SOLVABILITY CONDITIONS OF THE CAUCHY PROBLEM FOR TWO-DIMENSIONAL SYSTEMS OF LINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS WITH MONOTONE OPERATORS

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Abstract. We establish new efficient conditions sufficient for the unique solvability of the initial value problem for two-dimensional systems of linear functional differential equations with monotone operators.

Keywords: system of functional differential equations with monotone operators, initial value problem, solvability

MSC 2000: 34K06, 34K10

1. Introduction and notation

On the interval $[a,b]$ we consider the two-dimensional differential system

$$u_i'(t) = \sigma_{i1} l_{i1}(u_1)(t) + \sigma_{i2} l_{i2}(u_2)(t) + q_i(t) \qquad (i = 1,2) \tag{1.1}$$

with the initial conditions

$$u_1(a) = c_1, \qquad u_2(a) = c_2, \tag{1.2}$$

where $l_{ik}\colon C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ are linear nondecreasing operators, $\sigma_{ik} \in \{-1,1\}$, $q_i \in L([a,b];\mathbb{R})$, and $c_i \in \mathbb{R}$ $(i,k = 1,2)$. Under a solution of the problem (1.1), (1.2) we understand an absolutely continuous vector function

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$u = (u_1, u_2)^T \colon [a,b] \to \mathbb{R}^2$ satisfying (1.1) almost everywhere on $[a,b]$ and verifying also the initial conditions (1.2).

The problem of solvability of the Cauchy problem for linear functional differential equations and their systems has been studied by many authors (see, e.g., [1], [2], [3], [4], [5], [6], [9], [10], [11], [12], [13], [14], [15], [17] and references therein). At present, there are not but a few efficient conditions guaranteeing the unique solvability of the initial value problem for n-dimensional systems of functional differential equations, and most of them are available for the one-dimensional case only or for the systems with the so-called Volterra operators (see, e.g., [3], [4], [5], [12], [9], [6]). Let us mention that some efficient conditions guaranteeing the unique solvability of the problem studied can be found, e.g., in [11], [2], [14], [13], [10].

In this paper we establish new efficient conditions sufficient for the unique solvability of the problem (1.1), (1.2) with $\sigma_{11} = -1$ and $\sigma_{22} = -1$. The cases when $\sigma_{11} = \sigma_{22} = 1$ and $\sigma_{11}\sigma_{22} = -1$ are studied in [8] and [16], respectively.

The paper is based on the Fredholm property of the problem considered. The assumptions of Theorems 2.1–2.3 guarantee that the homogeneous problem corresponding to (1.1), (1.2) has only the trivial solution. To prove this, suitable a priori estimates are found for the maximal and minimal values of a possible nontrivial solution of the homogeneous problem indicated.

The integral conditions given in Theorems 2.1–2.3 are optimal in a certain sense; this is shown by counter-examples constructed in the last part of the paper.

The following notation is used throughout the paper:

(1) $\mathbb{R}$ is the set of all real numbers, $\mathbb{R}_+ = [0, +\infty[$.

(2) $C([a,b];\mathbb{R})$ is the Banach space of continuous functions $u \colon [a,b] \to \mathbb{R}$ equipped with the norm

$$\|u\|_C = \max\{|u(t)| \colon t \in [a,b]\}.$$

(3) $L([a,b];\mathbb{R})$ is the Banach space of Lebesgue integrable functions $h \colon [a,b] \to \mathbb{R}$ equipped with the norm

$$\|h\|_L = \int_a^b |h(s)|\,\mathrm{d}s.$$

(4) $L\big([a,b];\mathbb{R}_+\big) = \{h \in L([a,b];\mathbb{R}) \colon h(t) \geqslant 0 \text{ for a.a. } t \in [a,b]\}$.

(5) An operator $l \colon C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$ is said to be nondecreasing if the inequality

$$l(u_1)(t) \leqslant l(u_2)(t) \quad \text{for a.a.} \quad t \in [a,b]$$

holds for every functions $u_1, u_2 \in C([a,b];\mathbb{R})$ such that

$$u_1(t) \leqslant u_2(t) \quad \text{for } t \in [a,b].$$

(6) $\mathcal{P}_{ab}$ is the set of linear nondecreasing operators $l\colon C([a,b];\mathbb{R}) \to L([a,b];\mathbb{R})$.

In what follows, the equalities and inequalities with integrable functions are understood to hold almost everywhere.

2. Main results

In this section we present the main results of the paper. The proofs are given later, in Section 3. Theorems formulated below contain efficient conditions sufficient for the unique solvability of the problem (1.1), (1.2) with $\sigma_{11} = -1$ and $\sigma_{22} = -1$. Recall that the operators l_{ik} are supposed to be linear and nondecreasing, i.e., such that $l_{ik} \in \mathcal{P}_{ab}$ for $i,k = 1,2$.

Put

$$A_{ik} = \int_a^b l_{ik}(1)(s)\,\mathrm{d}s \quad \text{for } i,k = 1,2 \tag{2.1}$$

and

$$\varphi(s) = \begin{cases} 1 & \text{for } s \in [0,1[, \\ 1 - \frac{1}{4}(s-1)^2 & \text{for } s \in [1,3[. \end{cases} \tag{2.2}$$

First we consider the case when $\sigma_{12}\sigma_{21} > 0$.

Theorem 2.1. *Let $\sigma_{11} = -1$, $\sigma_{22} = -1$ and $\sigma_{12}\sigma_{21} > 0$. Let, moreover,*

$$A_{11} < 3, \qquad A_{22} < 3, \tag{2.3}$$

and

$$A_{12}A_{21} < \frac{1}{\omega}\varphi(A_{11})\varphi(A_{22}), \tag{2.4}$$

where

$$\omega = \max\{1, A_{11}(A_{22}-1), A_{22}(A_{11}-1)\}, \tag{2.5}$$

the numbers A_{ik} $(i,k = 1,2)$ are defined by (2.1) *and the function φ is given by* (2.2). *Then the problem* (1.1), (1.2) *has a unique solution.*

Remark 2.1. The strict inequalities (2.3) cannot be replaced by the nonstrict ones (see Example 4.1). Furthermore, the strict inequality (2.4) cannot be replaced by the nonstrict one provided $\omega = 1$ (see Examples 4.2–4.4).

R e m a r k 2.2. Let H_1 be the set of triplets $(x,y,z) \in \mathbb{R}^3_+$ satisfying

$$x < 3, \quad y < 3, \quad z < \frac{\varphi(x)\varphi(y)}{\max\bigl\{1, x(y-1), y(x-1)\bigr\}}$$

(see Fig. 2.1).

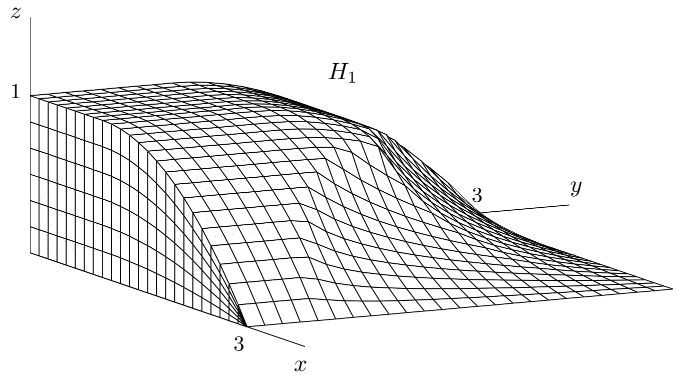

Fig. 2.1.

According to Theorem 2.1, the problem (1.1), (1.2) is uniquely solvable if $l_{ik} \in \mathcal{P}_{ab}$ $(i,k=1,2)$ are such that

$$\biggl(\int_a^b l_{11}(1)(s)\,\mathrm{d}s, \int_a^b l_{22}(1)(s)\,\mathrm{d}s, \int_a^b l_{12}(1)(s)\,\mathrm{d}s \int_a^b l_{21}(1)(s)\,\mathrm{d}s\biggr) \in H_1.$$

Now we consider the case when $\sigma_{12}\sigma_{21} < 0$.

Theorem 2.2. *Let $\sigma_{11} = -1$, $\sigma_{22} = -1$ and $\sigma_{12}\sigma_{21} < 0$. Let, moreover, the condition* (2.3) *be satisfied and*

$$A_{12}A_{21} < \frac{1}{\varrho}(3 - \max\{A_{11}, A_{22}\})\varphi(\min\{A_{11}, A_{22}\}), \tag{2.6}$$

where

$$\varrho = \max\{1, 3(A_{11}-1), 3(A_{22}-1)\}, \tag{2.7}$$

the numbers A_{ik} $(i,k=1,2)$ are defined by (2.1) *and the function φ is given by* (2.2). *Then the problem* (1.1), (1.2) *has a unique solution.*

R e m a r k 2.3. The strict inequalities (2.3) cannot be replaced by the nonstrict ones (see Example 4.1). Futhermore, the strict inequality (2.6) cannot be replaced by the nonstrict one provided $\varrho = 1$ and $\max\{A_{11}, A_{22}\} > 1$ (see Examples 4.5 and 4.6).

R e m a r k 2.4. Let H_2 be the set of triplets $(x, y, z) \in \mathbb{R}_+^3$ satisfying

$$x < 3, \quad y < 3, \quad z < \frac{\bigl(3 - \max\{x, y\}\bigr)\varphi\bigl(\min\{x, y\}\bigr)}{\max\bigl\{1, 3(x-1), 3(y-1)\bigr\}}$$

(see Fig. 2.2).

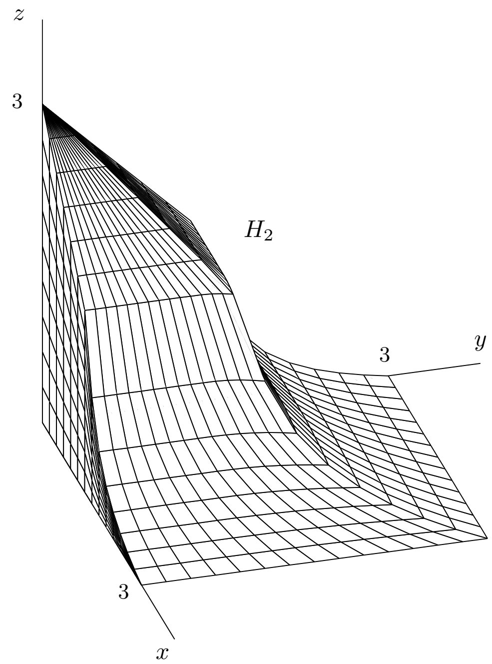

Fig. 2.2.

According to Theorem 2.2, the problem (1.1), (1.2) is uniquely solvable if $l_{ik} \in \mathcal{P}_{ab}$ $(i, k = 1, 2)$ are such that

$$\left(\int_a^b l_{11}(1)(s)\,\mathrm{d}s, \int_a^b l_{22}(1)(s)\,\mathrm{d}s, \int_a^b l_{12}(1)(s)\,\mathrm{d}s \int_a^b l_{21}(1)(s)\,\mathrm{d}s \right) \in H_2.$$

If $\max\{A_{11}, A_{22}\} \leqslant 1$ then the assumption (2.6) of Theorem 2.2 can be improved. For example, the next theorem improves Theorem 2.2 whenever $\max\{A_{11}, A_{22}\}$ is close to zero.

