PARALLELISMS BETWEEN DIFFERENTIAL AND DIFFERENCE EQUATIONS

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Abstract. The paper deals with the higher-order ordinary differential equations and the analogous higher-order difference equations and compares the corresponding fundamental concepts. Important dissimilarities appear for the moving frame method.

Keywords: ordinary differential equation, ordinary difference equation, first integral, symmetry, infinitesimal symmetry, moving frame

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Introduction

The parallelism between differential and difference equations was discussed in the inspirative article [1]. Retaining the notation [1], we propose an alternative approach where the interrelations become more transparent. Especially the fundamental concepts clarify: compare (e.g.) first integrals, shift operator, symmetries and infinitesimal transformations as stated in [1] with our conceptions introduced below. On the other hand, the common method of moving frames [2] not mentioned in [1] needs essential change for the case of the difference equations.

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1. Differential equation

We consider the equation

$$u^{(N)} = F(x, u, u', \dots, u^{(N-1)}) \qquad \Big(\, ' = \frac{\mathrm{d}}{\mathrm{d}x},\ u = u(x)\Big) \tag{1.1}$$

in the real domain, where F is a smooth function. Employing the jet space $\mathbf{M}$ with coordinates

$$\mathbf{M}\colon\ x, u^0, u^1, \dots, u^{N-1}, \tag{1.2}$$

equation (1.1) is expressed by the first order system

$$\frac{\mathrm{d}u^k}{\mathrm{d}x} = u^{k+1} \quad (k = 0, \dots, N-2), \qquad \frac{\mathrm{d}u^{N-1}}{\mathrm{d}x} = F(x, u^0, \dots, u^{N-1}) \tag{1.3}$$

and its *solution* is represented by the curve (the left-hand figure)

$$\mathbf{P}(x) = (x, u^0(x), \dots, u^{N-1}(x)) = (x, u(x), \dots, u^{(N-1)}(x)) \in \mathbf{M}. \tag{1.4}$$

Such a solution is uniquely determined by any of its points

$$\mathbf{P}(\bar{x}) = (\bar{x}, u^0(\bar{x}), \dots, u^{N-1}(\bar{x})) \in \mathbf{M} \quad (\text{fixed } \bar{x} \in \mathbb{R}). \tag{1.5}$$

A function

$$\Phi = \Phi(x, u^0, u^1, \dots, u^{N-1}) \tag{1.6}$$

on $\mathbf{M}$ is called the *first integral* of equation (1.1) if $\Phi(\mathbf{P}(x)) = \text{const.}$ for every solution (1.4). *It follows that the restriction of the function Φ to $x = \bar{x}$ can be prescribed.*

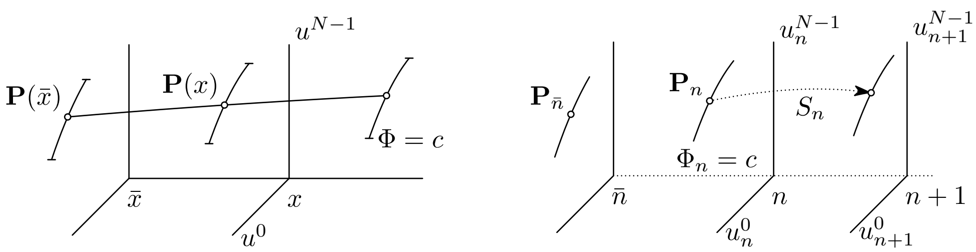

Fig. 1

2. Difference equation

We consider the equation

$$u_{n+N} = F_n(u_n, \dots, u_{n+N-1}) \quad (u_n \in \mathbb{R},\ n \in \mathbb{Z}) \tag{2.1}$$

where F_n are smooth functions. We suppose that it is exactly of the order N, that is, (2.1) can be equivalently expressed as $u_n = G_{n+1}(u_{n+1}, \dots, u_{n+N})$. After a formal adjustment of indices, we obtain the equation

$$u_{n-1} = G_n(u_n, \dots, u_{n+N-1}) \quad (u_n \in \mathbb{R},\ n \in \mathbb{Z}) \tag{2.2}$$

equivalent to (2.1).

