Mathematics of fluids in motion

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Contents

1 Continuum fluid mechanics

Our goal is to develop a *phenomenological theory* of fluid dynamics based on *observable* macroscopic quantities as mass density, velocity, internal energy etc. These will depend on the *time* t and the reference spatial coordinate $x \in R^N$. If not stated otherwise, the *Eulerian reference system* will be used attached to the physical domain occupied by the fluid in contrast with the *Lagrangean description* related to the hypothetical fluid particles and their trajectories - streamlines - in the physical space.

1.1 Mass density, velocity, mass conservation

The distribution of a fluid at a given time t is given by the *mass density* $\varrho = \varrho(t,x)$ - a non-negative scalar function such that the integral

$$\int_B \varrho(t,x) \ \mathrm{d}x = M(B)$$

gives the total mass of the fluid contained in a given set B at the time t.

The *motion* of (hypothetical) fluid particles is determined by the *velocity field* $\mathbf{u}(t,x) \in R^N$, typically $N = 1,2,3$. The trajectory of a bulk of fluid occupying at the initial instant $t = 0$ the set B is given by

$$\mathbf{X}(t,B),\ t \geq 0,\ \text{where } \frac{\partial}{\partial t}\mathbf{X}(t,x) = \mathbf{u}(t,\mathbf{X}(t,x)),\ \mathbf{X}(0,x) = x,\ x \in B.$$

The individual trajectories $t \mapsto \mathbf{X}(t,x)$ are called *streamlines*. The velocity field must enjoy certain regularity in the $x-$ variable for the streamlines to be well defined on a time interval I, specifically

$$\nabla_x \mathbf{u} \in L^1(I; L^\infty_{\rm loc}(R^N)).$$

1.1.1 Eulerian vs. Lagrangean description

Given a velocity field $\mathbf{u}$ generating a family of streamlines $\mathbf{X} = \mathbf{X}(t, x)$ in the physical space Ω, a quantity Q can be expressed in terms of the Eulerian variables as

$$Q = Q_E(t, x),\ t \in I,\ x \in \mathbf{X}(t, B),$$

or, in the Lagrangean variables

$$Q = Q_L(t, Y),\ t \in I,\ Y \in B,$$

where

$$Q_L(t, Y) = Q_E(t, \mathbf{X}(t, Y)) \text{ or } Q_L(t, \mathbf{X}^{-1}(t, x)) = Q_E(t, x).$$

In particular, the time derivative in the Lagrangean setting corresponds to the *material* derivative in the Euler coordinates:

$$\partial_t Q_L(t, Y) = \partial_t Q_E(t, \mathbf{X}(t, x)) + \nabla_x Q_E(t, \mathbf{X}(t, x)) \cdot \mathbf{u}(t, \mathbf{X}(t, x)).$$

At first glance it may seem that the Lagrangean description is simpler; whence more suitable for a mathematical treatment. On the other hand, transforming *spatial gradients* requires invertibility of the mapping $\mathbf{X}$ and therefore certain regularity of the velocity field that is often out of reach of the available analytical methods.

1.1.2 Mass transport

Consider a piece B of the physical space containing a fluid of the density ϱ moving with the (Eulerian) velocity $\mathbf{u} = \mathbf{u}(t, x)$.

The physical principle of MASS CONSERVATION asserts: The change of the total mass of the fluid contained in B during a time interval $t_1 < t_2$,

$$\int_B \varrho(t_2, x)\ \mathrm{d}x\ \mathrm{d}t - \int_B \varrho(t_1, x)\ \mathrm{d}x\ \mathrm{d}t,$$

equals

the total out/in flux of the mass through the boundary ∂_B,

$$-\int_{t_1}^{t_2} \int_{\partial B} \varrho \mathbf{u}(t, x) \cdot \mathbf{n}\ \mathrm{dS}_x\ \mathrm{d}t,$$

where $\mathbf{n}$ denotes the outer norm vector to ∂B. The above relation must hold for any time interval $[t_1, t_2]$ and any volume element B. Assuming that all quantities are sufficiently smooth, we may use the Gauss-Green theorem and perform the limit $t_2 \to t_1$ to deduce a differential form of the principle of mass conservation - *equation of continuity*:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0. \tag{1.1}$$

It is remarkable that the same principle can be derived from an apparently weaker statement

$$\int_I \int_\Omega [\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi] \ \mathrm{d}x \ \ \mathrm{d}t = 0 \text{ for any } \varphi \in C^1_c(I \times \Omega), \tag{1.2}$$

which is usually termed the *weak formulation* of (1.1). Indeed it is enough to take $\varphi_\varepsilon \in C^\infty_c(I \times \Omega)$ a suitable approximation of the characteristic function $1_{[t_1,t_2]\times B}$ and let $\varepsilon \to 0$. If ∂B is smooth, one can consider a family of Lipschitz functions

$$\varphi_\varepsilon(t,x) = \min\left\{\frac{1}{\varepsilon}\mathrm{dist}[x,\partial B]; 1\right\} \times \min\left\{\frac{1}{\varepsilon}\min\{t-t_1; t_2-t_1\}; 1\right\}.$$

1.2 Momentum equation

In order to determine the velocity field $\mathbf{u}$ we need a relation between the changes of the momentum $\varrho\mathbf{u}$ and the material forces acting on a volume element of the fluid. We consider two kinds of such forces: (i) *stress forces* acting on any surface element shared by two adjacent parts of the fluid, (ii) *bulk* or volumic forces. The stress is characterized by the *Cauchy stress tensor* $\mathbb{T}$ producing the stress $\mathbb{T} \cdot \mathbf{n}$ acting on a unit surface element characterized by a normal vector $\mathbf{n}$.

Mathematical formulation of NEWTON'S SECOND LAW reads

$$\begin{aligned}&\int_B \varrho \mathbf{u}(t_2,x) \ \ \mathrm{d}x - \int_B \varrho\mathbf{u}(t_1,x) \ \ \mathrm{d}x \\ &= -\int_{t_1}^{t_2}\int_{\partial B} \varrho\mathbf{u} \ \mathbf{u}\cdot\mathbf{n} \ \mathrm{dS}_x \, \mathrm{d}t + \int_{t_1}^{t_2}\int_{\partial B} \mathbb{T}\cdot\mathbf{n} \ \mathrm{dS}_x \, \mathrm{d}t + \int_{t_1}^{t_2}\int_B \varrho\mathbf{f} \, \mathrm{d}x \ \ \mathrm{d}t,\end{aligned}$$

for any B and any $t_1 < t_2$.

