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Game-theoretic characterization of the Gurariĭ space

Wiesław Kubiś

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GAME-THEORETIC CHARACTERIZATION OF THE GURARII SPACE

WIESŁAW KUBIŚ

Abstract. We present a simple and natural infinite game building an increasing chain of finite-dimensional Banach spaces. We show that one of the players has a strategy with the property that, no matter how the other player plays, the completion of the union of the chain is linearly isometric to the Gurariĭ space.

1. Introduction

We consider the following game. Namely, two players (called *Eve* and *Odd*) alternately choose finite-dimensional Banach spaces $E_0 \subseteq E_1 \subseteq E_2 \subseteq \cdots$ with no additional rules. For obvious reasons, Eve should start the game. The result is the completion of the chain $\bigcup_{n \in \mathbb{N}} E_n$. We shall denote this game by $\mathrm{BM}(\mathscr{B})$. This is in fact a special case of an abstract Banach-Mazur game studied recently in [4]. Main result:

Theorem 1. *There exists a unique, up to linear isometries, separable Banach space* $\mathbb{G}$ *such that Odd has a strategy* Σ *in* $\mathrm{BM}(\mathscr{B})$ *leading to* $\mathbb{G}$*, namely, the completion of every chain resulting from a play of* $\mathrm{BM}(\mathscr{B})$ *is linearly isometric to* $\mathbb{G}$*, assuming Odd uses strategy* Σ*, and no matter how Eve plays.*

Furthermore, $\mathbb{G}$ *is the Gurariĭ space.*

The result above may serve as a strong argument that the Gurariĭ space (see the definition below) should be considered as one of the classical Banach spaces. Indeed, Theorem 1 is completely elementary and can even be presented with no difficulties to undergraduate students who know the basic concepts of Banach space theory.

It turns out that the Gurariĭ space $\mathbb{G}$ (constructed by Gurariĭ in 1966) is not so well-known, even to people working in functional analysis. The reason might be that this is a Banach space constructed usually by some inductive set-theoretic arguments, without providing any concrete formula for the norm. Furthermore, the fact that $\mathbb{G}$ is actually unique up to linear isometries was proved by Lusky [6] only ten years after Gurariĭ's work [3]. Elementary proof of the uniqueness of $\mathbb{G}$ has been found recently by Solecki and the author [5]. Theorem 1 offers an alternative argument, still using the crucial lemma from [5].

In fact, uniqueness of a space $\mathbb{G}$ satisfying the assertion of Theorem 1 is almost trivial: If there were two Banach spaces G_0, G_1 in Theorem 1, then we can play the

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game so that Odd uses his strategy leading to G_1, while after the first move Eve uses Odd's strategy leading to G_2. Both players win, therefore G_1 is linearly isometric to G_2.

Below, after defining the Gurariĭ space, we show that it indeed satisfies the assertion of Theorem 1. Finally, we discuss other variants of the Banach-Mazur game, for example, playing with separable Banach spaces. Again, the Gurariĭ space is the unique object for which Odd has a winning strategy.

The *Gurariĭ space* is the unique separable Banach space $\mathbb{G}$ satisfying the following condition:

(G) For every $\varepsilon > 0$, for every finite-dimensional normed spaces $A \subseteq B$, for every isometric embedding $e\colon A \to \mathbb{G}$ there exists an extension $f\colon B \to \mathbb{G}$ that is an ε-isometric embedding, namely,

$$(1-\varepsilon)\|x\| \leq \|f(x)\| \leq (1+\varepsilon)\|x\|$$

for every $x \in B$.

As we have already mentioned, this space has been found by Gurariĭ [3] in 1966, yet its uniqueness was proved only ten years later by Lusky [6] using rather advanced methods (the method of representing matrices). Elementary proof can be found in [5]. According to [2, Thm. 2.7], the Gurariĭ space can be characterized by the following condition:

(H) For every $\varepsilon > 0$, for every finite-dimensional normed spaces $A \subseteq B$, for every isometric embedding $e\colon A \to \mathbb{G}$ there exists an isometric embedding $f\colon B \to \mathbb{G}$ such that $\|e - f \restriction A\| < \varepsilon$.