In order to obtain a parallel theory, we introduce discrete counterparts u_n^k to the derivatives $u^{(k)}(x)$. In full detail, we introduce spaces $\mathbf{M}_n$ with coordinates

$$\mathbf{M}_n\colon\ u_n^0, u_n^1, \dots, u_n^{N-1} \quad (n \in \mathbb{Z}). \tag{2.3}$$

Roughly speaking, the coordinate u_n^k corresponds to the variable u_{n+k} appearing in (2.1). It follows that equation (2.1) turns into the first order system

$$u_{n+1}^k = u_n^{k+1} \quad (k = 0, \dots, N-2), \qquad u_{n+1}^{N-1} = F_n(u_n^0, \dots, u_n^{N-1}) \quad (n \in \mathbb{Z}) \tag{2.4}$$

which may be regarded as a shift transformation $S_n\colon \mathbf{M}_n \to \mathbf{M}_{n+1}$. (More precisely, S_n is given by the formulae

$$S_n^* u_{n+1}^k = u_n^{k+1} \quad (k = 0, \dots, N-2), \qquad S_n^* u_{n+1}^{N-1} = F_n(u_n^0, \dots, u_n^{N-1}) \quad (n \in \mathbb{Z})$$

which are a mere transcription of (2.4). Alternatively, equation (2.2) is expressed by

$$u_{n-1}^0 = G_n(u_n^0, \dots, u_n^{N-1}), \quad u_{n-1}^1 = u_n^0, \dots, u_{n-1}^{N-1} = u_n^{N-2} \tag{2.5}$$

and may be regarded as a transformation $S_n^{-1}\colon \mathbf{M}_n \to \mathbf{M}_{n-1}$.) Then a *solution* is represented by a sequence of points

$$\mathbf{P}_n = (u_n^0, \dots, u_n^{N-1}) \in \mathbf{M}_n \quad (u_n^k \in \mathbb{R},\ n \in \mathbb{Z}) \tag{2.6}$$

satisfying (2.4) (hence $S_n \mathbf{P}_n = \mathbf{P}_{n+1}$, see the above right-hand figure). It follows that it is determined by any of its points

$$\mathbf{P}_{\bar n} = (u_{\bar n}^0, \dots, u_{\bar n}^{N-1}) \in \mathbf{M}_{\bar n} \quad (u_{\bar n}^k \in \mathbb{R},\ \text{fixed } \bar n \in \mathbb{Z}). \tag{2.7}$$

A sequence of functions

$$\Phi_n = \Phi_n(u_n^0, \dots, u_n^{N-1}) \quad (n \in \mathbb{Z}) \tag{2.8}$$

is called the *first integral* of equation (2.1) if the values $\Phi_n(\mathbf{P}_n)$ are independent of n for every solution (2.6). *It follows that the term* $\Phi_{\bar{n}}$ (fixed $\bar{n} \in \mathbb{Z}$) *can be prescribed.*

3. Symmetries of differential equation

We are interested in mappings Γ that preserve differential equation (1.1), however, reasonable counterparts for the difference equation (2.1) represent only the x-preserving mappings. In terms of coordinates (1.2), such *x-preserving symmetry* $\Gamma\colon \mathbf{M} \to \mathbf{M}$ is given by certain formulae

$$\Gamma^* x = x,\ \Gamma^* u^k = \hat{u}^k(x, u^0, \dots, u^{N-1}) \quad (k = 0, \dots, N-1) \tag{3.1}$$

and equations (1.3) are preserved if and only if

$$D\hat{u}^k = \hat{u}^{k+1} \quad (k = 0, \dots, N-2), \qquad D\hat{u}^{N-1} = F(x, \hat{u}^0, \dots, \hat{u}^{N-1}), \tag{3.2}$$

$$(D = \partial/\partial x + u^1 \partial/\partial u^0 + \dots + u^{N-1}\partial/\partial u^{N-2} + F\partial/\partial u^{N-1}).$$

