SOME INTEGRABILITY THEOREMS FOR MULTIPLE TRIGONOMETRIC SERIES

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Abstract. Several new integrability theorems are proved for multiple cosine or sine series.

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1. Introduction

Let $\mathbb{N}$ denote the set of all positive integers and let $\mathbb{N}_0 := \mathbb{N} \cup \{0\}$. We consider a double sequence $\{a_{j,k}\colon\ j,k \in \mathbb{N}_0\}$ of real numbers such that

$$\lim_{\max\{j,k\}\to\infty} a_{j,k} = 0 \qquad \text{and} \qquad \sum_{(j,k)\in\mathbb{N}_0^2} |\Delta_{\{1,2\}}(a_{j,k})| \text{ converges,}$$

where $\Delta_{\{1,2\}}(a_{j,k}) := a_{j,k} - a_{j+1,k} + a_{j+1,k+1} - a_{j,k+1}$, $j,k \in \mathbb{N}_0$. In [13] Móricz proved that if $(x,y) \in (0,\pi]^2$, then the double cosine series

$$\sum_{(j,k)\in\mathbb{N}_0^2} \lambda_j \lambda_k a_{j,k} \cos jx \cos ky, \quad \lambda_0 = 1/2;\ \lambda_j = 1,\ j \in \mathbb{N}$$

converges regularly in the sense of Hardy [3]. In addition, he proved that the following improper Riemann integral exists:

$$\lim_{\substack{(\varepsilon,\delta)\to(0,0)\\(\varepsilon,\delta)\in(0,\pi)^2}} \int_{[\varepsilon,\pi]\times[\delta,\pi]} \sum_{(j,k)\in\mathbb{N}_0^2} \lambda_j \lambda_k a_{j,k} \cos jx \cos ky \,\mathrm{d}(x,y). \tag{1}$$

In this paper we prove that the above result of Móricz is beyond the realm of Lebesgue integration; we establish the following integrability theorem for double cosine series:

Theorem 1.1. *Let* $\{a_{j,k}\colon\ (j,k)\in\mathbb{N}_0^2\}$ *be a double sequence of real numbers such that* $\lim_{\max\{j,k\}\to\infty} a_{j,k}=0$,

$$\Delta_{\{1,2\}}(a_{j,k})=0,\ (j,k)\in\mathbb{N}_0^2\setminus\{2^r\colon\ r\in\mathbb{N}\}^2$$

and

$$\sum_{(j,k)\in\mathbb{N}^2}\left\{|A_{j,k}|+\left\{\sum_{p=j}^{\infty}A_{p,k}^2\right\}^{\frac{1}{2}}+\left\{\sum_{q=k}^{\infty}A_{j,q}^2\right\}^{\frac{1}{2}}\right\}\ \textit{converges,}$$

where $A_{j,k}:=\Delta_{\{1,2\}}(a_{2^j,2^k})$, $(j,k)\in\mathbb{N}^2$.

Then $\sum_{(j,k)\in\mathbb{N}_0^2}\lambda_j\lambda_k a_{j,k}\cos jx\cos ky$ *is a double Fourier series if and only if*

$$\sum_{(j,k)\in\mathbb{N}^2}\left\{\sum_{p=j}^{\infty}\sum_{q=k}^{\infty}A_{p,q}^2\right\}^{\frac{1}{2}}\ \textit{converges.}$$

By modifying the proof of Theorem 1.1, we obtain the following result for double sine series.

Theorem 1.2. *Let* $\{b_{j,k}\colon\ j,k\in\mathbb{N}\}$ *be a double sequence of real numbers such that* $\lim_{\max\{j,k\}\to\infty} b_{j,k}=0$,

$$\Delta_{\{1,2\}}(b_{j,k})=0,\quad (j,k)\in\mathbb{N}^2\setminus\{2^r\colon\ r\in\mathbb{N}\}^2$$

and

$$\sum_{(j,k)\in\mathbb{N}^2}\left\{|B_{j,k}|+\left(\left(\sum_{p=j+1}^{\infty}B_{p,k}\right)^2+\sum_{p=j+1}^{\infty}B_{p,k}^2\right)^{\frac{1}{2}}+\left(\left(\sum_{q=k+1}^{\infty}B_{j,q}\right)^2+\sum_{q=k+1}^{\infty}B_{j,q}^2\right)^{\frac{1}{2}}\right\}$$

converges, where $B_{j,k}:=\Delta_{\{1,2\}}(b_{2^j,2^k})$, $(j,k)\in\mathbb{N}^2$.

Then $\sum_{(j,k)\in\mathbb{N}^2} b_{j,k}\sin jx\sin ky$ *is a double Fourier series if and only if*

$$\sum_{(j,k)\in\mathbb{N}^2}\left\{\sum_{p=j+1}^{\infty}\sum_{q=k+1}^{\infty}B_{p,q}^2+\left(\sum_{p=j+1}^{\infty}\sum_{q=k+1}^{\infty}B_{p,q}\right)^2\right.$$
$$\left.+\sum_{p=j+1}^{\infty}\left(\sum_{q=k+1}^{\infty}B_{p,q}\right)^2+\sum_{q=k+1}^{\infty}\left(\sum_{p=j+1}^{\infty}B_{p,q}\right)^2\right\}^{\frac{1}{2}}$$

converges.

This paper is organized as follows. In the next section we prove several integrability theorems for single cosine or sine series. In Section 3 we prove a higher-dimensional analogue of Theorem 1.1. In Section 4 we prove a higher-dimensional analogue of Theorem 1.2.

2. Some integrability theorems for single cosine or sine series

We begin with the following known result.

Theorem 2.1 (cf. [1, Theorem 5.27]). *Let $\{b_k\}_{k=1}^{\infty}$ be a sequence of real numbers such that $\sum\limits_{k=1}^{\infty}|b_k|$ converges. Then $\int\limits_0^{\pi}\Bigl|x^{-1}\sum\limits_{k=1}^{\infty}b_k\sin 2^k x\Bigr|\,\mathrm{d}x$ is finite if and only if $\sum\limits_{k=1}^{\infty}\Bigl\{\sum\limits_{j=k}^{\infty}b_j^2\Bigr\}^{\frac12}$ converges.*

We are now ready to state and prove an integrability theorem for one-dimensional cosine series.

Theorem 2.2. *Let $\{a_k\}_{k=0}^{\infty}$ be a null sequence of real numbers such that $a_k - a_{k+1} = 0$, $k\in\mathbb{N}\setminus\{2^r\colon r\in\mathbb{N}\}$ and $\sum\limits_{k=0}^{\infty}|a_k-a_{k+1}|$ converges. Then $\sum\limits_{k=0}^{\infty}\lambda_k a_k\cos kx$ is a Fourier series if and only if $\sum\limits_{k=1}^{\infty}\Bigl\{\sum\limits_{j=k}^{\infty}(a_{2^j}-a_{2^j+1})^2\Bigr\}^{\frac12}$ converges.*

P r o o f. Since a single summation by parts and our hypotheses yield

$$\sup_{x\in(0,\pi)}\Bigl|\sum_{k=0}^{\infty}\lambda_k a_k\cos kx-\sum_{k=0}^{\infty}(a_{2^k}-a_{2^k+1})\frac{\sin 2^k x}{2\tan\frac{x}{2}}\Bigr|\leqslant\frac12\sum_{k=0}^{\infty}|a_k-a_{k+1}|<\infty,$$

the result follows from Theorem 2.1. □

Let χ_A denote the characteristic function of a set A. The following example is an easy consequence of Theorem 2.2.

E x a m p l e 2.3. Let

$$a_0=a_1=\sum_{j=2}^{\infty}\frac{\chi_{\{2^r\colon r\in\mathbb{N}\}}(j)}{(\ln j)^{3/2}}\quad\text{and}\quad a_k=\sum_{j=k}^{\infty}\frac{\chi_{\{2^r\colon r\in\mathbb{N}\}}(j)}{(\ln j)^{3/2}},\quad k\in\mathbb{N}\setminus\{1\}.$$

Then $\sum\limits_{k=0}^{\infty}\lambda_k a_k\cos kx$ is not a Fourier series.

The next result involves absolutely convergent cosine series.

Theorem 2.4. *Let* $\{a_k\}_{k=1}^{\infty}$ *be a sequence of real numbers such that* $\sum\limits_{k=1}^{\infty} |a_k|$ *converges. Then* $\int\limits_0^{\pi} \Bigl| x^{-1} \sum\limits_{k=1}^{\infty} a_k (1 - \cos 2^k x) \Bigr| \, \mathrm{d}x$ *is finite if and only if*

$$\sum_{k=0}^{\infty} \biggl\{ \biggl(\sum_{j=k+1}^{\infty} a_j \biggr)^2 + \sum_{j=k+1}^{\infty} a_j^2 \biggr\}^{\frac{1}{2}}$$

converges.

