A NOTE ON k-c-SEMISTRATIFIABLE SPACES AND STRONG β-SPACES

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Abstract. Recall that a space X is a c-semistratifiable (CSS) space, if the compact sets of X are G_δ-sets in a uniform way. In this note, we introduce another class of spaces, denoting it by k-c-semistratifiable (k-CSS), which generalizes the concept of c-semistratifiable. We discuss some properties of k-c-semistratifiable spaces. We prove that a T_2-space X is a k-c-semistratifiable space if and only if X has a g function which satisfies the following conditions:

(1) For each $x \in X$, $\{x\} = \bigcap\{g(x,n)\colon n \in \mathbb{N}\}$ and $g(x,n+1) \subseteq g(x,n)$ for each $n \in \mathbb{N}$.

(2) If a sequence $\{x_n\}_{n\in\mathbb{N}}$ of X converges to a point $x \in X$ and $y_n \in g(x_n,n)$ for each $n \in \mathbb{N}$, then for any convergent subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$ we have that $\{y_{n_k}\}_{k\in\mathbb{N}}$ converges to x.

By the above characterization, we show that if X is a submesocompact locally k-c-semistratifiable space, then X is a k-c-semistratifible space, and the countable product of k-c-semistratifiable spaces is a k-c-semistratifiable space. If $X = \bigcup\{\mathrm{Int}(X_n)\colon n \in \mathbb{N}\}$ and X_n is a closed k-c-semistratifiable space for each n, then X is a k-c-semistratifiable space.

In the last part of this note, we show that if $X = \bigcup\{X_n\colon n \in \mathbb{N}\}$ and X_n is a closed strong β-space for each $n \in \mathbb{N}$, then X is a strong β-space.

Keywords: c-semistratifiable space, k-c-semistratifiable space, submesocompact space, g function, strong β-space

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1. Introduction

It is well known that the class of generalized metric spaces plays an important role in general topology. Some of the known generalized metric spaces are semistratifiable

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spaces ([3]), k-semistratifiable spaces ([15]) and stratifiable spaces ([2]). The notion of semistratifiable spaces was introduced by Creede in 1970 ([3]). In 1973, H. W. Martin introduced the class of CSS spaces ([16]).

Let $(X, \mathscr{T})$ be a topological space and let $\mathscr{C}$ be the collection of all compact sets of X. A space X is called a CSS *space*, if there is a function $U\colon \mathbb{N} \times \mathscr{C} \to \mathscr{T}$ such that for each $C \in \mathscr{C}$ and $n \in \mathbb{N}$ there is an open set $U(n, C)$ in X such that

(1) $C = \bigcap\{U(n, C)\colon n \in \mathbb{N}\}$;

(2) if $C \in \mathscr{C}$, $D \in \mathscr{C}$ and $C \subseteq D$ then $U(n, C) \subseteq U(n, D)$ for each $n \in \mathbb{N}$.

In other words, a CSS space is a space in which the compact subsets are G_δ-sets in a uniform way. Let $(X, \mathscr{T})$ be a CSS space and let U be a CSS function for X. If for any disjoint compact sets C and K of X there is $n \in \mathbb{N}$ such that $U(n, C) \cap K = \emptyset$, then X is called a k-CSS *space*. By the definitions, we know that every k-CSS space is a CSS space, and every k-semistratifiable space is a k-CSS space.

In this note, we mainly give a characterization of k-CSS spaces in terms of certain g functions. By this conclusion, we show that if X is a submesocompact locally k-CSS space, then X is a k-CSS space. We also show that the countable product of k-CSS spaces is a k-CSS space.

In [17] we showed that if $X = \bigcup\{X_n\colon n \in \mathbb{N}\}$ and X_n is a closed CSS space (or a semistratifiable space) for each $n \in \mathbb{N}$, then X is a CSS space (or a semistratifiable space). In this note we show that if $X = \bigcup\{\mathrm{Int}(X_n)\colon n \in \mathbb{N}\}$ and X_n is a closed k-CSS space for each $n \in \mathbb{N}$, then X is a k-CSS space.

In [7] the authors ask whether every F_σ-set of a β-space is a β-subspace. This question is answered in [18]. Since the article [18] is in Chinese and the referee suggested that we give a short description of the answer, we will give it in the last part of this note. In [17] we showed that if X is the countable union of closed β-subspaces, then X is a β-space. The class of strong β-spaces was introduced in [19] and was studied in [19] and [10] (see section 3 for the definition of strong β-spaces). In this note, we show that if $X = \bigcup\{X_n\colon n \in \mathbb{N}\}$ and X_n is a closed strong β-space for each $n \in \mathbb{N}$, then X is a strong β-space. Thus we have that a F_σ-set of a strong β-space is a strong β-subspace.

Let $(X, \mathscr{T})$ be a topological space. A g *function* is a function from $X \times \mathbb{N}$ into the topology $\mathscr{T}$ of X such that $x \in g(x, n+1) \subseteq g(x, n)$ for each $x \in X$ and $n \in \mathbb{N}$. We use g to denote a g function throughout this paper.

In this note, all spaces are assumed to be at least Hausdorff spaces. The set of all positive integers is denoted by $\mathbb{N}$. If $\mathscr{U}$ and $\mathscr{V}$ are families of subsets of X such that for each $U \in \mathscr{U}$ there exists $V \in \mathscr{V}$ such that $U \subseteq V$, then we write $\mathscr{U} \prec \mathscr{V}$. We put $\mathscr{U} \wedge \mathscr{V} = \{U \cap V\colon U \in \mathscr{U}$ and $V \in \mathscr{V}\}$. If X is a space, then we denote by $\mathscr{C}$ the collection of all compact subsets of X and by 2^X the collection of all closed subsets of X. If a sequence $\{x_n\}_{n\in\mathbb{N}}$ of X converges to a point x of X, then we denote it by $\{x_n\}_{n\in\mathbb{N}} \to x$. In notions and terminology we will follow [4], [8] and [13].

