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MOTION OF SPIRAL-SHAPED POLYGONAL CURVES BY NONLINEAR CRYSTALLINE MOTION WITH A ROTATING TIP MOTION

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Abstract. We consider a motion of spiral-shaped piecewise linear curves governed by a crystalline curvature flow with a driving force and a tip motion which is a simple model of a step motion of a crystal surface. We extend our previous result on global existence of a spiral-shaped solution to a linear crystalline motion for a power type nonlinear crystalline motion with a given rotating tip motion. We show that self-intersection of the solution curves never occurs and also show that facet extinction never occurs. Finally, we show that spiral-shaped solutions exist globally in time.

Keywords: curvature driven motion; crystalline curvature; spiral growth

MSC 2010: 34A34, 39A12, 74N05

1. Introduction

In [3], we propose a simple model of spiral growth on crystal surface and consider the linear crystalline motion $\beta(\boldsymbol{N}_j)V_j = U - H_j$, where V_j, $\boldsymbol{N}_j$ and H_j denote a normal velocity, a unit normal vector and a crystalline curvature of the j-th facet(edge) of the solution curve, respectively. Here, β (> 0) describes an anisotropy of a mobility of the crystal and U is a forced term. In this paper, we set $U > 0$ since we consider the case when the crystal grows. The detailed setting and definitions are mentioned in the next section. Here, in the wake of the terminology in crystal physics, we call each line-segment of the solution curve "facet". In [3], we treat a spiral-shaped solution with a given rotating tip motion and discuss the motion of the solution curves.

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In this short paper, we slightly extend the previous results for the nonlinear crystalline motion

$$\beta(\boldsymbol{N}_j)V_j = U - |H_j|^{\alpha-1}H_j. \tag{1.1}$$

Here, α is a positive parameter.

To solve the problem, we have to give a *boundary condition* at the tip, which is an end-point of the spiral. In this paper we give a movement of the tip concretely. For simplicity, we assume that the tip moves along a given closed convex curve and we discuss a time-global existence of spiral-shaped solution curves.

The paper is organized as follows: In the next section, we prepare some notation and definitions and introduce a setting of the tip motion and initial curve. In Section 3, we show that the solution curve does not intersect a given tip trajectory and that self-intersection of the solution curve does not occur. We also show that any facets never disappear during time evolution. In Section 4, we obtain that the spiral solution exists globally in time.

2. Preliminaries and our model

We first introduce admissible curves as the solutions, that is, we prepare a spacial class of spiral-shaped polygonal curves and consider the problem in this class. Note that we here treat only spiral-shaped curve, that is,

- ▷ the winding number of the curve may be more than one,
- ▷ a curvature at each point is always positive,
- ▷ the curve has no self-intersection.

Next, in order to define an *"admissible spiral"*, we introduce the Wulff shape. The notion of the Wulff shape has originally come from crystal physics and the Wulff shape describes an equilibrium shape of crystal. Let $\sigma = \sigma(\boldsymbol{n})$ be an interfacial energy density of a crystal. Here, $\boldsymbol{n}$ is a unit normal vector. Then, the Wulff shape is given by $\mathcal{W}_\sigma = \{\boldsymbol{z} \in \mathbb{R}^2;\ \boldsymbol{z}\cdot\boldsymbol{n} \leqslant \sigma(\boldsymbol{n}) \text{ for all } \boldsymbol{n} \in S^1\}$. By definition, $\mathcal{W}_\sigma$ is convex. If $\mathcal{W}_\sigma$ is a convex polygon, σ is called a *"crystalline"* energy. In this paper, we consider this case, that is, we suppose that $\mathcal{W}_\sigma$ is an N_σ-sided convex polygon. We denote by φ_j and l_j the normal angle and the length of the j-th facet of $\mathcal{W}_\sigma$ for $j = 0, 1, \ldots, N_\sigma - 1$, respectively. Without loss of generality, we set $0 = \varphi_0 < \varphi_1 < \ldots < \varphi_{N_\sigma-1}$ $(< 2\pi)$. Moreover, we set $\varphi_{j+mN_\sigma} = \varphi_j$ and $l_{j+mN_\sigma} = l_j$ for $j = 0, 1, \ldots, N_\sigma - 1$ and $m \in \mathbb{Z}$ and also set $\boldsymbol{N}_j = (\cos\varphi_j, \sin\varphi_j)$ and $\boldsymbol{T}_j = (-\sin\varphi_j, \cos\varphi_j)$ for $j \in \mathbb{Z}$.

