SOME ESTIMATES FOR THE FIRST EIGENVALUE OF THE STURM-LIOUVILLE PROBLEM WITH A WEIGHT INTEGRAL CONDITION

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Abstract. Let $\lambda_1(Q)$ be the first eigenvalue of the Sturm-Liouville problem

$$y'' - Q(x)y + \lambda y = 0, \quad y(0) = y(1) = 0, \quad 0 < x < 1.$$

We give some estimates for $m_{\alpha,\beta,\gamma} = \inf\limits_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q)$ and $M_{\alpha,\beta,\gamma} = \sup\limits_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q)$, where $T_{\alpha,\beta,\gamma}$ is the set of real-valued measurable on $[0,1]$ $x^\alpha(1-x)^\beta$-weighted L_γ-functions Q with non-negative values such that $\int_0^1 x^\alpha(1-x)^\beta Q^\gamma(x)\,\mathrm{d}x = 1$ $(\alpha, \beta, \gamma \in \mathbb{R}, \gamma \neq 0)$.

Keywords: first eigenvalue, Sturm-Liouville problem, weight integral condition

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We consider the Sturm-Liouville problem

$$y'' - Q(x)y + \lambda y = 0, \quad x \in (0,1), \tag{1}$$

$$y(0) = y(1) = 0, \tag{2}$$

where Q is a real-valued measurable on $[0,1]$ function with non-negative values such that the integral condition

$$\int_0^1 x^\alpha(1-x)^\beta Q^\gamma(x)\,\mathrm{d}x = 1 \ (\alpha, \beta, \gamma \in \mathbb{R}, \gamma \neq 0) \tag{3}$$

holds whenever Q belongs to the $x^\alpha(1-x)^\beta$-weighted L_γ-space. The set of all functions Q of this kind we denote by $T_{\alpha,\beta,\gamma}$.

By a *solution* of problem (1)–(2) we mean an absolutely continuous function y on the segment $[0,1]$ such that $y(0)=y(1)=0$; y' is absolutely continuous in the interval $(0,1)$; equality (1) holds almost everywhere in the interval $(0,1)$.

We study the dependence of the first eigenvalue λ_1 of problem (1)–(3) on the potential Q under different values of parameters α,β,γ. Our purpose is to give some estimates for

$$m_{\alpha,\beta,\gamma}=\inf_{Q\in T_{\alpha,\beta,\gamma}}\lambda_1(Q),\qquad M_{\alpha,\beta,\gamma}=\sup_{Q\in T_{\alpha,\beta,\gamma}}\lambda_1(Q).$$

Let H_Q be the closure of the set $C_0^\infty(0,1)$ in the norm $\|y\|^2_{H_Q}=\int_0^1(y'^2+Qy^2)\,\mathrm{d}x$, where $C_0^\infty(0,1)$ is the set of functions of $C^\infty(0,1)$ having their supports compactly embedded in $(0,1)$. Let Γ be the set of functions y from H_Q such that $\int_0^1 y^2\,\mathrm{d}x=1$.

Consider the functionals

$$R[Q,y]=\frac{\int_0^1(y'^2(x)+Q(x)y^2(x))\,\mathrm{d}x}{\int_0^1 y^2(x)\,\mathrm{d}x},\quad F[Q,y]=\int_0^1(y'^2(x)+Q(x)y^2(x))\,\mathrm{d}x.$$

Note that the values of R and F are bounded from below. Let us show that the first eigenvalue λ_1 of problem (1)–(2) can be found as

$$\lambda_1(Q)=\inf_{y\in H_Q,y\neq 0}R[Q,y]=\inf_{y\in\Gamma}F[Q,y].$$

Step 1. Let $Q\in T_{\alpha,\beta,\gamma}$ and $m=\inf\limits_{y\in\Gamma}F[Q,y]$. There exists $y\in\Gamma$ such that $F[Q,y]=m$.

For all functions $Q\in T_{\alpha,\beta,\gamma}$ and $y\in\Gamma$ one has $F[Q,y]=\int_0^1(y'^2+Qy^2)\,\mathrm{d}x=\|y\|^2_{H_Q}$. Let $\{y_k\}$ be a minimizing sequence of the functional $F[Q,y]$ in Γ. Then $F[Q,y_k]\leqslant m+1$ for all sufficiently large values of k. Hence $\|y_k(x)\|^2_{H_Q}=F[Q,y_k]\leqslant m+1$. Since $\{y_k\}$ is a bounded sequence in a separable Hilbert space H_Q, it contains a subsequence $\{z_k\}$, which converges weakly in the space H_Q to a function y. So we get $\|y\|^2_{H_Q}\leqslant m+1$.