2. On k-CSS spaces

We begin this section by showing how k-c-semistratifiable spaces are related to five types of spaces that have already been studied. The six types of spaces are defined based on different ways of writing certain closed sets as G_δ-sets in a uniform way. The definitions follow.

Definition 2.1. Let 2^X be the family of all closed subsets of a topological space $(X, \mathscr{T})$ and consider four properties that a function $U\colon \mathbb{N} \times 2^X \to \mathscr{T}$ may have:

(1) For each $F \in 2^X$ and each $n \in \mathbb{N}$, $U(n+1, F) \subseteq U(n, F)$ and if $F_1 \subseteq F_2$ are in 2^X then $U(n, F_1) \subseteq U(n, F_2)$ for each $n \in \mathbb{N}$.

(2) For each $F \in 2^X$, $F = \bigcap\{U(n, F)\colon\ n \in \mathbb{N}\} = \bigcap\{\overline{U(n, F)}\colon\ n \in \mathbb{N}\}$.

(3) For each $F \in 2^X$, $F = \bigcap\{U(n, F)\colon\ n \in \mathbb{N}\}$ and if K is compact and $K \cap F = \emptyset$, then for some $n \in \mathbb{N}$, $K \cap U(n, F) = \emptyset$.

(4) For each $F \in 2^X$, $F = \bigcap\{U(n, F)\colon\ n \in \mathbb{N}\}$.

A space $(X, \mathscr{T})$ that has a function U satisfying (1) and (2) is called *stratifiable* ([2]); a space with a function U satisfying (1) and (3) is called k-*semistratifiable* ([15]), and a space with a function U satisfying (1) and (4) is called *semistratifiable* ([3]).

Each class of spaces in Definition 2.1 has a well-developed theory. Next, we replace the collection 2^X by the collection $\mathscr{C}$ of all compact subsets of $(X, \mathscr{T})$ in each of properties (1) through (4).

Definition 2.2. Let $\mathscr{C}$ be the family of all compact subsets of a topological space $(X, \mathscr{T})$ and consider four properties that a function $U\colon \mathbb{N} \times \mathscr{C} \to \mathscr{T}$ may have:

(1) For each $C \in \mathscr{C}$ and each $n \in \mathbb{N}$, $U(n+1, C) \subseteq U(n, C)$ and if $C_1 \subseteq C_2$ are in $\mathscr{C}$ then $U(n, C_1) \subseteq U(n, C_2)$ for each $n \in \mathbb{N}$.

(2) For each $C \in \mathscr{C}$, $C = \bigcap\{U(n, C)\colon\ n \in \mathbb{N}\} = \bigcap\{\overline{U(n, C)}\colon\ n \in \mathbb{N}\}$.

(3) For each $C \in \mathscr{C}$, $C = \bigcap\{U(n, C)\colon\ n \in \mathbb{N}\}$ and if K is compact and $K \cap C = \emptyset$, then for some $n \in \mathbb{N}$, $K \cap U(n, C) = \emptyset$.

(4) For each $C \in \mathscr{C}$, $C = \bigcap\{U(n, C)\colon\ n \in \mathbb{N}\}$.

A c-*stratifiable* (CS) *space* ([11]) is a space that has a function $U\colon \mathbb{N} \times \mathscr{C} \to \mathscr{T}$ with properties (1) and (2). A k-c-*semistratifiable* (k-CSS) *space* is a space that has a function $U\colon \mathbb{N} \times \mathscr{C} \to \mathscr{T}$ with properties (1) and (3). A space is called c-*semistratifiable* (CSS) ([16]) if it has a function $U\colon \mathbb{N} \times \mathscr{C} \to \mathscr{T}$ with properties (1) and (4). The function U is called a CS *function* for X, k-CSS *function* for X, and CSS *function* for X, respectively.

The above definitions make the implications in the following diagram clear:

$$\begin{array}{ccccc} \text{stratifiable} & \Rightarrow & \text{k-semistratifiable} & \Rightarrow & \text{semistratifiable} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{c-stratifiable} & \Rightarrow & \text{k-c-semistratifiable} & \Rightarrow & \text{c-semistratifiable} \end{array}$$

We thank the referee for pointing out how to make it clear how naturally the topic of this paper fits into classical topology.

Theorem 2.3. *If X is a first countable* k-CSS *space, then X is a* CS *space.*

P r o o f. Let U be a k-CSS function for X. For each $F \in 2^X$, $F = \bigcap\{U(n,F)\colon n \in \mathbb{N}\}$. We claim that $F = \bigcap\{\overline{U(n,F)}\colon n \in \mathbb{N}\}$. Suppose there is a point $y \in \bigcap\{\overline{U(n,F)}\colon n \in \mathbb{N}\} \setminus F$. Let $\{V_n(y)\colon n \in \mathbb{N}\}$ be a decreasing neighborhood base of the point y in X. Since $y \notin F$ and F is closed in X, we have that $\{V_n(y)\setminus F\colon n \in \mathbb{N}\}$ is a decreasing neighborhood base of the point y in X. Thus $(V_n(y)\setminus F)\cap U(n,F) \neq \emptyset$ for each n. If $y_n \in (V_n(y)\setminus F)\cap U(n,F)$, then $\{y_n\}_{n\in\mathbb{N}} \to y$. Thus $\{y\}\cup\{y_n\colon n \in \mathbb{N}\} = C$ is a compact subset of X and $F\cap C = \emptyset$. Thus there is $n \in \mathbb{N}$ such that $U(n,F)\cap C = \emptyset$. This contradicts the fact that $y_n \in U(n,F)$. Thus $F = \bigcap\{\overline{U(n,F)}\colon n \in \mathbb{N}\}$, and hence U is a CS function for X. $\square$

