LACUNARY WEAK STATISTICAL CONVERGENCE

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(Received February 15, 2010)

Abstract. The aim of this work is to generalize lacunary statistical convergence to weak lacunary statistical convergence and $\mathcal{I}$-convergence to weak $\mathcal{I}$-convergence. We start by defining weak lacunary statistically convergent and weak lacunary Cauchy sequence. We find a connection between weak lacunary statistical convergence and weak statistical convergence.

Keywords: weak convergence, statistical convergence, lacunary sequence, lacunary statistical convergence

MSC 2010: 40A05, 46A25

1. Introduction

A number sequence (x_k) is statistically convergent to L provided that for every $\varepsilon > 0$,

$$\lim_n \frac{1}{n}|\{k \leqslant n\colon\ |x_k - L| \geqslant \varepsilon\}| = 0$$

where the vertical bars indicate the number of elements in the enclosed set [2], [11].

By a lacunary sequence we mean an increasing integer sequence $\theta = (k_r)$ such that $k_0 = 0$ and $h_r := k_r - k_{r-1} \to \infty$ as $r \to \infty$. Throughout this paper the intervals determined by θ will be denoted by $I_r := (k_{r-1}, k_r]$.

Let θ be a lacunary sequence; the number sequence (x_k) is lacunary statistically convergent to L provided that for every $\varepsilon > 0$,

$$\lim_r \frac{1}{h_r}|\{k \in I_r\colon\ |x_k - L| \geqslant \varepsilon\}| = 0$$

(see [3]). The space N_θ of N_θ-convergent sequences is defined by

$$N_\theta := \left\{(x_k)\colon \text{ for some } L,\ \lim_r \frac{1}{h_r}\sum_{k\in I_r}|x_k - L| = 0\right\}.$$

Let B be a Banach space, let (x_k) be a B-valued sequence, and $x \in B$.

1. The sequence (x_k) is weakly C_1-convergent to x provided that for any f in the continuous dual B^* of B,

$$\lim_n \frac{1}{n} \sum_{k=1}^{n} f(x_k - x) = 0.$$

2. The sequence (x_k) is weakly convergent to x provided that for any f in the continuous dual B^* of B,

$$\lim_k f(x_k - x) = 0.$$

In this case we write $\text{w-lim}\, x_k = x$.

3. The sequence (x_k) is norm statistically convergent to x provided that

$$\delta(\{k\colon \ \|x_k - x\| \geqslant \varepsilon\}) = 0$$

where $\delta(A) = \lim_n n^{-1}|\{k \leqslant n\colon \ k \in A\}|$.

4. The sequence (x_k) is weakly statistically convergent to x provided that for any f in the continuous dual B^* of B, the sequence $(f(x_k - x))$ is statistically convergent to 0 (see [1]).

2. Weakly lacunary statistically convergent sequence

Definition 1. Let B be a Banach space, let (x_k) be a B-valued sequence, θ a lacunary sequence and $x \in B$.

1. The sequence (x_k) is norm lacunary statistically convergent to x provided that

$$\delta_r(\{k\colon \ \|x_k - x\| \geqslant \varepsilon\}) = 0$$

where $\delta_r(A) = \lim_r h_r^{-1}|\{k \in I_r\colon \ k \in A\}|$.

2. The sequence (x_k) is weakly lacunary statistically convergent to x provided that for any f in the continuous dual B^* of B, the sequence $(f(x_k - x))$ is lacunary statistically convergent to 0.

3. The sequence (x_k) is weakly N_θ-convergent to x provided that, for any f in the continuous dual B^* of B, the sequence $(f(x_k - x))$ is N_θ-convergent to 0.

Let WS and WS_θ denote the sets of all weakly statistically convergent and weakly lacunary statistically convergent sequences, respectively.