Similarly to Section 1.1.2, we may deduce the differential form of Newton's second law

$$\partial_t(\varrho\mathbf{u}) + \mathrm{div}_x(\varrho\mathbf{u}\otimes\mathbf{u}) = \mathrm{div}_x\mathbb{T} + \varrho\mathbf{f}, \tag{1.3}$$

or its *weak formulation*

$$\int_I\int_\Omega [(\varrho\mathbf{u})\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi] \ \ \mathrm{d}x\,\mathrm{d}t = \int_I\int_\Omega \mathbb{T} : \nabla_x\varphi \ \ \mathrm{d}x\,\mathrm{d}t - \int_I\int_\Omega \varrho\mathbf{f}\cdot\varphi \ \ \mathrm{d}x \ \ \mathrm{d}t \tag{1.4}$$

for any $\varphi \in C^1_c(I \times \Omega; R^N)$.

1.3 Cauchy stress in fluids, examples of fluid equations

The system of equations (1.1), (1.3) is not closed, a description of the Cauchy stress $\mathbb{T}$ is needed in terms of ϱ, $\mathbf{u}$ or other quantities as the case may be. The following statement is often used as a *mathematical definition* of fluid.

A FLUID is characterized by STOKES' LAW

$$\mathbb{T} = \mathbb{S} - p\mathbb{I},$$

where $\mathbb{S}$ is *viscous* stress tensor and p a scalar quantity called *pressure*.

1.3.1 Inviscid fluids, compressible Euler system

We start with an example of inviscid (or perfect) fluids for which $\mathbb{S} = 0$. In the simplest case, the pressure is just a function of the density and we obtain

COMPRESSIBLE EULER SYSTEM

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.5}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = 0. \tag{1.6}$$

If $N = 1$, system (1.5, (1.6) can be rewritten in the Lagrangian (mass) coordinates by introducing new independent variables

$$[t, x] \mapsto [t, y(t, x) = \int_{-\infty}^x \varrho(t, z)\ \mathrm{d}z].$$

The Lagrangian velocity $v = v(t, y)$ satisfies

$$v\left(t, \int_{-\infty}^x \varrho(t, z)\ \mathrm{d}z\right) = u(t, x);$$

whence

$$\partial_t v - \varrho u \partial_y v = \partial_t u,\ \ \varrho \partial_y v = \partial_x u,$$

and (1.5) reads

$$\partial_t U - \partial_y v = 0, \tag{1.7}$$

while (1.6) gives rise to

$$\partial_t v + \partial_y p \left(\frac{1}{U} \right) = 0 \tag{1.8}$$

where $U = \frac{1}{\varrho}$ is the *specific volume*. Problem (1.7), (1.8) is called *p-system.*

The Euler system formally conserves energy. Taking the scalar product of (1.6) with $\mathbf{u}$ we obtain, by means of straightforward manipulation

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + p(\varrho) \right) \mathbf{u} \right] - p(\varrho) \mathrm{div}_x \mathbf{u} = 0. \tag{1.9}$$

Moreover, multiplying (1.5) on $b'(\varrho)$ we deduce the *renormalized equation of continuity*

$$\partial_t b(\varrho) + \mathrm{div}_x \left(b(\varrho) \mathbf{u} \right) + \left[b'(\varrho)\varrho - b(\varrho) \right] \ \mathrm{div}_x \mathbf{u} = 0, \tag{1.10}$$

which, together with (1.9), gives rise to the total energy balance

$$\partial_t \left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right) + \mathrm{div}_x \left[\left(\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) + p(\varrho) \right) \mathbf{u} \right] = 0, \text{ with } P(\varrho) = \varrho \int_1^{\varrho} \frac{p(z)}{z^2} \, \mathrm{d}z. \tag{1.11}$$

1.3.2 Viscous fluid, compressible Navier-Stokes system

We consider the simplest possible example of a viscous fluid, where the viscous stress tensor is a *linear* function of the velocity gradient. For isotropic fluid, the associated $\mathbb{S}$ is given by *Newton's rheological law*

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{N} \mathrm{div}_x \mathbf{u} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{1.12}$$

where μ and η are scalar quantities taken to be constant here for the sake of simplicity. Accordingly, we obtain

COMPRESSIBLE NAVIER-STOKES SYSTEM

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0, \tag{1.13}$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.14}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{N} \mathrm{div}_x \mathbf{u} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}. \tag{1.15}$$

Similarly to (1.11), we write the total energy balance

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right) + \mathrm{div}_x \left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) + p(\varrho) \right) \mathbf{u} - \mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{u} \right] \\ = -\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}. \tag{1.16}$$

As an *physically admissible* system should not produce energy, the source term $\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}$ must be non-negative for any admissible process, in particular

$$\mu \geq 0, \ \ \eta \geq 0.$$

We speak about *viscous fluids* if $\mu > 0$.

1.4 First law of thermodynamics - complete fluid systems

As we have seen in the preceding part, the description by means of purely "mechanical" quantities like ϱ and $\mathbf{u}$ is not complete from the physics point of view; the resulting compressible Navier-Stokes system dissipates (mechanical) energy. For the First law of thermodynamics to hold, one has to introduce a new quantity called (specific) *internal energy* e. Alternatively, we may consider the *absolute temperature* ϑ and suppose that both $p = p(\varrho, \vartheta)$ and $e = e(\varrho, \vartheta)$ are given functions of the state variables ϱ, ϑ. The functional dependence of p and e on ϱ and ϑ is called *equation of state* and characterizes the material properties of a given fluid.