Theorem 2.3. *Let* $\sigma_{11} = -1$, $\sigma_{22} = -1$ *and* $\sigma_{12}\sigma_{21} < 0$. *Let, moreover,*

$$A_{11} < 1, \qquad A_{22} < 1, \tag{2.8}$$

and

$$A_{12}A_{21} < \frac{\lambda}{\lambda\big(A_{11} + A_{22} - A_{11}A_{22}\big) + A_{11}A_{22} + 1} \tag{2.9}$$

where

$$\lambda = 4 + \big(\sqrt{1 - A_{11}} + \sqrt{1 - A_{22}}\big)^2 \tag{2.10}$$

and the numbers A_{ij} $(i, j = 1, 2)$ *are defined by* (2.1). *Then the problem* (1.1), (1.2) *has a unique solution.*

Remark 2.5. If $A_{11} = A_{22} = 0$ then the inequality (2.9) can be rewritten as

$$A_{12}A_{21} < 8,$$

which coincides with the assumptions of Theorem 2.2 in [8].

Remark 2.6. Let H_3 be the set of triplets $(x, y, z) \in \mathbb{R}^3_+$ satisfying

$$x < 1, \quad y < 1, \quad z < \frac{\lambda_0}{\lambda_0(x + y - xy) + xy + 1},$$

where

$$\lambda_0 = 4 + \big(\sqrt{1 - x} + \sqrt{1 - y}\big)^2$$

(see Fig. 2.3).

According to Theorem 2.3, the problem (1.1), (1.2) is uniquely solvable if $l_{ik} \in \mathcal{P}_{ab}$ $(i, k = 1, 2)$ are such that

$$\left(\int_a^b l_{11}(1)(s)\,\mathrm{d}s, \int_a^b l_{22}(1)(s)\,\mathrm{d}s, \int_a^b l_{12}(1)(s)\,\mathrm{d}s \int_a^b l_{21}(1)(s)\,\mathrm{d}s \right) \in H_3.$$

3. Proofs of the main results

In this section we shall prove the statements formulated above. Recall that the numbers A_{ij} $(i, j = 1, 2)$ are defined by (2.1) and the function φ is given by (2.2).

It is well-known from the general theory of boundary value problems for functional differential equations (see, e.g., [11], [7], [10], [15]) that the following lemma is true.

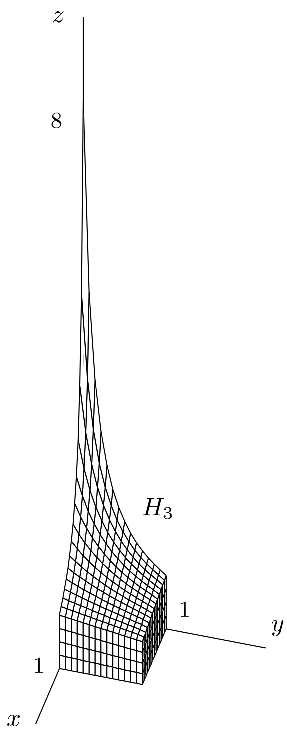

Fig. 2.3.

Lemma 3.1. *The problem* (1.1), (1.2) *is uniquely solvable if and only if the corresponding homogeneous problem*

$$u_i'(t) = \sigma_{i1} l_{i1}(u_1)(t) + \sigma_{i2} l_{i2}(u_2)(t) \qquad (i = 1, 2), \tag{3.1}$$

$$u_1(a) = 0, \qquad u_2(a) = 0 \tag{3.2}$$

has only the trivial solution.

In order to simplify the discussion in the proofs below, we formulate the following obvious lemma.

Lemma 3.2. $(u_1, u_2)^T$ *is a solution of the problem* (3.1), (3.2) *if and only if* $(u_1, -u_2)^T$ *is a solution of the problem*

$$v_i'(t) = (-1)^{i-1} \sigma_{i1} l_{i1}(v_1)(t) + (-1)^i \sigma_{i2} l_{i2}(v_2)(t) \qquad (i = 1, 2), \tag{3.3}$$

$$v_1(a) = 0, \qquad v_2(a) = 0. \tag{3.4}$$

Lemma 3.3. *Let the function* φ *be defined by* (2.2). *Then, for any* $0 \leqslant x \leqslant y < 3$, *the inequality*

$$(3 - y)\varphi(x) \leqslant (3 - x)\varphi(y) \tag{3.5}$$

is satisfied.

Proof. Let $0 \leqslant x \leqslant y < 3$ be arbitrary but fixed. It is clear that one of the following cases occurs:

(a) $0 \leqslant x \leqslant y \leqslant 1$ holds. Then

$$\big(3 - y\big)\varphi(x) = 3 - y \leqslant 3 - x = (3 - x)\varphi(y).$$

(b) $0 \leqslant x \leqslant 1$ and $1 < y < 3$. Then we have

$$3 - y \leqslant 2[1 - \tfrac{1}{4}(y - 1)^2].$$

Consequently,

$$\big(3 - y\big)\varphi(x) = 3 - y \leqslant 2[1 - \tfrac{1}{4}(y - 1)^2] \leqslant (3 - x)\varphi(y).$$

(c) $1 < x \leqslant y < 3$ is true. Then we obtain

$$\begin{aligned}\big(3 - y\big)\big[4 - (x - 1)^2\big] &= \big(3 - y\big)\big[2 + (x - 1)\big]\big[2 - (x - 1)\big] \\ &= (3 - y)(1 + x)(3 - x) \leqslant (3 - x)(1 + y)(3 - y) \\ &= \big(3 - x\big)\big[2 + (y - 1)\big]\big[2 - (y - 1)\big] = \big(3 - x\big)\big[4 - (y - 1)^2\big],\end{aligned}$$

i.e., the inequality (3.5) holds.

□

Now we are in position to prove Theorems 2.1–2.3.

Proof of Theorem 2.1. According to Lemmas 3.1 and 3.2, in order to prove the theorem it is sufficient to show that the system

$$u_1'(t) = -l_{11}(u_1)(t) + l_{12}(u_2)(t), \tag{3.6}$$

$$u_2'(t) = l_{21}(u_1)(t) - l_{22}(u_2)(t) \tag{3.7}$$

has only the trivial solution satisfying (3.2).

Suppose that, on the contrary, $(u_1, u_2)^T$ is a nontrivial solution of the problem (3.6), (3.7), (3.2). For $i = 1, 2$, we put

$$M_i = \max\big\{u_i(t)\colon\ t \in [a, b]\big\}, \quad m_i = -\min\big\{u_i(t)\colon\ t \in [a, b]\big\}. \tag{3.8}$$

Choose $\alpha_i, \beta_i \in [a,b]$ $(i=1,2)$ such that the equalities

$$u_1(\alpha_1) = M_1, \qquad u_1(\beta_1) = -m_1 \tag{3.9}$$

and

$$u_2(\alpha_2) = M_2, \qquad u_2(\beta_2) = -m_2 \tag{3.10}$$

are satisfied. Obviously, (3.2) guarantees

$$M_i \geqslant 0, \quad m_i \geqslant 0 \quad \text{for } i = 1,2.$$

Furthermore, for $i,k = 1,2$, we denote

$$B_{ik} = \int_a^{\min\{\alpha_i,\beta_i\}} l_{ik}(1)(s)\,\mathrm{d}s, \qquad D_{ik} = \int_{\min\{\alpha_i,\beta_i\}}^{\max\{\alpha_i,\beta_i\}} l_{ik}(1)(s)\,\mathrm{d}s. \tag{3.11}$$

It is clear that

$$B_{ik} + D_{ik} \leqslant A_{ik} \quad \text{for } i,k = 1,2. \tag{3.12}$$

For the sake of clarity we shall divide the discussion into the following cases.

(a) Neither of the functions u_1 and u_2 changes its sign and $u_1(t)u_2(t) \geqslant 0$ holds for $t \in [a,b]$. Then, without loss of generality, we can assume that

$$u_1(t) \geqslant 0, \quad u_2(t) \geqslant 0 \quad \text{for } t \in [a,b].$$

(b) Neither of the functions u_1 and u_2 changes its sign and $u_1(t)u_2(t) \leqslant 0$ holds for $t \in [a,b]$. Then, without loss of generality, we can assume that

$$u_1(t) \geqslant 0, \quad u_2(t) \leqslant 0 \quad \text{for } t \in [a,b].$$

(c) One of the functions u_1 and u_2 is of a constant sign and the other one changes its sign. Then, without loss of generality, we can assume that $u_1(t) \geqslant 0$ for $t \in [a,b]$.

(d) Both functions u_1 and u_2 change their signs. Then, without loss of generality, we can assume that $\alpha_1 < \beta_1$. Obviously, one of the following conditions is satisfied.

  (d1) $\beta_2 < \alpha_2$ and $D_{ii} \geqslant 1$ for some $i \in \{1,2\}$.

  (d2) $\beta_2 < \alpha_2$ and $D_{ii} < 1$ for $i = 1,2$.

(d3) $\beta_2 > \alpha_2$ and $D_{ii} \geqslant 1$ for some $i \in \{1,2\}$.

(d4) $\beta_2 > \alpha_2$ and $D_{ii} < 1$ for $i = 1,2$.

First we note that the function φ satisfies

$$\varphi(A_{ii}) \leqslant 1 - B_{ii}(D_{ii} - 1) \quad \text{for } i = 1,2. \tag{3.13}$$

Case (a): $u_1(t) \geqslant 0$ and $u_2(t) \geqslant 0$ for $t \in [a,b]$. Obviuously,

$$M_1 \geqslant 0, \quad M_2 \geqslant 0, \quad M_1 + M_2 > 0. \tag{3.14}$$

The integrations of (3.6) and (3.7) from a to α_1 and from a to α_2, respectively, in view of (3.8), (3.9), (3.10) and the assumptions $l_{ik} \in \mathcal{P}_{ab}$, result in

$$\begin{aligned} M_i &= (-1)^i \int_a^{\alpha_i} l_{i1}(u_1)(s)\,\mathrm{d}s + (-1)^{i-1} \int_a^{\alpha_i} l_{i2}(u_2)(s)\,\mathrm{d}s \\ &\leqslant M_{3-i} \int_a^{\alpha_i} l_{i3-i}(1)(s)\,\mathrm{d}s \leqslant M_{3-i} A_{i3-i} \quad (i = 1,2). \end{aligned} \tag{3.15}$$

By virtue of (3.14), (3.15) implies $M_1 > 0$, $M_2 > 0$ and $A_{12}A_{21} \geqslant 1$, which contradicts (2.4), because $\omega \geqslant 1$ and $0 < \varphi(A_{ii}) \leqslant 1$ for $i = 1,2$.

Case (b): $u_1(t) \geqslant 0$ and $u_2(t) \leqslant 0$ for $t \in [a,b]$. In view of the assumptions $l_{ij} \in \mathcal{P}_{ab}$ $(i,j = 1,2)$, (3.6) and (3.7) imply $u_1'(t) \leqslant 0$ for $t \in [a,b]$ and $u_2'(t) \geqslant 0$ for $t \in [a,b]$, respectively. Consequently, $u_1 \equiv 0$ and $u_2 \equiv 0$, which is a contradiction.