Proof. ($\Longleftarrow$) Suppose that $\sum\limits_{k=0}^{\infty} \Bigl\{ \Bigl(\sum\limits_{j=k+1}^{\infty} a_j \Bigr)^2 + \sum\limits_{j=k+1}^{\infty} a_j^2 \Bigr\}^{\frac{1}{2}}$ converges. We write $C_j(x) = 1 - \cos 2^j x$, $j \in \mathbb{N}$ and select any $N \in \mathbb{N}$. Then

$$\text{(2)} \quad \int_{\pi/2^{N+1}}^{\pi} \biggl| \sum_{j=1}^{\infty} \frac{a_j C_j(x)}{x} \biggr| \, \mathrm{d}x \leqslant \sum_{k=0}^{N} \int_{\pi/2^{k+1}}^{\pi/2^k} \biggl| \sum_{j=1}^{k} \frac{a_j C_j(x)}{x} \biggr| \, \mathrm{d}x + \sum_{k=0}^{N} \int_{\pi/2^{k+1}}^{\pi/2^k} \biggl| \sum_{j=k+1}^{\infty} \frac{a_j C_j(x)}{x} \biggr| \, \mathrm{d}x.$$

The first term on the right-hand side of (2) is bounded above by $\pi \sum\limits_{j=1}^{\infty} |a_j|$:

$$\sum_{k=0}^{N} \int_{\pi/2^{k+1}}^{\pi/2^k} \biggl| \sum_{j=1}^{k} \frac{a_j C_j(x)}{x} \biggr| \, \mathrm{d}x \leqslant \pi \sum_{k=1}^{\infty} \sum_{j=1}^{k} \biggl| \frac{a_j}{2^{k+1}} \biggr| 2^j = \pi \sum_{j=1}^{\infty} |a_j|.$$

The second term on the right-hand side of (2) is bounded above by $\sum\limits_{k=0}^{\infty} \Bigl\{ \Bigl(\sum\limits_{j=k+1}^{\infty} a_j \Bigr)^2 + \sum\limits_{j=k+1}^{\infty} a_j^2 \Bigr\}^{\frac{1}{2}}$:

$$\begin{aligned}
\sum_{k=0}^{N} \int_{\pi/2^{k+1}}^{\pi/2^k} \biggl| \sum_{j=k+1}^{\infty} \frac{a_j C_j(x)}{x} \biggr| \, \mathrm{d}x
&= \sum_{k=0}^{N} \int_{\pi/2}^{\pi} \biggl| \sum_{j=k+1}^{\infty} \frac{a_j C_{j-k}(x)}{x} \biggr| \, \mathrm{d}x \\
&\leqslant \sum_{k=0}^{N} \biggl\{ \biggl\{ \int_{\pi/2}^{\pi} \biggl| \sum_{j=k+1}^{\infty} a_j C_{j-k}(x) \biggr|^2 \, \mathrm{d}x \biggr\} \biggr\}^{\frac{1}{2}} \biggl\{ \int_{\pi/2}^{\pi} x^{-2} \, \mathrm{d}x \biggr\}^{\frac{1}{2}} \\
&\leqslant \sum_{k=0}^{\infty} \biggl\{ \biggl(\sum_{j=k+1}^{\infty} a_j \biggr)^2 + \sum_{j=k+1}^{\infty} a_j^2 \biggr\}^{\frac{1}{2}}.
\end{aligned}$$

As N is arbitrary, the above inequalities imply that

$$\sup_{N\in\mathbb{N}} \int_{\pi/2^{N+1}}^{\pi} \left|\sum_{k=1}^{\infty} \frac{a_k C_k(x)}{x}\right| \mathrm{d}x \leqslant \pi \sum_{k=1}^{\infty} |a_k| + \sum_{k=0}^{\infty} \left\{ \left(\sum_{j=k+1}^{\infty} a_j \right)^2 + \sum_{j=k+1}^{\infty} a_j^2 \right\}^{\frac{1}{2}}$$

and so $\int\limits_0^{\pi} \left| \sum\limits_{k=1}^{\infty} a_k C_k(x)/x \right| \mathrm{d}x$ is finite.

($\Longrightarrow$) Conversely, suppose that $\int\limits_0^{\pi} \left| \sum\limits_{k=1}^{\infty} a_k C_k(x)/x \right| \mathrm{d}x$ is finite. Since $\sum\limits_{k=1}^{\infty} |a_k|$ is assumed to be convergent, it suffices to prove that there exists a positive constant C such that

$$\text{(3)}\quad \sup_{N\in\mathbb{N}} \sum_{k=0}^{N} \left\{ \left(\sum_{j=k+1}^{\infty} a_j \right)^2 + \sum_{j=k+1}^{\infty} a_j^2 \right\}^{\frac{1}{2}} \leqslant \pi C \left\{ \int_0^{\pi} \left| \sum_{j=1}^{\infty} \frac{a_j C_j(x)}{x} \right| \mathrm{d}x + \pi \sum_{k=1}^{\infty} |a_k| \right\}.$$

According to [14, Chapter V, (8.20) Theorem], there exists a positive constant C such that

$$\left\{ \int_{\pi/2}^{\pi} \left| \sum_{j=k+1}^{\infty} a_j C_{j-k}(x) \right|^2 \mathrm{d}x \right\}^{\frac{1}{2}} \leqslant C \int_{\pi/2}^{\pi} \left| \sum_{j=k+1}^{\infty} a_j C_{j-k}(x) \right| \mathrm{d}x, \quad k \in \mathbb{N}$$

and so (3) holds:

$$\begin{aligned}
\sum_{k=0}^{N} &\left\{ \left(\sum_{j=k+1}^{\infty} a_j \right)^2 + \sum_{j=k+1}^{\infty} a_j^2 \right\}^{\frac{1}{2}} \\
&= \sum_{k=0}^{N} \left\{ \int_{\pi/2}^{\pi} \left| \sum_{j=k+1}^{\infty} a_j C_{j-k}(x) \right|^2 \mathrm{d}x \right\}^{\frac{1}{2}} \\
&\leqslant C \sum_{k=0}^{N} \int_{\pi/2}^{\pi} \left| \sum_{j=k+1}^{\infty} a_j C_{j-k}(x) \right| \mathrm{d}x \\
&= C \sum_{k=0}^{N} \int_{\pi/2^{k+1}}^{\pi/2^k} 2^k \left| \sum_{j=k+1}^{\infty} a_j C_j(x) \right| \mathrm{d}x \\
&\leqslant \pi C \sum_{k=0}^{N} \int_{\pi/2^{k+1}}^{\pi/2^k} \left| \sum_{j=k+1}^{\infty} \frac{a_j C_j(x)}{x} \right| \mathrm{d}x \\
&\leqslant \pi C \left\{ \sum_{k=0}^{N} \int_{\pi/2^{k+1}}^{\pi/2^k} \left| \sum_{j=1}^{\infty} \frac{a_j C_j(x)}{x} \right| \mathrm{d}x + \sum_{k=0}^{N} \int_{\pi/2^{k+1}}^{\pi/2^k} \left| \sum_{j=1}^{k} \frac{a_j C_j(x)}{x} \right| \mathrm{d}x \right\} \\
&\leqslant \pi \left\{ C \int_0^{\pi} \left| \sum_{j=1}^{\infty} \frac{a_j C_j(x)}{x} \right| \mathrm{d}x + \pi \sum_{k=1}^{\infty} |a_k| \right\}.
\end{aligned}$$

We are now ready to state and prove an integrability theorem for single sine series.

Theorem 2.5. *Let* $\{b_k\}_{k=1}^{\infty}$ *be a null sequence of real numbers such that* $b_k - b_{k+1} = 0$, $k \in \mathbb{N} \setminus \{2^r\colon\ r \in \mathbb{N}\}$ *and* $\sum\limits_{k=1}^{\infty} |b_k - b_{k+1}|$ *converges. Then* $\sum\limits_{k=1}^{\infty} b_k \sin kx$ *is a Fourier series if and only if*

$$\sum_{k=0}^{\infty} \left\{ \sum_{j=k+1}^{\infty} (b_{2^j} - b_{2^j+1})^2 + \left(\sum_{j=k+1}^{\infty} (b_{2^j} - b_{2^j+1}) \right)^2 \right\}^{\frac{1}{2}}$$

converges.