We rewrite the energy balance (1.16) as

$$\partial_t \left(\frac{1}{2}\varrho|\mathbf{u}|^2 \right) + \mathrm{div}_x \left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + p(\varrho, \vartheta) \right) \mathbf{u} - \mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{u} \right] \\ = -\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} + p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u} \tag{1.17}$$

and append the system by a similar relation for the internal energy

$$\partial_t \left(\varrho e(\varrho, \vartheta) \right) + \mathrm{div}_x \left[\varrho e(\varrho, \vartheta)\mathbf{u} \right] + \mathrm{div}_x \mathbf{q} = \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} - p(\varrho, \vartheta)\mathrm{div}_x \mathbf{u}, \tag{1.18}$$

where $\mathbf{q}$ is a new quantity that represents the diffuse flux of the internal energy. In accordance with the First law of thermodynamics, the total energy of the system must be conserved; whence

$$\partial_t \left[\varrho \left(\frac{1}{2}|\mathbf{u}|^2 + e(\varrho, \vartheta) \right) \right] \\ + \mathrm{div}_x \left[\varrho \left(\frac{1}{2}|\mathbf{u}|^2 + e(\varrho, \vartheta) \right) \mathbf{u} + p(\varrho, \vartheta)\mathbf{u} - \mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{u} + \mathbf{q} \right] = 0. \tag{1.19}$$

The internal energy flux very often coincides with the heat flux, the latter being given by *Fourier's law*

$$\mathbf{q} = -\kappa \nabla_x \vartheta. \tag{1.20}$$

Thus we have derived

NAVIER-STOKES-FOURIER SYSTEM

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.21}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{1.22}$$

$$\begin{aligned} &\partial_t \left[\varrho \left(\frac{1}{2} |\mathbf{u}|^2 + e(\varrho, \vartheta) \right) \right] \\ &\quad + \mathrm{div}_x \left[\varrho \left(\frac{1}{2} |\mathbf{u}|^2 + e(\varrho, \vartheta) \right) \mathbf{u} + p(\varrho, \vartheta)\mathbf{u} - \mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{u} + \mathbf{q} \right] = 0, \end{aligned} \tag{1.23}$$

with

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{N} \mathrm{div}_x \mathbf{u} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{1.24}$$

$$\mathbf{q} = -\kappa \nabla_x \vartheta. \tag{1.25}$$

The "inviscid" version is known as

COMPLETE EULER SYSTEM

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{1.26}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho, \vartheta) = 0, \tag{1.27}$$

$$\partial_t \left[\varrho \left(\frac{1}{2} |\mathbf{u}|^2 + e(\varrho, \vartheta) \right) \right] + \mathrm{div}_x \left[\varrho \left(\frac{1}{2} |\mathbf{u}|^2 + e(\varrho, \vartheta) \right) \mathbf{u} + p(\varrho, \vartheta)\mathbf{u} \right] = 0. \tag{1.28}$$

1.5 Second law of thermodynamics

The Second law of thermodynamics encodes the time *irreversibility* of the fluid evolution characteristic for viscous and heat conducting fluid. We recall the internal energy balance

$$\varrho \partial_t e(\varrho, \vartheta) + \varrho \mathbf{u} \cdot \nabla_x e(\varrho, \vartheta) + \mathrm{div}_x \mathbf{q} = \mathbb{S} : \nabla_x \mathbf{u} - p \mathrm{div}_x \mathbf{u},$$

where

$$\varrho \partial_t e(\varrho, \vartheta) + \varrho \mathbf{u} \cdot \nabla_x e(\varrho, \vartheta) = \varrho De(\varrho, \vartheta) \cdot [\partial_t \varrho, \partial_t \vartheta] + \varrho \mathbf{u} \cdot De(\varrho, \vartheta) \cdot [\nabla_x \varrho, \nabla_x \vartheta].$$

Suppose that e and p are interrelated through *Gibbs' equation*

$$\vartheta Ds(\varrho,\vartheta) = De(\varrho,\vartheta) + D\left(\frac{1}{\varrho}\right)p(\varrho,\vartheta), \tag{1.29}$$

where s is a new quantity called (specific) *entropy*. Accordingly,

$$\begin{aligned}
\varrho\partial_t e(\varrho,\vartheta) + \varrho\mathbf{u}\cdot\nabla_x e(\varrho,\vartheta) &= \varrho De(\varrho,\vartheta)\cdot[\partial_t\varrho,\partial_t\vartheta] + \varrho\mathbf{u}\cdot De(\varrho,\vartheta)\cdot[\nabla_x\varrho,\nabla_x\vartheta] \\
&= \vartheta\varrho Ds(\varrho,\vartheta)\cdot[\partial_t\varrho,\partial_t\vartheta] + \vartheta\varrho\mathbf{u}\cdot Ds(\varrho,\vartheta)\cdot[\nabla_x\varrho,\nabla_x\vartheta] \\
&+ p(\varrho,\vartheta)\frac{1}{\varrho}\left(\partial_t\varrho + \mathbf{u}\cdot\nabla_x\varrho\right) \\
&= \vartheta\left[\partial_t(\varrho s(\varrho,\vartheta)) + \mathrm{div}_x(\varrho s(\varrho,\vartheta)\mathbf{u})\right] - p(\varrho,\vartheta)\mathrm{div}_x\mathbf{u};
\end{aligned}$$

whence (1.18) can be rewritten as

$$\partial_t\left(\varrho s(\varrho,\vartheta)\right) + \mathrm{div}_x\left[\varrho s(\varrho,\vartheta)\mathbf{u}\right] + \frac{1}{\vartheta}\mathrm{div}_x\mathbf{q} = \frac{1}{\vartheta}\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u},$$

where, furthermore,

$$\frac{1}{\vartheta}\mathrm{div}_x\mathbf{q} = \mathrm{div}_x\left(\frac{1}{\vartheta}\mathbf{q}\right) + \frac{1}{\vartheta^2}\mathbf{q}\cdot\nabla_x\vartheta,$$

and, consequently, we end up with the *entropy balance equation*

$$\partial_t\left(\varrho s(\varrho,\vartheta)\right) + \mathrm{div}_x\left[\varrho s(\varrho,\vartheta)\mathbf{u}\right] + \mathrm{div}_x\left(\frac{1}{\vartheta}\mathbf{q}\right) = \frac{1}{\vartheta}\left[\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u} - \frac{\mathbf{q}\cdot\nabla_x\vartheta}{\vartheta}\right]. \tag{1.30}$$

If $\mathbf{q} = -\kappa\nabla_x\vartheta$ is given by Fourier's law, the *entropy production rate* represented by the right-hand side of (1.30) is non-negative as long as $\kappa \geq 0$. In other words, the entropy is produced in the course of the process in accordance with the Second law of thermodynamics.