Let us prove that the space H_Q is compactly embedded into the space $C(0,1)$. First we shall establish the boundedness of the corresponding operator of embedding. Note that the inequality $\|u\|_C\leqslant\|u'\|_{L_1}+(b-a)^{-1}\|u\|_{L_1}$ holds for any function $u(x)\in C[a,b]$. If $u(x)\in AC[0,1]$ and $u(0)=u(1)=0$, then

$$\begin{aligned}\|u\|_{L_1}&=\int_0^1|u|\,\mathrm{d}x=\int_0^1\left|\int_0^x u'\,\mathrm{d}x\right|\mathrm{d}x\\&\leqslant\int_0^1\left(\int_0^1|u'|\,\mathrm{d}x\right)\mathrm{d}x=\int_0^1|u'|\,\mathrm{d}x=\|u'\|_{L_1}.\end{aligned}$$

By the Hölder inequality we get

$$\|u\|_C \leqslant \|u'\|_{L_1} + \|u\|_{L_1} \leqslant 2\|u'\|_{L_1} \leqslant 2\|u'\|_{L_2} \leqslant 2\|u\|_{H_Q}. \tag{4}$$

The boundedness of this operator is proved.

Now let us prove the compactness of the operator of embedding. Let $M \in H_Q$ be a bounded set, i.e. there is a real number R such that $\|u\|_{H_Q} \leqslant R$ for all $u \in M$. We need to prove the precompactness of M in $C(0,1)$. By the Arzela theorem it suffices to prove that the set M is uniformly bounded and equicontinuous.

The set M is called uniformly bounded if there is a real number R_1 such that $|u(x)| \leqslant R_1$ for all $u \in M$ and $x \in [0,1]$. In virtue of (4) we have $|u(x)| \leqslant \|u\|_C \leqslant 2R = R_1$ for all $u \in M$ and $x \in [0,1]$.

Now let us prove that the set M is equicontinuous, i.e. for any $\varepsilon > 0$ one can find $\delta > 0$ such that $|u(x)-u(y)| < \varepsilon$ as $|x-y| < \delta$ for all $u \in M$. By the Newton-Leibniz formula we obtain: if $|x-y| < \delta = (\varepsilon R^{-1})^2$ then

$$|u(x)-u(y)| \leqslant \left| \int_x^y |u'(\xi)|\,\mathrm{d}\xi \right| \leqslant |x-y|^{\frac12}\|u'\|_{L_2} \leqslant |x-y|^{\frac12} R < \varepsilon \quad \text{for all } u \in M.$$

The space H_Q is compactly embedded into the space $C(0,1)$. Consequently, there is a converging in $C(0,1)$ subsequence $\{u_k\}$ of the sequence $\{z_k\}$. Since $C(0,1)$ is embedded into $L_p(0,1)$, where $p \geqslant 1$, then the sequence $\{u_k\}$ converges in $L_2(0,1)$ to a function $u \in L_2(0,1)$ such that $\int_0^1 u^2\,\mathrm{d}x = 1$.

Let us prove that the subsequence $\{u_k\}$ converges in H_Q. Since the functional F is quadric, we have the identity

$$F\Big[Q, \frac{y_k - y_l}{2}\Big] + F\Big[Q, \frac{y_k + y_l}{2}\Big] = \frac12 F[Q, y_k] + \frac12 F[Q, y_l].$$

Let $\varepsilon > 0$ and let k and l be so large that for u_k, u_l from the subsequence one has

$$F[Q,u_k] \leqslant m + \varepsilon, \quad F[Q,u_l] \leqslant m + \varepsilon, \quad \text{and} \quad \int_0^1 \Big(\frac{u_k - u_l}{2}\Big)^2 \mathrm{d}x \leqslant \varepsilon^2.$$

Hence,

$$\begin{aligned}\int_0^1 \Big(\frac{u_k + u_l}{2}\Big)^2 \mathrm{d}x &= \int_0^1 \Big(u_l + \frac{u_k - u_l}{2}\Big)^2 \mathrm{d}x \\ &\geqslant (1-\varepsilon)\int_0^1 u_l^2\,\mathrm{d}x - \frac{1}{\varepsilon}\int_0^1 \Big(\frac{u_k - u_l}{2}\Big)^2 \mathrm{d}x \geqslant (1-\varepsilon) - \varepsilon = 1 - 2\varepsilon.\end{aligned}$$

Therefore, $F[Q, \frac12(u_k + u_l)] \geqslant m(1-2\varepsilon)$ and $F[Q, \frac12(u_k - u_l)] \leqslant m + \varepsilon - m(1-2\varepsilon) = \varepsilon(1+2m)$. It means that the subsequence $\{u_k\}$ converges in H_Q. Since it converges in H_Q weakly to y, then the limit function of this subsequence in H_Q is equal to y too. Then, taking into account that the functional F is continuous in H_Q, we obtain $F[Q,y] = m$.