Proof. Since a single summation by parts and our hypotheses yield

$$\sup_{x\in(0,\pi)} \left| \sum_{k=1}^{\infty} b_k \sin kx - \sum_{k=1}^{\infty} (b_{2^k} - b_{2^k+1}) \frac{1-\cos 2^k x}{2\tan\frac{x}{2}} \right| \leqslant \frac{1}{2} \sum_{k=1}^{\infty} |b_k - b_{k+1}| < \infty,$$

the result follows from Theorem 2.4. □

Example 2.6. Let $b_1 = 0$, let

$$b_k = \sum_{j=k}^{\infty} (-1)^{\ln j/\ln 2} \frac{\chi_{\{2^r\colon\ r\in\mathbb{N}\}}(j)}{(\ln j)^{3/2}}, \quad k \in \mathbb{N} \setminus \{1\}$$

and let $B_j = b_{2^j} - b_{2^j+1}$ for $j = 1, 2, \ldots$. Then $\{b_n\}_{n=1}^{\infty}$ is a null sequence of bounded variation with

$$\sum_{k=0}^{\infty} \left\{ \sum_{j=k+1}^{\infty} B_j^2 + \left(\sum_{j=k+1}^{\infty} B_j \right)^2 \right\}^{\frac{1}{2}} \geqslant \sum_{k=0}^{\infty} \left\{ \sum_{j=k+1}^{\infty} \frac{1}{j^3} \right\}^{\frac{1}{2}} = \infty.$$

An application of Theorem 2.5 shows that $\sum\limits_{k=1}^{\infty} b_k \sin kx$ is not a Fourier series.

The main aim of this section is to establish a higher-dimensional analogue of Theorem 2.2.

Let $m \geqslant 2$ be a fixed positive integer. For any two m-tuples $\boldsymbol{p} := (p_1, \dots, p_m)$, $\boldsymbol{q} := (q_1, \dots, q_m)$ belonging to $\mathbb{N}_0^m := \prod_{i=1}^{m} \mathbb{N}_0$, we write $\boldsymbol{p} \leqslant \boldsymbol{q}$ if and only if $p_i \leqslant q_i$ for $i = 1, \dots, m$. If $\{u_{\boldsymbol{k}} \colon \boldsymbol{k} \in \mathbb{N}_0^m\}$ is a multiple sequence of real numbers, we write

$$\sum_{\boldsymbol{p} \leqslant \boldsymbol{k} \leqslant \boldsymbol{q}} u_{\boldsymbol{k}} := \sum_{k_1=p_1}^{q_1} \dots \sum_{k_m=p_m}^{q_m} u_{\boldsymbol{k}}, \quad \boldsymbol{p}, \boldsymbol{q} \in \mathbb{N}_0^m$$

and

$$\sum_{\boldsymbol{k} \geqslant \boldsymbol{p}} u_{\boldsymbol{k}} := \sum_{k_1=p_1}^{\infty} \dots \sum_{k_m=p_m}^{\infty} u_{\boldsymbol{k}}, \quad \boldsymbol{p} \in \mathbb{N}_0^m,$$

where an empty sum is taken to be zero. We also write $W' = \{1, \dots, m\} \setminus W$ ($W \subseteq \{1, \dots, m\}$).

Definition 3.1 (cf. [9], [10], [5]). Let $\{u_{\boldsymbol{k}} \colon \boldsymbol{k} \in \mathbb{N}_0^m\}$ be a multiple sequence of real numbers. The multiple series $\sum_{\boldsymbol{k} \in \mathbb{N}_0^m} u_{\boldsymbol{k}}$ *converges regularly* if for each $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}_0$ such that

$$\Big| \sum_{\boldsymbol{p} \leqslant \boldsymbol{k} \leqslant \boldsymbol{q}} u_{\boldsymbol{k}} \Big| < \varepsilon$$

for every $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{N}_0^m$ satisfying $\boldsymbol{q} \geqslant \boldsymbol{p}$ and $\max\{p_1, \dots, p_m\} \geqslant N(\varepsilon)$.

For any multiple sequence $\{u_{\boldsymbol{k}} \colon \boldsymbol{k} \in \mathbb{N}_0^m\}$ of real numbers, we write $\Delta_{\emptyset}(u_{\boldsymbol{k}}) = u_{\boldsymbol{k}}$,

$$\Delta_{\{j\}}(u_{\boldsymbol{k}}) = u_{k_1, \dots, k_{j-1}, k_j, k_{j+1}, \dots k_m} - u_{k_1, \dots, k_{j-1}, k_j+1, k_{j+1}, \dots, k_m}, \ j = 1, \dots, m$$

and $\Delta_{\{j_1, \dots, j_s\}}(u_{\boldsymbol{k}}) := \Delta_{\{j_1\}} \dots \Delta_{\{j_s\}}(u_{\boldsymbol{k}})$, $\{j_1, \dots, j_s\} \subseteq \{1, \dots, m\}$.

Set $\|\boldsymbol{x}\| := \max_{k=1,\dots,m} |x_k|$ ($\boldsymbol{x} := (x_1, \dots, x_m) \in \mathbb{R}^m$). We are now ready to state the following important generalized Dirichlet test for multiple series.

Theorem 3.2 (cf. [5, Theorem 2.3]). *Let $\{c_{\boldsymbol{k}} \colon \boldsymbol{k} \in \mathbb{N}_0^m\}$ be a multiple sequence of real numbers such that $\lim_{\|\boldsymbol{n}\| \to \infty} c_{\boldsymbol{n}} = 0$. If $\{x_{\boldsymbol{k}} \colon \boldsymbol{k} \in \mathbb{N}_0^m\}$ is a multiple sequence of real numbers and if*

$$\sum_{\boldsymbol{k} \in \mathbb{N}_0^m} \{|\Delta_{\{1,\dots,m\}}(c_{\boldsymbol{k}})|\} \Big\{ \max_{\boldsymbol{0} \leqslant \boldsymbol{r} \leqslant \boldsymbol{k}} \Big| \sum_{\boldsymbol{0} \leqslant \boldsymbol{j} \leqslant \boldsymbol{r}} x_{\boldsymbol{j}} \Big| \Big\} \text{ converges,}$$

then $\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} \Delta_{\{1,\ldots,m\}}(c_{\boldsymbol{k}}) \sum_{\boldsymbol{0}\leqslant\boldsymbol{j}\leqslant\boldsymbol{k}} x_{\boldsymbol{j}}$ *converges absolutely,* $\sum_{\boldsymbol{k}\in\mathbb{N}_0^m c_{\boldsymbol{k}}} x_{\boldsymbol{k}}$ *converges regularly and*

$$\lim_{\min\{n_1,\ldots,n_m\}\to\infty} \left| \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{n}} c_{\boldsymbol{k}} x_{\boldsymbol{k}} - \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{n}} \Delta_{\{1,\ldots,m\}}(c_{\boldsymbol{k}}) \sum_{\boldsymbol{0}\leqslant\boldsymbol{j}\leqslant\boldsymbol{k}} x_{\boldsymbol{j}} \right| = 0.$$

In order to prove a higher-dimensional analogue of Theorem 2.1, we need the following result.

Lemma 3.3. *Let* $\{b_{\boldsymbol{j}}\colon \boldsymbol{j}\in\mathbb{N}^m\}$ *be a multiple sequence of real numbers such that* $\sum_{\boldsymbol{j}\in\mathbb{N}^m} |b_{\boldsymbol{j}}|$ *converges.*

(i) *If* $\boldsymbol{k}\in\mathbb{N}_0^m$, *then*

$$\int_{[\pi/2,\pi]^m} \left| \sum_{\boldsymbol{j}\geqslant\boldsymbol{k}+\boldsymbol{1}} b_{\boldsymbol{j}} \prod_{i=1}^m \sin 2^{j_i-k_i} x_i \right| \mathrm{d}\boldsymbol{x} \leqslant \pi^m \left\{ \sum_{\boldsymbol{j}\geqslant\boldsymbol{k}+\boldsymbol{1}} b_{\boldsymbol{j}}^2 \right\}^{\frac{1}{2}}.$$

(ii) *There exists a constant* C_m *(depending only on* m*) such that*

$$\int_{[\pi/2,\pi]^m} \left| \sum_{\boldsymbol{j}\geqslant\boldsymbol{k}+\boldsymbol{1}} b_{\boldsymbol{j}} \prod_{i=1}^m \sin 2^{j_i-k_i} x_i \right| \mathrm{d}\boldsymbol{x} \geqslant C_m \left\{ \sum_{\boldsymbol{j}\geqslant\boldsymbol{k}+\boldsymbol{1}} b_{\boldsymbol{j}}^2 \right\}^{\frac{1}{2}}$$

for every $\boldsymbol{k}\in\mathbb{N}_0^m$.