We also recall the *infinitesimal symmetry*

$$\Gamma^* x = x, \quad \Gamma^* u^k = u^k + \varepsilon Q^k(x, u^0, \dots, u^{N-1}) + \dots \tag{3.3}$$

represented by the vector field

$$X = \sum Q^k(x, u^0, \dots, u^{N-1}) \frac{\partial}{\partial u^k} \tag{3.4}$$

on the space $\mathbf{M}$. The coefficients Q^k satisfy the identities

$$DQ^k = Q^{k+1} \quad (k = 0, \dots, N-2), \qquad DQ^{N-1} = \sum Q^k \frac{\partial F}{\partial u^k}.$$

Since symmetries are mappings permuting the solutions, it follows that *the restriction of symmetries to the fiber* $x = \bar{x}$ (fixed $\bar{x} \in \mathbb{R}$) *can be arbitrarily prescribed.*

There is a huge literature on the symmetry theory of differential equations. In particular, we refer to [2] for the moving frame method. The particular *pointwise subcase* when $\Gamma^* u^0 = \hat{u}^0(x, u^0)$ is supposed is not easier and nontrivial pointwise symmetries of a given equation (1.1) *need not exist.*

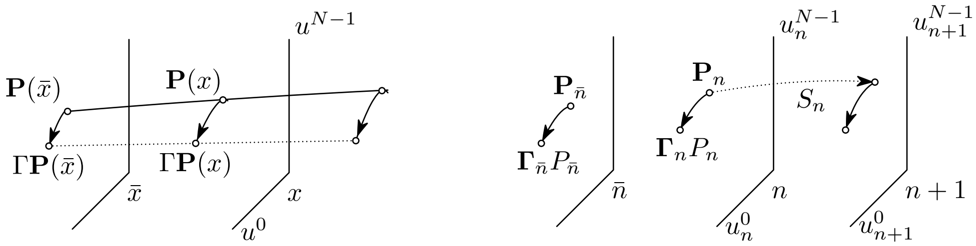

Fig. 2

4. Symmetries of difference equations

We introduce mappings $\Gamma_n\colon \mathbf{M}_n \to \mathbf{M}_n$ $(n \in \mathbb{Z})$ given by certain formulae

$$\Gamma_n^* u_n^k = \hat{u}_n^k(u_n^0, \ldots, u_n^{N-1}) \quad (k = 0, \ldots, N-1). \tag{4.1}$$

Then equations (2.4) are preserved if and only if the requirements

$$\hat{u}_{n+1}^k = \hat{u}_n^{k+1} \quad (k = 0, \ldots, N-2), \quad \hat{u}_{n+1}^{N-1} = F_n(\hat{u}_n^0, \ldots, \hat{u}_n^{N-1}) \tag{4.2}$$

are satisfied by virtue of (2.4). In full detail, we have the requirements

$$\hat{u}_{n+1}^k(u_n^1, \ldots, u_n^{N-1}, F_n) = \begin{cases} \hat{u}_n^{k+1} & (\text{if } k = 0, \ldots, N-2), \\ \Gamma_n^* F_n & (\text{if } k = N-1). \end{cases}$$

The sequence Γ_n $(n \in \mathbb{Z})$ represents a *symmetry* of difference equation (2.1). The relevant *infinitesimal symmetry*

$$\Gamma_n^* u_n^k = u_n^k + \varepsilon Q_n^k(u_n^0, \ldots, u_n^{N-1}) + \ldots \tag{4.3}$$

is analogously represented by the sequence of vector fields

$$X_n = \sum Q_n^k(u_n^0, \ldots, u_n^{N-1}) \frac{\partial}{\partial u_n^k} \quad (n \in \mathbb{Z}) \tag{4.4}$$

on the spaces $\mathbf{M}_n$. The coefficients Q_n^k satisfy the identities

$$Q_{n+1}^k(u_n^1, \ldots, u_n^{N-1}, F_n) = \begin{cases} Q_n^{k+1} & (\text{if } k = 0, \ldots, N-2), \\ \sum Q_n^l \partial F_n / \partial u^l & (\text{if } k = N-1). \end{cases}$$