Case (c): $u_1(t) \geqslant 0$ for $t \in [a,b]$ and u_2 changes its sign. Obviously, $m_1 = 0$ and

$$M_2 > 0, \qquad m_2 > 0. \tag{3.16}$$

Suppose that $\beta_2 < \alpha_2$ (the case when $\beta_2 > \alpha_2$ can be proved analogously). The integration of (3.6) from a to α_1, on account of (3.8), (3.9) and the assumptions $l_{11}, l_{12} \in \mathcal{P}_{ab}$, yields

$$\begin{aligned} M_1 &= -\int_a^{\alpha_1} l_{11}(u_1)(s)\,\mathrm{d}s + \int_a^{\alpha_1} l_{12}(u_2)(s)\,\mathrm{d}s \\ &\leqslant M_2 \int_a^{\alpha_1} l_{12}(1)(s)\,\mathrm{d}s \leqslant M_2 A_{12}. \end{aligned} \tag{3.17}$$

On the other hand, the integrations of (3.7) from a to β_2 and from β_2 to α_2, in view of (3.8), (3.10), (3.11) and the assumptions $l_{21}, l_{22} \in \mathcal{P}_{ab}$, result in

$$\begin{aligned} m_2 &= -\int_a^{\beta_2} l_{21}(u_1)(s)\,\mathrm{d}s + \int_a^{\beta_2} l_{22}(u_2)(s)\,\mathrm{d}s \\ &\leqslant M_2 \int_a^{\beta_2} l_{22}(1)(s)\,\mathrm{d}s = M_2 B_{22} \end{aligned} \tag{3.18}$$

and

$$\begin{aligned}(3.19)\quad M_2+m_2&=\int_{\beta_2}^{\alpha_2} l_{21}(u_1)(s)\,\mathrm{d}s-\int_{\beta_2}^{\alpha_2} l_{22}(u_2)(s)\,\mathrm{d}s\\&\leqslant M_1\int_{\beta_2}^{\alpha_2} l_{21}(1)(s)\,\mathrm{d}s+m_2\int_{\beta_2}^{\alpha_2} l_{22}(1)(s)\,\mathrm{d}s=M_1D_{21}+m_2D_{22},\end{aligned}$$

respectively.

It follows from (3.17) and (3.19) that

$$M_2\leqslant M_2A_{12}A_{21}+m_2(D_{22}-1). \tag{3.20}$$

Hence, by virtue of (2.4) and (3.16), (3.20) implies

$$0<M_2(1-A_{12}A_{21})\leqslant m_2(D_{22}-1). \tag{3.21}$$

Using (3.13), the relations (3.18) and (3.21) result in

$$\varphi(A_{22})\leqslant 1-B_{22}(D_{22}-1)\leqslant A_{12}A_{21},$$

which contradicts (2.4), because $\omega\geqslant 1$ and $0<\varphi(A_{11})\leqslant 1$.

Case (d): u_1 and u_2 change their signs and $\alpha_1<\beta_1$. Obviously,

$$M_i>0,\quad m_i>0\quad\text{for } i=1,2. \tag{3.22}$$

The integrations of (3.6) from a to α_1 and from α_1 to β_1, in view of (3.8), (3.9), (3.11) and the assumptions $l_{11},l_{12}\in\mathcal{P}_{ab}$, yield

$$\begin{aligned}(3.23)\quad M_1&=-\int_a^{\alpha_1} l_{11}(u_1)(s)\,\mathrm{d}s+\int_a^{\alpha_1} l_{12}(u_2)(s)\,\mathrm{d}s\\&\leqslant m_1\int_a^{\alpha_1} l_{11}(1)(s)\,\mathrm{d}s+M_2\int_a^{\alpha_1} l_{12}(1)(s)\,\mathrm{d}s=m_1B_{11}+M_2B_{12}\end{aligned}$$

and

$$\begin{aligned}(3.24)\quad M_1+m_1&=\int_{\alpha_1}^{\beta_1} l_{11}(u_1)(s)\,\mathrm{d}s-\int_{\alpha_1}^{\beta_1} l_{12}(u_2)(s)\,\mathrm{d}s\\&\leqslant M_1\int_{\alpha_1}^{\beta_1} l_{11}(1)(s)\,\mathrm{d}s+m_2\int_{\alpha_1}^{\beta_1} l_{12}(1)(s)\,\mathrm{d}s=M_1D_{11}+m_2D_{12}.\end{aligned}$$

Furthermore, under the assumption $\beta_2 < \alpha_2$, the integrations of (3.7) from a to β_2 and from β_2 to α_2, in view of (3.8), (3.10), (3.11) and the assumptions $l_{21}, l_{22} \in \mathcal{P}_{ab}$, result in

$$(3.25) \quad m_2 = -\int_a^{\beta_2} l_{21}(u_1)(s)\,\mathrm{d}s + \int_a^{\beta_2} l_{22}(u_2)(s)\,\mathrm{d}s \\ \leqslant m_1 \int_a^{\beta_2} l_{21}(1)(s)\,\mathrm{d}s + M_2 \int_a^{\beta_2} l_{22}(1)(s)\,\mathrm{d}s = m_1 B_{21} + M_2 B_{22}$$

and

$$(3.26) \quad M_2 + m_2 = \int_{\beta_2}^{\alpha_2} l_{21}(u_1)(s)\,\mathrm{d}s - \int_{\beta_2}^{\alpha_2} l_{22}(u_2)(s)\,\mathrm{d}s \\ \leqslant M_1 \int_{\beta_2}^{\alpha_2} l_{21}(1)(s)\,\mathrm{d}s + m_2 \int_{\beta_2}^{\alpha_2} l_{22}(1)(s)\,\mathrm{d}s = M_1 D_{21} + m_2 D_{22}.$$

If $\beta_2 > \alpha_2$, we obtain in a similar manner the inequalities

$$(3.27) \quad M_2 \leqslant M_1 B_{21} + m_2 B_{22},$$

$$(3.28) \quad M_2 + m_2 \leqslant m_1 D_{21} + M_2 D_{22}.$$

Now we are in position to discuss the cases (d1)–(d4).

Case (d1): $\beta_2 < \alpha_2$ and $D_{ii} \geqslant 1$ for some $i \in \{1, 2\}$. Suppose that $D_{22} \geqslant 1$ (the case $D_{11} \geqslant 1$ can be proved analogously). Using this assumption, from (3.25) and (3.26) we get

$$m_2 \leqslant m_1 B_{21} + M_1 B_{22} D_{21} + m_2 B_{22}(D_{22} - 1)$$

and

$$M_2 \leqslant M_1 D_{21} + m_1 B_{21}(D_{22} - 1) + M_2 B_{22}(D_{22} - 1).$$

Hence, in view of (3.13), the last two inequalities yield

$$(3.29) \quad m_2 \varphi(A_{22}) \leqslant m_1 B_{21} + M_1 B_{22} D_{21},$$

$$(3.30) \quad M_2 \varphi(A_{22}) \leqslant M_1 D_{21} + m_1 B_{21}(D_{22} - 1).$$

By virtue of (2.4) and (3.22), it follows from (3.23), (3.30) and (3.24), (3.29) that

$$(3.31) \quad 0 < M_1[\varphi(A_{22}) - B_{12} D_{21}] \leqslant m_1[\varphi(A_{22}) B_{11} + B_{12} B_{21}(D_{22} - 1)]$$

and

$$(3.32) \quad 0 < m_1[\varphi(A_{22}) - D_{12} B_{21}] \leqslant M_1[\varphi(A_{22})(D_{11} - 1) + D_{12} D_{21} B_{22}],$$

respectively. Combining (3.31) and (3.32), we get

$$
\begin{aligned}
\varphi^2(A_{22}) \leqslant{} & \varphi(A_{22})[B_{12}D_{21} + D_{12}B_{21}] \\
& - B_{12}D_{12}B_{21}D_{21}(1 - B_{22}(D_{22} - 1)) \\
& + \varphi(A_{22})[B_{12}B_{21}(D_{11} - 1)(D_{22} - 1) + D_{12}D_{21}B_{11}B_{22}] \\
& + \varphi^2(A_{22})B_{11}(D_{11} - 1).
\end{aligned} \tag{3.33}
$$

Since $1 - B_{ii}(D_{ii} - 1) \geqslant \varphi(A_{ii}) > 0$ for $i = 1, 2$ and

$$B_{12}D_{21} + D_{12}B_{21} \leqslant A_{12}A_{21} - B_{12}B_{21} - D_{12}D_{21}, \tag{3.34}$$

we obtain from (3.33) that

$$
\begin{aligned}
\varphi(A_{11})\varphi(A_{22}) \leqslant{} & A_{12}A_{21} + B_{12}B_{21}[(D_{11} - 1)(D_{22} - 1) - 1] \\
& + D_{12}D_{21}[B_{11}B_{22} - 1].
\end{aligned} \tag{3.35}
$$

If $(D_{11} - 1)(D_{22} - 1) \leqslant 1$ and $B_{11}B_{22} \leqslant 1$ then (3.35) implies

$$\varphi(A_{11})\varphi(A_{22}) \leqslant A_{12}A_{21},$$

which contradicts (2.4).

If $(D_{11}-1)(D_{22}-1) \leqslant 1$ and $B_{11}B_{22} > 1$ then, in view of (3.12) and the assumption $D_{22} \geqslant 1$, we obtain from (3.35) that

$$\varphi(A_{11})\varphi(A_{22}) \leqslant A_{12}A_{21}B_{11}B_{22} \leqslant A_{12}A_{21}B_{11}(A_{22} - D_{22}) \leqslant A_{12}A_{21}A_{11}(A_{22} - 1),$$

which contradicts (2.4).

If $(D_{11} - 1)(D_{22} - 1) > 1$ and $B_{11}B_{22} \leqslant 1$ then (3.35) results in

$$\varphi(A_{11})\varphi(A_{22}) \leqslant A_{12}A_{21}(D_{11} - 1)(D_{22} - 1) \leqslant A_{12}A_{21}A_{11}(A_{22} - 1),$$

which contradicts (2.4).

If $(D_{11} - 1)(D_{22} - 1) > 1$ and $B_{11}B_{22} > 1$ then (3.35) yields

$$
\begin{aligned}
\varphi(A_{11})\varphi(A_{22}) & \leqslant A_{12}A_{21}[(D_{11} - 1)(D_{22} - 1) + B_{11}B_{22} - 1] \\
& \leqslant A_{12}A_{21}[A_{11}(D_{22} - 1) + A_{11}B_{22}] \leqslant A_{12}A_{21}A_{11}(A_{22} - 1),
\end{aligned}
$$

which contradicts (2.4).

