TWO INEQUALITIES FOR SERIES AND SUMS

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(Received January 20, 1994)

Summary. In this paper we refine an inequality for infinite series due to Astala, Gehring and Hayman, and sharpen and extend a Hölder-type inequality due to Daykin and Eliezer.

Keywords: inequalities for series and sums, Hölder's inequality

MSC 1991: 26D15

1. An inequality for infinite series

In 1985 K. Astala and F.W. Gehring [1] presented a proof for the following inequality.

Proposition 1. *Suppose that $B \geqslant 1$ and (a_k) $(k = 1, 2, \ldots)$ is a sequence of non-negative real numbers such that*

$$\sum_{k=n}^{\infty} a_k \leqslant B a_n$$

for all $n \geqslant 1$. If $0 < p < 1$, then

$$\sum_{n=1}^{\infty} a_n^p \leqslant C^* a_1^p, \tag{1.1}$$

where

$$C^* = \frac{B^{2p}}{B^p - (B-1)^p}.$$

We note that Astala and Gehring applied this result to prove an interesting distortion property of quasi-conformal mappings. In 1986 W. K. Hayman [4] showed that in inequality (1.1) the constant C^* can be replaced by $C = B^p/\big(B^p-(B-1)^p\big)$ and that this constant is the best possible. Furthermore, Hayman established that the sign of equality holds in (1.1) (with C instead of C^*) if and only if $a_n = a_1(1-1/B)^{n-1}$ ($n = 1, 2, \ldots$). In 1988 G. Bennett [2] provided a very short and elegant proof of Hayman's version. In this section we show that a modification of Bennett's proof leads to a refinement of inequality (1.1) with C instead of C^* under the slightly restrictive assumption that all a_k's are positive.

Theorem 1. *Let (a_k) $(k = 1, 2, \ldots)$ be a sequence of positive real numbers, and suppose that there exists a constant $B > 1$ such that*

$$\alpha_n = \sum_{k=n}^{\infty} a_k \leqslant B a_n \tag{1.2}$$

holds for all $n \geqslant 1$. If $0 < p < 1$, then

$$D \sum_{n=1}^{\infty} a_n \alpha_n^{p-2}(Ba_n - \alpha_n) + \sum_{n=1}^{\infty} a_n^p \leqslant C a_1^p, \tag{1.3}$$

where

$$C = \frac{B^p}{B^p-(B-1)^p} \quad \text{and} \quad D = B^{1-p}\Big\{\frac{p(B-1)^{p-1}}{B^p-(B-1)^p} - 1\Big\}.$$

The sign of equality holds in (1.3) if and only if

$$a_n = a_1\Big(1-\frac{1}{B}\Big)^{n-1} \quad (n = 1, 2, \ldots).$$

P r o o f. Let $p \in (0,1)$; we denote by f the function

$$f(x) = \frac{1}{x}\big(1-(1-x)^p\big), \quad x \in (0,1).$$

Then we obtain

$$f(x) = \sum_{k=0}^{\infty} \binom{p}{k+1}(-1)^k x^k$$

and

$$f''(x) = \sum_{k=2}^{\infty} \binom{p}{k+1}(-1)^k k(k-1)x^{k-2}.$$

Since $(-1)^k\binom{p}{k+1} > 0$ for $k \geqslant 2$, we conclude $f''(x) > 0$ for all $x \in (0,1)$. Thus, f is strictly convex on $(0,1)$, which implies

$$f(x) \geqslant f(y) + (x-y)f'(y) \tag{1.4}$$

for all x, $y \in (0,1)$, with equality holding if and only if $x = y$. Setting $x = a_n/\alpha_n$ and $y = 1/B$ in (1.4) we get after simple calculation

$$\alpha_n^p - \alpha_{n+1}^p \geqslant \alpha_n^{p-1} a_n f\Big(\frac{1}{B}\Big) + \alpha_n^{p-1} a_n \Big(\frac{a_n}{\alpha_n} - \frac{1}{B}\Big) f'\Big(\frac{1}{B}\Big). \tag{1.5}$$

Form (1.2) and (1.5) we obtain

$$\begin{aligned} f\Big(\frac{1}{B}\Big) B^{p-1} \sum_{n=1}^{\infty} a_n^p &\leqslant f\Big(\frac{1}{B}\Big) \sum_{n=1}^{\infty} a_n \alpha_n^{p-1} \\ &\leqslant \sum_{n=1}^{\infty} (\alpha_n^p - \alpha_{n+1}^p) - f'\Big(\frac{1}{B}\Big) \sum_{n=1}^{\infty} \alpha_n^{p-1} a_n \Big(\frac{a_n}{\alpha_n} - \frac{1}{B}\Big). \end{aligned} \tag{1.6}$$

Further, from (1.6) and (1.2) we get

$$\begin{aligned} &B^{-p} \Big[f'\Big(\frac{1}{B}\Big) \Big/ f\Big(\frac{1}{B}\Big)\Big] \sum_{n=1}^{\infty} a_n \alpha_n^{p-2} (a_n B - \alpha_n) + \sum_{n=1}^{\infty} a_n^p \\ &\qquad \leqslant \frac{B^{1-p}}{f(1/B)} \alpha_1^p \leqslant \frac{B}{f(1/B)} a_1^p. \end{aligned}$$

Since

$$B^{-p} \frac{f'(1/B)}{f(1/B)} = B^{1-p} \Big[\frac{p(B-1)^{p-1}}{B^p - (B-1)^p} - 1\Big]$$

and

$$\frac{B}{f(1/B)} = \frac{B^p}{B^p - (B-1)^p},$$

we obtain inequality (1.3). Moreover, we conclude that the sign of equality is valid in (1.3) if and only if $a_n/\alpha_n = 1/B$ for all $n \geqslant 1$, and this is true if and only if $a_n = a_1(1-1/B)^{n-1}$ for all $n \geqslant 1$. $\square$

Remark. Since $D \geqslant 0$ and $Ba_n - \alpha_n \geqslant 0$ $(n \geqslant 1)$, it follows that (1.3) refines inequality (1.1) with C instead ofC^*.