Since symmetries are just the mappings that permute the solutions, it follows that *the data* $\Gamma_{\bar{n}}$ and $Q_{\bar{n}}^k$ (fixed $\bar{n} \in \mathbb{Z}$) *can be arbitrarily prescribed.*

We will mention in more detail the particular *pointwise subcase* when the transformation formula

$$\Gamma_n^* u_n^0 = g_n(u_n^0) \; (= g_n(u_n)) \quad (n \in \mathbb{Z}, \text{ abbreviation } g_n = \hat{u}_n^0) \tag{4.5}$$

is supposed. Then the remarkable identities

$$\begin{gathered} \Gamma_n^* u_n^k = \Gamma_n^* u_{n+k}^0 = g_{n+k}(u_{n+k}^0) = g_{n+k}(u_n^k) \quad (k = 0, \dots, N-1), \\ \Gamma_n^* F_n = \hat{u}_{n+1}^{N-1} = g_{n+N}(u_{n+1}^{N-1}) = g_{n+N}(F_n) \end{gathered} \tag{4.6}$$

follow from (4.2) and (2.4).

We refer to [1] for the infinitesimal pointwise symmetries. Instead we will mention the alternative method of moving frames. It is rather dissimilar from the common theory of symmetries of differential equations [2].

5. Moving frames for pointwise symmetries

Functions g_{n+k}, g_{n+N} in identities (4.6) are unknown. Therefore identities (4.6) can be expressed by saying that Γ_n transforms every level set $u_n^k = c_n^k$, $F_n = c_n$ (c_n^k, c_n are constants) again into such a level set. Alternatively: every equation $\mathrm{d}u_n^k = 0$, $\mathrm{d}F_n = 0$ is preserved. Still otherwise: *let us consider spaces* $\mathbf{N}_n$ *with coordinates and differential forms*

$$\begin{aligned} &\mathbf{N}_n \colon \; u_n^0, \dots, u_n^{N-1}, a_n^0, \dots, a_n^{N-1}, a_n \quad (n \in \mathbb{Z}), \\ &\alpha_n^k = a_n^k \, \mathrm{d}u_n^k \quad (k = 0, \dots, N-1), \quad \alpha_n = a_n \, \mathrm{d}F_n. \end{aligned} \tag{5.1}$$

Then

$$\Gamma_n^* \alpha_n^k = \alpha_n^k \quad (k = 0, \dots, N-1), \quad \Gamma_n^* \alpha_n = \alpha_n \tag{5.2}$$

if the transformation formulae (4.5), (4.6) *are appropriately completed by the additional coordinates.* One can directly verify that the additional transformation formulae (in fact needless in the sequel) ensuring the invariance (5.2) are

$$\Gamma_n^* a_n^k = \frac{a_n^k}{g'_{n+k}(u_n^k)} \; (k = 0, \dots, N-1), \quad \Gamma_n^* a_n = \frac{a_n}{g'_{n+N}(F_n)}. \tag{5.3}$$

Identities (5.2) express the invariance of the differential forms α_n^k, α_n and we will search for more invariants in order to determine the mapping $\Gamma_n\colon \mathbf{N}_n \to \mathbf{N}_n$. The indices $n \in \mathbb{Z}$ may be kept fixed in this section.

First of all, we have

$$\alpha_n = \sum A_n^k \alpha_n^k \quad \left(A_n^k = \frac{a_n}{a_n^k}\partial F_n/\partial u_n^k\right)$$

and it follows that $A_n^k = \Gamma_n^* A_n^k$ are invariants. Assume $\partial F_n/\partial u_n^k \neq 0$ for a certain k. Since the level set $A_n^k = 1$ is preserved, we may restrict our calculations to this level set. Alternatively, the coordinate a_n^k in (5.1) may be omitted and the coefficient a_n^k in (5.1) is replaced by $a_n \partial F_n/\partial u_n^k$. (This is the common procedure in the moving frame method.) Repeatedly applying this reduction, we have the invariant forms

$$\alpha_n^k = \begin{cases} a_n \partial F_n/\partial u_n^k \cdot \mathrm{d}u_n^k & (\text{if } \partial F_n/\partial u_n^k \neq 0) \\ a_n^k \,\mathrm{d}u_n^k & (\text{if } \partial F_n/\partial u_n^k = 0) \end{cases} \tag{6.1}$$

on the "reduced" space $\mathbf{N}_n$ (some coordinates a_n^k are not occuring here).