S t e p 2. Let $y(x) \in \Gamma$ and $F[Q,y] = m = \inf\limits_{y\in\Gamma} F[Q,y]$. Then

$$-y'' + Qy - \lambda y = 0,$$

where $\lambda = m$ is the minimal eigenvalue of the Sturm-Liouville problem (1)–(2).

First we note that $m = \inf\limits_{y\in H_Q, y\neq 0} R[Q,y]$. We have that the minimum of the functional $F[Q,y]$ is equal to m under the condition $\int_0^1 y^2\,\mathrm{d}x = 1$.

Let $u(x)$ be an element of H_Q. Consider two functions of $t \in \mathbb{R}$

$$g(t) = \int_0^1 ((y' + tu')^2(x) + Q(x)(y+tu)^2(x))\,\mathrm{d}x, \quad h(t) = \int_0^1 (y+tu)^2\,\mathrm{d}x.$$

If $h(0) = 1$ then $g(t) \geqslant g(0) = m$, i.e. the function g has the minimal value at $t = 0$ under the condition $h(0) = 1$. Therefore, $g'(0) + \lambda_1 h'(0) = 0$, where λ_1 is a real number. Let $\lambda = -\lambda_1$. It means that for all $u(x) \in H_Q$ the equality $\int_0^1 (y'u' + Qyu)\,\mathrm{d}x = \lambda \int_0^1 yu\,\mathrm{d}x$ holds. In particular, if $u = y$, then we obtain $\lambda = m$. Consequently, $\int_0^1 (y'u' + Qyu - myu)\,\mathrm{d}x = 0$.

This equality is valid for all $u \in C_0^\infty(0,1)$. It implies the existence of the generalized derivative of the function y' such that

$$-y(x)'' + Q(x)y(x) - my(x) = 0. \tag{5}$$

By the method of averaging one can obtain a sequence $\{y_k(x)\}$ of $C_0^\infty(0,1)$ functions with the following properties: 1) $\{y_k(x)\}$ converges uniformly in the space H_Q to the function y; 2) the sequence $\{Qy_k(x)\}$ also converges uniformly in H_Q to the function Qy. Then the sequence $\{y_k(x)''\}$ converges uniformly in this space to the function y''. Therefore the equality (5) holds almost everywhere in $(0,1)$. Moreover, $y(0) = y(1) = 0$.

Thus y is a solution of the Sturm-Liouville problem (1)–(2) with the eigenvalue $\lambda = m$. For any solution z of this problem we have $\int_0^1 ({z'}^2(x) + Q(x)z^2(x))\,\mathrm{d}x = \lambda \int_0^1 z^2\,\mathrm{d}x$; then in virtue of (5) we obtain the relation $\lambda \geqslant m$. Consequently, m is the minimal eigenvalue.

The following theorems give some estimates for $m_{\alpha,\beta,\gamma}$ and $M_{\alpha,\beta,\gamma}$.

Theorem 1.

(1) *If* $\gamma > 0$, *then* $m_{\alpha,\beta,\gamma} = \pi^2$.

(2) *If* $\gamma < 0$, *then* $m_{\alpha,\beta,\gamma} < +\infty$.

Theorem 2.

(1) *If* $\gamma < 0$ *and* $0 < \gamma < 1$, *then* $M_{\alpha,\beta,\gamma} = +\infty$.

(2) *If* $\gamma \geqslant 1$, *then* $M_{\alpha,\beta,\gamma} < +\infty$.

P r o o f of Theorem 1. We emphasize that in virtue of Friedrichs' inequality the following relations hold for all $Q \in T_{\alpha,\beta,\gamma}$:

$$\lambda_1(Q) = \inf_{y \in H_Q, y \neq 0} \frac{\int_0^1 (y'^2(x) + Q(x)y^2(x))\,\mathrm{d}x}{\int_0^1 y^2(x)\,\mathrm{d}x} \geqslant \inf_{y \in H_Q, y \neq 0} \frac{\int_0^1 y'^2(x)\,\mathrm{d}x}{\int_0^1 y^2(x)\,\mathrm{d}x} = \pi^2.$$

Hence, $m_{\alpha,\beta,\gamma} \geqslant \pi^2$.