2 Various concepts of solutions to the equations and systems of mathematical fluid dynamics

A typical problem for an evolutionary equation is the initial-value or Cauchy problem. Knowing the state of the system at an initial time, say $t = 0$, solve the evolutionary equation for this initial data. If the physical system (fluid) is confined to a bounded domain, the boundary behavior must be prescribed. To avoid the difficulties created by the influence boundaries on the fluid motion, we focus on problems posed on the whole physical space R^N or on the periodic (flat) torus

$$\mathcal{T}^N = \left([0,1]|_{\{0,1\}}\right)^N.$$

In the latter case, we simply imposed the periodic boundary conditions, that may be seen as a useful though not very realistic approximation. Evolutionary problems are expected to be *well posed*:

- Solution exist for any physically admissible choice of the initial data;
- solutions are uniquely determined by the initial data;
- solutions depend in a continuous way on the initial data.

Our goal is to address these issues for the evolutionary equations arising in fluid dynamics.

2.1 Classical - strong solutions

Ideally, a system of partial differential equations should admit strong or classical solutions possessing all the necessary derivatives and enjoying certain continuity to attain the initial data. Sometimes we speak about strong solutions if all derivatives involved in equations exist in the generalized sense as integrable functions. Once the *existence* of a classical solution is established, the questions of uniqueness and/or continuous dependence on the data are usually an easy task given the high regularity of solutions. Unfortunately, classical solutions of most problems in mathematical fluid dynamics are known to exist only on short time intervals - *local eistence* - or globally in time but for the initial data that are sufficiently close to an equilibrium solution - *global existence for "small" data.*

2.1.1 Local existence for the compressible Euler and Navier-Stokes system

Consider the Euler/Navier-Stokes system describing the motion of a compressible fluid introduced in Section 1.3.2:

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0, \tag{2.1}$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}), \tag{2.2}$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{N} \mathrm{div}_x \mathbf{u} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \tag{2.3}$$

with $\mu = \eta = 0$ for the inviscid (Euler) system, and $\mu > 0$ for the viscous (Navier-Stokes) fluid. For the sake of simplicity, we consider the periodic boundary conditions, meaning the physical space $\Omega = \mathcal{T}^N$ is the flat torus, and prescribe the initial state

$$\varrho(0, \cdot) = \varrho_0, \ \mathbf{u}(0, \cdot) = \mathbf{u}_0, \tag{2.4}$$

where ϱ_0 and $\mathbf{u}_0$ are as smooth as needed, and $\varrho_0 > 0$ to the degenerate vacuum regime. Problem (2.1–2.4) is known to be locally well posed in the Sobolev scale $W^{m,2}(\Omega)$ for m large enough, see e.g. Majda [8], Kleinerman and Majda [7].

Theorem 2.1. *Let* $p \in C^\infty(0,\infty)$, $p'(\varrho) > 0$ *for* $\varrho > 0$, $\mu \geq 0$, $\eta \geq 0$, *and*

$$\varrho_0 > 0, \varrho_0 \in W^{m,2}(\Omega),\ \mathbf{u}_0 \in W^{m,2}(\Omega; R^N) \text{ for } m > \left[\frac{N}{2}\right] + 1.$$

Then there exists a positive time $T > 0$ *such that problem (2.1–2.4) admits a strong solution* $[\varrho, \mathbf{u}]$ *in* $(0,T) \times \Omega$, *unique in the class*

$$\begin{aligned} &\varrho \in C([0,T]; W^{m,2}(\Omega)),\ \partial_t \varrho \in C([0,T]; W^{m-1,2}(\Omega)), \\ &\mathbf{u} \in C([0,T]; W^{m,2}(\Omega; R^N)),\ \partial_t \mathbf{u} \in C([0,T]; W^{m-k,2}(\Omega; R^N)), \end{aligned}$$

$k = 1$ *if* $\mu = \eta = 0$, $k = 2$ *if* $\mu > 0$ *or* $\eta > 0$.

Remark 2.2. *Obviously, the solution* $[\varrho, \mathbf{u}]$ *is as smooth as we wish thanks to the Sobolev embedding relation*

$$W^k(\Omega) \hookrightarrow C(\Omega),\ k > \left[\frac{N}{2}\right].$$

2.1.2 Global existence for small initial data

The Navier-Stokes system admits global-in-time solutions provided the initial data are close enough to an equilibrium state. Here, we present a possible result in this direction that was essentially shown by Matsumura and Nishida [10], [9] for $N = 3$.

Theorem 2.3. *Let $p \in C^\infty(0,\infty)$, $p'(\varrho) > 0$ for $\varrho > 0$, $\mu > 0$, $\eta \geq 0$. Let a positive constant $\overline{\varrho} > 0$ be given.*

Then there exists $\varepsilon > 0$ such that for any initial data

$$\varrho_0 \in W^{3,2}(\Omega),\ \mathbf{u}_0 \in W^{3,2}(\Omega; R^3),$$

$$\|\varrho_0 - \overline{\varrho}\|_{W^{3,2}(\Omega)} + \|\mathbf{u}_0\|_{W^{3,2}(\Omega;R^3)} < \varepsilon$$

the Navier–Stokes problem (2.1–2.4) admits a unique strong solution $[\varrho, \mathbf{u}]$ defined on the time interval $(0,\infty)$,

$$\begin{aligned} &\varrho \in C([0,T]; W^{3,2}(\Omega)),\ \partial_t \varrho \in C([0,T]; W^{2,2}(\Omega)), \\ &\mathbf{u} \in C([0,T]; W^{3,2}(\Omega; R^3)),\ \partial_t \mathbf{u} \in C([0,T]; W^{1,2}(\Omega; R^3)) \end{aligned}$$

such that

$$\varrho(t,\cdot) \to \overline{\varrho} \text{ in } W^{3,2}(\Omega),\ \mathbf{u}(t,\cdot) \to 0 \text{ in } W^{3,2}(\Omega; R^3) \text{ as } t \to \infty.$$

Remark 2.4. *Solutions in Theorem 2.3 are more regular provided we increase accordingly the regularity of the initial data.*

A similar result for the inviscid Euler system is not available. As we shall see below, solutions of the compressible Euler system develop singularities in a finite lap of time for a fairly generic class of smooth and even small initial data.