3. Weak lacunary statistically Cauchy sequence

In [5], Fridy and Orhan defined the lacunary statistical Cauchy sequence for a complex number sequence (x_k) as follows:
Let θ be a lacunary sequence. The sequence (x_k) is said to be lacunary statistically Cauchy if there is a subsequence $(x_{k'(r)})$ of x such that $k'(r) \in I_r$ for each r, $\lim x_{k'(r)} = x$ and for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} |\{k \in I_r \colon |x_k - x_{k'(r)}| \geqslant \varepsilon\}| = 0.$$

Now we will give the definition of the weakly lacunary statistically Cauchy sequence for a B-valued sequence (x_k).

Definition 2. Let B be a Banach space, (x_k) a B-valued sequence, θ a lacunary sequence and $x \in B$. The sequence (x_k) is weakly lacunary statistically Cauchy if there is a subsequence $(x_{k'(r)})$ of (x_k) such that $k'(r) \in I_r$ for each r, w-lim $x_{k'(r)} = x$, and for any f in the continuous dual B^* of B and for every $\varepsilon > 0$

$$\lim_r \frac{1}{h_r} |\{k \in I_r \colon |f(x_k - x_{k'(r)})| \geqslant \varepsilon\}| = 0.$$

Theorem 3. *A sequence (x_k) is weakly lacunary statistically convergent if and only if (x_k) is a weakly lacunary statistically Cauchy sequence.*

Proof. Let (x_k) be a weakly lacunary statistically Cauchy sequence. Then for every $\varepsilon > 0$ we have

$$\begin{aligned} &|\{k \in I_r \colon |f(x_k - x)| \geqslant \varepsilon\}| \\ &\quad \leqslant \left|\left\{k \in I_r \colon |f(x_k - x_{k'(r)})| \geqslant \frac{\varepsilon}{2}\right\}\right| + \left|\left\{k \in I_r \colon |f(x_{k'(r)} - x)| \geqslant \frac{\varepsilon}{2}\right\}\right|, \end{aligned}$$

hence we get that the sequence (x_k) is weakly lacunary statistically convergent. Let (x_k) be weakly lacunary statistically convergent to x and write $M_j = \{k \in \mathbb{N} \colon |f(x_k - x)| < 1/j\}$ for each $j \in \mathbb{N}$, $M_j \supseteq M_{j+1}$ and $|M_j \cap I_r|/h_r \to 1$ as $r \to \infty$. Choose m_1 such that $r \geqslant m_1$ implies $|M_1 \cap I_r|/h_r > 0$, i.e., $M_1 \cap I_r \neq \emptyset$. Next choose $m_1 < m_2$ such that $r \geqslant m_2$ implies $M_2 \cap I_r \neq \emptyset$. Then for each r satisfying $m_1 \leqslant r \leqslant m_2$, choose $k'(r) \in I_r$ such that $k'(r) \in I_r \cap M_1$. In this way, choose $m_{l+1} > m_l$ such that $r > m_{l+1}$ implies $M_{l+1} \cap I_r \neq \emptyset$. Then for all r satisfying $m_l \leqslant r < m_{l+1}$, choose $k'(r) \in I_r \cap M_l$, i.e.,

$$|f(x_{k'(r)} - x)| < \frac{1}{l}.$$

Hence we get $k'(r) \in I_r$ for every r, and $\text{w-lim}\, x_{k'(r)} = x$. Also, we have, for every $\varepsilon > 0$,

$$\begin{aligned}\frac{1}{h_r}&|\{k \in I_r\colon\ |f(x_k - x_{k'(r)})| \geqslant \varepsilon\}|\\&\leqslant \frac{1}{h_r}\left|\left\{k \in I_r\colon\ |f(x_{k(r)} - x)| \geqslant \frac{\varepsilon}{2}\right\}\right| + \frac{1}{h_r}\left|\left\{k \in I_r\colon\ |f(x_{k'(r)} - x)| \geqslant \frac{\varepsilon}{2}\right\}\right|,\end{aligned}$$

whence (x_k) is a weakly lacunary statistically Cauchy sequence. □

4. Inclusion theorems

In this section we first give a theorem that provides the relation between weak N_θ- and weak lacunary statistical convergences. We also study the inclusions between weak statistical convergence and weak lacunary statistical convergence.