Proof. Part (i) is obvious. The proof of part (ii) is similar to that of [2, Theorem 3.7.4] and [11, Lemmas 1 and 2]. □

A two-dimensional analogue of Theorem 2.1 is given in [12]. The next theorem is a higher-dimensional analogue of Theorem 2.1.

Theorem 3.4. *Let* $\{b_{\boldsymbol{k}}\colon \boldsymbol{k}\in\mathbb{N}^m\}$ *be a multiple sequence of real numbers such that*

$$\sum_{\Gamma\subset\{1,\ldots,m\}} \sum_{\boldsymbol{k}\in\mathbb{N}^m} \left\{ \sum_{\substack{\boldsymbol{r}\in\mathbb{N}^m \\ r_i\geqslant k_i \forall i\in\Gamma \\ r_l=k_l \forall l\in\Gamma'}} b_{\boldsymbol{r}}^2 \right\}^{\frac{1}{2}} < \infty. \tag{4}$$

Then $\int_{[0,\pi]^m} \left| \sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i=1}^m \sin(2^{k_i} x_i)/x_i \right| \mathrm{d}\boldsymbol{x}$ *is finite if and only if*

$$\sum_{\boldsymbol{k}\in\mathbb{N}^m} \left\{ \sum_{\boldsymbol{r}\geqslant\boldsymbol{k}} b_{\boldsymbol{r}}^2 \right\}^{\frac{1}{2}} < \infty. \tag{5}$$

P r o o f. We write $S_j(x) = \sin 2^j(x)$, $j \in \mathbb{N}$. Clearly, it suffices to prove that

$$\sup_{\boldsymbol{N}\in\mathbb{N}_0^m} \int_{\prod_{i=1}^m [\pi/2^{N_i+1},\pi]} \Big| \sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i=1}^m \frac{S_{k_i}(x_i)}{x_i} \Big| \,\mathrm{d}\boldsymbol{x} < \infty \tag{6}$$

$$\iff \sum_{\boldsymbol{k}\in\mathbb{N}^m} \Big\{ \sum_{\boldsymbol{r}\geqslant k} b_{\boldsymbol{r}}^2 \Big\}^{\frac{1}{2}} < \infty.$$

For each $\boldsymbol{N} \in \mathbb{N}_0^m$ we have

$$\int_{\prod_{i=1}^m [\pi/2^{N_i+1},\pi]} \Big| \sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}} \prod_{i=1}^m x_i^{-1} S_{k_i}(x_i) \Big| \,\mathrm{d}\boldsymbol{x}$$

$$\leqslant \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} \int_{\prod_{i=1}^m [\pi/2^{k_i+1},\pi/2^{k_i}]} \Big| \sum_{\boldsymbol{1}\leqslant\boldsymbol{j}\leqslant\boldsymbol{k}} b_{\boldsymbol{j}} \prod_{i=1}^m x_i^{-1} S_{j_i}(x_i) \Big| \,\mathrm{d}\boldsymbol{x}$$

$$+ \sum_{\emptyset\neq\Gamma\subset\{1,\dots,m\}} \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} \int_{\prod_{i=1}^m [\pi/2^{k_i+1},\pi/2^{k_i}]} \Big| \sum_{\substack{j_i\geqslant k_i+1 \forall i\in\Gamma \\ 1\leqslant j_l\leqslant k_l \forall l\in\Gamma'}} b_{\boldsymbol{j}} \prod_{i=1}^m x_i^{-1} S_{j_i}(x_i) \Big| \,\mathrm{d}\boldsymbol{x}$$

$$+ \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} \int_{\prod_{i=1}^m [\pi/2^{k_i+1},\pi/2^{k_i}]} \Big| \sum_{\boldsymbol{j}\geqslant k+1} b_{\boldsymbol{j}} \prod_{i=1}^m x_i^{-1} S_{j_i}(x_i) \Big| \,\mathrm{d}\boldsymbol{x}$$

$$=: S_{\boldsymbol{N},\emptyset} + \sum_{\emptyset\neq\Gamma\subset\{1,\dots,m\}} S_{\boldsymbol{N},\Gamma} + S_{\boldsymbol{N},\{1,\dots,m\}}.$$

The sum $S_{\boldsymbol{N}},\emptyset$ is bounded above by $\pi^m \sum\limits_{\boldsymbol{k}\in\mathbb{N}^m} |b_{\boldsymbol{k}}|$:

$$S_{\boldsymbol{N},\emptyset} := \sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} \int_{\prod_{i=1}^m [\pi/2^{k_i+1},\pi/2^{k_i}]} \Big| \sum_{\boldsymbol{1}\leqslant\boldsymbol{j}\leqslant\boldsymbol{k}} b_{\boldsymbol{j}} \prod_{i=1}^m x_i^{-1} S_{j_i}(x_i) \Big| \,\mathrm{d}\boldsymbol{x} \tag{7}$$

$$\leqslant \sum_{\boldsymbol{1}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} \Big(\prod_{i=1}^m \frac{\pi}{2^{k_i+1}} \Big) \sum_{\boldsymbol{1}\leqslant\boldsymbol{j}\leqslant\boldsymbol{k}} \Big| b_j \prod_{i=1}^m 2^{j_i} \Big|$$

$$\leqslant \pi^m \sum_{\boldsymbol{k}\in\mathbb{N}^m} |b_{\boldsymbol{k}}|.$$

Next, we fix a non-empty set $\Gamma \subset \{1,\dots,m\}$. For each $\boldsymbol{k} \in \mathbb{N}_0^m$ we write

$$\Omega_1(\boldsymbol{k},\Gamma) = \{\boldsymbol{j} \in \mathbb{N}_0^m\colon\ j_i = k_i \ \forall\ i \in \Gamma,\ \text{and } 1 \leqslant j_l \leqslant k_l \ \forall\ l \in \Gamma'\},$$

$$\Omega_2(\boldsymbol{k},\Gamma) = \{\boldsymbol{r} \in \mathbb{N}_0^m\colon\ r_i \geqslant k_i + 1 \ \forall\ i \in \Gamma,\ \text{and } r_l = k_l \ \forall\ l \in \Gamma'\},$$

and apply a $\#(\Gamma)$-dimensional version of Lemma 3.3(i) to get

$$\begin{aligned}
&\int_{\prod_{i=1}^m[\pi/2^{k_i+1},\pi/2^{k_i}]}\Bigg|\sum_{\substack{j_i\geqslant k_i+1\forall i\in\Gamma\\ 1\leqslant j_l\leqslant k_l\forall l\in\Gamma'}} b_{\boldsymbol{j}}\prod_{i=1}^m x_i^{-1}S_{j_i}(x_i)\Bigg|\,\mathrm{d}\boldsymbol{x}\\
&\quad\leqslant\sum_{\boldsymbol{j}\in\Omega_1(\boldsymbol{k},\Gamma)}A_\Gamma(\boldsymbol{j})\int_{\prod_{i\in\Gamma}[\pi/2^{k_i+1},\pi/2^{k_i}]}\Bigg|\sum_{\boldsymbol{r}\in\Omega_2(\boldsymbol{j},\Gamma)} b_{\boldsymbol{r}}\prod_{i\in\Gamma}x_i^{-1}S_{r_i}(x_i)\Bigg|\prod_{i\in\Gamma}\mathrm{d}x_i\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\Big(\text{where } A_\Gamma(\boldsymbol{j}):=\prod_{l\in\Gamma'}\frac{\pi 2^{j_l}}{2^{k_l+1}}\Big)\\
&\quad\leqslant\sum_{\boldsymbol{j}\in\Omega_1(\boldsymbol{k},\Gamma)}A_\Gamma(\boldsymbol{j})\int_{\prod_{i\in\Gamma}[\pi/2,\pi]}\Bigg|\sum_{\boldsymbol{r}\in\Omega_2(\boldsymbol{j},\Gamma)} b_{\boldsymbol{r}}\prod_{i\in\Gamma}x_i^{-1}S_{r_i-k_i}(x_i)\Bigg|\prod_{i\in\Gamma}\mathrm{d}x_i\\
&\quad\leqslant\pi^m\sum_{\boldsymbol{j}\in\Omega_1(\boldsymbol{k},\Gamma)}A_\Gamma(\boldsymbol{j})\Bigg\{\sum_{\boldsymbol{r}\in\Omega_2(\boldsymbol{j},\Gamma)}b_{\boldsymbol{r}}^2\Bigg\}^{1/2}
\end{aligned}$$

and hence

$$\begin{aligned}
S_{\boldsymbol{N},\Gamma}&:=\sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}}\int_{\prod_{i=1}^m[\pi/2^{k_i+1},\pi/2^{k_i}]}\Bigg|\sum_{\substack{j_i\geqslant k_i+1\forall i\in\Gamma\\ 1\leqslant j_l\leqslant k_l\forall l\in\Gamma'}} b_{\boldsymbol{j}}\prod_{i=1}^m x_i^{-1}S_{j_i}(x_i)\Bigg|\,\mathrm{d}\boldsymbol{x}\\
&\leqslant\pi^m\sum_{\boldsymbol{k}\in\mathbb{N}^m}\Bigg\{\sum_{\substack{r_i\geqslant k_i\forall i\in\Gamma\\ r_l=k_l\forall l\in\Gamma'}}b_{\boldsymbol{r}}^2\Bigg\}^{\frac12}.
\end{aligned}\tag{8}$$