2.2 Weak solutions

Weak solutions satisfy the equations in the sense of distributions. A single equation in the weak formulation is replaced by an (infinite) family of integral identities satisfied by sufficiently smooth test functions. The weak formulation of the Euler/Navier Stokes system (1.13), (1.14) reads

$$\int_0^T \int_\Omega [\varrho \partial_t \varphi + \varrho \mathbf{u} \cdot \nabla_x \varphi] \ \mathrm{d}x \ \mathrm{d}t = 0,$$

for any test function $\varphi \in C^1_c((0,T) \times \Omega)$,

$$\int_0^T \int_\Omega [\varrho \mathbf{u} \cdot \partial_t \varphi + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \varphi + p(\varrho) \mathrm{div}_x \varphi] \ \mathrm{d}x \, \mathrm{d}t = \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \ \mathrm{d}x \ \mathrm{d}t$$

for any test function $\varphi \in C^1_c((0,T) \times \Omega; R^N)$.

Note that for the Euler system $\mathbb{S} \equiv 0$ and therefore less regularity for the weak solution is required. It follows from the weak formulation that

$$t \mapsto \int_\Omega \varrho(t,\cdot)\varphi \ \mathrm{d}x \text{ and } t \mapsto \int_\Omega (\varrho\mathbf{u})(t,\cdot)\cdot\varphi \ \mathrm{d}x$$

can be seen as *continuous* scalar function of time for any test function φ. Accordingly, it is more convenient to write the weak formulation in the form

$$\left[\int_\Omega \varrho\varphi \ \mathrm{d}x\right]_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2}\int_\Omega [\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi] \ \mathrm{d}x \ \mathrm{d}t,$$

for any $0 \le \tau_1 \le \tau_2 \le T$ and for any test function $\varphi \in C^1_c([0,T]\times\Omega)$,

$$\begin{aligned}\left[\int_\Omega \varrho\mathbf{u}\cdot\varphi \ \mathrm{d}x\right]_{t=\tau_1}^{t=\tau_2} &= \int_0^T\int_\Omega [\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u}:\nabla_x\varphi + p(\varrho)\mathrm{div}_x\varphi] \ \mathrm{d}x\,\mathrm{d}t \quad (2.5)\\ &- \int_0^T\int_\Omega \mathbb{S}(\nabla_x\mathbf{u}):\nabla_x\varphi \ \mathrm{d}x \ \mathrm{d}t\end{aligned}$$

for any $0 \le \tau_1 \le \tau_2 \le T$ and for any test function $\varphi \in C^1_c([0,T]\times\Omega; R^N)$.

Remark 2.5. *Note that the weak formulation (2.5) already includes satisfaction of the initial conditions. Observe that it is more natural to formulate the initial state in terms of the momentum* $\varrho\mathbf{u}$ *rather that the velocity* $\mathbf{u}$ *as the former is always* weakly *continuous.*

2.2.1 Weak solutions to the compressible Euler system

The class of weak solutions is obviously larger than that of classical solutions as basically no smoothness is required. The price to pay is a dramatic lost of well posedness as the following result shows (see Chiodaroli [1], EF [6]).

Theorem 2.6. *Let* $p \in C^3(0,\infty)$ *and let* $T > 0$ *be given. Let the initial data belong to the class*

$$\varrho_0 \in C^3(\Omega),\ \varrho_0 > 0,\ \mathbf{u}_0 \in C^3(\Omega; R^N),\ \Omega = \mathcal{T}^N,\ N = 2,3.$$

Then the compressible Euler system (1.5), (1.6 admits infinitely many *weak solutions emanating from the initial state* $[\varrho_0, \mathbf{u}_0$ *and belonging to the class*

$$\varrho \in L^\infty((0,T)\times\Omega),\ \varrho > 0,\ \mathbf{u} \in L^\infty((0,T)\times\Omega; R^N).$$

Theorem 2.6 can be shown by the method of *convex integration* developed recently in the context of fluid mechanics by DeLellis and Székelyhidi [4], [3], [5]. It shows that on one hand the class of weak solutions, at leats for the inviscid fluid flows, is large enough to provide positive existence results, on the other hand it is too large to ensure uniqueness.

Remark 2.7. *We emphasize that results similar to Theorem 2.6 for* viscous *fluids, meaning for the Navier-Stokes system, are not known.*

2.3 Dissipative (weak) solutions

The solutions, the existence of which is claimed in Theorem 2.6, are produced in a rather non-constructive way by adding oscillatory components to a quantity called sub-solution. As a result, they typically produce energy, meaning violate the energy balance (1.11). Thus we may try to eliminate them by adding a weak counterpart of the energy balance as an integral part of the definition of weak solution.

We say the $[\varrho, \mathbf{u}]$ is a *dissipative* weak solution to the Euler/Navier Stokes system if, in addition to the weak formulation (2.5), the total *energy inequality*

$$\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) \right] (\tau, \cdot) \ \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t \leq \int_\Omega \left[\frac{1}{2}\varrho_0|\mathbf{u}_0|^2 + P(\varrho_0) \right] \ \mathrm{d}x \tag{2.6}$$

holds for a.a. $\tau \in (0, T)$.

Remark 2.8. *Recall that for the Euler system* $\mathbb{S}(\nabla_x \mathbf{u}) \equiv 0$ *and (2.6) simply says that the total energy of the system is non-increasing in time.*

Remember that the Navier-Stokes system is already dissipative so adding inequality in (2.6) does not violate any physical principle. In addition, we have extrapolated this argument also to the "conservative" Euler system.

2.4 Relative energy

The dissipative solutions introduced in the previous section satisfy an extended version of the energy inequality (2.6) known as *relative energy inequality*. We introduce the *relative energy functional*

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U} \right) = \int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u} - \mathbf{U}|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r) \right] \ \mathrm{d}x. \tag{2.7}$$

If

$$\varrho \mapsto p(\varrho) \text{ is strictly increasing in } (0, \infty),$$

then the pressure potential P is strictly convex and $\mathcal{E}$ represents a kind of (non-symmetric) distance function between $[\varrho, \mathbf{u}]$ and $[r, \mathbf{U}]$.