Let Γ be a polygonal curve. We denote by $\mathcal{F}_j$ the j-th facet of Γ. Namely, $\mathcal{F}_j$ is the line segment from p_{j-1} to p_j, where $p_j = (x_j, y_j)$ is the j-th vertex of Γ. We denote by d_j the length of $\mathcal{F}_j$, that is, $d_j = |p_j - p_{j-1}|$.

We say that a polygonal curve Γ is an *admissible spiral* if (i) Γ has no self-intersection, (ii) the outward unit normal vector of $\mathcal{F}_j$ coincides with $\boldsymbol{N}_j$ and (iii) d_j is positive for all j. For each facet of an admissible spiral, the crystalline curvature H_j is given by $H_j := l_j/d_j$. Note that the crystalline curvature is defined for more general curves. (See [2] and its references.) In this paper we only treat spiral curves, thus, each facet of admissible spirals has a positive curvature.

Here we will introduce our target problem. We first give a tip motion. According to the modeling in [3], the tip does not move in the normal direction and only moves in the tangential direction and rotates along a certain convex closed curve, named the *tip trajectory*. We set the tip trajectory as follows: (i) The tip trajectory is a closed convex N_σ-sided polygonal curve, (ii) the normal angle of the j-th line-segment $\mathcal{F}_j^{\mathrm{tip}}$ is φ_j, and (iii) $l_j^{\mathrm{tip}} > l_j/U^{1/\alpha}$, where l_j^{tip} is the length of $\mathcal{F}_j^{\mathrm{tip}}$. Note that the third condition means that if we take a tip trajectory as an initial curve, then each facet of the solution curve moves outward by (1.1), that is, the crystal surface is under a growth mode. We denote by P_j^{tip} the j-th vertex of the tip trajectory and also extend the subscripts j like φ_j for all $j \in \mathbb{Z}$. Note that $\mathcal{F}_j^{\mathrm{tip}}$ is the line-segment from P_{j-1}^{tip} to P_j^{tip}. We suppose that the tip moves along the tip trajectory anticlockwise with a given speed $v(\boldsymbol{T}_j) > 0$ on each $\mathcal{F}_j^{\mathrm{tip}}$. That is, the tip moves from P_{j-1}^{tip} to P_j^{tip} with a speed $v(\boldsymbol{T}_j)$ during a time period $\tau_j := l_j^{\mathrm{tip}}/v(\boldsymbol{T}_j)$.

Now we set an initial curve $\Gamma_0 = \sum_{j=0}^{M+1} \mathcal{F}_j(0)$ as follows:

- ▷ $\sum_{j=0}^{M} \mathcal{F}_j(0)$ is an admissible spiral.
- ▷ The normal vector of $\mathcal{F}_{M+1}(0)$ is $\boldsymbol{N}_{M+1}$.
- ▷ $d_{M+1}(0) = 0$. That is, $p_{M+1}(0) = p_M(0) = P_M^{\mathrm{tip}}$.
- ▷ Γ_0 never intersects the interior domain enclosed by the tip trajectory.
- ▷ $d_0 = \infty$.

Here p_{M+1} is the *tip* and we call $\mathcal{F}_{M+1}$ the *tip part* of the spiral. According to the modeling in [3], we suppose that the normal velocity at the tip part is zero, that is, $V_{M+1} = 0$. Thus, by the setting of the tip motion, p_{M+1} moves only in the $\boldsymbol{T}_{M+1}$ direction. Note that the normal angle of $\mathcal{F}_{M+1}$ is φ_{M+1} and thus $\Gamma(t) = \sum_{j=0}^{M+1} \mathcal{F}_j(t)$ is admissible if all $d_j(t)$'s are positive and $\Gamma(t)$ has no self-intersections.