Theorem 2.4. *If $(X,\mathscr{T})$ is a* k-CSS *space and $Y \subseteq X$, then Y is a* k-CSS *space.*

P r o o f. Let U be a k-CSS function for X. For each compact subset C of the subspace Y, we know that C is a compact subset of X. Put $U_1(n,C) = U(n,C)\cap Y$; hence $U_1(n,C)$ is an open subset of the subspace Y. We can see that the function $U_1\colon \mathbb{N}\times\mathscr{C}_Y \to \mathscr{T}\cap Y$ is a k-CSS function for Y, where $\mathscr{C}_Y$ is the family of all compact subsets of the subspace Y. $\square$

Theorem 2.5. *Let X be a* k-CSS *space. If $f\colon X \to Y$ is a perfect map, then Y is a* k-CSS *space.*

P r o o f. Let $\mathscr{C}_X$ denote the collection of all compact subsets of X and let $\mathscr{C}_Y$ denote the collection of all compact subsets of Y. Let U be a k-CSS function for X. For each $K \in \mathscr{C}_Y$, we know that $f^{-1}(K) \in \mathscr{C}_X$ since f is a perfect map. Let $U_1(n,K) = Y \setminus f(X \setminus U(n,f^{-1}(K)))$ for each $n \in \mathbb{N}$.

Since $f^{-1}(K) \subseteq U(n,f^{-1}(K))$, we have $K \subseteq U_1(n,K)$ and $U_1(n,K)$ is an open subset of Y for each $n \in \mathbb{N}$. So $K \subseteq \bigcap\{U_1(n,K)\colon n \in \mathbb{N}\}$. If $y \notin K$, then $f^{-1}(y)\cap f^{-1}(K) = \emptyset$. Thus there exists $x \in f^{-1}(y)$ such that $x \notin f^{-1}(K)$. Since $f^{-1}(K) = \bigcap\{U(n,f^{-1}(K))\colon n \in \mathbb{N}\}$, there is $n \in \mathbb{N}$ such that $x \notin U(n,f^{-1}(K))$. Hence $y \notin Y \setminus f(X \setminus U(n,f^{-1}(K))) = U_1(n,K)$, so $K = \bigcap\{U_1(n,K)\colon n \in \mathbb{N}\}$. If

$C \in \mathscr{C}_Y, K \in \mathscr{C}_Y$ and $K \cap C = \emptyset$, then $f^{-1}(K) \cap f^{-1}(C) = \emptyset$. Thus there exists $n \in \mathbb{N}$ such that $U(n, f^{-1}(K)) \cap f^{-1}(C) = \emptyset$. Thus $U_1(n, K) \cap C = \emptyset$. It is obvious that $U_1(n, K_1) \subseteq U_1(n, K_2)$ if $K_1 \in \mathscr{C}_Y$, $K_2 \in \mathscr{C}_Y$, and $K_1 \subseteq K_2$. Hence Y is a k-CSS space. □

Theorem 2.6. *Let $(X, \mathscr{T})$ be a space. The space $(X, \mathscr{T})$ is a* k-CSS *space if and only if there exists a function $F\colon \mathbb{N} \times \mathscr{T}' \to 2^X$, where $\mathscr{T}' = \{V\colon X \setminus V$ is a compact subset of X and $V \in \mathscr{T}\}$ is such that the following conditions holds for each $V \in \mathscr{T}'$:*

(1) *$\bigcup\{F(n, V)\colon n \in \mathbb{N}\} = V$ and $F(n, V) \subseteq F(n+1, V)$ for each $n \in \mathbb{N}$;*
(2) *if $V_1, V_2 \in \mathscr{T}'$ and $V_1 \subseteq V_2$, then $F(n, V_1) \subseteq F(n, V_2)$ for each $n \in \mathbb{N}$;*
(3) *if C is a compact subset of X and $C \subseteq V$ for some $V \in \mathscr{T}'$, then there exists $n \in \mathbb{N}$ such that $C \subseteq F(n, V)$.*

Proof. $\Rightarrow$: Let U be a k-CSS function for X. For each $V \in \mathscr{T}'$, we know that $X \setminus V$ is a compact subset of X. Thus $X \setminus V = \bigcap\{U(n, X \setminus V)\colon n \in \mathbb{N}\}$, and hence $V = \bigcup\{X \setminus U(n, X \setminus V)\colon n \in \mathbb{N}\}$. If $F(n, V) = X \setminus U(n, X \setminus V)$ for each $n \in \mathbb{N}$, then $F(n, V)$ is closed in X. Thus both the conditions (1) and (2) of the theorem are satisfied. Since C is compact, $C \subseteq V$, and $V \in \mathscr{T}'$, we have that $X \setminus V$ is a compact set and $C \cap (X \setminus V) = \emptyset$. Thus there is $n \in \mathbb{N}$ such that $C \cap U(n, X \setminus V) = \emptyset$, i.e., $C \subseteq X \setminus U(n, X \setminus V) = F(n, V)$.