Case (d2): $\beta_2 < \alpha_2$ and $D_{ii} < 1$ for $i = 1, 2$. We first note that

$$B_{11}B_{22} \leqslant (A_{11} - D_{11})B_{22} = (A_{11} - 1)B_{22} + (1 - D_{11})B_{22} \tag{3.36}$$

and also

$$B_{11}B_{22} \leqslant (A_{22} - D_{22})B_{11} = (A_{22} - 1)B_{11} + (1 - D_{22})B_{11}. \tag{3.37}$$

By virtue of (3.22), we get from the inequalities (3.24) and (3.26)

$$m_1 \leqslant m_2 D_{12} \tag{3.38}$$

and

$$M_2 \leqslant M_1 D_{21}. \tag{3.39}$$

Therefore, in view of (2.4) and (3.22), the relations (3.24), (3.25), (3.39) and (3.23), (3.39) result in

$$0 < m_1(1 - D_{12}B_{21}) \leqslant M_1[D_{12}D_{21}B_{22} - (1 - D_{11})] \tag{3.40}$$

and

$$0 < M_1(1 - B_{12}D_{21}) \leqslant m_1 B_{11}, \tag{3.41}$$

respectively. Combining (3.36), (3.40), (3.41) and taking the inequality $D_{12}D_{21} \leqslant 1$ into account, we get

$$(1 - B_{12}D_{21})(1 - D_{12}B_{21}) \leqslant D_{12}D_{21}(A_{11} - 1)B_{22} + (B_{22} - B_{11})(1 - D_{11}). \tag{3.42}$$

On the other hand, by virtue of (2.4) and (3.22), the relations (3.23), (3.26), (3.38) and (3.25), (3.38) imply

$$0 < M_2(1 - B_{12}D_{21}) \leqslant m_2[D_{12}D_{21}B_{11} - (1 - D_{22})] \tag{3.43}$$

and

$$0 < m_2(1 - D_{12}B_{21}) \leqslant M_2 B_{22}, \tag{3.44}$$

respectively. Combining (3.37), (3.43), (3.44) and taking the inequality $D_{12}D_{21} \leqslant 1$ into account, we obtain

$$(1 - B_{12}D_{21})(1 - D_{12}B_{21}) \leqslant D_{12}D_{21}(A_{22} - 1)B_{11} + (B_{11} - B_{22})(1 - D_{22}). \tag{3.45}$$

First suppose that $B_{22} \leqslant B_{11}$. Then, by virtue of (3.34), the inequality (3.42) yields

$$
\begin{aligned}
1 &\leqslant B_{12}D_{21} + D_{12}B_{21} + D_{12}D_{21}(A_{11}-1)B_{22} \\
&\leqslant A_{12}A_{21} + D_{12}D_{21}[(A_{11}-1)B_{22}-1].
\end{aligned} \tag{3.46}
$$

If $(A_{11}-1)B_{22} \leqslant 1$ then (3.46) implies $1 \leqslant A_{12}A_{21}$, which contradicts (2.4), because $0 < \varphi(A_{ii}) \leqslant 1$ for $i = 1, 2$.

If $(A_{11}-1)B_{22} > 1$ then (3.46) yields

$$
1 \leqslant A_{12}A_{21}(A_{11}-1)B_{22} \leqslant A_{12}A_{21}(A_{11}-1)A_{22},
$$

which contradicts (2.4), because $0 < \varphi(A_{ii}) \leqslant 1$ for $i = 1, 2$.

Now suppose that $B_{22} > B_{11}$. Then, by virtue of (3.34), the inequality (3.45) results in

$$
\begin{aligned}
1 &\leqslant B_{12}D_{21} + D_{12}B_{21} + D_{12}D_{21}(A_{22}-1)B_{11} \\
&\leqslant A_{12}A_{21} + D_{12}D_{21}[(A_{22}-1)B_{11}-1].
\end{aligned} \tag{3.47}
$$

If $(A_{22}-1)B_{11} \leqslant 1$ then (3.47) implies $1 \leqslant A_{12}A_{21}$, which contradicts (2.4), because $0 < \varphi(A_{ii}) \leqslant 1$ for $i = 1, 2$.

If $(A_{22}-1)B_{11} > 1$ then (3.47) yields

$$
1 \leqslant A_{12}A_{21}(A_{22}-1)B_{11} \leqslant A_{12}A_{21}(A_{22}-1)A_{11},
$$

which contradicts (2.4), because $0 < \varphi(A_{ii}) \leqslant 1$ for $i = 1, 2$.

Case (d3): $\beta_2 > \alpha_2$ and $D_{ii} \geqslant 1$ for some $i \in \{1, 2\}$. Suppose that $D_{22} \geqslant 1$ (the case $D_{11} \geqslant 1$ can be proved analogously). In a similar manner as in the case (d1), combining (3.23), (3.24) and (3.27), (3.28) we get

$$
\begin{aligned}
\varphi(A_{11})\varphi(A_{22}) \leqslant A_{12}A_{21} &+ D_{12}B_{21}[B_{11}(D_{22}-1)-1] \\
&+ B_{12}D_{21}[B_{22}(D_{11}-1)-1].
\end{aligned} \tag{3.48}
$$

If $B_{11}(D_{22}-1) \leqslant 1$ and $B_{22}(D_{11}-1) \leqslant 1$ then (3.48) implies

$$
\varphi(A_{11})\varphi(A_{22}) \leqslant A_{12}A_{21},
$$

which contradicts (2.4).

If $B_{11}(D_{22}-1) \leqslant 1$ and $B_{22}(D_{11}-1) > 1$ then we obtain from (3.48) that

$$\varphi(A_{11})\varphi(A_{22}) \leqslant A_{12}A_{21}B_{22}(D_{11}-1) \leqslant A_{12}A_{21}A_{22}(A_{11}-1),$$

which contradicts (2.4).

If $B_{11}(D_{22}-1) > 1$ and $B_{22}(D_{11}-1) \leqslant 1$ then (3.48) implies at

$$\varphi(A_{11})\varphi(A_{22}) \leqslant A_{12}A_{21}B_{11}(D_{22}-1) \leqslant A_{12}A_{21}A_{11}(A_{22}-1),$$

which contradicts (2.4).

If $B_{11}(D_{22}-1) > 1$ and $B_{22}(D_{11}-1) > 1$ then (3.48) yields

$$\begin{aligned}\varphi(A_{11})\varphi(A_{22}) &\leqslant A_{12}A_{21}[B_{11}(D_{22}-1) + (D_{11}-1)B_{22} - 1] \\ &\leqslant A_{12}A_{21}[A_{11}(D_{22}-1) + A_{11}B_{22}] \leqslant A_{12}A_{21}A_{11}(A_{22}-1),\end{aligned}$$

which contradicts (2.4).

Case (d4): $\beta_2 > \alpha_2$ and $D_{ii} < 1$ for $i = 1, 2$. The inequalities (3.24) and (3.28) result in

$$m_1 \leqslant m_2 D_{12}, \qquad m_2 \leqslant m_1 D_{21}.$$

Hence, we get

$$1 \leqslant D_{12}D_{21} \leqslant A_{12}A_{21},$$

which contradicts (2.4), because $0 < \varphi(A_{ii}) \leqslant 1$ for $i = 1, 2$.

The contradictions obtained in (a)–(d) prove that the problem (3.6), (3.7), (3.2) has only the trivial solution. $\square$

Proof of Theorem 2.2. According to Lemmas 3.1 and 3.2, in order to prove the theorem it is sufficient to show that the system

$$u_1'(t) = -l_{11}(u_1)(t) + l_{12}(u_2)(t), \tag{3.49}$$

$$u_2'(t) = -l_{21}(u_1)(t) - l_{22}(u_2)(t) \tag{3.50}$$

has only the trivial solution satisfying (3.2).

Suppose that, on the contrary, $(u_1, u_2)^T$ is a nontrivial solution of the problem (3.49), (3.50), (3.2). Define numbers M_i, m_i $(i = 1, 2)$ by (3.8) and choose $\alpha_i, \beta_i \in [a, b]$ $(i = 1, 2)$ such that the equalities (3.9) and (3.10) are satisfied. Furthermore, let numbers B_{ij}, D_{ij} $(i, j = 1, 2)$ be given by (3.11). It is clear that (3.2) guarantees

$$M_i \geqslant 0, \quad m_i \geqslant 0 \quad \text{for } i = 1, 2.$$

For the sake of clarity we shall divide the discussion into the following cases.

(a) Neither of the functions u_1 and u_2 changes its sign. According to Lemma 3.2, we can assume without loss of generality that

$$u_1(t) \geqslant 0, \quad u_2(t) \geqslant 0 \quad \text{for } t \in [a,b].$$

(b) One of the functions u_1 and u_2 is of a constant sign and the other one changes its sign. According to Lemma 3.2, we can assume without loss of generality that $u_1(t) \geqslant 0$ for $t \in [a,b]$.

(c) Both functions u_1 and u_2 change their signs. According to Lemma 3.2, we can assume without loss of generality that $\alpha_1 < \beta_1$ and $\beta_2 < \alpha_2$. Obviously, one of the following conditions is satisfied:

(c1) $D_{ii} \geqslant 1$ for some $i \in \{1,2\}$,

(c2) $D_{ii} < 1$ for $i = 1,2$ and

(c2.1) $m_1 D_{21} \leqslant m_2 B_{22}$,

(c2.2) $M_1 \leqslant M_2 D_{12}$,

(c2.3) $m_1 D_{21} > m_2 B_{22}$ and $M_1 > M_2 D_{12}$.

First we note that (3.13) is true and, by virtue of Lemma 3.3, the assumption (2.6) can be rewritten as

$$\varrho A_{12} A_{21} < (3 - A_{ii})\varphi\big(A_{3-i3-i}\big) \quad \text{for } i = 1,2. \tag{3.51}$$

Case (a): $u_1(t) \geqslant 0$ and $u_2(t) \geqslant 0$ for $t \in [a,b]$. In view of the assumptions $l_{21}, l_{22} \in \mathcal{P}_{ab}$, (3.50) implies $u_2'(t) \leqslant 0$ for $t \in [a,b]$. Therefore, $u_2 \equiv 0$ and, by virtue of the assumption $l_{11} \in \mathcal{P}_{ab}$, (3.49) yields $u_1'(t) \leqslant 0$ for $t \in [a,b]$. Consequently, $u_1 \equiv 0$ as well, which is a contradiction.

Case (b): $u_1(t) \geqslant 0$ for $t \in [a,b]$ and u_2 changes its sign. Obviously, (3.16) is true, $M_1 \geqslant 0$ and $m_1 = 0$. Suppose that $\alpha_2 < \beta_2$ (the case $\alpha_2 > \beta_2$ can be proved analogously). The integration of (3.49) from a to α_1, in view of (3.8), (3.9) and the assumptions $l_{11}, l_{12} \in \mathcal{P}_{ab}$, yields (3.17).