Since $\boldsymbol{N}\in\mathbb{N}_0^m$ is arbitrary, we infer from (4), (7) and (8) that

$$\sup_{\boldsymbol{N}\in\mathbb{N}_0^m}\Bigg\{S_{\boldsymbol{N},\emptyset}+\sum_{\emptyset\neq\Gamma\subset\{1,\dots,m\}}S_{\boldsymbol{N},\Gamma}\Bigg\}<\infty.\tag{9}$$

Finally, an application of Lemma 3.3 gives

$$\sup_{\boldsymbol{N}\in\mathbb{N}_0^m}S_{\boldsymbol{N},\{1,\dots,m\}}<\infty\iff\sum_{\boldsymbol{k}\in\mathbb{N}^m}\Bigg\{\sum_{\boldsymbol{r}\geqslant\boldsymbol{k}}b_{\boldsymbol{r}}^2\Bigg\}^{\frac12}<\infty\tag{10}$$

and so (6) follows from (9) and (10).

The following theorem is a higher-dimensional analogue of Theorem 2.2.

Theorem 3.5. *Let* $\{a_{\boldsymbol{k}}\colon \boldsymbol{k}\in\mathbb{N}_0^m\}$ *be a multiple sequence of real numbers such that* $\lim\limits_{\|\boldsymbol{n}\|\to\infty} a_{\boldsymbol{n}}=0$, $\Delta_{\{1,\dots,m\}}(a_{\boldsymbol{k}})=0$ $(\boldsymbol{k}\in\mathbb{N}_0^m\setminus\{2^r\colon r\in\mathbb{N}\}^m)$ *and*

$$\sum_{\Gamma\subset\{1,\dots,m\}}\sum_{\boldsymbol{k}\in\mathbb{N}^m}\Bigg\{\sum_{\substack{\boldsymbol{r}\in\mathbb{N}^m\\ r_i\geqslant k_i\forall i\in\Gamma\\ r_l=k_l\forall l\in\Gamma'}}(\Delta_{\{1,\dots,m\}}(a_{2^{r_1},\dots,2^{r_m}}))^2\Bigg\}^{\frac12}<\infty. \tag{11}$$

Then $\sum\limits_{\boldsymbol{k}\in\mathbb{N}_0^m} a_{\boldsymbol{k}}\prod\limits_{i=1}^m\lambda_{k_i}\cos k_ix_i$ *is a multiple Fourier series if and only if*

$$\sum_{\boldsymbol{k}\in\mathbb{N}^m}\Bigg\{\sum_{\boldsymbol{r}\geqslant\boldsymbol{k}}(\Delta_{\{1,\dots,m\}}(a_{2^{r_1},\dots,2^{r_m}}))^2\Bigg\}^{\frac12}<\infty.$$

Proof. Let $A_{\boldsymbol{k}}:=\Delta_{\{1,\dots,m\}}(a_{2^{k_1},\dots,2^{k_m}})$ $(\boldsymbol{k}\in\mathbb{N}^m)$. Then, for each $\boldsymbol{x}\in(0,\pi]^m$, an application of Theorem 3.2 yields

$$\begin{aligned}\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} a_{\boldsymbol{k}}\prod_{i=1}^m\lambda_{k_i}\cos k_ix_i&=\sum_{\boldsymbol{k}\in\mathbb{N}^m}A_{\boldsymbol{k}}\prod_{i=1}^m\frac12\Big(\frac{\sin 2^{k_i}x_i\cos\frac{x_i}{2}}{\sin\frac12x_i}+\cos 2^{k_i}x_i\Big)\\&=\sum_{\Gamma\subseteq\{1,\dots,m\}}\sum_{\boldsymbol{k}\in\mathbb{N}^m}A_{\boldsymbol{k}}\prod_{i\in\Gamma}\frac{\sin 2^{k_i}x_i\cos\frac12x_i}{2\sin\frac12x_i}\prod_{l\in\Gamma'}\frac{\cos 2^{k_l}x_l}{2}.\end{aligned}$$

It is now clear that the theorem is a consequence of Theorem 3.4, since the proof of (9) gives

$$\begin{aligned}&\max_{\Gamma\subset\{1,\dots,m\}}\int_{[0,\pi]^m}\Bigg|\sum_{\boldsymbol{k}\in\mathbb{N}^m}A_{\boldsymbol{k}}\prod_{i\in\Gamma}\frac{\sin 2^{k_i}x_i\cos\frac12x_i2}{\sin\frac12x_i}\prod_{l\in\Gamma'}\cos 2^{k_l}x_l\Bigg|\,\mathrm{d}\boldsymbol{x}\\&\leqslant\pi^m\max_{\Gamma\subset\{1,\dots,m\}}\sum_{\substack{k_i=1\forall i\in\Gamma\\ k_l\geqslant1\forall l\in\Gamma'}}\int_{[0,\pi]^{\operatorname{card}(\Gamma)}}\Bigg|\sum_{\substack{r_i\geqslant1\forall i\in\Gamma\\ r_l=k_l\forall l\in\Gamma'}}A_{\boldsymbol{r}}\prod_{i\in\Gamma}\frac{\sin 2^{r_i}x_i}{\tan\frac12x_i}\Bigg|\prod_{i\in\Gamma}\mathrm{d}x_i\\&\leqslant(2\pi^2)^m\max_{\Gamma\subset\{1,\dots,m\}}\sum_{\boldsymbol{k}\in\mathbb{N}^m}\Bigg\{\sum_{\substack{r_i\geqslant k_i\forall i\in\Gamma\\ r_l=k_l\forall l\in\Gamma'}}A_{\boldsymbol{r}}^2\Bigg\}^{\frac12}\end{aligned}$$

and (11) holds.

Example 3.6. Let $\{b_{j,k}\colon (j,k)\in\mathbb{N}_0^2\}$ be a double sequence of real numbers such that $b_{j,k}=0$, $(j,k)\in\mathbb{N}_0^2\setminus\{2^r\colon r\in\mathbb{N}\}^2$ and $b_{2^j,2^k}=(j+k)^{-3}$, $(j,k)\in\mathbb{N}^2$.

Then the double series $\sum\limits_{(j,k)\in\mathbb{N}_0^2}\Bigl(\sum\limits_{r=j}^{\infty}\sum\limits_{s=k}^{\infty}b_{r,s}\Bigr)\lambda_j\lambda_k\cos jx\cos ky$ converges regularly for every $(x,y)\in(0,\pi]^2$. However,

$$\sum_{(j,k)\in\mathbb{N}_0^2}\Bigl(\sum_{r=j}^{\infty}\sum_{s=k}^{\infty}b_{r,s}\Bigr)\lambda_j\lambda_k\cos jx\cos ky \tag{12}$$

is not a double Fourier series.

Proof. This is a consequence of Theorems 3.2 and 3.5. □

Remark 3.7. Let $\{b_{j,k}\colon (j,k)\in\mathbb{N}^2\}$ be given as in Example 3.6 and let

$$f(x,y)=\begin{cases}\sum\limits_{(j,k)\in\mathbb{N}_0^2}\Bigl(\sum\limits_{r=j}^{\infty}\sum\limits_{s=k}^{\infty}b_{r,s}\Bigr)\lambda_j\lambda_k\cos jx\cos ky & \text{if } (x,y)\in(0,\pi]^2,\\ 0 & \text{if } (x,y)\in[0,\pi]^2\setminus(0,\pi]^2.\end{cases}$$

An application of [8, Corollary 6.4] shows that the function f is Henstock-Kurzweil (i.e. Perron) integrable on $[0,\pi]^2$, and (12) is the Henstock-Kurzweil Fourier series of f.

More details on the Henstock-Kurzweil integral can be found in [6], [7] and references therein.