Secondly, clearly

$$\mathrm{d}\alpha_n = \mathrm{d}a_n \wedge \mathrm{d}F_n = \beta_n \wedge \alpha_n \quad \left(\beta_n = \frac{\mathrm{d}a_n}{a_n} + b_n \,\mathrm{d}F_n\right) \tag{6.2}$$

where b_n is regarded as a new variable. Since this β_n is the most general differential form satisfying (6.2), it follows that $\beta_n = \Gamma_n^* \beta_n$ is invariant. The invariance is ensured if

$$\Gamma_n^* b_n = \frac{1}{g'_{n+N}(F_n)}\left(b_n + a_n \frac{g''_{n+N}(F_n)}{g'_{n+N}(F_n)}\right)$$

by using (4.6) and (5.3). (The formula will be in fact needless.)

Thirdly, assuming $\partial F_n/\partial u_n^k \neq 0$ and $\partial F_n/\partial u_n^l \neq 0$, (6.1) gives

$$\mathrm{d}\alpha_n^k = \left(\beta_n + \frac{1}{a_n}\sum (A_n^{kl} - b_n)\alpha_n^l\right) \wedge \alpha_n^k \quad \left(A_n^{kl} = \frac{\partial^2 F_n/\partial u_n^k \partial u_n^l}{\partial F_n/\partial u_n^k \cdot \partial F_n/\partial u_n^l}\right) \tag{6.3}$$

with help of (6.2). The coefficients are invariant. Since every level set $A_n^{kl} - b_n = 0$ with $k \neq l$ is preserved, we may restrict the calculations to the level set and put $b_n = A_n^{kl}$ for a certain appropriate k, l (where $k \neq l$).

Fourthly, after this restriction, we have the invariant form

$$\mathrm{d}\beta_n = \mathrm{d}A_n^{kl} \wedge \mathrm{d}F_n = \frac{1}{(a_n)^2}\sum \mathcal{A}_n^{rs} \alpha^r \wedge \alpha^s \tag{6.4}$$

where

$$\frac{1}{(a_n)^2}\mathcal{A}_n^{rs} = \frac{1}{(a_n)^2}\Big(\frac{\partial A_n^{kl}/\partial u_n^r}{\partial F_n/\partial u_n^s} - \frac{\partial A_n^{kl}/\partial u_n^s}{\partial F_n/\partial u_n^r}\Big) \tag{6.5}$$

are invariant coefficients. If $\mathcal{A}_n^{rs} \neq 0$, the level set $\mathcal{A}_n^{rs}/(a_n)^2 = 1$ may be employed in order to determine a_n which clarifies the form β_n. Then $\beta_n = \sum I_n^k \alpha_n^k$, with the true invariant functions I_n^k depending only on the primary variables $u_n^0, \ldots, u_n^{N-1}$.

In particular, we have proved the following assertion needful below.

Theorem 6.1. *Assume $\partial F_n/\partial u_n^k \neq 0$ ($n \in \mathbb{Z};\ k = 0, \ldots, N-1$). Then the pointwise symmetry Γ_n ($n \in \mathbb{Z}$) given by (4.5) preserves all differential forms*

$$\alpha_n^k = a_n \frac{\partial F_n}{\partial u_n^k}\,\mathrm{d}u_n^k,\ \beta_n = \frac{\mathrm{d}a_n}{a_n} + \frac{\partial^2 F_n/\partial u_n^k \partial u_n^l}{\partial F_n/\partial u_n^k \cdot \partial F_n/\partial u_n^l} \quad (n \in \mathbb{Z};\ k, l = 0, \ldots, N-1).$$

Here the additional parameters a_n are subject to formulae (5.3_2).