On the other hand, the integrations of (3.50) from a to α_2 and from α_2 to β_2, in view of (3.8), (3.10) and the assumptions $l_{21}, l_{22} \in \mathcal{P}_{ab}$, result in

$$\begin{aligned} M_2 &= -\int_a^{\alpha_2} l_{21}(u_1)(s)\,\mathrm{d}s - \int_a^{\alpha_2} l_{22}(u_2)(s)\,\mathrm{d}s \\ &\leqslant m_2 \int_a^{\alpha_2} l_{22}(1)(s)\,\mathrm{d}s = m_2 B_{22} \end{aligned} \tag{3.52}$$

and

$$\begin{aligned} M_2 + m_2 &= \int_{\alpha_2}^{\beta_2} l_{21}(u_1)(s)\,\mathrm{d}s + \int_{\alpha_2}^{\beta_2} l_{22}(u_2)(s)\,\mathrm{d}s \\ &\leqslant M_1 \int_{\alpha_2}^{\beta_2} l_{21}(1)(s)\,\mathrm{d}s + M_2 \int_{\alpha_2}^{\beta_2} l_{22}(1)(s)\,\mathrm{d}s = M_1 D_{21} + M_2 D_{22}, \end{aligned} \tag{3.53}$$

respectively. By virtue of (3.16), combining (3.17), (3.52), and (3.53), we get

$$3 - A_{22} \leqslant 1 + \frac{M_2}{m_2} + \frac{m_2}{M_2} - B_{22} - D_{22} \leqslant \frac{M_1}{M_2} D_{21} \leqslant A_{12} A_{21},$$

which contradicts (3.51), because $\varrho \geqslant 1$ and $0 < \varphi(A_{11}) \leqslant 1$.

C a s e (c): u_1 and u_2 change their signs, $\alpha_1 < \beta_1$, and $\beta_2 < \alpha_2$. Obviously, (3.22) is true. The integrations of (3.49) from a to α_1 and from α_1 to β_1, in view of (3.8), (3.9) and the assumptions $l_{11}, l_{12} \in \mathcal{P}_{ab}$, imply (3.23) and (3.24). On the other hand, the integrations of (3.50) from a to β_2 and from β_2 to α_2, on account of (3.8), (3.10) and the assumptions $l_{21}, l_{22} \in \mathcal{P}_{ab}$, result in

$$\begin{aligned} m_2 &= \int_a^{\beta_2} l_{21}(u_1)(s)\,\mathrm{d}s + \int_a^{\beta_2} l_{22}(u_2)(s)\,\mathrm{d}s \\ &\leqslant M_1 \int_a^{\beta_2} l_{21}(1)(s)\,\mathrm{d}s + M_2 \int_a^{\beta_2} l_{22}(1)(s)\,\mathrm{d}s = M_1 B_{21} + M_2 B_{22} \end{aligned} \tag{3.54}$$

and

$$\begin{aligned} M_2 + m_2 &= -\int_{\beta_2}^{\alpha_2} l_{21}(u_1)(s)\,\mathrm{d}s - \int_{\beta_2}^{\alpha_2} l_{22}(u_2)(s)\,\mathrm{d}s \\ &\leqslant m_1 \int_{\beta_2}^{\alpha_2} l_{21}(1)(s)\,\mathrm{d}s + m_2 \int_{\beta_2}^{\alpha_2} l_{22}(1)(s)\,\mathrm{d}s = m_1 D_{21} + m_2 D_{22}. \end{aligned} \tag{3.55}$$

By virtue of (3.22), the relations (3.23), (3.24) and (3.54), (3.55) yield

$$3 - B_{11} - D_{11} \leqslant 1 + \frac{M_1}{m_1} + \frac{m_1}{M_1} - B_{11} - D_{11} \leqslant \frac{M_2}{m_1} B_{12} + \frac{m_2}{M_1} D_{12} \tag{3.56}$$

and

$$3 - B_{22} - D_{22} \leqslant 1 + \frac{M_2}{m_2} + \frac{m_2}{M_2} - B_{22} - D_{22} \leqslant \frac{M_1}{M_2} B_{21} + \frac{m_1}{m_2} D_{21}, \tag{3.57}$$

respectively.

C a s e (c1): $D_{ii} \geqslant 1$ for some $i \in \{1, 2\}$. Suppose that $D_{11} \geqslant 1$ (the case $D_{22} \geqslant 1$ can be proved analogously). Using this assumption and combining (3.23) and (3.24), we get

$$M_1 \leqslant M_1 B_{11}(D_{11} - 1) + m_2 B_{11} D_{12} + M_2 B_{12}$$

and

$$m_1 \leqslant m_1 B_{11}(D_{11} - 1) + M_2(D_{11} - 1) B_{12} + m_2 D_{12}.$$

Hence, in view of (3.13), the last two inequalities yield

$$M_1\varphi(A_{11}) \leqslant m_2B_{11}D_{12} + M_2B_{12}, \tag{3.58}$$

$$m_1\varphi(A_{11}) \leqslant M_2(D_{11}-1)B_{12} + m_2D_{12}. \tag{3.59}$$

By virtue of the assumption $D_{11} \geqslant 1$, it follows from (3.54), (3.58 and (3.55), (3.59) that

$$M_1[\varphi(A_{11}) - B_{11}D_{12}B_{21}] \leqslant M_2[B_{11}B_{22}D_{12} + B_{12}] \tag{3.60}$$

and

$$m_1[\varphi(A_{11}) - (D_{11}-1)B_{12}D_{21}] \leqslant m_2[(D_{11}-1)(D_{22}-1)B_{12} + D_{12}], \tag{3.61}$$

respectively. Note that, in view of (3.12) and the condition $D_{11} \geqslant 1$, the assumption (3.51) guarantees

$$\begin{aligned} B_{11}D_{12}B_{21} &\leqslant (A_{11}-1)A_{12}A_{21} < \frac{3-A_{22}}{3}\varphi(A_{11}) \leqslant \varphi(A_{11}), \\ (D_{11}-1)B_{12}D_{21} &\leqslant (A_{11}-1)A_{12}A_{21} < \frac{3-A_{22}}{3}\varphi(A_{11}) \leqslant \varphi(A_{11}). \end{aligned} \tag{3.62}$$

Consequently, we get from (3.57), (3.60) and (3.61) that

$$\begin{aligned} &(3-B_{22}-D_{22})[\varphi(A_{11}) - B_{11}D_{12}B_{21}][\varphi(A_{11}) - (D_{11}-1)B_{12}D_{21}] \\ &\qquad \leqslant [B_{11}B_{22}D_{12}B_{21} + B_{12}B_{21}][\varphi(A_{11}) - (D_{11}-1)B_{12}D_{21}] \\ &\qquad\quad + [(D_{11}-1)(D_{22}-1)B_{12}D_{21} + D_{12}D_{21}][\varphi(A_{11}) - B_{11}D_{12}B_{21}] \\ &\qquad \leqslant \varphi(A_{11})[B_{12}B_{21} + D_{12}D_{21} + B_{11}B_{22}D_{12}B_{21} \\ &\qquad\quad + (D_{11}-1)(D_{22}-1)B_{12}D_{21}]. \end{aligned} \tag{3.63}$$

On the other hand,

$$\begin{aligned} &(3-B_{22}-D_{22})[\varphi(A_{11}) - B_{11}D_{12}B_{21}][\varphi(A_{11}) - (D_{11}-1)B_{12}D_{21}] \\ &\qquad \geqslant (3-A_{22})\varphi(A_{11})^2 - \varphi(A_{11})(3-B_{22}-D_{22})B_{11}D_{12}B_{21} \\ &\qquad\quad - \varphi(A_{11})(3-B_{22}-D_{22})(D_{11}-1)B_{12}D_{21}. \end{aligned} \tag{3.64}$$

By virtue of (3.12), the inequality

$$B_{12}B_{21} + D_{12}D_{21} \leqslant A_{12}A_{21} - D_{12}B_{21} - B_{12}D_{21} \tag{3.65}$$

is true. Consequently, (3.63) and (3.64) imply

$$\begin{aligned}(3-A_{22})\varphi(A_{11}) \leqslant A_{12}A_{21} + D_{12}B_{21}[(3-D_{22})B_{11}-1] \\ + B_{12}D_{21}[(2-B_{22})(D_{11}-1)-1].\end{aligned} \tag{3.66}$$

If $(3-D_{22})B_{11} \leqslant 1$ and $(2-B_{22})(D_{11}-1) \leqslant 1$ then (3.66) yields

$$(3-A_{22})\varphi(A_{11}) \leqslant A_{12}A_{21},$$

which contradicts (3.51).

If $(3-D_{22})B_{11} \leqslant 1$ and $(2-B_{22})(D_{11}-1) > 1$ then (3.66) results in

$$(3-A_{22})\varphi(A_{11}) \leqslant A_{12}A_{21}(2-B_{22})(D_{11}-1) \leqslant 3(A_{11}-1)A_{12}A_{21},$$

which contradicts (3.51).

If $(3-D_{22})B_{11} > 1$ and $(2-B_{22})(D_{11}-1) \leqslant 1$ then, in view of (3.12) and the assumption $D_{11} \geqslant 1$, we obtain from (3.66) that

$$(3-A_{22})\varphi(A_{11}) \leqslant A_{12}A_{21}(3-D_{22})B_{11} \leqslant 3A_{12}A_{21}(A_{11}-D_{11}) \leqslant 3(A_{11}-1)A_{12}A_{21},$$

which contradicts (3.51).

If $(3-D_{22})B_{11} > 1$ and $(2-B_{22})(D_{11}-1) > 1$ then (3.66) arrives at

$$\begin{aligned}(3-A_{22})\varphi(A_{11}) &\leqslant A_{12}A_{21}[(3-D_{22})B_{11} + (2-B_{22})(D_{11}-1)-1] \\ &\leqslant A_{12}A_{21}[3B_{11} + 3(D_{11}-1)] \leqslant 3(A_{11}-1)A_{12}A_{21},\end{aligned}$$

which contradicts (3.51).

Case (c2): $D_{ii} < 1$ for $i = 1, 2$. By virtue of (3.22), the inequalities (3.24) and (3.55) result in

$$m_1 \leqslant m_2 D_{12} \tag{3.67}$$

and

$$M_2 \leqslant m_1 D_{21}, \tag{3.68}$$

respectively.

Case (c2.1): $m_1 D_{21} \leqslant m_2 B_{22}$. Combining (3.54), (3.55) and taking (3.12) into account, we get

$$\begin{aligned}m_2 &\leqslant M_1 B_{21} + m_1 B_{22} D_{21} + m_2 B_{22}(D_{22}-1) \\ &\leqslant M_1 B_{21} + m_1(A_{22}-D_{22})D_{21} + m_2 B_{22}(D_{22}-1) \\ &= M_1 B_{21} + m_1(A_{22}-1)D_{21} + (1-D_{22})[m_1 D_{21} - m_2 B_{22}].\end{aligned}$$

Consequently,

$$m_2 \leqslant M_1 B_{21} + m_1(A_{22} - 1)D_{21}. \tag{3.69}$$

If $A_{22} \leqslant 1$ then (3.56), (3.68), and (3.69) imply

$$3 - A_{11} \leqslant 3 - B_{11} - D_{11} \leqslant B_{12}D_{21} + D_{12}B_{21} \leqslant A_{12}A_{21},$$

which contradicts (3.51), because $0 < \varphi(A_{22}) \leqslant 1$.