4. An integrability theorem for multiple sine series

The main aim of this section is to prove a higher-dimensional analogue of Theorem 2.4. We need

Lemma 4.1. *Let $\{a_{\boldsymbol{k}}\colon \boldsymbol{k}\in\mathbb{N}_0^m\}$ be a multiple sequence of real numbers such that $\sum\limits_{\boldsymbol{k}\in\mathbb{N}_0^m}|a_{\boldsymbol{k}}|$ converges. Then there exists a constant $\widetilde{C}_m$ (depending only on m) such that*

$$\int_{[\pi/2,\pi]^m}\Bigl|\sum_{\boldsymbol{j}\geqslant\boldsymbol{k}+\boldsymbol{1}}a_{\boldsymbol{j}}\prod_{i=1}^{m}\frac{1-\cos 2^{j_i-k_i}x_i}{x_i}\Bigr|\,\mathrm{d}\boldsymbol{x}$$
$$\geqslant\widetilde{C}_m\Bigl\{\int_{[\pi/2,\pi]^m}\Bigl|\sum_{\boldsymbol{j}\geqslant\boldsymbol{k}+\boldsymbol{1}}a_{\boldsymbol{j}}\prod_{i=1}^{m}\frac{1-\cos 2^{j_i-k_i}x_i}{x_i}\Bigr|^2\,\mathrm{d}\boldsymbol{x}\Bigr\}^{\frac12}$$

for every $\boldsymbol{k}\in\mathbb{N}_0^m$.

Proof. The proof is similar to that of [2, Theorem 3.7.4] and [11, Lemmas 1 and 2]. □

The following theorem is a higher-dimensional analogue of Theorem 2.4.

Theorem 4.2. *Let $\{a_{\boldsymbol{k}}\colon \boldsymbol{k}\in\mathbb{N}_0^m\}$ be a multiple sequence of real numbers such that*

$$(13)\qquad \sum_{\Gamma\subset\{1,\dots,m\}}\sum_{\boldsymbol{k}\in\mathbb{N}_0^m}\Bigg\{\sum_{S\subseteq\Gamma}\sum_{\substack{j_i\geqslant k_i+1\forall i\in S\\ j_l=k_l\forall l\in S'}}\Bigg(\sum_{\substack{r_i=j_i\forall i\in S\cup\Gamma'\\ r_l\geqslant j_l+1\forall l\in\Gamma\setminus S}} a_{\boldsymbol{r}}\Bigg)^2\Bigg\}^{\frac12}<\infty.$$

Then $\int_{[0,\pi]^m}\Big|\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} a_{\boldsymbol{k}}\prod_{i=1}^m(1-\cos 2^{k_i}x_i)/x_i\Big|\,\mathrm{d}\boldsymbol{x}$ *is finite if and only if*

$$\sum_{\boldsymbol{k}\in\mathbb{N}_0^m}\Bigg\{\sum_{S\subseteq\{1,\dots,m\}}\sum_{\substack{j_i\geqslant k_i+1\forall i\in S\\ j_l=k_l\forall l\in S'}}\Bigg(\sum_{\substack{r_i=j_i\forall i\in S\\ r_l\geqslant j_l+1\forall l\in S'}} a_{\boldsymbol{r}}\Bigg)^2\Bigg\}^{\frac12}<\infty.$$

P r o o f. Let $C_j(x):=1-\cos 2^j(x)$, $j\in\mathbb{N}_0$. For each $\boldsymbol{N}\in\mathbb{N}_0^m$ we have

$$\begin{aligned}
&\int_{\prod_{i=1}^m[\pi/2^{N_i+1},\pi]}\Big|\sum_{\boldsymbol{k}\in\mathbb{N}_0^m} a_{\boldsymbol{k}}\prod_{i=1}^m x_i^{-1}C_{k_i}(x_i)\Big|\,\mathrm{d}\boldsymbol{x}\\
&\quad\leqslant\sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}}\int_{\prod_{i=1}^m[\pi/2^{k_i+1},\pi/2^{k_i}]}\Big|\sum_{\boldsymbol{0}\leqslant\boldsymbol{j}\leqslant\boldsymbol{k}} a_{\boldsymbol{j}}\prod_{i=1}^m x_i^{-1}C_{j_i}(x_i)\Big|\,\mathrm{d}\boldsymbol{x}\\
&\qquad+\sum_{\emptyset\neq\Gamma\subset\{1,\dots,m\}}\sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}}\int_{\prod_{i=1}^m[\pi/2^{k_i+1},\pi/2^{k_i}]}\Big|\sum_{\substack{j_i\geqslant k_i+1\forall i\in\Gamma\\ 0\leqslant j_l\leqslant k_l\forall l\in\Gamma'}} a_{\boldsymbol{j}}\prod_{i=1}^m x_i^{-1}C_{j_i}(x_i)\Big|\,\mathrm{d}\boldsymbol{x}\\
&\qquad+\sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}}\int_{\prod_{i=1}^m[\pi/2^{k_i+1},\pi/2^{k_i}]}\Big|\sum_{\boldsymbol{j}\geqslant\boldsymbol{k}+\boldsymbol{1}} a_{\boldsymbol{j}}\prod_{i=1}^m x_i^{-1}C_{j_i}(x_i)\Big|\,\mathrm{d}\boldsymbol{x}\\
&\quad=:C_{\boldsymbol{N},\emptyset}+\sum_{\emptyset\neq\Gamma\subset\{1,\dots,m\}}C_{\boldsymbol{N},\Gamma}+C_{\boldsymbol{N},\{1,\dots,m\}}.
\end{aligned}$$

The rest of the proof is similar to that of Theorem 3.4. The term $C_{\boldsymbol{N},\emptyset}$ is bounded above by $\pi^m\sum_{\boldsymbol{k}\in\mathbb{N}_0^m}|a_{\boldsymbol{k}}|$:

$$(14)\qquad C_{\boldsymbol{N},\emptyset}\leqslant\sum_{\boldsymbol{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}}\Bigg(\prod_{i=1}^m\frac{\pi}{2^{k_i+1}}\Bigg)\sum_{\boldsymbol{0}\leqslant\boldsymbol{j}\leqslant\boldsymbol{k}}\Big|a_{\boldsymbol{j}}\prod_{i=1}^m 2^{j_i}\Big|\leqslant\pi^m\sum_{\boldsymbol{k}\in\mathbb{N}_0^m}|a_{\boldsymbol{k}}|.$$

Next, we fix a non-empty set $\Gamma\subset\{1,\dots,m\}$. For each $\boldsymbol{k}\in\mathbb{N}_0^m$ we follow the proof of Theorem 3.4 by setting

$$\Omega_2(\boldsymbol{k},\Gamma)=\{\boldsymbol{r}\in\mathbb{N}_0^m\colon\ r_i\geqslant k_i+1\ \forall\ i\in\Gamma,\ \text{and}\ r_l=k_l\ \forall\ l\in\Gamma'\}$$

and

$$\Omega_3(\boldsymbol{k},\Gamma)=\{\boldsymbol{j}\in\mathbb{N}_0^m:\ j_i=k_i\ \forall\ i\in\Gamma,\ \text{and } 0\leqslant j_l\leqslant k_l\ \forall\ l\in\Gamma'\}$$