$\Leftarrow$: Suppose $F\colon \mathbb{N} \times \mathscr{T}' \to 2^X$ is a function which satisfies the conditions (1), (2), and (3) of the theorem. If K is a compact subset of X, then $X \setminus K \in \mathscr{T}'$. Thus $\bigcup\{F(n, X \setminus K)\colon n \in \mathbb{N}\} = X \setminus K$, and hence we have $\bigcap\{X \setminus F(n, X \setminus K)\colon n \in \mathbb{N}\} = K$. Let $U(n, K) = X \setminus F(n, X \setminus K)$ for each n. We can easily show that U is a k-CSS function for X. □

Theorem 2.7. *If $X = Y_1 \cup Y_2$, Y_1 and Y_2 are closed* k-CSS *subspaces of X, then X is a* k-CSS *space.*

Proof. Let U_1 and U_2 be k-CSS functions for Y_1 and Y_2, respectively. For each compact subset F of X and $n \in \mathbb{N}$, if we set $U(n, F) = X \setminus [(Y_1 \setminus U_1(n, F \cap Y_1)) \cup (Y_2 \setminus U_2(n, F \cap Y_2))]$, then $F \subseteq U(n, F)$ and $U(n, F)$ is an open subset of X. If E is a compact subset of X and $E \cap F = \emptyset$, then there is $n \in \mathbb{N}$ such that $U_1(n, F \cap Y_1) \cap (E \cap Y_1) = \emptyset$, and there is $m \in \mathbb{N}$ such that $U_2(m, F \cap Y_2) \cap (E \cap Y_2) = \emptyset$. If $l \geqslant \max\{n, m\}$, then we have $U_1(l, F \cap Y_1) \cap (E \cap Y_1) = \emptyset$ and $U_2(l, F \cap Y_2) \cap (E \cap Y_2) = \emptyset$. So $U(l, F) \cap E = \emptyset$. We can show that the function U satisfies the other conditions of a k-CSS function. Hence U is a k-CSS function for X. □

In 1986 ([5]) and in 1988 ([12]), Z. M. Gao and S. Lin gave a characterization for k-semistratifiable spaces by a g function. In what follows, we give a characterization for k-CSS spaces by a g function.

If X is a compact CSS space, then X is a compact semistratifiable space. Thus we have the following lemma.

Lemma 2.8. *If X is a compact* CSS *space, then X is a metrizable space.*

Lemma 2.9 ([1]). *Let $(X, \mathscr{T})$ be a space. The space $(X, \mathscr{T})$ is a* CSS *space if and only if there is a function $g\colon X \times \mathbb{N} \to \mathscr{T}$, such that*
(1) *$\bigcap\{g(x,n)\colon n \in \mathbb{N}\} = \{x\}$ and $g(x,n+1) \subseteq g(x,n)$ for each $n \in \mathbb{N}$;*
(2) *if a sequence $\{x_n\}_{n\in\mathbb{N}}$ of X converges to y, then $\bigcap\{g(x_n,n)\colon n \in \mathbb{N}\} \subseteq \{y\}$.*
This g function is called a CSS *g function for X.*

Theorem 2.10. *A topological space $(X, \mathscr{T})$ is* k-CSS *if and only if there is a function $g\colon X \times \mathbb{N} \to \mathscr{T}$ which satisfies the following conditions:*
(1) *for each $x \in X$, $\{x\} = \bigcap\{g(x,n)\colon n \in \mathbb{N}\}$ and $g(x,n+1) \subseteq g(x,n)$ for each $n \in \mathbb{N}$;*
(2) *if a sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to x and $y_n \in g(x_n,n)$ for each $n \in \mathbb{N}$, then any convergent subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ of $\{y_n\}_{n\in\mathbb{N}}$ converges to x.*

Proof. $\Rightarrow$: Let U be a k-CSS function for X. For each $x \in X$ and $n \in \mathbb{N}$, we define $g(x,n) = U(n,\{x\})$. Thus $g(x,n+1) \subseteq g(x,n)$ for each $n \in \mathbb{N}$ and $\{x\} = \bigcap\{g(x,n)\colon n \in \mathbb{N}\}$. We assume that the sequences $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ satisfy the condition (2) of the theorem. Let $\{y_{n_k}\}_{k\in\mathbb{N}}$ be a convergent subsequence of $\{y_n\}_{n\in\mathbb{N}}$. We will show that $\{y_{n_k}\}_{k\in\mathbb{N}} \to x$. Suppose $\{y_{n_k}\}_{k\in\mathbb{N}} \to y$ and $y \neq x$. Without loss of generality, we can assume that $y_{n_k} \neq x_m$ for each m and n_k. Since $\{x_{n_k}\}_{k\in\mathbb{N}} \to x$, we know that $K_1 = \{x_{n_k}\colon k \in \mathbb{N}\} \cup \{x\}$ and $K_2 = \{y_{n_k}\colon k \in \mathbb{N}\} \cup \{y\}$ are compact sets and $K_1 \cap K_2 = \emptyset$. Therefore there is $m \in \mathbb{N}$ such that $U(m,K_1) \cap K_2 = \emptyset$. For each $n_k > m$, we have $g(x_{n_k},n_k) = U(n_k,\{x_{n_k}\}) \subseteq U(m,\{x_{n_k}\}) \subseteq U(m,K_1)$. Since $y_{n_k} \in K_2$, we have $y_{n_k} \notin U(m,K_1)$ and hence we have $y_{n_k} \notin g(x_{n_k},n_k)$. This is a contradiction with $y_{n_k} \in g(x_{n_k},n_k)$. Hence $\{y_{n_k}\}_{k\in\mathbb{N}} \to x$.

$\Leftarrow$: Suppose X has a g function which satisfies the conditions (1) and (2).

We first show that X is a CSS space.

Let $\{x_n\}_{n\in\mathbb{N}}$ converge to x. If $z \in \bigcap\{g(x_n,n)\colon n \in \mathbb{N}\}$, then $z \in g(x_n,n)$ for each $n \in \mathbb{N}$. If $y_n = z$ for each $n \in \mathbb{N}$, then the sequence $\{y_n\}_{n\in\mathbb{N}}$ is convergent. By the condition (2), we have $\{y_n\}_{n\in\mathbb{N}} \to x$, so $z = x$. Thus $\bigcap\{g(x_n,n)\colon n \in \mathbb{N}\} \subseteq \{x\}$, and hence X is a CSS space by Lemma 2.9.

Let C be any non-empty compact subset of X. We set $U(n,C) = \bigcup\{g(x,n)\colon x \in C\}$. Let D be a compact subset of X and $C \cap D = \emptyset$. We will show that there is

$n \in \mathbb{N}$ such that $U(n, C) \cap D = \emptyset$. By Lemma 2.8 we have that C and D are two metrizable subspaces of X.