Seeing that

$$
\begin{aligned}
&\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) \\
&= \int_\Omega \left(\frac{1}{2}\varrho |\mathbf{u}|^2 + P(\varrho) \right) \ \mathrm{d}x - \int_\Omega \varrho \mathbf{u} \cdot \mathbf{U} \ \mathrm{d}x + \int_\Omega \varrho |\mathbf{U}|^2 \ \mathrm{d}x - \int_\Omega P'(r) \varrho \ \mathrm{d}x + \int_\Omega p(r) \ \mathrm{d}x
\end{aligned}
$$

we easily observe that wa may evaluate the time difference

$$
\left[\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right) \right]_{t=0}^{t=\tau}
$$

as soon as $[\varrho, \mathbf{u}]$ is a *dissipative* (weak) solution to the Euler/Navier-Stokes system, and $[r, \mathbf{U}]$ is a pair of sufficiently smooth “test functions”, $r > 0$. Indeed we successively compute

- $$
\left[\int_\Omega \left(\frac{1}{2}\varrho |\mathbf{u}|^2 + P(\varrho) \right) \ \mathrm{d}x \right]_{t=0}^{t=\tau} + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \ \mathrm{d}t \le 0
$$
  in accordance with the energy inequality (2.6), keeping in mind that $\mathbb{S} \equiv 0$ for the Euler system;

- taking $\mathbf{U}$ as a test function in the weak formulation of the momentum equation:
$$
\begin{aligned}
\left[\int_\Omega \varrho \mathbf{u} \cdot \mathbf{U} \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2} &= \int_0^T \int_\Omega \left[\varrho \mathbf{u} \cdot \partial_t \mathbf{U} + \varrho \mathbf{u} \otimes \mathbf{u} : \nabla_x \mathbf{U} + p(\varrho) \mathrm{div}_x \mathbf{U} \right] \ \mathrm{d}x \, \mathrm{d}t \\
&\quad - \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t;
\end{aligned}
$$

- taking $\frac{1}{2}|\mathbf{U}|^2$ and $P'(r)$ as test functions in the weak formulation of the equation of continuity:
$$
\left[\int_\Omega \varrho \frac{1}{2} |\mathbf{U}|^2 \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2} \int_\Omega \left[\varrho \mathbf{U} \cdot \partial_t \mathbf{U} + \varrho \mathbf{u} \cdot \mathbf{U} \cdot \nabla_x \mathbf{U} \right] \ \mathrm{d}x \ \mathrm{d}t,
$$
  and
$$
\left[\int_\Omega \varrho P'(r) \ \mathrm{d}x \right]_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2} \int_\Omega \left[\varrho P''(r) \partial_t r + P''(r) \varrho \mathbf{u} \cdot \nabla_x r \right] \ \mathrm{d}x \ \mathrm{d}t.
$$

Summing up the previous observations we obtain

RELATIVE ENERGY INEQUALITY

$$
\begin{aligned}
\left[\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right)\right]_{t=0}^{t=\tau} &+ \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \ \mathrm{d}x \\
&\leq \int_0^\tau \int_\Omega \varrho \left[\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U}\right] \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{U} - p(\varrho) \mathrm{div}_x \mathbf{U}\right] \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_{\tau_1}^{\tau_2} \int_\Omega \left[(\varrho - r) P''(r) \partial_t r + P''(r) \varrho \mathbf{u} \cdot \nabla_x r\right] \ \mathrm{d}x \ \mathrm{d}t,
\end{aligned}
\tag{2.8}
$$

where we have used the identity

$$
p'(r) = r P''(r).
$$

3 Weak vs. strong solutions

Our goal in this section is to show the *weak-strong uniqueness* property for the dissipative solutions of the Euler/Navier-Stokes system. In other words, the weak and strong solutions emanating from the same initial data coincided as long as the latter exists.

3.1 Weak-strong uniqueness

Suppose that the Euler/Navier-Stokes system admits a strong solution $[\varrho = r, \mathbf{u} = \mathbf{U}]$ on a time interval $(0, T)$, $r > 0$. Let $[\varrho, \mathbf{u}]$ be a dissipative weak solution of the same problem with the same initial data. Taking $[r, \mathbf{U}]$ as test functions in the relative energy inequality (2.8), and using the fact that

$$
\partial_t \mathbf{U} + \mathbf{U} \cdot \nabla_x \mathbf{U} = -\frac{1}{r} \nabla_x p(r) + \frac{1}{r} \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}),
$$

we obtain

$$
\begin{aligned}
\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right)(\tau) &+ \int_0^\tau \int_\Omega (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \\
&\leq \int_0^\tau \int_\Omega \varrho |\nabla_x \mathbf{U}| |\mathbf{u} - \mathbf{U}|^2 \ \mathrm{d}x \\
&- \int_0^\tau \int_\Omega \varrho P''(r) \nabla_x r \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega [(\varrho - r) P''(r) \partial_t r + P''(r) \varrho \mathbf{u} \cdot \nabla_x r + p(\varrho) \mathrm{div}_x \mathbf{U}] \ \mathrm{d}x \ \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\left(1 - \frac{\varrho}{r}\right) \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) \cdot (\mathbf{u} - \mathbf{U}) \right] \ \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
$$

Furthermore, as r satisfies

$$
\partial_t r + \mathbf{U} \cdot \nabla_x \mathbf{r} = -r \mathrm{div}_x \mathbf{U},
$$

we get, performing several by-parts integrations,

$$
\begin{aligned}
&- \int_0^\tau \int_\Omega \varrho P''(r) \nabla_x r \cdot (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_\Omega [(\varrho - r) P''(r) \partial_t r + P''(r) \varrho \mathbf{u} \cdot \nabla_x r + p(\varrho) \mathrm{div}_x \mathbf{U}] \ \mathrm{d}x \ \mathrm{d}t \\
&= - \int_0^\tau \int_\Omega [(\varrho - r) P''(r) \partial_t r + P''(r) \varrho \mathbf{U} \cdot \nabla_x r + p(\varrho) \mathrm{div}_x \mathbf{U}] \ \mathrm{d}x \ \mathrm{d}t \\
&= - \int_0^\tau \int_\Omega \mathrm{div}_x \mathbf{U} \Big(p(\varrho) - p'(r)(\varrho - r) - p(r) \Big) \ \mathrm{d}x \ \mathrm{d}t.
\end{aligned}
$$