Theorem 4. *Let θ be a lacunary sequence; then (x_k) is weakly N_θ-convergent to x if and only if (x_k) is weakly lacunary statistically convergent to x.*

P r o o f. If $\varepsilon > 0$ and (x_k) is weakly N_θ-convergent to x, we can write

$$\begin{aligned}\lim_r \frac{1}{h_r}\sum_{k\in I_r}|f(x_k - x)| &\geqslant \lim_r \frac{1}{h_r}\sum_{\substack{k\in I_r\\ |f(x_k-x)|\geqslant\varepsilon}}|f(x_k - x)|\\&\geqslant \varepsilon|\{k \in I_r\colon\ |f(x_k - x)| \geqslant \varepsilon\}|,\end{aligned}$$

so (x_k) is weakly lacunary statistically convergent to x.

Conversely, suppose that (x_k) is weakly lacunary statistically convergent to x. Since $f \in B^*$, f is bounded, say $|f(x_k - x)| \leqslant K$ for all k. Given $\varepsilon > 0$, we get

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r}|f(x_k - x)| &= \frac{1}{h_r}\sum_{\substack{k\in I_r\\ |f(x_k-x)|\geqslant\varepsilon}}|f(x_k - x)| + \frac{1}{h_r}\sum_{\substack{k\in I_r\\ |f(x_k-x)|<\varepsilon}}|f(x_k - x)|\\&\leqslant \frac{K}{h_r}|\{k \in I_r\colon\ |f(x_k - x)| \geqslant \varepsilon\}| + \varepsilon,\end{aligned}$$

so (x_k) is weakly N_θ-convergent to x. □

Theorem 5. *For any lacunary sequence* θ, $\mathrm{WS}\text{-}\lim x_k = x$ *implies* $\mathrm{WS}_\theta\text{-}\lim x_k = x$ *if and only if* $\liminf_r k_r/k_{r-1} > 1$.

P r o o f. k_r/k_{r-1} will be denoted by q_r. If $\liminf\limits_r q_r > 1$ there exist $\eta > 0$ such that $1+\eta \leqslant q_r$ for all sufficiently large r, which implies that

$$\frac{h_r}{k_r} \geqslant \frac{1}{1+\eta}.$$

If $x_k \to x(\mathrm{WS})$, then for every $\varepsilon > 0$ and for sufficiently large r we have

$$\begin{aligned}\frac{1}{k_r}|\{k \leqslant k_r \colon\ |f(x_k - x)| \geqslant \varepsilon\}| &\geqslant \frac{1}{k_r}|\{k \in I_r \colon\ |f(x_k - x)| \geqslant \varepsilon\}| \\ &\geqslant \frac{\eta}{1+\eta}\frac{1}{h_r}|\{k \in I_r \colon\ |f(x_k - x)| \geqslant \varepsilon\}|;\end{aligned}$$

this proves sufficiency. Conversely, if we suppose that $\liminf_r q_r = 1$, then following the idea in [4], we can find a sequence (x_k) such that $(x_k) \notin \mathrm{WS}_\theta$ but $(x_k) \in \mathrm{WS}$. □

Theorem 6. *For any lacunary sequence* θ, $\mathrm{WS}_\theta\text{-}\lim x_k = x$ *implies* $\mathrm{WS}\text{-}\lim x_k = x$ *if and only if* $\limsup_r k_r/k_{r-1} < \infty$.