Suppose $U(n, C) \cap D \neq \emptyset$ for each $n \in \mathbb{N}$. Then there exists $x_n \in C$ such that $g(x_n, n) \cap D \neq \emptyset$ for each $n \in \mathbb{N}$. We choose a point y_n such that $y_n \in g(x_n, n) \cap D$ for each $n \in \mathbb{N}$. Since $\{x_n \colon n \in \mathbb{N}\} \subseteq C$ and C is metrizable, the sequence $\{x_n\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{x_{n_k}\}_{k\in\mathbb{N}}$. If we assume that $\{x_{n_k}\}_{k\in\mathbb{N}} \to x$, then $x \in C$. Without loss of generality, we assume that $\{n_k\}_{k\in\mathbb{N}}$ is an increasing sequence. For any $k \in \mathbb{N}$, if $n \in \mathbb{N}$ and $n_{k-1} \leqslant n < n_k (n_0 = 0)$, then we set $x_n^* = x_{n_k}$, and hence $g(x_n^*, n) = g(x_{n_k}, n)$ and $g(x_{n_k}, n_k) \subseteq g(x_{n_k}, n)$. If $n_{k-1} \leqslant n < n_k (n_0 = 0)$ then we set $y_n^* = y_{n_k}$, and hence $y_n^* = y_{n_k} \in g(x_{n_k}, n_k) \subseteq g(x_{n_k}, n) = g(x_n^*, n)$. The sequence $\{x_n^*\}_{n\in\mathbb{N}}$ converges to x. Since the compact subspace D is metrizable and $y_n^* \in D$ for each $n \in \mathbb{N}$, the sequence $\{y_n^*\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{y_{n_m}^*\}_{m\in\mathbb{N}}$. For each $n \in \mathbb{N}$, $y_n^* \in g(x_n^*, n)$, we know that the sequence $\{y_{n_m}^*\}_{m\in\mathbb{N}}$ converges to x. Since D is closed, we have $x \in D$. So $x \in C \cap D$, which contradicts $C \cap D = \emptyset$. Thus there exists $n \in \mathbb{N}$ such that $U(n, C) \cap D = \emptyset$. It is obvious that the function U satisfies the other conditions of a k-CSS function for X. Hence X is a k-CSS space. □

The g function which appears in Theorem 2.10 is called a k-CSS *g function* for X.

In [17] it is proved that the countable product of CSS spaces is a CSS space. The following theorem shows that k-CSS spaces also have the countable product property.

Theorem 2.11. *Let $X = \prod\limits_{n\in\mathbb{N}} X_n$. If X_n is a* k-CSS *space for each $n \in \mathbb{N}$, then X is a* k-CSS *space.*

P r o o f. For each $i \in \mathbb{N}$, since X_i is a k-CSS space, let g_i be a k-CSS function for X_i which satisfies the conditions of Theorem 2.10. For any $x = (x_i \colon i \in \mathbb{N}) \in X$ and $n \in \mathbb{N}$ we define $g(x, n) = \prod\limits_{i\leqslant n} g_i(x_i, n) \times \prod\limits_{i>n} X_i$. It is clear that $\{x\} = \bigcap\{g(x, n) \colon n \in \mathbb{N}\}$ and $g(x, n+1) \subseteq g(x, n)$ for each $n \in \mathbb{N}$. Let $x_n = (x_n^i \colon i \in \mathbb{N}) \in X$, $y_n = (y_n^i \colon i \in \mathbb{N}) \in g(x_n, n)$ for each $n \in \mathbb{N}$, and let the sequence $\{x_n\}_{n\in\mathbb{N}}$ converge to $z = (z_i \colon i \in \mathbb{N}) \in X$. Assume the sequence $\{y_n\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$. For each $i \in \mathbb{N}$ and each $n \geqslant i$ we have $y_n^i \in g_i(x_n^i, n)$. Since the sequence $\{x_n\}_{n\in\mathbb{N}}$ converges to z, the sequence $\{x_n^i\}_{n\in\mathbb{N}}$ converges to z_i. Since g_i is a k-CSS g function for X_i and the subsequence $\{y_{n_k}^i\}_{k\in\mathbb{N}}$ is convergent, so $\{y_{n_k}^i\}_{k\in\mathbb{N}}$ converges to z_i. So the sequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ converges to z. Hence X is a k-CSS space. □

In [1] it is proved that in the class of submetacompact spaces, locally CSS is equivalent to CSS. In what follows, we show that in the class of submesocompact spaces, locally k-CSS is equivalent to k-CSS.

Definition 2.12 ([14]). A space X is called a *submesocompact* space if for any open cover $\mathscr{U}$ of X there is a sequence of open covers $\{\mathscr{U}_n \colon n \in \mathbb{N}\}$ such that $\mathscr{U}_n$ is an open refinement of $\mathscr{U}$, i.e. $\mathscr{U}_n \prec \mathscr{U}$ for each $n \in \mathbb{N}$, and for each compact set C of X there is $n \in \mathbb{N}$ such that $|\{V \in \mathscr{U}_n \colon V \cap C \neq \emptyset\}| < \omega$.

Definition 2.13. A space X is called *locally* k-CSS if for each $x \in X$ there exists a neighborhood V_x of x such that V_x is a k-CSS subspace of X.

Theorem 2.14. *Let X be a submesocompact space. If X is a locally* k-CSS *space, then X is a* k-CSS *space.*

Proof. Let $\mathscr{W} = \{W(\alpha) \colon \alpha \in \Lambda\}$ be an open cover of X such that $W(\alpha)$ is a k-CSS subspace for each $\alpha \in \Lambda$, where Λ is a well ordered set. Let g_α be a k-CSS function for the subspace $W(\alpha)$ as described in Theorem 2.10. Let $\mathscr{C}$ be the collection of all compact subsets of X.