to obtain

$$\begin{aligned}
&\int_{\prod_{i=1}^m[\pi/2^{k_i+1},\pi/2^{k_i}]}\Big|\sum_{\substack{j_i\geqslant k_i+1\forall i\in\Gamma\\ 0\leqslant j_l\leqslant k_l\forall l\in\Gamma'}} a_{\boldsymbol{j}}\prod_{i=1}^m x_i^{-1}C_{j_i}(x_i)\Big|\,\mathrm{d}\boldsymbol{x}\\
&\quad\leqslant\sum_{\boldsymbol{j}\in\Omega_3(\boldsymbol{k},\Gamma)}A_\Gamma(\boldsymbol{j})\int_{\prod_{i\in\Gamma}[\pi/2,\pi]}\Big|\sum_{\boldsymbol{r}\in\Omega_2(\boldsymbol{j},\Gamma)}a_{\boldsymbol{r}}\prod_{i\in\Gamma}x_i^{-1}C_{r_i-k_i}(x_i)\Big|\prod_{i\in\Gamma}\mathrm{d}x_i\\
&\qquad\qquad\qquad\qquad\qquad\qquad(\text{where } A_\Gamma(\boldsymbol{j}):=\prod_{l\in\Gamma'}\frac{\pi 2^{j_l}}{2^{k_l+1}})\\
&\quad\leqslant\sum_{\boldsymbol{j}\in\Omega_3(\boldsymbol{k},\Gamma)}A_\Gamma(\boldsymbol{j})\Big\{\int_{\prod_{i\in\Gamma}[\frac{\pi}{2},\pi]}\Big|\sum_{\boldsymbol{r}\in\Omega_2(\boldsymbol{j},\Gamma)}a_{\boldsymbol{r}}\prod_{i\in\Gamma}(C_{r_i-k_i}(x_i))\Big|^2\prod_{i\in\Gamma}\mathrm{d}x_i\Big\}^{\frac12}\\
&\quad=\sum_{\boldsymbol{j}\in\Omega_3(\boldsymbol{k},\Gamma)}A_\Gamma(\boldsymbol{j})\Big\{\int_{\prod_{i\in\Gamma}[\frac{\pi}{2},\pi]}\Big|\sum_{S\subseteq\Gamma}\sum_{\boldsymbol{r}\in\Omega_2(\boldsymbol{j},\Gamma)}a_{\boldsymbol{r}}\prod_{i\in S}(C_{r_i-k_i}(x_i)-1)\Big|^2\prod_{i\in\Gamma}\mathrm{d}x_i\Big\}^{\frac12}\\
&\quad\leqslant\sum_{\boldsymbol{j}\in\Omega_3(\boldsymbol{k},\Gamma)}A_\Gamma(\boldsymbol{j})\Big\{\sum_{S\subseteq\Gamma}\int_{\prod_{i\in\Gamma}[\pi/2,\pi]}\Big|\sum_{\boldsymbol{r}\in\Omega_2(\boldsymbol{j},\Gamma)}a_{\boldsymbol{r}}\prod_{i\in S}(-\cos 2^{r_i-k_i}x_i)\Big|^2\prod_{i\in\Gamma}\mathrm{d}x_i\Big\}^{\frac12}\\
&\quad\leqslant\pi^m\sum_{\boldsymbol{j}\in\Omega_3(\boldsymbol{k},\Gamma)}A_\Gamma(\boldsymbol{j})\Big\{\sum_{S\subseteq\Gamma}\sum_{\substack{r_i\geqslant j_i+1\forall i\in S\\ r_l=j_l\forall l\in S'}}\Big(\sum_{\substack{q_i=r_i\forall i\in S\cup\Gamma'\\ q_l\geqslant r_l+1\forall l\in\Gamma\setminus S}}a_{\boldsymbol{q}}\Big)^2\Big\}^{\frac12}.
\end{aligned}$$

Therefore

$$C_{\boldsymbol{N},\Gamma}\leqslant\pi^m\sum_{\boldsymbol{k}\in\mathbb{N}_0^m}\Big\{\sum_{S\subseteq\Gamma}\sum_{\substack{j_i\geqslant k_i+1\forall i\in S\\ j_l=k_l\forall l\in S'}}\Big(\sum_{\substack{r_i=j_i\forall i\in S\cup\Gamma'\\ r_l\geqslant j_l+1\forall l\in\Gamma\setminus S}}a_{\boldsymbol{r}}\Big)^2\Big\}^{\frac12}. \tag{15}$$

Since $\boldsymbol{N}\in\mathbb{N}_0^m$ is arbitrary, it follows from (13), (14) and (15) that

$$\sup_{\boldsymbol{N}\in\mathbb{N}_0^m}\Big\{C_{\boldsymbol{N},\emptyset}+\sum_{\emptyset\neq\Gamma\subset\{1,\dots,m\}}C_{\boldsymbol{N}},\Gamma\Big\}<\infty. \tag{16}$$

In view of (16), it remains to prove that

$$\begin{aligned}
&\sup_{\boldsymbol{N}\in\mathbb{N}_0^m}C_{\boldsymbol{N},\{1,\dots,m\}}<\infty\\
&\quad\Longleftrightarrow\sum_{\boldsymbol{k}\in\mathbb{N}_0^m}\Big\{\sum_{S\subseteq\{1,\dots,m\}}\sum_{\substack{j_i\geqslant k_i+1\forall i\in S\\ j_l=k_l\forall l\in S'}}\Big(\sum_{\substack{r_i=j_i\forall i\in S\\ r_l\geqslant j_l+1\forall l\in S'}}a_{\boldsymbol{r}}\Big)^2\Big\}^{\frac12}<\infty.
\end{aligned} \tag{17}$$

But it is clear that (17) holds, since $\boldsymbol{N} \in \mathbb{N}_0^m$ implies

$$\begin{aligned}
&\sum_{\mathbf{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} \int_{\prod_{i=1}^m [\pi/2^{k_i+1},\pi/2^{k_i}]} \left| \sum_{\boldsymbol{j}\geqslant\boldsymbol{k}+1} a_{\boldsymbol{j}} \prod_{i=1}^m x_i^{-1} C_{j_i}(x_i) \right| \mathrm{d}\boldsymbol{x} \\
&\quad\leqslant \sum_{\mathbf{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} \int_{\prod_{i=1}^m [\frac{\pi}{2},\pi]} \left| \sum_{\boldsymbol{j}\geqslant\boldsymbol{k}+1} a_{\boldsymbol{j}} \prod_{i=1}^m x_i^{-1} C_{j_i-k_i}(x_i) \right| \mathrm{d}\boldsymbol{x} \\
&\quad\leqslant \pi^m \sum_{\boldsymbol{k}\in\mathbb{N}_0^m} \left\{ \int_{\prod_{i=1}^m [\pi/2,\pi]} \left| \sum_{\boldsymbol{j}\geqslant\boldsymbol{k}+1} a_{\boldsymbol{j}} \prod_{i=1}^m C_{j_i-k_i}(x_i) \right|^2 \mathrm{d}\boldsymbol{x} \right\}^{\frac{1}{2}} \\
&\quad= \pi^m \sum_{\boldsymbol{k}\in\mathbb{N}_0^m} \left\{ \int_{\prod_{i=1}^m [\pi/2,\pi]} \left| \sum_{S\subseteq\{1,\dots,m\}} \sum_{\boldsymbol{j}\geqslant\boldsymbol{k}+\mathbf{1}} a_{\boldsymbol{j}} \prod_{i\in S} (C_{j_i-k_i}(x_i)-1) \right|^2 \mathrm{d}\boldsymbol{x} \right\}^{\frac{1}{2}} \\
&\quad= \pi^m \sum_{\boldsymbol{k}\in\mathbb{N}_0^m} \left\{ \sum_{S\subseteq\{1,\dots,m\}} \int_{\prod_{i=1}^m [\pi/2,\pi]} \left| \sum_{\boldsymbol{j}\geqslant\boldsymbol{k}+\mathbf{1}} a_{\boldsymbol{j}} \prod_{i\in S} (C_{j_i-k_i}(x_i)-1) \right|^2 \mathrm{d}\boldsymbol{x} \right\}^{\frac{1}{2}} \\
&\quad= \pi^m \sum_{\boldsymbol{k}\in\mathbb{N}_0^m} \left\{ \sum_{S\subseteq\{1,\dots,m\}} \sum_{\substack{j_i\geqslant k_i+1\forall i\in S \\ j_l=k_l\forall l\in S'}} \left(\sum_{\substack{r_i=j_i\forall i\in S \\ r_l\geqslant j_l+1\forall l\in S'}} a_{\boldsymbol{r}} \right)^2 \right\}^{\frac{1}{2}}
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{\mathbf{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} \int_{\prod_{i=1}^m [\pi/2^{k_i+1},\pi/2^{k_i}]} \left| \sum_{\boldsymbol{j}\geqslant\boldsymbol{k}+1} a_{\boldsymbol{j}} \prod_{i=1}^m x_i^{-1} C_{j_i}(x_i) \right| \mathrm{d}\boldsymbol{x} \\
&\quad= \sum_{\mathbf{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} \int_{\prod_{i=1}^m [\pi/2,\pi]} \left| \sum_{\boldsymbol{j}\geqslant\boldsymbol{k}+\mathbf{1}} a_{\boldsymbol{j}} \prod_{i=1}^m x_i^{-1} C_{j_i-k_i}(x_i) \right| \mathrm{d}\boldsymbol{x} \\
&\quad\geqslant \widetilde{C}_m \sum_{\mathbf{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} \left\{ \int_{\prod_{i=1}^m [\pi/2,\pi]} \left| \sum_{\boldsymbol{j}\geqslant\boldsymbol{k}+\mathbf{1}} a_{\boldsymbol{j}} \prod_{i=1}^m C_{j_i-k_i}(x_i) \right|^2 \mathrm{d}\boldsymbol{x} \right\}^{\frac{1}{2}} \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(by Lemma 4.1)} \\
&\quad\geqslant \widetilde{C}_m \sum_{\mathbf{0}\leqslant\boldsymbol{k}\leqslant\boldsymbol{N}} \left\{ \sum_{S\subseteq\{1,\dots,m\}} \sum_{\substack{j_i\geqslant k_i+1\forall i\in S \\ j_l=k_l\forall l\in S'}} \left(\sum_{\substack{r_i=j_i\forall i\in S \\ r_l\geqslant j_l+1\forall l\in S'}} a_{\boldsymbol{r}} \right)^2 \right\}^{\frac{1}{2}}.
\end{aligned}$$

The next theorem is a higher-dimensional analogue of Theorem 2.5.