P r o o f. If $\limsup_r q_r < \infty$, then there is a $K > 0$ such that $q_r < K$ for all r. Suppose that $x_k \to x(\mathrm{WS}_\theta)$, and let $M_r = |\{k \in I_r \colon\ |f(x_k - x)| \geqslant \varepsilon\}|$. Since WS_θ-$\lim x_k = x$, given $\varepsilon > 0$, there is an $r_0 \in \mathbb{N}$ such that $M_r/h_r < \varepsilon$ for all $r > r_0$. Now let $M = \max\{M_r \colon\ 1 \leqslant r \leqslant r_0\}$ and let n be any integer satisfying $k_{r-1} < n \leqslant k_r$. Then we can write

$$\begin{aligned}\frac{1}{n}|\{k \leqslant n \colon\ |f(x_k - x)| \geqslant \varepsilon\}| &\leqslant \frac{1}{k_{r-1}}|\{k \leqslant k_r \colon\ |f(x_k - x)| \geqslant \varepsilon\}| \\ &= \frac{1}{k_{r-1}}\{M_1 + M_2 + \ldots + M_{r_0} + M_{r_0+1} + \ldots + M_r\} \\ &\leqslant \frac{M}{k_{r-1}}r_0 + \frac{1}{k_{r-1}}\left\{h_{r_0+1}\frac{M_{r_0+1}}{h_{r_0+1}} + \ldots + h_r\frac{M_r}{h_r}\right\} \\ &\leqslant \frac{r_0 M}{k_{r-1}} + \frac{1}{k_{r-1}}\left(\sup_{r>r_0}\frac{M_r}{h_r}\right)\{h_{r_0+1} + \ldots + h_r\} \\ &\leqslant \frac{r_0 M}{k_{r-1}} + \varepsilon\frac{k_r - k_{r_0}}{k_{r-1}} \\ &\leqslant \frac{r_0 M}{k_{r-1}} + \varepsilon q_r \\ &\leqslant \frac{r_0 M}{k_{r-1}} + \varepsilon K,\end{aligned}$$

and the sufficiency follows immediately.

Conversely, if we suppose that $\limsup_r q_r = \infty$, then following the idea in [4], we can find a sequence (x_k) such that $(x_k) \notin \mathrm{WS}$ but $(x_k) \in \mathrm{WS}_\theta$. □

Combining Theorems 5 and 6 we get

Theorem 7. *Let* θ *be a lacunary sequence; then* $\mathrm{WS} = \mathrm{WS}_\theta$ *if and only if* $1 < \liminf_r k_r/k_{r-1} \leqslant \limsup_r k_r/k_{r-1} < \infty$.

Theorem 8. *If* $x \in \mathrm{WS} \cap \mathrm{WS}_\theta$, *then* $\mathrm{WS}_\theta\text{-}\lim x = \mathrm{WS}\text{-}\lim x$.

Proof. Suppose $\mathrm{WS}\text{-}\lim x = x$ and $\mathrm{WS}_\theta\text{-}\lim x = y$ and $x \neq y$. For $\varepsilon < \frac{1}{2}|x-y|$ we get

$$\lim_n \frac{1}{n}|\{k \leqslant n \colon |f(x_k - y)| \geqslant \varepsilon\}| = 1.$$

Consider the k_mth term of the weak statistical limit expression $n^{-1}|\{k \leqslant n \colon |f(x_k - y)| \geqslant \varepsilon\}|$:

$$(1)\qquad \frac{1}{k_m}\left|\left\{k \in \bigcup_{r=1}^{m} I_r \colon |f(x_k - y)| \geqslant \varepsilon\right\}\right| = \frac{1}{k_m}\sum_{r=1}^{m}|\{k \in I_r \colon |f(x_k - y)| \geqslant \varepsilon\}| = \frac{1}{\sum_{r=1}^{m} h_r}\sum_{r=1}^{m} h_r t_r,$$

where $t_r = h_r^{-1}|\{k \in I_r \colon |f(x_k - y)| \geqslant \varepsilon\}| \to 0$ because $\mathrm{WS}_\theta\text{-}\lim x = y$. Since θ is a lacunary sequence, (1) is a regular weighted mean transform of t_r, and therefore it, too, tends to zero as $m \to \infty$. Also, since this is a subsequence of $\{n^{-1}|\{k \leqslant n \colon |f(x_k - y)| \geqslant \varepsilon\}|\}$, we infer that

$$\lim_n \frac{1}{n}|\{k \leqslant n \colon |f(x_k - y)| \geqslant \varepsilon\}| \neq 1,$$

and this contradiction shows that we can't have $x \neq y$. □

5. Weak strong almost convergence and weak lacunary statistical convergence

The idea of almost convergence was introduced by Lorentz [9]. Later Maddox [10] and (independently) Freedman at al. [6] introduced the notion of the strong almost convergence. Now we will introduce the notions of weakly almost convergence and weakly strong almost convergence for sequences in a Banach space.