Since X is a submesocompact space, there is a sequence $\{\mathscr{U}_n \colon n \in \mathbb{N}\}$ of open covers of X such that $\mathscr{U}_n \prec \mathscr{W}$ for each $n \in \mathbb{N}$, and for any $C \in \mathscr{C}$ there exists $n \in \mathbb{N}$ such that $|\{U \colon U \in \mathscr{U}_n \text{ and } U \cap C \neq \emptyset\}| < \omega$.

For any $n \in \mathbb{N}, \alpha \in \Lambda$, we set $V(n,\alpha) = \bigcup\{U \colon U \in \mathscr{U}_n, U \subseteq W(\alpha), U \nsubseteq W(\beta)$ for each $\beta < \alpha\}$; then $V(n,\alpha) \subseteq W(\alpha)$.

For any $x \in X$ there is $U \in \mathscr{U}_n$ such that $x \in U$. If $\alpha = \min\{\beta \colon U \subseteq W(\beta)$ and $\beta \in \Lambda\}$, then $U \subseteq V(n,\alpha)$. Thus $x \in V(n,\alpha)$. So we have that if $\mathscr{V}_n = \{V(n,\alpha) \colon \alpha \in \Lambda\}$ then the collection $\mathscr{V}_n$ is an open cover of X for each n, and for any $C \in \mathscr{C}$ there exists $n \in \mathbb{N}$ such that $|\{V \colon V \in \mathscr{V}_n, V \cap C \neq \emptyset\}| < \omega$. For each $x \in X$ there is $n \in \mathbb{N}$ such that $\operatorname{ord}(x,\mathscr{V}_n) < \omega$, where $\operatorname{ord}(x,\mathscr{V}_n) = |\{V \colon V \in \mathscr{V}_n, x \in V\}|$. For each $x \in X$ and $n \in \mathbb{N}$, we define $A(x,n)$ as follows: If $\operatorname{ord}(x,\mathscr{V}_n) < \omega$, then set $A(x,n) = \{\alpha \in \Lambda \colon x \in V(n,\alpha)\}$; if $\operatorname{ord}(x,\mathscr{V}_n) \geqslant \omega$ then we choose any $\alpha \in \Lambda$ such that $x \in V(n,\alpha)$ and set $A(x,n) = \{\alpha\}$.

By the definition of $A(x,n)$, we know that it is a finite set. Now we set $g(x,n) = \bigcap\{g_\alpha(x,n) \cap (\bigcap\{V(j,\alpha) \colon j \leqslant n\}) \colon \alpha \in \bigcup\{A(x,i) \colon i \leqslant n\}\}$. Thus $g(x,n)$ is an open subset of X and $x \in g(x,n+1) \subseteq g(x,n)$ for each $n \in \mathbb{N}$.

If $\alpha(x) \in A(x,1)$, then $g(x,n) \subseteq g_{\alpha(x)}(x,n)$ for each $n \in \mathbb{N}$. Thus $x \in \bigcap\{g(x,n) \colon n \in \mathbb{N}\} \subseteq \bigcap\{g_{\alpha(x)}(x,n) \colon n \in \mathbb{N}\} = \{x\}$.

Let $\{x_n\}_{n\in\mathbb{N}}$ and $\{y_n\}_{n\in\mathbb{N}}$ be two sequences, such that $\{x_n\}_{n\in\mathbb{N}} \to x$ and $y_n \in g(x_n,n)$ for each $n \in \mathbb{N}$. Let $\{y_{n_k}\}_{k\in\mathbb{N}}$ be any convergent subsequence of $\{y_n\}_{n\in\mathbb{N}}$. In what follows, we will show that $\{y_{n_k}\}_{k\in\mathbb{N}} \to x$.

We assume $\{y_{n_k}\}_{k\in\mathbb{N}} \to y$. If $K_1 = \{y_{n_k} \colon k \in \mathbb{N}\} \cup \{y\}$ and $K_2 = \{x_n \colon n \in \mathbb{N}\} \cup \{x\}$, then $K = K_1 \cup K_2$ is compact.

Thus there is $l \in \mathbb{N}$ such that $|\{U \colon U \in \mathscr{U}_l, U \cap K \neq \emptyset\}| < \omega$, and hence $|\{V \colon V \in \mathscr{V}_l, V \cap K \neq \emptyset\}| < \omega$. So $\operatorname{ord}(x_n,\mathscr{V}_l) < \omega$ for each $n \in \mathbb{N}$.

Since $X = \bigcup \mathscr{V}_l$, there is $V(l,\alpha) \in \mathscr{V}_l$ such that $x \in V(l,\alpha)$. Since $\{x_n\}_{n\in\mathbb{N}} \to x$, there exists $n_1 \in \mathbb{N}$ such that $x_n \in V(l,\alpha)$ for each $n \geqslant n_1$. Since $\operatorname{ord}(x_n, \mathscr{V}_l) < \omega$ and $x_n \in V(l,\alpha)$ for $n \geqslant n_1$, we have $\alpha \in A(x_n, l)$ and hence $y_n \in g(x_n, n) \subseteq g_\alpha(x_n, n)$ if $n \geqslant n_1$. Since $\{x_n \colon n \geqslant n_1\} \to x$ and the sequence $\{y_n\}_{n\in\mathbb{N}}$ has a convergent subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$, and $\{x_n \colon n \geqslant n_1\} \subseteq V(l,\alpha) \subseteq W(\alpha)$, we know that $\{y_{n_k} \colon n_k \geqslant n_1\} \to x$ by the k-CSS property of $W(\alpha)$. So $y = x$, hence X is a k-CSS space. $\square$

Corollary 2.15. *If X is a paracompact locally* k-CSS *space, then X is a* k-CSS *space.*