Theorem 4.3. *Let $\{b_{\boldsymbol{k}}\colon \boldsymbol{k}\in\mathbb{N}^m\}$ be a multiple sequence of real numbers such that* $\lim\limits_{\|\boldsymbol{n}\|\to\infty} b_{\boldsymbol{n}}=0$, $\Delta_{\{1,\dots,m\}}(b_{\boldsymbol{k}})=0$ $(\boldsymbol{k}\in\mathbb{N}^m\setminus\{2^r\colon r\in\mathbb{N}\}^m)$ *and*

$$\sum_{\Gamma\subset\{1,\dots,m\}}\sum_{\boldsymbol{k}\in\mathbb{N}^m}\Biggl\{\sum_{S\subseteq\Gamma}\sum_{\substack{j_i\geqslant k_i+1\,\forall i\in S\\ j_l=k_l\,\forall l\in S'}}\Biggl(\sum_{\substack{r_i=j_i\,\forall i\in S\cup\Gamma'\\ r_l\geqslant j_l+1\,\forall l\in\Gamma\setminus S}}B_{\boldsymbol{r}}\Biggr)^2\Biggr\}^{\frac12}<\infty,$$

where $B_{\boldsymbol{r}}:=\Delta_{\{1,\dots,m\}}(b_{2^{r_1},\dots,2^{r_m}})$ $(\boldsymbol{r}\in\mathbb{N}^m)$. *Then* $\int_{[0,\pi]^m}\Bigl|\sum\limits_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}}\prod\limits_{i=1}^m\sin k_ix_i\Bigr|\,\mathrm{d}\boldsymbol{x}$ *is finite if and only if*

$$\sum_{\boldsymbol{k}\in\mathbb{N}^m}\Biggl\{\sum_{S\subseteq\{1,\dots,m\}}\sum_{\substack{j_i\geqslant k_i+1\,\forall i\in S\\ j_l=k_l\,\forall l\in S'}}\Biggl(\sum_{\substack{r_i=j_i\,\forall i\in S\\ r_l\geqslant j_l+1\,\forall l\in S'}}B_{\boldsymbol{r}}\Biggr)^2\Biggr\}^{\frac12}<\infty.$$

P r o o f. For each $j\in\mathbb{N}$ we set $C_j(x):=1-\cos 2^jx$ and $S_j(x):=\sin 2^jx$. Then, for each $\boldsymbol{x}\in(0,\pi)^m$, we deduce from Theorem 3.2 that

$$\begin{aligned}&\sum_{\boldsymbol{k}\in\mathbb{N}^m} b_{\boldsymbol{k}}\prod_{i=1}^m\sin k_ix_i\\
&=\sum_{\boldsymbol{k}\in\mathbb{N}^m}B_{\boldsymbol{k}}\prod_{i=1}^m\Bigl(\frac{C_{k_i}(x_i)}{2\tan\frac12x_i}+S_{k_i}(x_i)\Bigr)\\
&=\sum_{\boldsymbol{k}\in\mathbb{N}^m}B_{\boldsymbol{k}}\prod_{i=1}^m\frac{C_{k_i}(x_i)}{2\tan\frac12x_i}+\sum_{\Gamma\subset\{1,\dots,m\}}\sum_{\boldsymbol{k}\in\mathbb{N}^m}B_{\boldsymbol{k}}\prod_{i\in\Gamma}\frac{C_{k_i}(x_i)}{2\tan\frac12x_i}\prod_{i\in\Gamma'}S_{k_i}(x_i).\end{aligned}$$

Next, we follow the proof of (16) to find a constant D_m (depending only on m) such that

$$\begin{aligned}&\max_{\Gamma\subset\{1,\dots,m\}}\int_{[0,\pi]^m}\Bigl|\sum_{\boldsymbol{k}\in\mathbb{N}^m}B_{\boldsymbol{k}}\Bigl(\prod_{i\in\Gamma}\frac{C_{k_i}(x_i)}{2\tan\frac12x_i}\Bigr)\prod_{i\in\Gamma'}S_{k_i}(x_i)\Bigr|\,\mathrm{d}\boldsymbol{x}\\
&\quad\leqslant\max_{\Gamma\subset\{1,\dots,m\}}D_m\sum_{\boldsymbol{k}\in\mathbb{N}^m}\Biggl\{\sum_{S\subseteq\Gamma}\sum_{\substack{j_i\geqslant k_i+1\,\forall i\in S\\ j_l=k_l\,\forall l\in S'}}\Biggl(\sum_{\substack{r_i=j_i\,\forall i\in S\cup\Gamma'\\ r_l\geqslant j_l+1\,\forall l\in\Gamma\setminus S}}B_{\boldsymbol{k}}\Biggr)^2\Biggr\}^{\frac12}<\infty.\end{aligned}$$

An application of Theorem 4.2 gives the result.

E x a m p l e 4.4. Let

$$a_{j,k}=\begin{cases}\dfrac{(-1)^{r+s}}{(r+s)^3} & \text{if } (j,k)\in\{(2^r,2^s)\colon (r,s)\in\mathbb{N}^2\},\\ 0 & \text{otherwise,}\end{cases}$$

and let $b_{j,k}=\sum\limits_{r=j}^{\infty}\sum\limits_{s=k}^{\infty}a_{r,s}$, $(j,k)\in\mathbb{N}^2$. Then the double series $\sum\limits_{(j,k)\in\mathbb{N}^2} b_{j,k}\sin jx\sin ky$ converges regularly on $[0,\pi]^2$. However, it is not a double Fourier series.

P r o o f. The first assertion is a consequence of Theorem 3.2. The other assertion follows from Theorem 4.3, since $\lim_{\max\{j,k\}\to\infty} b_{j,k}=0$,

$$\sum_{(j,k)\in\mathbb{N}^2}\left\{a_{j,k}^2+\left(\sum_{p=j+1}^{\infty}a_{p,k}\right)^2+\sum_{p=j+1}^{\infty}a_{p,k}^2+\left(\sum_{q=k+1}^{\infty}a_{j,q}\right)^2+\sum_{q=k+1}^{\infty}a_{j,q}^2\right\}^{\frac12}$$
$$\leqslant\sum_{(j,k)\in\mathbb{N}^2}\left\{\frac{2}{(j+k)^6}+\sum_{p=j+1}^{\infty}\frac{1}{(p+k)^6}+\frac{2}{(j+k)^6}\right\}^{\frac12}<\infty$$

and

$$\sum_{(j,k)\in\mathbb{N}^2}\left\{\sum_{p=j+1}^{\infty}\sum_{q=k+1}^{\infty}a_{p,q}^2+\left(\sum_{p=j+1}^{\infty}\sum_{q=k+1}^{\infty}a_{p,q}\right)^2\right.$$
$$\left.+\sum_{p=j+1}^{\infty}\left(\sum_{q=k+1}^{\infty}a_{p,q}\right)^2+\sum_{q=k+1}^{\infty}\left(\sum_{p=j+1}^{\infty}a_{p,q}\right)^2\right\}^{\frac12}$$
$$\geqslant\sum_{(j,k)\in\mathbb{N}^2}\left\{\sum_{p=j+1}^{\infty}\sum_{q=k+1}^{\infty}a_{p,q}^2\right\}^{\frac12}=\infty.$$

R e m a r k 4.5. Let $\{b_{j,k}\colon (j,k)\in\mathbb{N}^2\}$ be given as in Example 4.4. k An application of [8, Theorem 8.1] shows that the function $(x,y)\mapsto\sum\limits_{(j,k)\in\mathbb{N}^2} b_{j,k}\sin jx\sin ky$ is Henstock-Kurzweil (i.e. Perron) integrable on $[0,\pi]^2$. Furthermore, we deduce from [13, Theorem 2] that $\sum\limits_{(j,k)\in\mathbb{N}^2} b_{j,k}\sin jx\sin ky$ is the Henstock-Kurzweil Fourier series of the function $(x,y)\mapsto\sum\limits_{(j,k)\in\mathbb{N}^2} b_{j,k}\sin jx\sin ky$.

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