Finally we mention the short time existence and uniqueness of the solution. Notice that $\sum_{j=0}^{M} \mathcal{F}_j$ is governed by (1.1). Then we can obtain the system of ordinary differential equations for the length $d_j(t)$ as follows:

$$\dot{d}_j(t) = -(\cot(\theta_{j+1}-\theta_j)+\cot(\theta_j-\theta_{j-1}))V_j + \frac{1}{\sin(\theta_j-\theta_{j-1})}V_{j-1} + \frac{1}{\sin(\theta_{j+1}-\theta_j)}V_{j+1}, \quad \text{for } j=1,2,\ldots,M,\ t>0. \tag{2.1}$$

Here θ_j denotes the normal angle of $\mathcal{F}_j$. Thus, by the standard theory of a system of ordinary differential equations, we can show that the solution $\{d_j(t)\}$ of (2.1) exists uniquely in a short time interval and $d_j(t)$ for each t is positive since $d_j(0)$ is positive for $j=0,1,\ldots,M$. On the other hand, $d_{M+1}(t)$ is governed by $\dot{d}_{M+1}(t) = v(\boldsymbol{T}_{M+1})+V_M/\sin(\theta_{M+1}-\theta_M)$. From the 4-th condition on Γ_0, we have $d_M(0)>l_M^{\mathrm{tip}}$ and thus $V_M(0)>0$. Then we have that $d_{M+1}(t)$ is positive in a short time. Note that $p_j(t)=p_{j+1}(t)-d_{j+1}(t)\boldsymbol{T}_{j+1}$ for $j=M,M-1,\ldots,0$ and $\Gamma(t)=\sum_{j=0}^{M+1}\mathcal{F}_j(t)$, where $\mathcal{F}_j(t)=\{\lambda p_{j-1}(t)+(1-\lambda)p_j(t);\ \lambda\in[0,1]\}$ for $j=1,2,\ldots,M+1$ and $\mathcal{F}_0(t)=\{p_0(t)-\lambda\boldsymbol{T}_0;\ \lambda>0\}$. We also have that $\Gamma(t)$ has no self-intersections in a short time since Γ_0 has no self-intersections. Thus, it is shown that the solution curve $\Gamma(t)$ is admissible in a short time interval. However, there are two possibilities how to break the admissibility of the solution. One is a self-intersection of $\Gamma(t)$ and the other is a facet-extinction in finite time. In the next section, we treat these issues. Suppose that the above singularities do not appear in $[0,\tau_{M+1}]$. Then the tip reaches the next vertex of the tip trajectory, that is, $p_{M+1}(\tau_{M+1})=P_{M+1}^{\mathrm{tip}}$, and changes its developing direction from $\boldsymbol{T}_{M+1}$ to $\boldsymbol{T}_{M+2}$. (See Figure 1 (iii) and (iv).) At $t=\tau_{M+1}$, we also add a new vertex p_{M+2} and a new facet $\mathcal{F}_{M+2}$ to the solution curve $\Gamma(\tau_{M+1})$ as $p_{M+2}(\tau_{M+1})=p_{M+1}(\tau_{M+1})=P_{M+1}^{\mathrm{tip}}$ and solve the problem with the initial curve $\Gamma(\tau_{M+1})$. Note that the system size of (1.1) increases by 1. If we can repeat this procedure, then we finally obtain the time-global existence of the spiral solution.

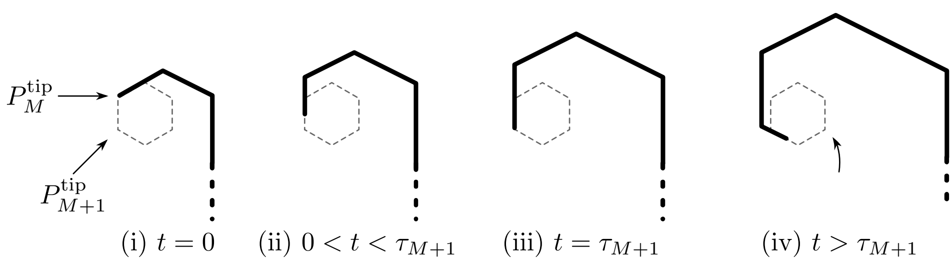

Figure 1. Time evolution of admissible spiral.