Theorem 2.16. *If $X = \bigcup\{\operatorname{Int}(X_n) \colon n \in \mathbb{N}\}$ and X_n is a closed* k-CSS *space for each $n \in \mathbb{N}$, then X is a* k-CSS *space.*

P r o o f. By Theorem 2.7, we can assume that $X_m \subseteq X_{m+1}$ for each $m \in \mathbb{N}$. For each $m \in \mathbb{N}$, let U_m be a k-CSS function for X_m. For any compact set F of X and $m \in \mathbb{N}$, we have that $F \cap X_m = \bigcap\{U_m(n, F \cap X_m) \colon n \in \mathbb{N}\}$, where $U_m(n, F \cap X_m)$ is an open subset of X_m. Since X_m is closed in X, the set $U_m^*(n, F \cap X_m) = U_m(n, F \cap X_m) \cup (X \setminus X_m)$ is an open subset of X. We know that $F \subseteq U_m^*(n, F \cap X_m)$ for each $m \in \mathbb{N}$ and $n \in \mathbb{N}$.

For each $n \in \mathbb{N}$, we define $U(n, F) = \bigcap\{U_m^*(n, F \cap X_m) \colon m \leqslant n\}$. Thus $F \subseteq U(n, F)$ for each $n \in \mathbb{N}$ and $U(n, F)$ is an open subset of X. For any compact subset C of X, if $C \cap F = \emptyset$ then there is $m \in \mathbb{N}$ such that $C \subseteq \bigcup\{\operatorname{Int}(X_i) \colon i \leqslant m\} \subseteq X_m$. Thus there is $n \in \mathbb{N}$ such that $C \cap U_m(n, F \cap X_m) = \emptyset$. Thus $C \cap U_m^*(n, F \cap X_m) = \emptyset$. We can assume that $n > m$. Thus $C \cap (\bigcap\{U_k^*(n, F \cap X_k) \colon k \leqslant m\}) = \emptyset$, and hence $C \cap U(n, F) = \emptyset$. If F and E are two compact sets and $F \subseteq E$, we have $U(n, F) \subseteq U(n, E)$. Thus X is a k-CSS space. $\square$

Recall that a space X is a *σ-space* if X has a σ-discrete closed network. It is known that any stratifiable space is a σ-space, and more generally so is any k-semistratifiable space ([6]). Every σ-space is semistratifiable, and hence every σ-space is perfect.

In [11] it is pointed out that the space in Example 6.2 which appears in [11] is a Hausdorff CS space which is neither regular nor perfect. Such a space is CS but not semistratifiable. Thus such a space is not a σ-space.

Every Moore space is a semistratifiable space. So we have:

P r o b l e m 2.17. Is every Moore space a k-CSS space?

P r o b l e m 2.18. Is locally k-CSS equivalent to k-CSS in the class of submeta-compact spaces?

We know that c-stratifiable $\Rightarrow$ k-c-semistratifiable $\Rightarrow$ c-semistratifiable.

We have not found an appropriate example to show the first arrow cannot be reversed.

In [11] it is pointed out that the space in Example 6.3 which appears in [11] is a T_1 and CSS first countable space, but it is not CS. Such a space is not k-CSS by Theorem 2.3.

We have known that large classes of topological spaces are CSS.

Recall that a space X has a *quasi-$G_\delta(2)$-diagonal* provided there is a sequence $\{\mathscr{U}_n\colon n \in \mathbb{N}\}$ of collections of open sets with the property that, given distinct points $x, y \in X$, there is n with $x \in \mathrm{St}^2(x, \mathscr{U}_n) \subseteq X \setminus \{y\}$.

Proposition 19 (cf. [17]). *If X has a quasi-$G_\delta(2)$-diagonal, then X is a* CSS *space.*

Proposition 20 (cf. [16]). *The space $(X, \mathscr{T})$ is* CSS *provided one of the following conditions holds:*

(1) *X is a $\sigma^\sharp$-space, i.e., X has a σ-closure-preserving collection $\mathscr{C}$ of closed sets with the property that if $x \neq y$ are points of X, then there is $C \in \mathscr{C}$ such that $x \in C$ and $y \notin C$;*

(2) *X has a G^*_δ-diagonal.*

Problem 2.21. Let X be a space and let one of the following conditions hold:

(1) X has a quasi-$G_\delta(2)$-diagonal;

(2) X is a $\sigma^\sharp$-space;

(3) X has a G^*_δ-diagonal.

Is the space X k-CSS?

3. On strong β-spaces

A space X is called a *β-space* ([9]) if there is a function $g\colon X \times \mathbb{N} \to \mathscr{T}$, where $\mathscr{T}$ denotes the topology of X, satisfying

(1) $x \in g(x, n+1) \subseteq g(x, n)$ for each $x \in X$ and $n \in \mathbb{N}$;

(2) if $y \in \bigcap\limits_{n\in\mathbb{N}} g(x_n, n)$ for some $y \in X$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ has a cluster point.

The function g is called a *β function* for X.

A space $(X, \mathscr{T})$ is called a *strong β-space* ([19]) if there is a function $g\colon X \times \mathbb{N} \to \mathscr{T}$, satisfying

(1) $x \in g(x, n+1) \subseteq g(x, n)$ for each $x \in X$ and $n \in \mathbb{N}$;

(2) if $y \in \bigcap\limits_{n\in\mathbb{N}} g(x_n, n)$ for some $y \in X$, then $\{x_n\}_{n\in\mathbb{N}}$ has a compact closure.

The function g is called a *strong β function* for X.