Actually, in the proof of equivalence (G)$\iff$(H) one has to use the crucial lemma from [5].

By a *chain* of normed spaces we mean a sequence $\{E_n\}_{n\in\mathbb{N}}$ such that each E_n is a normed space, $E_n \subseteq E_{n+1}$ and the norm of E_{n+1} restricted to E_n coincides with that of E_n for every $n \in \mathbb{N}$.

2. Proof of Theorem 1

Let us fix a separable Banach space $\mathbb{G}$ satisfying (H). We do not assume a priori that it is uniquely determined, therefore the arguments below will also show the uniqueness of $\mathbb{G}$. Odd's strategy in $\mathrm{BM}(\mathscr{B})$ can be described as follows.

Fix a countable set $\{v_n\}_{n\in\mathbb{N}}$ linearly dense in $\mathbb{G}$. Let E_0 be the first move of Eve. Odd finds an isometric embedding $f_0\colon E_0 \to \mathbb{G}$ and finds $E_1 \supseteq E_0$ together with an isometric embedding $f_1\colon E_1 \to \mathbb{G}$ extending f_0 and such that $v_0 \in f_1[E_1]$.

Suppose now that $n = 2k > 0$ and E_n was the last move of Eve. We assume that a linear isometric embedding $f_{n-1}\colon E_{n-1} \to \mathbb{G}$ has been fixed. Using (H) we choose a linear isometric embedding $f_n\colon E_n \to \mathbb{G}$ such that $f_n \restriction E_{n-1}$ is 2^{-k}-close to f_{n-1}. Extend f_n to a linear isometric embedding $f_{n+1}\colon E_{n+1} \to \mathbb{G}$ so that $E_{n+1} \supseteq E_n$ and $f_{n+1}[E_{n+1}]$ contains all the vectors $v_0, \dots, v_k$. The finite-dimensional space E_{n+1} is Odd's move. This finishes the description of Odd's strategy.

Let $\{E_n\}_{n\in\mathbb{N}}$ be the chain of finite-dimensional normed spaces resulting from a fixed play, when Odd was using the strategy described above. In particular, Odd has recorded a sequence $\{f_n\colon E_n \to \mathbb{G}\}_{n\in\mathbb{N}}$ of linear isometric embeddings such that $f_{2n+1} \restriction E_{2n-1}$ is 2^{-n}-close to f_{2n-1} for each $n \in \mathbb{N}$. Let $G_\infty = \bigcup_{n\in\mathbb{N}} G_n$. For each $x \in G_\infty$ the sequence $\{f_n(x)\}_{n\in\mathbb{N}}$ is Cauchy, therefore we can set $f_\infty(x) = \lim_{n\to\infty} f_n(x)$, thus defining a linear isometric embedding $f_\infty\colon G_\infty \to \mathbb{G}$. The assumption that $f_{2n+1}[E_{2n+1}]$ contains all the vectors $v_0,\dots,v_n$ ensures that $f_\infty[E_\infty]$ is linearly dense in $\mathbb{G}$. Finally, f_∞ extends to a linear isometry from the completion of G_∞ onto $\mathbb{G}$. This completes the proof of Theorem 1.

3. Final remarks

Below we collect some comments around Theorem 1.

Universality. It has been noticed by Gurariĭ that $\mathbb{G}$ contains isometric copies of all separable Banach spaces. In fact, the space $\mathbb{G}$ can be constructed in such a way that it contains any prescribed separable Banach space, e.g., the space $C([0,1])$, which is well-known to be universal. The paper [5] contains a more direct and elementary proof of the isometric universality of $\mathbb{G}$. The main result of this note offers yet another direct proof (cf. [4, Thm. 10]).