Definition 9. Let B be a Banach space, (x_k) be a B-valued sequence and let f be in the continuous dual B^* of B. Sequence (x_k) is said to be weakly almost convergent to x if

$$\lim_n \frac{1}{n}\sum_{i=m+1}^{m+n} f(x_i - x) = 0$$

uniformly in m.

Definition 10. Let B be a Banach space, (x_k) be a B-valued sequence and let f be in the continuous dual B^* of B. Sequence (x_k) is said to be weakly strongly almost convergent to x if

$$\lim_n \frac{1}{n} \sum_{i=m+1}^{m+n} |f(x_i - x)| = 0$$

uniformly in m.

Let WN_θ, WS_θ, WAC and [WAC] denote the sets of all weakly N_θ-convergent, all weakly statistically convergent, all weakly almost convergent and all weakly strongly almost convergent sequences, respectively.

Lemma 11. $[\mathrm{WAC}] = \bigcap\limits_{\theta\in\mathcal{L}} \mathrm{WN}_\theta$.

Proof is similar to the proof of Theorem 3.1 in [6].

Theorem 12. *If $\mathcal{L}$ denotes the set of all lacunary sequences, then*

$$[\mathrm{WAC}] = \bigcap_{\theta\in\mathcal{L}} \mathrm{WS}_\theta.$$

Proof. By Lemma 12 and Theorem 4, we have

$$[\mathrm{WAC}] = \bigcap_{\theta\in\mathcal{L}} \mathrm{WN}_\theta = \bigcap_{\theta\in\mathcal{L}} \mathrm{WS}_\theta.$$

□

6. Weak $\mathcal{I}$-convergence

The concept of the $\mathcal{I}$-convergence is a generalization of statistical convergence and is based on the notion of the ideal $\mathcal{I}$ of subsets of the set $\mathbb{N}$ of positive integers. A non-void class $\mathcal{I} \subseteq 2^{\mathbb{N}}$ is called an ideal if $\mathcal{I}$ is additive (i.e., $A, B \in \mathcal{I} \Rightarrow A \cup B \in \mathcal{I}$) and hereditary (i.e., $A \in \mathcal{I}$ and $B \subseteq A \Rightarrow B \in \mathcal{I}$).

An ideal $\mathcal{I}$ is said to be non-trivial if $\mathcal{I} \neq 2^{\mathbb{N}}$. A non-trivial ideal $\mathcal{I}$ is said to be admissible if $\mathcal{I}$ contains every finite subset of $\mathbb{N}$. For any ideal $\mathcal{I}$ there is a filter $\mathcal{F}(\mathcal{I})$ corresponding to $\mathcal{I}$, given by

$$\mathcal{F}(\mathcal{I}) = \{K \subseteq \mathbb{N} \colon \mathbb{N} \setminus K \in \mathcal{I}\}.$$

Definition 13. Let B be a Banach space, let (x_k) be a B-valued sequence, and $x \in B$. The sequence (x_k) is norm $\mathcal{I}$-convergent to x provided that

$$\{k \in \mathbb{N}\colon \ \|x_k - x\| \geqslant \varepsilon\} \in \mathcal{I}.$$

Definition 14. Let B be a Banach space, let f be in the continuous dual B^* of B, let (x_k) be a B-valued sequence, and $x \in B$. The sequence (x_k) is weakly $\mathcal{I}$-convergent to x provided that

$$\{k \in \mathbb{N}\colon \ |f(x_k - x)| \geqslant \varepsilon\} \in \mathcal{I}.$$

If $\mathcal{I} = \mathcal{I}_{\text{fin}}$ the ideal of all finite subsets of $\mathbb{N}$, we have the usual weak convergence.