7. Examples

(i) *The linear equation.* Let us consider an equation

$$u_{n+N} = C_n^0 u_n + \ldots + C_n^{N-1} u_{n+N-1} \quad (C_n^k \in \mathbb{R}, C_n^0 \neq 0),$$

therefore $F_n = \sum C_n^k u_n^k$ in terms of variables (2.3). We have invariant forms

$$\alpha_n^k = a_n C_n^k\,\mathrm{d}u_n^k \quad (\text{if } C_n^k \neq 0), \qquad \beta_n = \frac{\mathrm{d}a_n}{a_n};$$

see (6.1), (6.2) where $b_n = A_n^{kl} = 0$ identically. The invariance equations (5.2) with $\Gamma_n^* u_n^k = g_{n+k}(u_n^k)$ together with the additional invariance $\Gamma_n^* \beta_n = \beta_n$ read $\Gamma_n^* a_n \cdot C_n^k\,\mathrm{d}g_{n+k} = a_n C_n^k\,\mathrm{d}u_n^k$, $\mathrm{d}\Gamma_n^* \ln a_n = \mathrm{d}\ln a_n$ whence

$$\Gamma_n^* a_n = E_n a_n,\ g_{n+k}(u_n^k) = \frac{1}{E_n}(u_n^k + D_n^k) \quad (E_n, D_n^k \in \mathbb{R})$$

and it follows that $E_n = E$ is independent of n and $D_n^k = D_{n+k}$. In terms of original variables, we have the substitution $u_{n+k} \mapsto g_{n+k}(u_{n+k}) = (u_{n+k} + D_{n+k})/E$ which can be simplified as

$$u_n \mapsto g_n(u_n) = \frac{1}{E}(u_n + D_n) \quad (n \in \mathbb{Z}).$$

This is indeed a symmetry if the recurrence $D_{n+N} = \sum C_n^k D_{n+k}$ $(n \in \mathbb{Z})$ is satisfied. It follows that *the point symmetries depend on the choice of* N *constants* $E \neq 0$, $D_0, \ldots, D_{N-1}$.

(ii) *Nontrivial invariants.* Let us consider the equation

$$u_{n+N} = u_n + \ln G_n \qquad (G_n = u_n + \ldots + u_{n+N-1}).$$

In terms of variables (2.3), we have invariant forms

$$\alpha_n^0 = a_n\Big(1 + \frac{1}{G_n}\Big)\,\mathrm{d}u_n^0, \qquad \alpha_n^k = a_n \frac{1}{G_n}\, u_n^k \quad (k = 1, \ldots, N-1),$$

and β_n which will be needless. One can then find that all coefficients to appear (e.g., $A_n^{kl}, b_n, \mathcal{A}_n^{kl}, I_n^k$) depend only on functions G_n and we may introduce the *invariance requirement* $\Gamma_n^* G_n = G_n$. This implies $\Gamma_n^*\,\mathrm{d}u_n^k = \mathrm{d}u_n^k$ whence

$$g_{n+k}(u_n^k) = \Gamma_n^* u_n^k = u_n^k + E_{n+k}.$$

In terms of original variables, we have the substitution $u_{n+k} \mapsto g_{n+k}(u_{n+k}) = u_{n+k} + E_{n+k}$ which can be simplified to

$$u_n \mapsto g_n(u_n) = u_n + E_n.$$

This is indeed a symmetry if the *compatible requirements*

$$E_{n+N} = E_n, E_n + \ldots + E_{n+N-1} = 0$$

equivalent to the *single recurrence*

$$E_{n+1} + \ldots + E_{n+N} = 0 \quad (n \in \mathbb{Z})$$

are satisfied. It follows that *the point symmetries depend on the choice of* $N-1$ *constants.* One can however see that a slight change of data may provide a difference equation without any point symmetries, see also [4] for a quite general discussion if $N = 2$ is supposed.