Lemma 3.1. *If $X = X_1 \cup X_2$, where X_i is a closed strong β-subspace of X for $i \in \{1,2\}$, then X is a strong β-space.*

Proof. For each $i \in \{1,2\}$, the subspace X_i is a strong β-space, and hence there exists a function g_i described in the definition of the strong β-space. For each $x \in X_1 \setminus X_2$ and each $n \in \mathbb{N}$, let $g(x,n) = g_1(x,n) \setminus X_2$. Thus $x \in g(x,n)$ and $g(x,n)$ is an open subset of X for each $x \in X_1 \setminus X_2$ and $n \in \mathbb{N}$. Similarly, let $g(x,n) = g_2(x,n) \setminus X_1$ if $x \in X_2 \setminus X_1$.

If $x \in X_1 \cap X_2$ then let $g(x,n) = (g_1(x,n) \cup (X \setminus X_1)) \cap (g_2(x,n) \cup (X \setminus X_2))$ for each $n \in \mathbb{N}$. Thus $g(x,n)$ is an open subset of X for each $x \in X_1 \cap X_2$ and each $n \in \mathbb{N}$.

For each $x \in X$, if $x \in X_2 \setminus X_1$ then $x \notin g(y,n)$ for each $y \in X_1 \setminus X_2$ and each $n \in \mathbb{N}$. Similarly, we have that $x \notin g(y,n)$ for each $n \in \mathbb{N}$, if $x \in X_1 \setminus X_2$ and $y \in X_2 \setminus X_1$.

For each $x \in X$ and each $n \in \mathbb{N}$, we have that $x \in g(x, n+1) \subseteq g(x,n)$. If $y \in \bigcap\limits_{n\in\mathbb{N}} g(x_n, n)$ for some $y \in X$, then there is $i \in \{1,2\}$ such that $y \in X_i$. We assume $i = 1$. Thus $x_n \notin X_2 \setminus X_1$, and hence $x_n \in X_1$ for each $n \in \mathbb{N}$. Thus $y \in \bigcap\limits_{n\in\mathbb{N}} g_1(x_n, n)$. So the closure of the sequence $\{x_n\}_{n\in\mathbb{N}}$ is compact. Thus X is a strong β-space. $\square$

Theorem 3.2. *If $X = \bigcup\{X_n \colon n \in \mathbb{N}\}$ and X_n is a closed strong β-space for each $n \in \mathbb{N}$, then X is a strong β-space.*

Proof. For each $m \in \mathbb{N}$, we can assume that $X_m \subseteq X_{m+1}$ by Lemma 3.1. For each $m \in \mathbb{N}$, let g_m be a strong β function for X_m. For each $x \in X$, let $m(x) = \min\{m \colon m \in \mathbb{N} \text{ and } x \in X_m\}$. For each $n \in \mathbb{N}$, let $g(x,n) = (g_{m(x)}(x,n) \cup (X \setminus X_{m(x)})) \setminus X_{m(x)-1}$ if $n \leqslant m_x$; otherwise let

$$g(x,n) = \Bigg[\bigcap_{m(x)\leqslant k\leqslant n} (g_k(x,n) \cup (X \setminus X_k))\Bigg] \setminus X_{m(x)-1}.$$

Thus $g(x,n)$ is an open subset of X and $x \in g(x,(n+1)) \subseteq g(x,n)$ for each $x \in X$ and each $n \in \mathbb{N}$.

If $x \in \bigcap\limits_{n\in\mathbb{N}} g(x_n, n)$ for some $x \in X$, then $x_n \in X_{m(x)}$ for each $n \in \mathbb{N}$. If $n \geqslant m(x)$ then $g(x_n, n) \cap X_{m(x)} \subseteq g_{m(x)}(x,n)$. Thus $x \in \bigcap\{g_{m(x)}(x_n, n) \colon n \geqslant m(x)\}$. So the closure of $\{x_n \colon n \geqslant m(x)\}$ is compact. Thus X is a strong β-space. $\square$

We know that every closed subspace of a strong β-space is a strong β-space. So we have the following corollary.

Corollary 3.3. *If $(X, \mathscr{T})$ is a strong β-space, then every F_σ-subset of X is a strong β-subspace.*

In [7] the authors ask whether every F_σ-subset of a β-space is a β-subspace. This question is answered in [18]. A short description of the answer follows.

Lemma 3.4. *If $X = X_1 \cup X_2$, where X_i is a closed β-subspace of X for $i \in \{1, 2\}$, then X is a β-space.*

P r o o f. The proof is analogous to the proof of Lemma 3.1. □

Theorem 3.5 (cf. [18, Theorem 1])**.** *If $(X, \mathscr{T})$ is a β-space, then every F_σ-subset of X is a β-subspace.*

P r o o f. Let $U = \bigcup\{F_n \colon n \in \mathbb{N}\}$, where F_n is closed in X for each $n \in \mathbb{N}$. Thus F_n is a β-subspace for each n. By Lemma 3.4, we can assume that $F_n \subseteq F_{n+1}$ for each $n \in \mathbb{N}$. Let $g\colon X \times \mathbb{N} \to \mathscr{T}$ be a β function for X. For each $x \in U$, let $m(x)$ be the smallest index such that $x \in F_{m(x)}$. Thus $x \notin F_{m(x)-1}$, where $F_0 = \emptyset$. We define $g'(x, n) = (U \cap g(x, n)) \setminus F_{m(x)-1}$. If $x \in g'(y_n, n)$ for each $n \in \mathbb{N}$, then $y_n \in F_{m(x)}$ for each n. Thus $x \in g(y_n, n)$ for each $n \in \mathbb{N}$, and hence $\{y_n\}_{n\in\mathbb{N}}$ has a cluster point y in X. Since $y_n \in F_{m(x)}$ for each $n \in \mathbb{N}$ and $F_{m(x)}$ is closed in X, we have that $y \in F_{m(x)}$. Thus $y \in U$. So the function $g'\colon U \times \mathbb{N} \to \mathscr{T} \cap U$ is a β function for the subspace U, and hence U is a β-subspace. □

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