In this section, we prove three lemmas. The proofs are similar to or almost the same as that in [3]. For reader's convenience, we give proofs.

We first show that self-intersection of the solution curves never occurs as long as all facets of the solution exist.

We divide the argument into two parts. First we show an inner facet of the spiral never catches up an outer facet.

Lemma 3.1. $\sum_{j=0}^{M} \mathcal{F}_j(t)$ *has no self-intersections as long as all facets exist.*

Proof. Suppose that a self-intersection of the solution curve first occurs at $t = t^* > 0$. Then there are 3 possible self-intersection patterns: (1) Facet-Facet contact: $\mathcal{F}_{j_0}$ contacts $\mathcal{F}_{j_1}$, (2) Vertex-Facet contact: p_{j_0} contacts $\mathcal{F}_{j_1}$, (3) Vertex-Vertex contact: p_{j_0} contacts p_{j_0}.

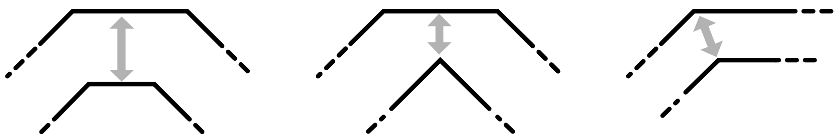

Figure 2. Three contact patterns: (1) Facet-Facet, (2) Vertex-Facet and (3) Vertex-Vertex.

The case (2) never occurs because of the admissibility of $\Gamma(t)$ and the positivity of the curvature of each facet. If the case (3) occurs, then both adjacent facets of each vertex contact at $t = t^*$. Thus, it is sufficient to discuss the Facet-Facet contact case. Suppose that $\mathcal{F}_{j_0}$ contacts $\mathcal{F}_{j_1}$. Note that $\boldsymbol{N}_{j_0} = \boldsymbol{N}_{j_1}$. Let us consider the case $j_0 < j_1$. That is, $\mathcal{F}_{j_0}$ and $\mathcal{F}_{j_1}$ are, resectively, the outer and inner parts of the spiral. If $d_{j_0}(t^*) < d_{j_1}(t^*)$, then $\mathcal{F}_{j_1}$ intersects $\Gamma(t)$ before $t = t^*$. This contradicts the definition of t^*. Thus, $d_{j_0}(t^*) \geqslant d_{j_1}(t^*)$. If $d_{j_0}(t^*) > d_{j_1}(t^*)$, then $V_{j_0} > V_{j_1}$ near $t = t^*$ and thus the distance between the facets in the normal direction increases. Hence, such a contact is impossible. Thus, we have $d_{j_0}(t^*) = d_{j_1}(t^*)$, that is, $\mathcal{F}_{j_0}$ and $\mathcal{F}_{j_1}$ are completely overlapping each other. By the admissibility of the solution curve, both adjacent facets also have overlaps and we can repeatedly apply the above argument and obtain that both the adjacent facets also overlap each other completely. Therefore, we finally obtain that $\mathcal{F}_{j_0 \pm i}(t^*) = \mathcal{F}_{j_1 \pm i}(t^*)$ and $d_{j_0 \pm i}(t^*) = d_{j_1 \pm i}(t^*)$ for $i = 0, 1, 2, \ldots$, from which it follows that there exists an infinite-length facet $\mathcal{F}_k$ except another infinite-length facet $\mathcal{F}_0$. That is, p_k or p_{k-1} moves away to infinity in finite time t^*. However, this is impossible since all V_j's are bounded above by $U / \min_{0 \leqslant j < N_\sigma} \beta(\boldsymbol{N}_j)$. Therefore, we have the assertion. □

Next we show that the regular step part $\sum_{j=0}^{M} \mathcal{F}_j(t)$ never intersects the tip trajectory. Due to this result, the tip does not touch the regular step part of the spiral.

