{C}_{i+1} \subseteq \sigma\mathcal{C}_\omega$. Now for each $i < \omega$ we will find a functor $K_i$ such that

- $K_i\colon \mathcal{C}_i \to \sigma\mathcal{C}_{i+1}$,
- there is a natural transformation $\nu$ for inclusion and $K_i$, i.e., $\nu_x : x \to K_i(x)$ for $x \in \mathcal{C}_i$,
- for every $x \in \mathcal{C}_i$ every $\mathcal{C}_i-$type over $x$ is realized in $\nu_x[x]$.

The functor $K_i$ extends with all its properties (as in the case of Katětov functors) to $K_i\colon \sigma\mathcal{C}_i \to \sigma\mathcal{C}_{i+1}$. Once we have this, let us put

$$K_\omega = ... \circ K_i \circ ... \circ K_1 \circ K_0\colon \sigma\mathcal{C}_0 \to \sigma\mathcal{C}_\omega.$$

The functor $K_\omega$ is correctly defined thanks to the natural transformations for the functors $K_i$, and there is a natural transformation from the inclusion $\sigma\mathcal{C}_0 \subseteq \sigma\mathcal{C}_\omega$ to $K_\omega$. Moreover, we have that $K_\omega(X) = \operatorname{Flim}(\mathcal{C}_\omega) \simeq \operatorname{Flim}(\mathcal{C}_0)$, which proves our claim.

To finish the proof it is enough to describe $K_0$. Let us put $K := K_0$, $\mathcal{C} := \mathcal{C}_0$, $Q := Q_0$, $\mathcal{D} := \mathcal{C}_1$, $P := Q_1$ and fix any $p \in P \setminus Q$. Let us define $K$ on objects. Take any $x \in \mathcal{C}$. Assume that we have fixed a linear ordering $<_Q$ on $Q$ isomorphic to the natural numbers. Denote by $\mathcal{O}_x$ the set of all partial proper one-point extensions of $x$ from $\mathcal{C}$. We will use the same letter for a type and for its realization. To every $\xi \in \mathcal{O}_x$ let $\operatorname{supp}(\xi) \subseteq x$ denote the support of $\xi$, that is, the substructure of $x$ to which $\xi$ is added. Let $K(x) = x \cup \mathcal{O}_x$. Given an embedding $e\colon x \to y$, define $K(e)$ in the obvious way, namely, a partial type $\xi$ is mapped to the corresponding partial type with support $e[\operatorname{supp}(\xi)]$. Next we turn $K(x)$ to be an element of $\sigma\mathcal{D}$ in such a way that $K(e)$ remains an embedding of structures whenever $e$ is an embedding.

Extend the coloring by putting $c(v, \xi) = p$ for $v \in x \setminus \operatorname{supp}(\xi)$ and extend the ordering on $\xi$ by declaring $\xi < v$ for $v \in x \setminus \operatorname{supp}(\xi)$ whenever it is consistent with the ordering on $\operatorname{supp}(\xi) \cup \{\xi\}$. In the next step we define a linear ordering on $K(x)$. Given $\xi \neq \psi \in \mathcal{O}_x$, let us define $\xi < \psi$ if one of the following conditions is satisfied.

(1) there is $v \in x$ such that $\xi < v < \psi$,

(2) condition (1) fails and $|\operatorname{supp}(\xi)| < |\operatorname{supp}(\psi)|$,

(3) conditions (1), (2) fail and the biggest $v \in \operatorname{supp}(\xi) \triangle \operatorname{supp}(\psi)$ satisfies $v \in \operatorname{supp}(\xi)$,

(4) none of the above is satisfied and for the biggest $v \in \operatorname{supp}(\xi) = \operatorname{supp}(\psi)$ for which $c_\xi((v,\xi)) \neq c_\psi((v,\psi))$ we have that $c_\xi((v,\xi)) <_Q c_\psi((v,\psi))$.