Denote by $\mathcal{I}_\delta$ the class of all $K \subset \mathbb{N}$ with

$$\delta(K) = \lim_{n\to\infty} \frac{1}{n} |\{k \leqslant n\colon \ k \in K\}| = 0,$$

then $\mathcal{I}_\delta$ is a non-trivial admissible ideal, and the $\mathcal{I}_\delta$-convergence coincides with the weak statistical convergence.

Denote by $\mathcal{I}_\theta$ the class of all $K \subset \mathbb{N}$ with

$$\delta_r(K) = \lim_{r\to\infty} \frac{1}{h_r} |\{k \in I_r\colon \ k \in K\}| = 0,$$

then $\mathcal{I}_\theta$ is a non-trivial admissible ideal, $\mathcal{I}_\theta$-convergence coincides with the weak lacunary statistical convergence.

Definition 15. Let B be a Banach space, (x_k) a B-valued sequence and let f be in the continuous dual B^* of B, and $x \in B$. The sequence (x_k) is weakly $\mathcal{I}^*$-convergent to x if and only if there exists a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \subseteq \mathbb{N}$, $M \in \mathcal{F}(\mathcal{I})$ such that $\lim_k f(x_{m_k} - x) = 0$.

Let $\mathcal{WI}$ and $\mathcal{WI}^*$ denote the sets of all weakly $\mathcal{I}$-convergent and all weakly $\mathcal{I}^*$-convergent sequences, respectively.

Theorem 16. *Let $\mathcal{I}$ be an admissible ideal. If $\mathcal{WI}^*$-$\lim x_k = x$, then $\mathcal{WI}$-$\lim x_k = x$.*

P r o o f. By assumption there is a set $L \in \mathcal{I}$ such that for $M = \mathbb{N} \setminus L = \{m_1 < m_2 < \ldots < m_k < \ldots\}$ we have

$$\lim_k f(x_{k_m} - x) = 0. \tag{2}$$

Let $\varepsilon > 0$. By (2), there exists $k_0 \in \mathbb{N}$ such that $|f(x_{m_k} - x)| < \varepsilon$ for each $k > k_0$. Then since $\mathcal{I}$ is admissible, we get

$$\{k \in \mathbb{N}\colon \ |f(x_{m_k} - x)| \geqslant \varepsilon\} \subset L \cup \{m_1 < m_2 < \ldots < m_{k_0}\} \in \mathcal{I}.$$

□

Definition 17 (see [8]). An admissible ideal $\mathcal{I} \subset 2^{\mathbb{N}}$ is said to satisfy the condition (AP) if for every countable family of mutually disjoint sets $\{A_1, A_2, \ldots\}$ belonging to $\mathcal{I}$ there exists a countable family of sets $\{B_1, B_2, \ldots\}$ such that $A_j \bigtriangleup B_j$ is a finite set for $j \in \mathbb{N}$ and $B = \bigcup\limits_{j=1}^{\infty} B_j \in \mathcal{I}$.

Theorem 18. *Let $\mathcal{I} \subset 2^{\mathbb{N}}$ be an admissible ideal. If the ideal $\mathcal{I}$ has property* (AP), *then for an arbitrary sequence $(x_k) \in X$, $\mathcal{WI}$-$\lim(x_k) = x$ implies $\mathcal{WI}^*$-$\lim(x_k) = x$.*