Lemma 3.2. *For $t > 0$, $\sum_{j=0}^{M} \mathcal{F}_j(t)$ never touches the tip trajectory as long as all facets exist.*

P r o o f. From the setting of Γ_0, $\mathcal{F}_M(0)$ has an overlap with the tip trajectory. Generally, there is a possibility that some facets of Γ_0 have an overlap with the tip trajectory. Let $\mathcal{F}_k(0)$ ($k \leqslant M$) be such a facet, that is, $\mathcal{F}_k(0) \cap \mathcal{F}_k^{\mathrm{tip}} \neq \emptyset$. Note that $\mathcal{F}_k^{tip}$ and l_k^{tip} are N_σ-periodic. Then, by admissibility of the curves and the 4-th condition on Γ_0, we have $d_k(0) \geqslant l_k^{\mathrm{tip}}$. Thus,

$$\beta(\boldsymbol{N}_k)V_k(0) = U - (l_k/d_k(0))^\alpha \geqslant U - (l_k/l_k^{\mathrm{tip}})^\alpha > 0.$$

Therefore, $\mathcal{F}_k$ moves outward for some while near $t = 0$ and thus it cannot keep the overlap with the tip trajectory.

Let $t^* > 0$ be the first time when some facets contact the tip trajectory. There are 3 possible contact patterns.

(1) Facet-Facet contact: $\mathcal{F}_{j_0}(t^*)$ contacts $\mathcal{F}_{j_0}^{\mathrm{tip}}$ from outside of the tip trajectory.

(2) Vertex-Facet contact: (2i) p_{j_0} contacts $\mathcal{F}_{i_0}^{\mathrm{tip}}$ or (2ii) $\mathcal{F}_{j_0}$ contacts $P_{i_0}^{\mathrm{tip}}$.

(3) Vertex-Vertex contact: p_{j_0} contacts $P_{j_0}^{\mathrm{tip}}$.

Notice that the contact pattern in Figure 3 never occurs. (See Remark 3.1.) We first exclude the case (2). If the case (2i) occurs, then Γ is locally concave around p_{j_0}, that is, $\theta_{j_0} > \theta_{j_0+1}$. This contradicts our setting. If the case (2ii) occurs, then we have $\varphi_{i_0} < \theta_{j_0} < \varphi_{i_0+1}$. This means that the Wulff shape has a facet whose normal angle lies between φ_{i_0} and φ_{i_0+1}. However, the i_0-th facet and the (i_0+1)-st facet of the Wulff shape are adjacent to each other. Thus, we have a contradiction. Therefore, the case (2) does not occur.

When the case (3) occurs, Facet-Facet contact occurs at least for one adjacent facet of the vertex. Thus, we only consider the case (1). Then, we again apply the above estimate on the velocity and get $V_{j_0}(t) > 0$ near $t = t^*$. Thus, the distance between $\mathcal{F}_{j_0}$ and $\mathcal{F}_{j_0}^{\mathrm{tip}}$ in the normal direction increases near $t = t^*$. That is, the contact between $\mathcal{F}_{j_0}$ and $\mathcal{F}_{j_0}^{\mathrm{tip}}$ is impossible. Hence, we have the assertion. $\square$

R e m a r k 3.1. If the situations in Figure 3 occur, then self-intersection occurs since $d_0 = \infty$. This fact contradicts Lemma 3.1. Thus, these situation never occur.

From the above two lemmas, we conclude that $\Gamma(t)$ never has any self-intersections.

We next show that all facets exist for $0 < t \leqslant \tau_{M+1}$. Note that only $\mathcal{F}_{M+1}$ has zero-length. We first show that no facets never disappear in $(0, \tau_{M+1}]$.

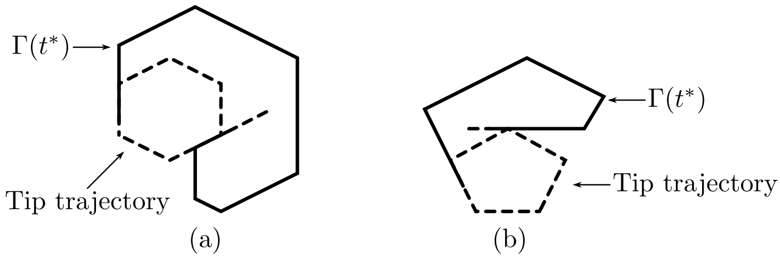

Figure 3. Impossible intersection patterns.