(iii) *The zero curvature examples.* Let us introduce a large class of equations

$$u_{n+N} = \varphi_n(p_n^0(u_n) + \ldots + p_n^{N-1}(u_{n+N-1})) \qquad (\varphi_n' \neq 0,\ {p_n^k}' \neq 0)$$

where φ_n, p_n^k are smooth functions. In terms of variables (2.3), we have

$$F_n = \varphi_n\Big(\sum p_n^k(u_n^k)\Big), \quad A_n^{kl} = \frac{\varphi_n''}{(\varphi_n')^2} = b_n, \quad \mathcal{A}_n^{kl} = 0.$$

We recall the invariant forms

$$\alpha_n^k = a_n\varphi_n' p_n^{k\,\prime}\,\mathrm{d}u_n^k = a_n\varphi_n'\,\mathrm{d}p_n^k, \quad \beta_n = \frac{\mathrm{d}a_n}{a_n} + \frac{\varphi_n''}{(\varphi_n')^2}\,\mathrm{d}\varphi_n = \mathrm{d}(\ln a_n\varphi_n').$$

The invariance of the forms β_n implies

$$\Gamma_n^*(a_n\varphi_n') = E_n a_n\varphi_n'$$

for appropriate nonvanishing constants $E_n \in \mathbb{R}$. Then the invariance of the forms α_n^k reads

$$E_n a_n\varphi_n'\Gamma_n^*\,\mathrm{d}p_n^k = a_n\varphi_n'\,\mathrm{d}p_n^k$$

whence

$$E_n\Gamma_n^* p_n^k = p_n^k + D_n^k \quad (k = 0,\dots,N-1)$$

for appropriate constants $D_n^k \in \mathbb{R}$. However,

$$\Gamma_n^* p_n^k = p_n^k(\Gamma_n^* u_n^k) = p_n^k(g_{n+k})$$

and therefore $E_n = E$ is independent of n and $D_n^k = D_{n+k}$. Altogether we have the substitution

$$\text{(7.1)} \quad u_{n+k} \mapsto g_{n+k}(u_{n+k}) = \left(p_n^k\right)^{-1}\left(\frac{1}{E}(p_n^k(u_{n+k}) + D_{n+k})\right) \quad (k = 0,\dots,N-1)$$

(with the inverse function $\left(p_n^k\right)^{-1}$) in terms of original variables. In particular,

$$\text{(7.2)} \qquad u_n \mapsto g_n(u_n) = \left(p_n^0\right)^{-1}\left(\frac{1}{E}(p_n^0(u_n) + D_n)\right)$$

and this result is quite reasonable in applications. We have an overdetermined system of requirements. *If requirements* (7.1), (7.2) *are compatible, they determine all possible point symmetries.*

(iv) *Continuation with* $N = 2$. Let us mention two intentionally simple examples

$$u_{n+2} = \frac{u_n u_{n+1}}{u_n + u_{n+1}}, \qquad u_{n+2} = \frac{1 - u_n u_{n+1}}{u_n + u_{n+1}}.$$

They can be transcribed as

$$u_{n+2} = \frac{1}{1/u_n + 1/u_{n+1}}, \qquad u_{n+2} = \frac{1}{\tan(\arctan u_n + \arctan u_{n+1})},$$

respectively. The choice of functions $\varphi_n = \varphi$, $p_n^k = p$ (independent of n, k) is obvious here and formula (7.2) may be applied. We obtain substitutions

$$u_n \mapsto \frac{1}{E^{-1}(1/u_n + D_n)}, \qquad u_n \mapsto \frac{1}{\tan E^{-1}(\arctan u_n + D_n)} \qquad (n \in \mathbb{Z})$$

and *they are true symmetries if and only if the recurrence* $D_{n+2} = D_n + D_{n+1}$ $(n \in \mathbb{Z})$ *is satisfied* (direct verification). The same formulae were obtained in [1] with the use of infinitesimal transformations and rather lengthy calculations.

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