Therefore, suppose that

$$A_{22} > 1. \tag{3.70}$$

Then, using (3.24) in (3.69), we obtain

$$m_2 \leqslant M_1 B_{21} + M_1(A_{22} - 1)(D_{11} - 1)D_{21} + m_2(A_{22} - 1)D_{12}D_{21},$$

i.e.,

$$m_2[1 - (A_{22} - 1)D_{12}D_{21}] \leqslant M_1[B_{21} - (A_{22} - 1)(1 - D_{11})D_{21}. \tag{3.71}$$

Note that the assumption (3.51) guarantees

$$(A_{22} - 1)D_{12}D_{21} \leqslant (A_{22} - 1)A_{12}A_{21} < \frac{3 - A_{22}}{3}\varphi(A_{11}) < 1.$$

Consequently, we get from (3.56), (3.68) and (3.71) that

$$\begin{aligned}(3 - B_{11} - D_{11})&[1 - (A_{22} - 1)D_{12}D_{21}] \\ &\leqslant [1 - (A_{22} - 1)D_{12}D_{21}]B_{12}D_{21} \\ &\quad + D_{12}B_{21} - (A_{22} - 1)(1 - D_{11})D_{12}D_{21} \\ &\leqslant B_{12}D_{21} + D_{12}B_{21} - (A_{22} - 1)(1 - D_{11})D_{12}D_{21}.\end{aligned} \tag{3.72}$$

By virtue of the inequality

$$B_{12}D_{21} + D_{12}B_{21} \leqslant A_{12}A_{21} - B_{12}B_{21} - D_{12}D_{21}, \tag{3.73}$$

(3.72) implies

$$3 - A_{11} \leqslant A_{12}A_{21} + D_{12}D_{21}[(A_{22} - 1)(2 - B_{11}) - 1]. \tag{3.74}$$

If $(A_{22}-1)(2-B_{11}) \leqslant 1$ then (3.74) results in

$$3 - A_{11} \leqslant A_{12}A_{21},$$

which contradicts (3.51), because $0 < \varphi(A_{22}) \leqslant 1$.

If $(A_{22}-1)(2-B_{11}) > 1$ then (3.74) yields

$$3 - A_{11} \leqslant A_{12}A_{21}(A_{22}-1)(2-B_{11}) \leqslant 3(A_{22}-1)A_{12}A_{21},$$

which contradicts (3.51), because $0 < \varphi(A_{22}) \leqslant 1$.

Case (c2.2): $M_1 \leqslant M_2 D_{12}$. Using (3.67), we get from (3.57) that

$$3 - A_{22} \leqslant 3 - B_{22} - D_{22} \leqslant D_{12}B_{21} + D_{12}D_{21} = D_{12}(B_{21}+D_{21}) \leqslant A_{12}A_{21},$$

which contradicts (3.51), because $0 < \varphi(A_{11}) \leqslant 1$.

Case (c2.3): $m_1 D_{21} > m_2 B_{22}$ and $M_1 > M_2 D_{12}$. It follows from the relation (3.67) that $D_{12} > 0$, because $m_i > 0$ for $i = 1, 2$. Therefore, we have

$$\frac{M_2}{M_1} < \frac{1}{D_{12}}. \tag{3.75}$$

Note also that (3.67) and the assumption $m_1 D_{21} > m_2 B_{22}$ guarantee

$$D_{12}D_{21} > B_{22}. \tag{3.76}$$

It follows from (3.54) and (3.75) that

$$\frac{m_2}{M_1} \leqslant B_{21} + \frac{M_2}{M_1}B_{22} \leqslant B_{21} + \frac{B_{22}}{D_{12}}. \tag{3.77}$$

Finally, (3.56), (3.68) and (3.77) result in

$$3 - A_{11} \leqslant 3 - B_{11} - D_{11} \leqslant B_{12}D_{21} + D_{12}B_{21} + B_{22}.$$

Using (3.73) and (3.76) in the last inequality, we get

$$3 - A_{11} \leqslant A_{12}A_{21} - B_{12}B_{21} - D_{12}D_{21} + B_{22} \leqslant A_{12}A_{21},$$

which contradicts (3.51), because $0 < \varphi(A_{22}) \leqslant 1$.

The contradictions obtained in (a)–(c) prove that the problem (3.49), (3.50), (3.2) has only the trivial solution. $\square$

P r o o f o f T h e o r e m 2.3. If $A_{12}A_{21} < 1$ then the validity of the theorem follows immediately from Theorem 2.2. Therefore, suppose in the sequel that

$$A_{12}A_{21} \geqslant 1. \tag{3.78}$$

According to Lemmas 3.1 and 3.2, in order to prove the theorem it is sufficient to show that the problem (3.49), (3.50), (3.2) has only the trivial solution.

Suppose that, on the contrary, $(u_1, u_2)^T$ is a nontrivial solution of the problem (3.49), (3.50), (3.2). Define numbers M_i, m_i $(i = 1, 2)$ by (3.8) and choose $\alpha_i, \beta_i \in [a, b]$ $(i = 1, 2)$ such that the equalities (3.9) and (3.10) are satisfied. Furthermore, let numbers B_{ij}, D_{ij} $(i, j = 1, 2)$ be given by (3.11). It is clear that (3.2) guarantees

$$M_i \geqslant 0, \quad m_i \geqslant 0 \quad \text{for } i = 1, 2.$$

For the sake of clarity we shall divide the discussion into the following cases.

(a) Neiher of the functions u_1 and u_2 changes its sign. According to Lemma 3.2, we can assume without loss of generality that

$$u_1(t) \geqslant 0, \quad u_2(t) \geqslant 0 \quad \text{for } t \in [a, b].$$

(b) One of the functions u_1 and u_2 is of a constant sign and the other one changes its sign. According to Lemma 3.2, we can assume without loss of generality that $u_1(t) \geqslant 0$ for $t \in [a, b]$. Obviously, one of the following conditions is satisfied:

(b1) $\alpha_2 < \beta_2$,

(b2) $\alpha_2 > \beta_2$.

(c) Both functions u_1 and u_2 change their signs. According to Lemma 3.2, we can assume without loss of generality that $\alpha_1 < \beta_1$ and $\beta_2 < \alpha_2$.

First we note that, in view of (2.8), the inequality (2.9) guarantees

$$\begin{aligned} A_{ii}A_{12}A_{21} &\leqslant [A_{ii} + (1 - A_{ii})A_{3-i3-i}]A_{12}A_{21} \\ &= (A_{11} + A_{22} - A_{11}A_{22})A_{12}A_{21} < 1 \quad \text{for } i = 1, 2. \end{aligned} \tag{3.79}$$

Now we are in position to discuss the cases (a)–(c).

C a s e (a): $u_1(t) \geqslant 0$ and $u_2(t) \geqslant 0$ for $t \in [a, b]$. In view of the assumptions $l_{21}, l_{22} \in \mathcal{P}_{ab}$, (3.50) implies $u_2'(t) \leqslant 0$ for $t \in [a, b]$. Therefore, $u_2 \equiv 0$ and, by virtue of the assumption $l_{11} \in \mathcal{P}_{ab}$, (3.49) implies $u_1'(t) \leqslant 0$ for $t \in [a, b]$. Consequently, $u_1 \equiv 0$ as well, which is a contradiction.

C a s e (b): $u_1(t) \geqslant 0$ for $t \in [a, b]$ and u_2 changes its sign. Obviously, $m_1 = 0$ and (3.16) is true. The integration of (3.49) from a to α_1, in view of (3.8), (3.9), and the assumptions $l_{11}, l_{12} \in \mathcal{P}_{ab}$, yield (3.17).

C a s e (b1): $\alpha_2 < \beta_2$. The integrations of (3.50) from a to α_2 and from α_2 to β_2, in view of (3.8), (3.10) and the assumptions $l_{21}, l_{22} \in \mathcal{P}_{ab}$, yield (3.52) and (3.53), respectively. Using (2.8), (3.17) and (3.52) in the relation (3.53), we get

$$0 < m_2 \leqslant M_1 D_{21} \leqslant M_2 A_{12} A_{21} \leqslant m_2 B_{22} A_{12} A_{21}.$$

Hence we get $1 \leqslant A_{22} A_{12} A_{21}$, which contradicts (3.79).

C a s e (b2): $\alpha_2 > \beta_2$. The integration of (3.50) from β_2 to α_2, on account of (3.8), (3.10) and the assumptions $l_{21}, l_{22} \in \mathcal{P}_{ab}$, yields

$$\begin{aligned} M_2 + m_2 &= -\int_{\beta_2}^{\alpha_2} l_{21}(u_1)(s)\,\mathrm{d}s - \int_{\beta_2}^{\alpha_2} l_{22}(u_2)(s)\,\mathrm{d}s \\ &\leqslant m_2 \int_{\beta_2}^{\alpha_2} l_{22}(1)(s)\,\mathrm{d}s \leqslant m_2 A_{22}. \end{aligned} \tag{3.80}$$

By virtue of (2.8) and (3.16), (3.80) implies

$$0 < M_2 \leqslant m_2(A_{22} - 1) < 0,$$

a contradiction.

C a s e (c): u_1 and u_2 change their signs, $\alpha_1 < \beta_1$ and $\beta_2 < \alpha_2$. Obviously, (3.22) is true. The integrations of (3.49) from a to α_1 and from α_1 to β_1, in view of (3.8), (3.9) and the assumptions $l_{11}, l_{12} \in \mathcal{P}_{ab}$, imply (3.23) and (3.24). On the other hand, the integrations of (3.50) from a to β_2 and from β_2 to α_2, on account of (3.8), (3.10) and the assumptions $l_{21}, l_{22} \in \mathcal{P}_{ab}$, result in (3.54) and (3.55).

By virtue of (2.8) and (3.22), from the inequalities (3.23), (3.24) and (3.54), (3.55) we get

$$0 < \frac{M_1}{M_2} + \frac{M_1}{m_2}(1 - D_{11}) + \frac{m_1}{m_2} \leqslant A_{12} + \frac{m_1}{M_2} B_{11} \tag{3.81}$$

and

$$0 < \frac{m_2}{M_1} + \frac{M_2}{m_1} + \frac{m_2}{m_1}(1 - D_{22}) \leqslant A_{21} + \frac{M_2}{M_1} B_{22}, \tag{3.82}$$

respectively.

On the other hand, in view of (2.8), the inequalities (3.55) and (3.24) imply

$$m_1 \leqslant m_2 D_{12}, \qquad M_2 \leqslant m_1 D_{21}. \tag{3.83}$$

Combining (3.83) and (3.54), we get

$$M_2 \leqslant m_2 D_{12} D_{21} \leqslant M_1 A_{12} A_{21}^2 + M_2 A_{22} A_{12} A_{21},$$

i.e.,

$$M_2(1 - A_{22}A_{12}A_{21}) \leqslant M_1 A_{12} A_{21}^2. \tag{3.84}$$

Furthermore, combining (3.54), (3.23) and (3.83), we obtain

$$\begin{aligned} m_1 &\leqslant m_2 D_{12} \leqslant M_1 A_{12} A_{21} + M_2 A_{22} A_{12} \\ &\leqslant m_1 A_{11} A_{12} A_{21} + M_2 A_{12}^2 A_{21} + M_2 A_{22} A_{12}, \end{aligned}$$

i.e.,

$$m_1(1 - A_{11}A_{12}A_{21}) \leqslant M_2 A_{12}(A_{12}A_{21} + A_{22}). \tag{3.85}$$

Now, (3.84) and (3.85) yield

$$\begin{aligned} A_{12} + \frac{m_1}{M_2} B_{11} &\leqslant \frac{(1 + A_{11}A_{22})A_{12}}{1 - A_{11}A_{12}A_{21}}, \\ A_{21} + \frac{M_2}{M_1} B_{22} &\leqslant \frac{A_{21}}{1 - A_{22}A_{12}A_{21}}, \end{aligned} \tag{3.86}$$

because the condition (3.79) is true.