Lemma 3.3. *No facet-extinctions occur in* $(0, \tau_{M+1}]$.

Here and in the sequel we denote by C various positive constants and $A \sim B$ means that $c_1 A \leqslant B \leqslant c_2 A$ for some positive constants c_1 and c_2. We also introduce the following notation: $\mathcal{L}_k(t) := \{z \in \mathbb{R}^2;\ (p_k(t) - z) \cdot \boldsymbol{N}_k = 0\}$. Note that $\mathcal{L}_k(t)$ contains the k-th facet $\mathcal{F}_k(t)$ and $p_k(t)$ is the vertex of $\mathcal{F}_k$.

Proof. We first note that if $\liminf_{t \to T} d_j(t) = 0$ for some $T > 0$ and j, then we can show $\lim_{t \to T} d_j(t) = 0$ in the same manner as in [2]. In the case $M = 0$, then $\Gamma(t) = \mathcal{F}_0(t) + \mathcal{F}_1(t)$. From the fact that $V_0(t) = U/\beta(\boldsymbol{N}_0)$ and $V_1(t) = 0$, these two facets exist for $t \in [0, \tau_{M+1}]$. Thus, we consider the case $M \geqslant 1$.

Assume that there exist $T \in [0, \tau_{M+1}]$ and $j_0 \in \{0, 1, 2, \ldots, M+1\}$ such that $d_{j_0}(T) = 0$ and $d_j(t) > 0$ for all j and $0 < t < T$.

We first exclude the case $j_0 = M + 1$, that is, the tip part disappears. In this case, $p_M(T) = p_{M+1}(T)$ and thus $\mathcal{F}_M(t)$ intersects an interior region of the tip trajectory at $t \in (T - \delta, T)$ for some $\delta > 0$. This is a contradiction. Thus, $\mathcal{F}_{M+1}$ remains.

We also show that $\mathcal{F}_0$ remains. Note that $d_0(0) = \infty$. If $\mathcal{F}_0$ disappears at $t = T$, then $y_0(t) \to -\infty$ as $t \to T$. Recall that $p_0(t) = (x_0(t), y_0(t))$ and $p_0(t) \in \mathcal{L}_1(t)$. By Lemma 3.2 and the fact that $\Gamma(t)$ is spiral, $\mathcal{F}_1(t)$ lies in the region above the line $\mathcal{L}_1^{\mathrm{tip}}$, where $\mathcal{L}_1^{\mathrm{tip}}$ is the line containing $\mathcal{F}_1^{\mathrm{tip}}$. That is, $\mathcal{F}_1(t) \subset R := \{z \in \mathbb{R}^2;\ (z - P_1^{\mathrm{tip}}) \times \boldsymbol{N}_1 > 0\}$. Since $V_0(t) = U/\beta(\boldsymbol{N}_0)$ is constant, we have $x_0(0) \leqslant x_0(t) \leqslant x_0(0) + V_0 T$ for $0 \leqslant t \leqslant T$. Due to this fact and $\lim_{t \to T} y_0(t) = -\infty$, there exists $T' \in (0, T)$ such that $(p_0(t) - P_1^{\mathrm{tip}}) \cdot \boldsymbol{N}_1 < 0$ for $t > T'$. That is, $\mathcal{F}_1(t) \not\subset R$ for $t > T'$, which leads to a contradiction.

Note that there is a possibility that more than two facets disappear at the same time. Then we can find a consecutive extinction part $\bigcup_{k=j_1+1}^{j_2-1} \mathcal{F}_k$ which includes the j_0-th facet and $\mathcal{F}_{j_1}$ and $\mathcal{F}_{j_2}$ remain. If $\theta_{j_2} - \theta_{j_1} > \pi$, then self-intersection of $\Gamma(t)$ occurs before $t = T$ since $\mathcal{F}_{j_1}$ and $\mathcal{F}_{j_2}$ have positive lengths. Thus, $0 < \theta_{j_2} - \theta_{j_1} \leqslant \pi$. In the case when $\theta_{j_2} - \theta_{j_1} = \pi$, $\mathcal{L}_{j_1}$ and $\mathcal{L}_{j_2}$ are parallel. Note that the tip trajectory and the subarc $\sum_{j=j_2+1}^{M+1} \mathcal{F}_j(t)$ lie between $\mathcal{L}_{j_1}$ and $\mathcal{L}_{j_2}$ by geometry. However, the distance between $\mathcal{L}_{j_1}$ and $\mathcal{L}_{j_2}$ tends to zero as $t \to T$. Then the solution curve intersects the tip trajectory and this leads to a contradiction.