Thus we conclude that

$$
\begin{aligned}
\mathcal{E}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right)(\tau) &+ \int_0^\tau \int_\Omega (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \\
&\leq \int_0^\tau \int_\Omega \varrho |\nabla_x \mathbf{U}| |\mathbf{u} - \mathbf{U}|^2 \ \mathrm{d}x \\
&- \int_0^\tau \int_\Omega \mathrm{div}_x \mathbf{U} \Big(p(\varrho) - p'(r)(\varrho - r) - p(r) \Big) \ \mathrm{d}x \ \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\left(1 - \frac{\varrho}{r}\right) \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) \cdot (\mathbf{u} - \mathbf{U}) \right] \ \mathrm{d}x \, \mathrm{d}t.
\end{aligned}
\tag{3.1}
$$

3.1.1 Weak-strong uniqueness for the Euler system

Relation (3.1) implies $\varrho = r$, $\mathbf{u} = \mathbf{U}$ by means of the standard Gronwall lemma. More specifically, in the case of the Euler system, the strong solution must be at least globally

Lipschitz and we need

$$p(\varrho) - p'(r)(\varrho - r) - p(r) \text{ to be dominated by } P(\varrho) - P'(r)(\varrho - r) - P(r).$$

Note that this is true provided, for instance,

$$p \in C[0,\infty) \cap C^2(0,\infty), \ p'(\varrho) > 0 \text{ for } \varrho > 0, \ \liminf_{\varrho \to \infty} p'(\varrho) > 0, \ \liminf_{\varrho \to \infty} \frac{P(\varrho)}{p(\varrho)} > 0. \quad (3.2)$$

We have shown the following result.

Theorem 3.1. *Let the pressure p satisfy (3.2). Let the compressible Euler system (1.5), (1.6) admits a solution* $[\tilde{\varrho}, \tilde{\mathbf{u}}]$, $\tilde{\varrho} > 0$ *that is Lipschitz in* $[0,T] \times \Omega$, $\Omega = \mathcal{T}^N$, $N = 1, 2, 3$. *Let* $[\varrho, \mathbf{u}]$, $\varrho \geq 0$ *be a dissipative weak solution of the same problem,*

$$\varrho(0,\cdot) = \tilde{\varrho}(0,\cdot), \ \varrho\mathbf{u}(0,\cdot) = \tilde{\varrho}\tilde{\mathbf{u}}(0,\cdot).$$

Then

$$\varrho = \tilde{\varrho}, \ \mathbf{u} = \tilde{\mathbf{u}} \ \textit{a.a. in } (0,T) \times \Omega.$$

3.1.2 Weak-strong uniqueness for the Navier-Stokes system

If viscosity is present, we have to handle the last integral in (3.1), namely

$$\int_0^\tau \int_\Omega \left[\left(1 - \frac{\varrho}{r}\right) \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) \cdot (\mathbf{u} - \mathbf{U}) \right] \ \mathrm{d}x \, \mathrm{d}t.$$

Clearly, this integral can be dominated by the relative entropy functional on the set, where

$$0 < \underline{\varrho} \leq \varrho \leq \overline{\varrho},$$

where $\underline{\varrho}$, $\overline{\varrho}$ are positive constants.

Now, we distinguish two cases: (i) ϱ is large $\varrho \geq \overline{\varrho}$, (ii) ϱ is small $0 \leq \varrho \leq \underline{\varrho}$. If ϱ is large, specifically $\overline{\varrho} > 2\max\{r\}$, we get

$$|\left(1 - \frac{\varrho}{r}\right)(\mathbf{u} - \mathbf{U})| \leq c_1 |\sqrt{\varrho}||\sqrt{\varrho}(\mathbf{u} - \mathbf{U})| \leq c_2 \left(\varrho + \varrho|\mathbf{u} - \mathbf{U}|^2\right),$$

where the right-hand side is dominated by the relative energy functional.

If ϱ is small, specifically $\varrho \leq \underline{\varrho} < \frac{1}{2}\min\{r\}$, then

$$
\begin{aligned}
&\left| \int_{\varrho \leq \underline{\varrho}} \left[\left(1 - \frac{\varrho}{r}\right) \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U}) \cdot (\mathbf{u} - \mathbf{U}) \right] \ \mathrm{d}x \ \mathrm{d}t \right| \\
&\leq \delta \int_0^\tau \int_\Omega |\mathbf{u} - \mathbf{U}|^2 \ \mathrm{d}x + c(\delta) \int_{\varrho \leq \underline{\varrho}} |\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U})|^2 \left(1 - \frac{\varrho}{r}\right)^2 \ \mathrm{d}x \ \mathrm{d}t \\
&\leq \delta \int_0^\tau \int_\Omega \varrho |\mathbf{u} - \mathbf{U}|^2 + (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \\
&+ c(\delta) \int_{\varrho \leq \underline{\varrho}} |\mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{U})|^2 \left(1 - \frac{\varrho}{r}\right)^2 \ \mathrm{d}x \ \mathrm{d}t
\end{aligned}
$$

for any $\delta > 0$, where the last integral can be absorbed by the left-hand side of (3.1) provided $\delta > 0$ is small enough.

We have shown an analogue of Theorem 3.1 for the Navier-Stokes system.

Theorem 3.2. *Let the pressure p satisfy (3.2). Let the compressible Navier-Stokes system (1.13–1.15) admits a solution $[\tilde{\varrho}, \tilde{\mathbf{u}}]$, $\tilde{\varrho} > 0$ that is Lipschitz in $[0,T] \times \Omega$, $\Omega = \mathcal{T}^N$,*

$$\mathbf{u} \in L^1(0,T; W^{2,\infty}(\Omega; R^N)), \ N = 1,2,3.$$

Let $[\varrho, \mathbf{u}]$, $\varrho \geq 0$ be a dissipative weak solution of the same problem,

$$\varrho(0,\cdot) = \tilde{\varrho}(0,\cdot), \ \varrho\mathbf{u}(0,\cdot) = \tilde{\varrho}\tilde{\mathbf{u}}(0,\cdot).$$

Then

$$\varrho = \tilde{\varrho}, \ \mathbf{u} = \tilde{\mathbf{u}} \ \textit{a.a. in } (0,T) \times \Omega.$$

3.2 Wild dissipative solutions of the Euler system

In the light of the weak-strong uniqueness result established in Theorem 3.2, it may seem that imposing the energy inequality eliminates the “wild” solutions, the existence of which is claimed in Theorem 2.6. However, this is true only to certain extent as shown in the following result, see [6].