It follows from (3.81), (3.82) and (3.86) that

$$\begin{aligned} &\frac{(1 + A_{11}A_{22})A_{12}A_{21}}{(1 - A_{11}A_{12}A_{21})(1 - A_{22}A_{12}A_{21})} \\ &\geqslant \frac{m_2}{M_2} + \frac{M_1}{m_1} + \frac{M_1 m_2}{M_2 m_1}(1 - D_{22}) + 1 - D_{11} + \frac{M_1 M_2}{m_1 m_2}(1 - D_{11}) \\ &\quad + \frac{M_1}{m_1}(1 - D_{11})(1 - D_{22}) + \frac{m_1}{M_1} + \frac{M_2}{m_2} + 1 - D_{22}. \end{aligned} \tag{3.87}$$

Using the relation

$$x + y \geqslant 2\sqrt{xy} \quad \text{for } x \geqslant 0,\ y \geqslant 0,$$

we get

$$\frac{M_1 m_2}{M_2 m_1}(1 - D_{22}) + \frac{M_1 M_2}{m_1 m_2}(1 - D_{11}) \geqslant 2\frac{M_1}{m_1}\sqrt{(1 - D_{11})(1 - D_{22})}, \tag{3.88}$$

$$\begin{aligned} &\frac{M_1}{m_1} + 2\frac{M_1}{m_1}\sqrt{(1 - D_{11})(1 - D_{22})} + \frac{M_1}{m_1}(1 - D_{11})(1 - D_{22}) \\ &\quad = \frac{M_1}{m_1}\bigl(1 + \sqrt{(1 - D_{11})(1 - D_{22})}\bigr)^2, \end{aligned} \tag{3.89}$$

$$\frac{M_1}{m_1}\bigl(1 + \sqrt{(1 - D_{11})(1 - D_{22})}\bigr)^2 + \frac{m_1}{M_1} \geqslant 2\bigl(1 + \sqrt{(1 - D_{11})(1 - D_{22})}\bigr), \tag{3.90}$$

and

$$\frac{m_2}{M_2} + \frac{M_2}{m_2} \geqslant 2. \tag{3.91}$$

Now, in view of (3.88)–(3.91), (3.87) implies

$$\begin{aligned}(3.92)\qquad &\frac{(1 + A_{11}A_{22})A_{12}A_{21}}{(1 - A_{11}A_{12}A_{21})(1 - A_{22}A_{12}A_{21})}\\ &\qquad \geqslant 2 + 2\big(1 + \sqrt{(1 - D_{11})(1 - D_{22})}\big) + 1 - D_{11} + 1 - D_{22}\\ &\qquad = 4 + \big(\sqrt{1 - D_{11}} + \sqrt{1 - D_{22}}\big)^2\\ &\qquad \geqslant 4 + \big(\sqrt{1 - A_{11}} + \sqrt{1 - A_{22}}\big)^2 = \lambda.\end{aligned}$$

Therefore, using (3.79) and the inequality (3.78), we get

$$\begin{aligned}(1 + A_{11}A_{22})A_{12}A_{21} &\geqslant \lambda[1 - (A_{11} + A_{22})A_{12}A_{21} + A_{11}A_{22}(A_{12}A_{21})^2]\\ &\geqslant \lambda[1 - (A_{11} + A_{22} - A_{11}A_{22})A_{12}A_{21}],\end{aligned}$$

which contradicts (2.9).

The contradictions obtained in (a)–(c) prove that the problem (3.49), (3.50), (3.2) has only the trivial solution. □

4. Counter-examples

In this section, counter-examples are constructed verifying that the results obtained above are optimal in a certain sense.

Example 4.1. Let $\sigma_{ij} \in \{-1, 1\}$, $h_{ij} \in L\big([a, b]; \mathbb{R}_+\big)$ $(i, j = 1, 2)$ be such that

$$\sigma_{11} = -1, \qquad \int_a^b h_{11}(s)\,\mathrm{d}s \geqslant 3.$$

It is clear that there exist $t_0 \in\,]a, b[$ and $t_1 \in\,]t_0, b]$ such that

$$\int_a^{t_0} h_{11}(s)\,\mathrm{d}s = 1, \qquad \int_{t_0}^{t_1} h_{11}(s)\,\mathrm{d}s = 2.$$

Let operators $l_{ij} \in \mathcal{P}_{ab}$ $(i, j = 1, 2)$ be defined by

$$l_{ij}(v)(t) \overset{\text{def}}{=} h_{ij}(t)v\big(\tau_{ij}(t)\big) \quad \text{for } t \in [a, b],\ v \in C([a, b]; \mathbb{R}), \tag{4.1}$$

where $\tau_{12}(t)=a$, $\tau_{21}(t)=a$, $\tau_{22}(t)=a$ for $t\in[a,b]$, and

$$\tau_{11}(t)=\begin{cases} t_1 & \text{for } t\in[a,t_0[,\\ t_0 & \text{for } t\in[t_0,b].\end{cases}$$

Put

$$u(t)=\begin{cases} \displaystyle\int_a^t h_{11}(s)\,\mathrm{d}s & \text{for } t\in[a,t_0[,\\ \displaystyle 1-\int_{t_0}^t h_{11}(s)\,\mathrm{d}s & \text{for } t\in[t_0,b].\end{cases}$$

It is easy to verify that $(u,0)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i\equiv 0$ and $c_i=0$ $(i=1,2)$.

An analogous example can be constructed for the case

$$\sigma_{22}=-1,\qquad \int_a^b h_{22}(s)\,\mathrm{d}s\geqslant 3.$$

This example shows that the constant 3 on the right-hand side of the inequalities in (2.3) is optimal and cannot be weakened.

Example 4.2. Let $\sigma_{ii}=-1$, $\sigma_{i3-i}=1$ for $i=1,2$ and let $h_{ij}\in L\bigl([a,b];\mathbb{R}_+\bigr)$ $(i=1,2)$ be such that

$$\int_a^b h_{11}(s)\,\mathrm{d}s\leqslant 1,\qquad \int_a^b h_{22}(s)\,\mathrm{d}s\leqslant 1,\qquad \int_a^b h_{12}(s)\,\mathrm{d}s\int_a^b h_{21}(s)\,\mathrm{d}s\geqslant 1.$$

It is clear that there exists $t_0\in\,]a,b]$ satisfying

$$\int_a^{t_0} h_{12}(s)\,\mathrm{d}s\int_a^{t_0} h_{21}(s)\,\mathrm{d}s=1.$$

Let operators $l_{ij}\in\mathcal{P}_{ab}$ $(i,j=1,2)$ be defined by (4.1), where $\tau_{ii}(t)=a$ and $\tau_{i3-i}(t)=t_0$ for $t\in[a,b]$ $(i=1,2)$. Put

$$u_1(t)=\int_a^t h_{12}(s)\,\mathrm{d}s,\quad u_2(t)=\int_a^{t_0} h_{12}(s)\,\mathrm{d}s\int_a^t h_{21}(s)\,\mathrm{d}s\quad \text{for } t\in[a,b].$$

It is easy to verify that $(u_1,u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i\equiv 0$ and $c_i=0$ $(i=1,2)$.

This example shows that the strict inequality (2.4) in Theorem 2.1 cannot be replaced by the nonstrict one provided $\max\{A_{11},A_{22}\}\leqslant 1$.

E x a m p l e 4.3. Let $\sigma_{ii} = -1$, $\sigma_{i3-i} = 1$ for $i = 1, 2$ and let functions $h_{11}, h_{22} \in L\big([a,b]; \mathbb{R}_+\big)$ be such that

$$\int_a^b h_{11}(s)\,\mathrm{d}s \leqslant 1, \qquad 1 < \int_a^b h_{22}(s)\,\mathrm{d}s < 3. \tag{4.2}$$

Obviously, there exists $t_0 \in \,]a,b[$ such that

$$\int_a^{t_0} h_{22}(s)\,\mathrm{d}s = \frac{\int_a^b h_{22}(s)\,\mathrm{d}s - 1}{2}.$$

Furthermore, we choose $h_{12}, h_{21} \in L\big([a,b]; \mathbb{R}_+\big)$ with the properties

$$h_{21}(t) = 0 \quad \text{for } t \in [t_0, b]$$

and

$$\int_a^b h_{12}(s)\,\mathrm{d}s \int_a^b h_{21}(s)\,\mathrm{d}s \geqslant 1 - \frac{1}{4}\Big(\int_a^b h_{22}(s)\,\mathrm{d}s - 1\Big)^2.$$

It is clear that there exists $t_1 \in \,]a,b]$ satisfying

$$\int_a^{t_1} h_{12}(s)\,\mathrm{d}s \int_a^{t_0} h_{21}(s)\,\mathrm{d}s = 1 - \frac{1}{4}\Big(\int_a^b h_{22}(s)\,\mathrm{d}s - 1\Big)^2.$$

Let operators $l_{ij} \in \mathcal{P}_{ab}$ $(i,j = 1,2)$ be defined by (4.1), where $\tau_{11}(t) = a$, $\tau_{12}(t) = t_0$, $\tau_{21}(t) = t_1$ for $t \in [a,b]$, and

$$\tau_{22}(t) = \begin{cases} b & \text{for } t \in [a, t_0[, \\ t_0 & \text{for } t \in [t_0, b]. \end{cases} \tag{4.3}$$

Put

$$u_1(t) = \int_a^t h_{12}(s)\,\mathrm{d}s \quad \text{for } t \in [a,b],$$

$$u_2(t) = \begin{cases} \displaystyle\int_a^{t_1} h_{12}(s)\,\mathrm{d}s \int_a^t h_{21}(s)\,\mathrm{d}s & \\ \displaystyle\quad + \frac{1}{2}\Big(\int_a^b h_{22}(s)\,\mathrm{d}s - 1\Big)\int_a^t h_{22}(s)\,\mathrm{d}s & \text{for } t \in [a, t_0[, \\ \displaystyle 1 - \int_{t_0}^t h_{22}(s)\,\mathrm{d}s & \text{for } t \in [t_0, b]. \end{cases}$$

It is easy to verify that $(u_1, u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$ $(i = 1, 2)$.

An analogous example can be constructed for the case

$$1 < \int_a^b h_{11}(s)\,\mathrm{d}s < 3, \qquad \int_a^b h_{22}(s)\,\mathrm{d}s \leqslant 1. \tag{4.4}$$

This example shows that the strict inequality (2.4) in Theorem 2.1 cannot be replaced by the nonstrict one provided $\min\{A_{11}, A_{22}\} \leqslant 1$, $\max\{A_{11}, A_{22}\} > 1$ and $\omega = 1$.