Namely, fix a separable Banach space X and fix a chain $\{X_n\}_{n\in\mathbb{N}} \subseteq X$ of finite-dimensional spaces whose union is dense in X. We describe a strategy of Eve that leads to an isometric embedding of X into $\mathbb{G}$. Specifically, Eve starts with $E_0 := X_0$ and records the identity embedding $e_0\colon X_0 \to E_0$. Once Odd has chosen E_n with $n = 2k+1$, having recorded a linear isometric embedding $e_k\colon X_k \to E_{n-1}$, Eve finds $E_{n+1} \supseteq E_n$ such that there is a linear isometric embedding $e_{k+1}\colon X_{k+1} \to E_{n+1}$ extending e_k. This is her response to E_n. The only missing ingredient showing that such a strategy is possible is the *amalgamation property* of finite-dimensional normed spaces:

Lemma 2. *Let $f\colon Z \to X$, $g\colon Z \to Y$ be linear isometric embeddings of Banach spaces. Then there are a Banach space W and linear isometric embeddings $f'\colon X \to W$, $g'\colon Y \to W$ such that $f' \circ f = g' \circ g$. If X, Y are finite-dimensional then so is W.*

The above lemma belongs to the folklore and can be found in several texts, e.g., [2] or [1].

In any case, when Eve uses the strategy described above and Odd uses a strategy leading to the Gurariĭ space, Eve constructs a linear isometric embedding $e\colon X \to \mathbb{G}$ such that $e \restriction X_n = e_n$ for every $n \in \mathbb{N}$. This shows that $\mathbb{G}$ is isometrically universal in the class of all separable Banach spaces.

Playing with separable spaces. It is natural to ask what happens if both players are allowed to choose infinite-dimensional separable Banach spaces. As it happens, in this case Odd has a very simple tactic (i.e. a strategy depending only on the last Eve's move) again leading to the Gurariĭ space. This follows immediately from the following

Proposition 3 ([2, Lemma 3.3])**.** *Let $\{G_n\}_{n\in\mathbb{N}}$ be a chain of Banach spaces such that each G_n is linearly isometric to the Gurariĭ space. Then the completion of the union $\bigcup_{n\in\mathbb{N}} G_n$ is linearly isometric to the Gurariĭ space.*

Thus, knowing that $\mathbb{G}$ contains isometric copies of all separable Banach spaces, Odd can always choose a space linearly isometric to $\mathbb{G}$, so that the resulting chain consists of Gurariĭ spaces.

Other variants of the game. It is evident that the Banach-Mazur game considered in this note can be played with other mathematical structures. The work [4] discusses this game in model theory, showing that Odd has a winning strategy leading to the so-called *Fraïssé limit* of a class of structures (which exists under some natural assumptions). Another natural variant of this game appears when finite-dimensional normed spaces are replaced by finite metric spaces. Almost the same arguments as in the proof of Theorem 1 show that Odd has a strategy leading to the *Urysohn space* [7], the unique complete separable metric space $\mathbb{U}$ satisfying the following condition:

(U) For every finite metric spaces $A \subseteq B$, every isometric embedding $e\colon A \to \mathbb{U}$ can be extended to an isometric embedding $f\colon B \to \mathbb{U}$.

It turns out that $\mathbb{U}$ is uniquely determined by a weaker condition (analog of (H)) asserting that f is ε-close to e with arbitrarily small $\varepsilon > 0$, not necessarily extending e. An analog of Theorem 1 is rather obvious; the proof is practically the same as in the case of normed spaces, simply replacing all phrases "finite-dimensional" by "finite" and deleting all adjectives "linear".

Strategies vs. tactics. The proof of Theorem 1 actually gives a Markov strategy, that is, a strategy depending only on the step n and the last move of Eve. In the modified game, playing with separable spaces, Odd has a tactic, that is, a strategy depending on the last Eve's move only (such a strategy is called *stationary*). We do not know whether Odd has a winning tactic in the Banach-Mazur game played with finite-dimensional normed spaces or finite metric spaces, where "winning" means obtaining the Gurariĭ space or the Urysohn space, respectively.

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Institute of Mathematics, Czech Academy of Sciences (CZECH REPUBLIC)

Faculty of Natural Sciences, College of Science, Cardinal Stefan Wyszyński University, Warsaw (POLAND)