In 1968 D.E. Daykin and C.J. Eliezer [3] presented several interesting inequalities which are closely related to the classical Hölder inequality

$$\sum_{k=1}^{n} a_k b_k \leqslant \left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q} \qquad (p, q > 0;\ 1/p + 1/q = 1). \tag{2.1}$$

One of their results is

Proposition 2. *Let p, q, a_k and b_k $(k = 1, \ldots, n)$ be positive real numbers. If $1/p + 1/q < 1$, then*

$$\begin{aligned}\left(\sum_{k=1}^{n} a_k b_k\right)^{(1/p)+(1/q)} \leqslant \frac{1}{2}\Bigg[&\left(\sum_{k=1}^{n} a_k^p\right)^{1/p} \left(\sum_{k=1}^{n} b_k^q\right)^{1/q} \\ &+ \left(\sum_{k=1}^{n} a_k^{2-p} b_k^2\right)^{1/p} \left(\sum_{k=1}^{n} a_k^2 b_k^{2-q}\right)^{1/q}\Bigg].\end{aligned} \tag{2.2}$$

We note that this theorem has been quoted in the well-known book [5, p. 53] and in the recently published monograph [6, p. 104]. The proof for (2.2) given by Daykin and Eliezer is quite complicated. Apparently, it has been overlooked that an application of the Cauchy-Schwarz inequality provides not only a very short and simple proof but leads also to a refinement of (2.2). Moreover, it turns out that the assumption "$1/p + 1/q < 1$" can be dropped.

Theorem 2. *If p, q, a_k and b_k $(k = 1, \ldots, n)$ are positive real numbers, then*

$$\left(\sum_{k=1}^{n} a_k b_k\right)^{(1/p)+(1/q)} \leqslant \left(\sum_{k=1}^{n} a_k^p \sum_{k=1}^{n} a_k^{2-p} b_k^2\right)^{1/(2p)} \left(\sum_{k=1}^{n} b_k^q \sum_{k=1}^{n} a_k^2 b_k^{2-q}\right)^{1/(2q)}. \tag{2.3}$$

The sign of equality holds in (2.3) *if and only if there exist real numbers C_1 and C_2 such that $a_k^{p-1} = C_1 b_k$ and $b_k^{q-1} = C_2 a_k$ $(k = 1, \ldots, n)$.*

P r o o f. Applying the Cauchy-Schwarz inequality we obtain

$$\sum_{k=1}^{n} a_k^p \sum_{k=1}^{n} a_k^{2-p} b_k^2 = \sum_{k=1}^{n} (a_k^{p/2})^2 \sum_{k=1}^{n} (a_k^{1-p/2} b_k)^2 \geqslant \left(\sum_{k=1}^{n} a_k b_k\right)^2. \tag{2.4}$$

Since $p > 0$, this implies

$$\left(\sum_{k=1}^{n} a_k^p \sum_{k=1}^{n} a_k^{2-p} b_k^2\right)^{1/(2p)} \geqslant \left(\sum_{k=1}^{n} a_k b_k\right)^{1/p}. \tag{2.5}$$

Similarly, we get

$$\left(\sum_{k=1}^{n} b_k^q \sum_{k=1}^{n} a_k^2 b_k^{2-q}\right)^{1/(2q)} \geqslant \left(\sum_{k=1}^{n} a_k b_k\right)^{1/q}. \tag{2.6}$$

Form (2.5) and (2.6) we obtain inequality (2.3). The sign of equality holds in (2.3) if and only if it holds in (2.5) and (2.6), and this is true if and only if there exist real numbers C_1 and C_2 such that $a_k^{p/2} = C_1 a_k^{1-p/2} b_k$ and $b_k^{q/2} = C_2 a_k b_k^{1-q/2}$ $(k = 1, \ldots, n)$. □

Remarks. 1) If $p = q = 2$, then (2.3) reduces to the Cauchy-Schwarz inequality.

2) Using the arithmetic mean-geometric mean inequality $(xy)^{1/2} \leqslant \frac{1}{2}(x+y)$, we conclude that (2.3) is sharper than (2.2).

3) It is natural to ask whether the two upper bounds for $\sum_{k=1}^{n} a_k b_k$ which are given in (2.1) and (2.3) can be compared if $p,\ q > 0$, $1/p + 1/q = 1$. Let us denote the bounds given in (2.1) and (2.3) by B_1 and B_2, respectively. If $p = q = 2$, then $B_1 = B_2$. Let us assume that $p > 2 > q > 1$ and $n \geqslant 2$. Then we obtain for all sufficiently small a_1 that $B_1 < B_2$. On the other hand, if we set $a_1 = \ldots = a_n = 1$, $b_1 = 1$ and $b_2 = \ldots = b_n = \varepsilon > 0$, then we get for all sufficiently small ε that $B_1 > B_2$. Hence, the two bounds cannot be compared in general.

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