Theorem 3.3. *Let $\Omega = \mathcal{T}^N$, $N = 2,3$, $T > 0$, and let $p \in C^3(0,\infty)$. Let*

$$\varrho_0 \in C^3(\Omega),\ \ \varrho_0 > 0 \ \textit{in}\ \Omega,$$

be a given density distribution.

Then there exists u_0,

$$u_0 \in L^\infty(\Omega; R^N)$$

such that the corresponding initial-value problem for the Euler system (1.5), (1.6) admits infinitely many dissipative (weak) solutions in $(0,T)\times\Omega$.

Remark 3.4. *Note carefully that there is no contradiction with the weak-strong uniqueness result as the initial velocity* $\mathbf{u}_0$ is not *regular. We also repeat that a similar result in the context of viscous fluids, specifically for the compressible Navier-Stokes, is not available.*

3.2.1 Admissible weak solutions to the compressible Euler system

In the context of hyperbolic conservation law, it is customary to strengthen the global energy inequality to a local one introduced in (1.11). We may say that $[\varrho, \mathbf{u}]$ is an *admissible weak solution* to the Euler system if

$$\begin{gathered}
\left[\int_\Omega \varrho\varphi \ \mathrm{d}x\right]_{t=\tau_1}^{t=\tau_2} = \int_{\tau_1}^{\tau_2}\int_\Omega \left[\varrho\partial_t\varphi + \varrho\mathbf{u}\cdot\nabla_x\varphi\right] \ \mathrm{d}x \ \mathrm{d}t, \\
\text{for any } 0 \le \tau_1 \le \tau_2 \le T \text{ and for any test function } \varphi \in C^1_c([0,T]\times\Omega), \\
\left[\int_\Omega \varrho\mathbf{u}\cdot\varphi \ \mathrm{d}x\right]_{t=\tau_1}^{t=\tau_2} = \int_0^T\int_\Omega \left[\varrho\mathbf{u}\cdot\partial_t\varphi + \varrho\mathbf{u}\otimes\mathbf{u} : \nabla_x\varphi + p(\varrho)\mathrm{div}_x\varphi\right] \ \mathrm{d}x\,\mathrm{d}t \\
\text{for any } 0 \le \tau_1 \le \tau_2 \le T \text{ and for any test function } \varphi \in C^1_c([0,T]\times\Omega; R^N) \\
\left[\int_\Omega \left[\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right]\varphi \ \mathrm{d}x\right]_{t=0}^{t=\tau} \ge \int_0^\tau\int_\Omega \left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)\partial_t\varphi\right] \ \mathrm{d}x\,\mathrm{d}t \\
+ \int_0^\tau\int_\Omega \left[\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho) + p(\varrho)\right)\mathbf{u}\cdot\nabla_x\varphi\right] \ \mathrm{d}x\,\mathrm{d}t \\
\text{for a.a. } 0 \le \tau \le T \text{ and for any test function } \varphi \in C^1_c([0,T]\times\Omega),\ \ \varphi \ge 0.
\end{gathered} \tag{3.3}$$

Note that the notion of admissible weak solution is stronger than the dissipative weak solution as a *local* version of the energy inequality is required. Still the following result holds, see Chiodaroli [1].

Theorem 3.5. *Let $\Omega = \mathcal{T}^N$, $N = 2, 3$, and let $p \in C^3(0, \infty)$. Let*

$$\varrho_0 \in C^3(\Omega),\ \ \varrho_0 > 0 \ \textit{in}\ \Omega,$$

be a given density distribution.

Then there exist $T > 0$ and u_0,

$$u_0 \in L^\infty(\Omega; R^N),$$

such that the corresponding initial-value problem for the Euler system (1.5), (1.6) admits infinitely many admissible weak solutions in $(0, T) \times \Omega$.

The previous result is local in time. However, we also have the following, see [6].

Theorem 3.6. *Let $\Omega = \mathcal{T}^N$, $N = 2, 3$, $p \in C^3(0, \infty)$, $\overline{\varrho} > 0$, and $T > 0$ be given.*

Then there exists $\varepsilon > 0$ such that for any

$$\varrho_0 \in C^2(\Omega),\ \ |\varrho_0 - \overline{\varrho}| < \varepsilon \ \textit{in}\ \Omega$$

there is u_0,

$$u_0 \in L^\infty(\Omega; R^N),$$

such that the corresponding initial-value problem for the Euler system (1.5), (1.6) admits infinitely many admissible weak solutions in $(0, T) \times \Omega$.

In the light of the above results it may still seem that one could save the game by taking smooth initial data and working in the class of admissible solutions for the Euler system. Unfortunately, even in this case the presence of "wild solutions" cannot be avoided, see Chiodaroli, DeLellis, Kreml [2].

Theorem 3.7. *Let* $\Omega = R^2$*,* $N = 2, 3$*,* $p \in C^3(0, \infty)$*.*

Then there exist such initial data

$$\varrho_0 \in W^{1,\infty}(\Omega),\ \mathbf{u}_0 \in W^{1,\infty}(\Omega; R^2),\ \varrho_0 > 0 \text{ uniformly in } R^2$$

such that the corresponding initial-value problem for the Euler system (1.5), (1.6) admits infinitely many admissible weak solutions in $(0, \infty) \times \Omega$*.*

Remark 3.8. *Note that the initial data belong to the class, where the weak-strong uniqueness holds. Thus the corresponding solution is smooth (Lipschitz) on some interval* $[0, T_{\rm crit})$*, looses regularity at* $T_{\rm crit}$ *and bifurcates into branches of admissible " wild" solutions after the critical time.*

4 Stability of the solution set for viscous fluid flows

In this section, we focus on the viscous fluid flows governed by the compressible Navier-Stokes system (1.13–1.15). Our goal is to identify stability properties of the solution set, in particular, the available *a priori* bounds and weak sequential stability of families of solutions.

4.1 A priori bounds

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