We finally consider the case $0 < \theta_{j_2} - \theta_{j_1} < \pi$. We can show that this case does not happen by the same argument as in [2]. Set $Q := \{j_1 + 1, j_1 + 2, \dots, j_2 - 1\}$. Thus, we can define the meeting point $p^* := \lim_{t \to T} p_i(t)$ for $i \in Q \cup \{j_2\}$. Let $q(t)$ be the intersection point of two lines $\mathcal{L}_{j_1}$ and $\mathcal{L}_{j_2}$. Define two functions $h_1(t)$ and $h_2(t)$ by $h_1(t) := (q(t) - p^*) \cdot \boldsymbol{N}_{j_0}$ and $h_2(t) := (p_{j_0}(t) - p^*) \cdot \boldsymbol{N}_{j_0}$, respectively. Note that $h_1(T) = h_2(T) = 0$. From $d_{j_0}(T) = 0$ we have $V_{j_0}(t) \to -\infty$ as $t \to T$. Thus, $V_{j_0}(t) < 0$ in $t \in (T - \delta, T)$ for some $\delta > 0$. Then, by geometry, $h_1(t) > h_2(t) > 0$ at $t \in (T - \delta, T)$. Since $d_{j_1}(t)$ and $d_{j_2}(t)$ remain positive in $[0, T]$, we have $V_{j_1}(t)$ and $V_{j_2}(t)$ remain bounded in $[0, T]$. Thus, the velocity of $q(t)$ is bounded, that is, $\dot{h}_1(t)$ is bounded. On the other hand, since $V_{j_0}(t) \to -\infty$ as $t \to T$, we have $\dot{h}_2(t) \to -\infty$ as $t \to T$. Hence, the function $h_1(t) - h_2(t)$ increases near $t = T$. This leads to a contradiction.

Therefore, we conclude that no facet-extinctions ever occur. □

4. Main results

From the results in the previous section and the fact that all facets remain in a bounded region in a finite time since their velocities are bounded above, we have the following:

Theorem 4.1. *The solution curve exists in $[0, \tau_{M+1}]$. Moreover, the solution curve is an admissible spiral and does not intersect the tip trajectory for $t \in [0, \tau_{M+1}]$.*

As mentioned in the last part of Section 2, at $t = \tau_{M+1}$, we add a new vertex p_{M+2} and a new facet $\mathcal{F}_{M+2}$ to $\Gamma(\tau_{M+1})$ as $p_{M+2}(\tau_{M+1}) = p_{M+1}(\tau_{M+1}) = P_{M+1}^{\mathrm{tip}}$. We also denote by $\Gamma(\tau_{M+1})$ this polygonal curve. Note that $d_{M+2} = 0$ and the subarc $\sum_{j=0}^{M+1} \mathcal{F}_j(\tau_{M+1})$ is an admissible spiral, thus $\Gamma(\tau_{M+1})$ satisfies the conditions on the initial curve for our problem in Section 2. Therefore, we can apply the above theorem and obtain a solution for $t \in [\tau_{M+1}, \tau_{M+1} + \tau_{M+2}]$. We repeat this procedure. Let $T_k := \sum_{i=1}^{k} \tau_{M+k}$ for $k = 1, 2, 3, \dots$. Note that the zero-length new facet is created at each T_k, that is, the number of the facets diverges to infinity as $t \to \infty$. We easily obtain that the winding number of the solution curve also diverges to infinity as $t \to \infty$ since the tip rotates anticlockwise along the tip trajectory at $t = T_k$ for all $k = 1, 2, 3, \dots$. Thus, we obtain the following:

Theorem 4.2. *The spiral-shaped solution exists uniquely and globally in time. Moreover, the number of the facets and the winding number of the solution curve diverge to infinity as $t \to \infty$.*

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