It remains to define a coloring on $K(x)$ extending the coloring of $x$. In order to do this, let us define an equivalence relation on pairs of elements from $\mathcal{O}_x$. We say that $\{\xi_0 < \psi_0\} \sim \{\xi_1 < \psi_1\}$ if there is an isomorphism between $\operatorname{supp}(\xi_i) \cup \operatorname{supp}(\psi_i)$ where $i \in \{0,1\}$ whose extension to $\{\xi_i < \psi_i\}$ remains an isomorphism. It is clear from the definition that if $\{\xi_0 < \psi_0\} \sim \{\xi_1 < \psi_1\}$ then $\operatorname{supp}(\xi_0) \simeq \operatorname{supp}(\xi_1)$ and $\operatorname{supp}(\psi_0) \simeq \operatorname{sup}(\psi_1)$. Furthermore, the isomorphisms are unique, because of the linear orderings. It follows immediately:

**Claim 1.** *For $f\colon x \to y$ we have that $\{\xi_0 < \psi_0\} \sim \{\xi_1 < \psi_1\}$ in $\mathcal{O}_x$ iff $\{K(f)(\xi_0) < K(f)(\psi_0)\} \sim \{K(f)(\xi_1) < K(f)(\psi_1)\}$ in $\mathcal{O}_y$.*

Let us now color the equivalence classes by induction on the size of $x$. Assume that for all sets of size $< n$ we have already defined the coloring and take $x$ such that $|x| = n$. For the equivalence class of a pair $\{\xi_0 < \psi_0\}$ use the color $r \in P$ if there is an embedding $f : y \to x$ such that $\{\xi_0 < \psi_0\}$ is in the image of $K(f)$ and their preimage is colored by $r$. This is well-defined by Claim 1. Color the remaining equivalence classes by different (not already used) colors so that infinitely many colors in $P$ are still left.

We see that whenever $f$ is an embedding of $\mathcal{C}$-structures then $K(f)$ respects both the ordering and the coloring. To finish the proof we must show that there are no monochromatic triangles in $K(x)$. Assume that there is one $\xi < \psi < \mu$. This means that all pairs are in the same equivalence class. There are isomorphisms $i, j$ which witness that for $\{\xi < \psi\} \sim \{\xi < \mu\}$ and $\{\xi < \mu\} \sim \{\psi < \mu\}$. We know that then $i$ is identity on $\operatorname{supp}(\xi)$ and $j$ is identity on $\operatorname{supp}(\mu)$. Because the triangle is not degenerated, there is $v \in \operatorname{supp}(\psi) \setminus \operatorname{supp}(\mu)$, for such $v$ it holds that $w = i(v) \neq v$ and $j(i(v)) = j(w) = w \neq v$. There must be some $z \in \operatorname{supp}(\xi)$ such that $j(z) = v$. Then we have that $c(\{v,z\}) = c(\{w,z\})$ which is witnessed by $i$, and $c(\{v,w\}) = c(\{z,w\})$ which is witnessed by $j$. This is a contradiction, because then $\{v,w,z\}$ is a monochromatic triangle in $x \in \mathcal{C}$. □

# 3 Final remarks

Our result suggests the following question: Is there a Fraïssé class without a faithful functor like in Theorem 1?

It can be easily verified that if $\sigma\mathcal{C}$ has push-outs, then for all $x \in \mathcal{C}$, all one-point extensions $\langle x, t\rangle \in \mathcal{C}$ and all embeddings $f : x \to \operatorname{Flim}\mathcal{C}$ there is a one-point homogenenous extension $\langle \operatorname{Flim}\mathcal{C}, t\rangle$ of $\operatorname{Flim}\mathcal{C}$ over $\langle x, t\rangle$. That follows from the fact that the extension of $\operatorname{Flim} \mathcal{C}$ can be defined to be push out of the following diagram

$$
\begin{array}{ccc}
\langle x, t\rangle & \xrightarrow{i_0} & \langle \operatorname{Flim} \mathcal{C}, t\rangle \\
{\scriptstyle i}\uparrow & & \uparrow{\scriptstyle f_0} \\
x & \xrightarrow[f]{} & \operatorname{Flim} \mathcal{C}
\end{array}
$$

one can easily verify that it is really an homogeneous one-point extension. This construction works for arbitrary object $X \in \sigma\mathcal{C}$ instead of $\operatorname{Flim} \mathcal{C}$. Let us finally state that the category of locally finite countable groups does not have a push out which brings us to the following concrete open question.

**Question 1.** Is there a Katětov functor for the Fraïssé class of finite groups? What can we say about one-point homogeneous extensions in this category?

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