Example 4.4. Let $\sigma_{ii} = -1$, $\sigma_{i3-i} = 1$ for $i = 1, 2$ and let functions $h_{11}, h_{22} \in L\big([a,b]; \mathbb{R}_+\big)$ be such that

$$1 < \int_a^b h_{ii}(s)\,\mathrm{d}s < 3 \quad \text{for } i = 1, 2.$$

Obviously, there exist $t_1, t_2 \in\,]a,b[$ satisfying

$$\int_a^{t_i} h_{ii}(s)\,\mathrm{d}s = \frac{\int_a^b h_{ii}(s)\,\mathrm{d}s - 1}{2} \quad \text{for } i = 1, 2.$$

Furthermore, we choose $h_{12}, h_{21} \in L\big([a,b]; \mathbb{R}_+\big)$ with the properties

$$h_{12}(t) = 0 \quad \text{for } t \in [t_1, b], \qquad h_{21}(t) = 0 \quad \text{for } t \in [a, t_2],$$

and

$$\begin{aligned}\int_a^b h_{12}(s)\,\mathrm{d}s \int_a^b h_{21}(s)\,\mathrm{d}s \\ \geqslant \left[1 - \frac{1}{4}\left(\int_a^b h_{11}(s)\,\mathrm{d}s - 1\right)^2\right]\left[1 - \frac{1}{4}\left(\int_a^b h_{22}(s)\,\mathrm{d}s - 1\right)^2\right].\end{aligned}$$

It is clear that there exists $\alpha \in\,]0,1]$ such that

$$\begin{aligned}\alpha \int_a^{t_1} h_{12}(s)\,\mathrm{d}s \int_{t_2}^b h_{21}(s)\,\mathrm{d}s \\ = \left[1 - \frac{1}{4}\left(\int_a^b h_{11}(s)\,\mathrm{d}s - 1\right)^2\right]\left[1 - \frac{1}{4}\left(\int_a^b h_{22}(s)\,\mathrm{d}s - 1\right)^2\right].\end{aligned}$$

Put

$$u_1(t)=\begin{cases}\dfrac{\int_a^b h_{11}(s)\,\mathrm{d}s-1}{2}\displaystyle\int_a^t h_{11}(s)\,\mathrm{d}s+\dfrac{\alpha\int_{t_2}^b h_{21}(s)\,\mathrm{d}s\int_a^t h_{12}(s)\,\mathrm{d}s}{1-\frac{1}{4}(\int_a^b h_{22}(s)\,\mathrm{d}s-1)^2} & \text{for } t\in[a,t_1[,\\ 1-\displaystyle\int_{t_1}^t h_{11}(s)\,\mathrm{d}s & \text{for } t\in[t_1,b],\end{cases}$$

$$u_2(t)=\begin{cases}-\dfrac{\int_{t_2}^b h_{21}(s)\,\mathrm{d}s}{1-\frac{1}{4}(\int_a^b h_{22}(s)\,\mathrm{d}s-1)^2}\displaystyle\int_a^t h_{22}(s)\,\mathrm{d}s & \text{for } t\in[a,t_2[,\\ \displaystyle\int_{t_2}^t h_{21}(s)\,\mathrm{d}s+\dfrac{\int_{t_2}^b h_{21}(s)\,\mathrm{d}s\int_a^{t_2} h_{22}(s)\,\mathrm{d}s(\int_{t_2}^t h_{22}(s)\,\mathrm{d}s-1)}{1-\frac{1}{4}(\int_a^b h_{22}(s)\,\mathrm{d}s-1)^2} & \text{for } t\in[t_2,b].\end{cases}$$

Since $u_2(t_2)<0$ and $u_2(b)>0$, there exists $t_0\in\,]t_2,b]$ satisfying $u_2(t_0)=\alpha u_2(b)$. Let operators $l_{ij}\in\mathcal{P}_{ab}$ $(i,j=1,2)$ be defined by (4.1), where $\tau_{12}(t)=t_0$, $\tau_{21}(t)=t_1$ for $t\in[a,b]$, and

$$\begin{aligned}\tau_{11}(t)&=\begin{cases}b & \text{for } t\in[a,t_1[,\\ t_1 & \text{for } t\in[t_1,b],\end{cases}\\ \tau_{22}(t)&=\begin{cases}b & \text{for } t\in[a,t_2[,\\ t_2 & \text{for } t\in[t_2,b].\end{cases}\end{aligned}\tag{4.5}$$

It is easy to verify that $(u_1,u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i\equiv 0$ and $c_i=0$ $(i=1,2)$.

This example shows that the strict inequality (2.4) in Theorem 2.1 cannot be replaced by the nonstrict one provided $\min\{A_{11},A_{22}\}>1$ and $\omega=1$.

Example 4.5. Let $\sigma_{11}=-1$, $\sigma_{12}=1$, $\sigma_{21}=-1$, $\sigma_{22}=-1$ and let $h_{11},h_{22}\in L\big([a,b];\mathbb{R}_+\big)$ be such that (4.2) holds. Obviously, there exists $t_0\in\,]a,b[$ such that

$$\int_a^{t_0} h_{22}(s)\,\mathrm{d}s=1.$$

Furthermore, we choose $h_{12},h_{21}\in L\big([a,b];\mathbb{R}_+\big)$ with the properties

$$h_{21}(t)=0\quad\text{for } t\in[a,t_0]$$

and

$$\int_a^b h_{12}(s)\,\mathrm{d}s\int_a^b h_{21}(s)\,\mathrm{d}s\geqslant 3-\int_a^b h_{22}(s)\,\mathrm{d}s.$$

It is clear that there exists $t_1\in\,]a,b]$ satisfying

$$\int_a^{t_1} h_{12}(s)\,\mathrm{d}s\int_{t_0}^b h_{21}(s)\,\mathrm{d}s=2-\int_{t_0}^b h_{22}(s)\,\mathrm{d}s.$$

Let operators $l_{ij} \in \mathcal{P}_{ab}$ $(i, j = 1, 2)$ be defined by (4.1), where $\tau_{11}(t) = a$, $\tau_{12}(t) = t_0$, $\tau_{21}(t) = t_1$ for $t \in [a, b]$, and τ_{22} is given by (4.3). Put

$$u_1(t) = \int_a^t h_{12}(s)\,\mathrm{d}s \quad \text{for } t \in [a, b],$$
$$u_2(t) = \begin{cases} \displaystyle\int_a^t h_{22}(s)\,\mathrm{d}s & \text{for } t \in [a, t_0[,\\ \displaystyle 1 - \int_a^{t_1} h_{12}(s)\,\mathrm{d}s \int_{t_0}^t h_{21}(s)\,\mathrm{d}s - \int_{t_0}^t h_{22}(s)\,\mathrm{d}s & \text{for } t \in [t_0, b]. \end{cases}$$

It is easy to verify that $(u_1, u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i \equiv 0$ and $c_i = 0$ $(i = 1, 2)$.

An analogous example can be constructed for the case when the functions $h_{11}, h_{22} \in L([a, b]; \mathbb{R}_+)$ satisfy (4.4).

This example shows that the strict inequality (2.6) in Theorem 2.2 cannot be replaced by the nonstrict one provided $\min\{A_{11}, A_{22}\} \leqslant 1$, $\max\{A_{11}, A_{22}\} > 1$ and $\varrho = 1$.

Example 4.6. Let $\sigma_{11} = -1$, $\sigma_{12} = 1$, $\sigma_{21} = -1$, $\sigma_{22} = -1$ and let $h_{11}, h_{22} \in L([a, b]; \mathbb{R}_+)$ be such that

$$1 < \int_a^b h_{11}(s)\,\mathrm{d}s \leqslant \int_a^b h_{22}(s)\,\mathrm{d}s < 3.$$

Obviously, there exist $t_1, t_2 \in \,]a, b[$ satisfying

$$\int_a^{t_1} h_{11}(s)\,\mathrm{d}s = \frac{\int_a^b h_{11}(s)\,\mathrm{d}s - 1}{2}, \qquad \int_a^{t_2} h_{22}(s)\,\mathrm{d}s = 1.$$

Furthermore, we choose $h_{12}, h_{21} \in L([a, b]; \mathbb{R}_+)$ with the properties

$$h_{12}(t) = 0 \quad \text{for } t \in [t_1, b], \qquad h_{21}(t) = 0 \quad \text{for } t \in [a, t_2],$$

and

$$\int_a^b h_{12}(s)\,\mathrm{d}s \int_a^b h_{21}(s)\,\mathrm{d}s \geqslant \left(3 - \int_a^b h_{22}(s)\,\mathrm{d}s\right)\left[1 - \frac{1}{4}\left(\int_a^b h_{11}(s)\,\mathrm{d}s - 1\right)^2\right].$$

It is clear that there exist $\alpha \in \,]0, 1]$ and $t_0 \in \,]a, t_2]$ such that

$$\alpha \int_a^{t_1} h_{12}(s)\,\mathrm{d}s \int_{t_2}^b h_{21}(s)\,\mathrm{d}s = \left(2 - \int_{t_2}^b h_{22}(s)\,\mathrm{d}s\right)\left[1 - \frac{1}{4}\left(\int_a^b h_{11}(s)\,\mathrm{d}s - 1\right)^2\right]$$

and

$$\int_a^{t_0} h_{22}(s)\,\mathrm{d}s = \alpha.$$

Let operators $l_{ij} \in \mathcal{P}_{ab}$ $(i,j=1,2)$ be defined by (4.1), where $\tau_{12}(t)=t_0$, $\tau_{21}(t)=t_1$ for $t\in[a,b]$, and τ_{11},τ_{22} are given by (4.5). Put

$$u_1(t) = \begin{cases} \dfrac{\alpha_0}{2\int_{t_2}^b h_{21}(s)\,\mathrm{d}s}\displaystyle\int_a^t h_{11}(s)\,\mathrm{d}s + \alpha\int_a^t h_{12}(s)\,\mathrm{d}s & \text{for } t\in[a,t_1[,\\ \dfrac{2-\int_{t_2}^b h_{22}(s)\,\mathrm{d}s}{\int_{t_2}^b h_{21}(s)\,\mathrm{d}s}\Big(1-\displaystyle\int_{t_1}^t h_{11}(s)\,\mathrm{d}s\Big) & \text{for } t\in[t_1,b], \end{cases}$$

$$u_2(t) = \begin{cases} \displaystyle\int_a^t h_{22}(s)\,\mathrm{d}s & \text{for } t\in[a,t_2[,\\ 1-\dfrac{\alpha\int_a^{t_1} h_{12}(s)\,\mathrm{d}s\int_{t_2}^t h_{21}(s)\,\mathrm{d}s}{1-\frac14\big(\int_a^b h_{11}(s)\,\mathrm{d}s-1\big)^2} - \displaystyle\int_{t_2}^t h_{22}(s)\,\mathrm{d}s & \text{for } t\in[t_2,b], \end{cases}$$

where

$$\alpha_0 = \Big(2-\int_{t_2}^b h_{22}(s)\,\mathrm{d}s\Big)\Big(\int_a^b h_{11}(s)\,\mathrm{d}s-1\Big).$$

It is easy to verify that $(u_1,u_2)^T$ is a nontrivial solution of the problem (1.1), (1.2) with $q_i\equiv 0$ and $c_i=0$ $(i=1,2)$.

An analogous example can be constructed for the case when

$$1<\int_a^b h_{22}(s)\,\mathrm{d}s \leqslant \int_a^b h_{11}(s)\,\mathrm{d}s < 3.$$

This example shows that the strict inequality (2.6) in Theorem 2.2 cannot be replaced by the nonstrict one provided $\min\{A_{11},A_{22}\}>1$ and $\varrho=1$.

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