P r o o f. Suppose that $\mathcal{I}$ satisfies condition (AP). Let $\mathcal{WI}$-$\lim(x_k) = x$. Then $\{k \in \mathbb{N}\colon |f(x_{k_m} - x)| \geqslant \varepsilon\} \in \mathcal{I}$ for $\varepsilon > 0$. Put $A_1 = \{k \in \mathbb{N}\colon |f(x_{k_m} - x)| \geqslant 1\}$ and $A_k = \{k \in \mathbb{N}\colon 1/k \leqslant |f(x_{k_m} - x)| \leqslant 1/(k+1)\}$ for $k \geqslant 2$, $k \in \mathbb{N}$. Obviously $A_i \cap A_j = \varphi$ for $i \neq j$. By condition (AP) there exists a sequence of sets $(B_k)_{k\in\mathbb{N}}$ such that $A_j \bigtriangleup B_j$ are finite sets for $j \in \mathbb{N}$ and $B = \bigcup\limits_{j=1}^{\infty} B_j \in \mathcal{I}$. It is sufficient to prove that for $M = \mathbb{N} \setminus B$ we have

$$\lim_{\substack{k\to\infty\\ k\in M}} f(x_k - x) = 0. \tag{3}$$

Let $\xi > 0$. Choose $k \in \mathbb{N}$ such that $1/(k+1) < \xi$. Then $\{k \in \mathbb{N}\colon |f(x_k - x)| \geqslant \xi\} \subset \bigcup\limits_{j=1}^{n+1} A_j$. Since $A_j \bigtriangleup B_j$, $j = 1, 2, \ldots, n+1$ are finite sets there exists $k_0 \in \mathbb{N}$ such that

$$\bigcup_{j=1}^{n+1} B_j \cap \{k \in \mathbb{N}\colon\ k > k_0\} = \bigcup_{j=1}^{n+1} A_j \cap \{k \in \mathbb{N}\colon\ k > k_0\}. \tag{4}$$

If $k > k_0$ and $k \notin B$, then $k \notin \bigcup\limits_{j=1}^{n+1} B_j$ and by (4), $k \notin \bigcup\limits_{j=1}^{n+1} A_j$. But then $|f(x_k - x)| < 1/(k+1) < \xi$; so (3) holds. $\square$

7. Weak $\mathcal{I}$-limit points and weak $\mathcal{I}$-cluster points

Definition 19. Let B be a Banach space, (x_k) a B-valued sequence, let f be in the continuous dual B^* of B and $x \in B$.

(a) An element $x \in X$ is said to be a weak $\mathcal{I}$-limit point of (x_k) provided that there exists a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \subseteq \mathbb{N}$ such that $M \notin \mathcal{I}$ and $\lim\limits_{k\to\infty, k\in M} f(x_k - x) = 0$.

(b) An element $x \in X$ is said to be a weak $\mathcal{I}$-cluster point of (x_k) if and only if for each $\varepsilon > 0$ we have $\{k \in \mathbb{N}\colon\ |f(x_k - x)| < \varepsilon\} \notin \mathcal{I}$.

Let $\mathcal{WI}(\Lambda_x)$ and $\mathcal{WI}(\Gamma_x)$ denote the sets of all $\mathcal{WI}$-limit and $\mathcal{WI}$-cluster points of x, respectively.

Theorem 20. *Let $\mathcal{I}$ be an admissible ideal. Then for each sequence $(x_k) \in B$ we have $\mathcal{WI}(\Lambda_x) \subset \mathcal{WI}(\Gamma_x)$.*

P r o o f. Let $x \in \mathcal{WI}(\Lambda_x)$. Then there exists a set $M = \{m_1 < m_2 < \ldots < m_k < \ldots\} \notin \mathcal{I}$ such that

$$\lim_{k\to\infty} f(x_{m_k} - x) = 0. \tag{5}$$

Take $\vartheta > 0$. According to (5) there exists $k_0 \in \mathbb{N}$ such that for $k > k_0$ we have $|f(x_{m_k} - x)| < \vartheta$. Hence $\{k \in \mathbb{N}\colon |f(x_k - x)| < \vartheta\} \supset M \setminus \{m_1, m_2, \ldots, m_{k_0}\}$ and $\{k \in \mathbb{N}\colon |f(x_k - x)| < \vartheta\} \notin \mathcal{I}$, which means that $x \in \mathcal{WI}(\Gamma_x)$. □

A c k n o w l e d g e m e n t. The author is grateful to the referee for his kind remarks which improved the presentation of the paper.

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