1) Let $\gamma > 0$, α, β be arbitrary real numbers. We prove that $m_{\alpha,\beta,\gamma} = \pi^2$. Consider the functions

$$Q_{\theta,\alpha,\beta,\gamma}(x) = \begin{cases} 0, & x \in (0,\theta); \\ ((1-\theta)x^{\alpha}(1-x)^{\beta})^{-1/\gamma}, & x \in [\theta, 1), \end{cases}$$

$$y_\theta(x) = \begin{cases} \sin \pi x/\theta, & x \in (0,\theta); \\ 0, & x \in [\theta, 1), \ \theta \to 1-0. \end{cases}$$

Then we have $\int_0^1 Q_{\theta,\alpha,\beta,\gamma}(x) y_\theta(x)^2\,\mathrm{d}x = 0$ and the integral condition holds. Since $\int_0^1 y_\theta(x)^2\,\mathrm{d}x = \frac{1}{2}\theta$, $\int_0^1 y_\theta'(x)^2\,\mathrm{d}x = \frac{1}{2}\pi^2/\theta$, we obtain

$$\lim_{\theta \to 1-0} R[Q_\theta, y_\theta] = \lim_{\theta \to 1-0} \frac{\frac{1}{2}(\pi^2/\theta)}{(\frac{1}{2}\theta)} = \pi^2$$

and $m_{\alpha,\beta,\gamma} = \inf\limits_{Q \in T_{\alpha,\beta,\gamma}} \lambda_1(Q) \leqslant \pi^2$. Therefore, $m_{\alpha,\beta,\gamma} = \pi^2$.

2.1) First we suppose that $\gamma < 0$, $\beta \geqslant 0$, $\alpha > 2\gamma - 1$. Consider the function $Q_\theta(x) = Cx^{-(\alpha+1)/\gamma + \theta/\gamma}(1-x)^{-\beta/\gamma}$, where θ is a positive real number such that $\alpha \geqslant 2\gamma - 1 + \theta$. We take the constant C such that $\int_0^1 Q_\theta(x)^\gamma x^\alpha (1-x)^\beta\,\mathrm{d}x = 1$, i.e. $C = \theta^{1/\gamma}$. By the Hardy inequality we obtain

$$\int_0^1 Q_\theta(x) y^2\,\mathrm{d}x = C \int_0^1 \frac{(1-x)^{-\beta/\gamma} y^2}{x^{(\alpha+1-\theta)/\gamma}}\,\mathrm{d}x \leqslant C \int_0^1 x^{-2} y^2\,\mathrm{d}x \leqslant 4C \int_0^1 y'^2\,\mathrm{d}x.$$

Then it follows from $C = \theta^{1/\gamma}$ that $m_{\alpha,\beta,\gamma} \leqslant (1 + 4(\alpha - 2\gamma + 1)^{1/\gamma})\pi^2$.

2.2) Suppose that $\gamma < 0$, $\beta \geqslant 0$ and $\alpha \leqslant 2\gamma - 1$. Consider the functions

$$Q_{\varepsilon,\alpha,\beta,\gamma}(x) = \varepsilon x^{-\alpha/\gamma}(1-x)^{-\beta/\gamma} x^{(\varepsilon^\gamma - 1)/\gamma} \text{ and } y_1(x) = \begin{cases} x^\theta, & 0 \leqslant x \leqslant \frac{1}{2}; \\ (1-x)^\theta, & \frac{1}{2} < x \leqslant 1, \end{cases}$$

where θ is a real number such that $2\theta - \alpha/\gamma + (\varepsilon^\gamma - 1)/\gamma > -1$ and $2\theta > 1$. Denote $\int_0^1 y_1'^2\,\mathrm{d}x = C_1$, $\int_0^1 y_1^2\,\mathrm{d}x = C_2$, $\int_0^1 x^{-\alpha/\gamma} x^{(\varepsilon^\gamma-1)/\gamma} y_1^2\,\mathrm{d}x = C_3$. Then $R[Q_{\varepsilon,\alpha,\beta,\gamma}, y_1] = (C_1 + \varepsilon C_3)/C_2$ and $m_{\alpha,\beta,\gamma} \leqslant C_1/C_2$. The case $\alpha \geqslant 0$, $\beta < 0$ is symmetric to the case $\beta \geqslant 0$, $\alpha < 0$.

2.3) Now we assume that $\gamma < 0$, $2\gamma - 1 < \alpha < 0$ and $2\gamma - 1 < \beta < 0$. Consider the function

$$Q_{\theta,\alpha,\beta,\gamma}(x) = \begin{cases} Cx^{-(\alpha+1)/\gamma+\theta/\gamma}(1-x)^{-\beta/\gamma}, & 0 < x < \frac{1}{2}; \\ Cx^{-\alpha/\gamma}(1-x)^{-(\beta+1)/\gamma+\theta/\gamma}, & \frac{1}{2} \leqslant x < 1, \end{cases}$$

where θ is a positive real number such that $\alpha \geqslant 2\gamma - 1 + \theta$. By the Hardy inequality

$$\int_0^1 Q_{\theta,\alpha,\beta,\gamma} y^2(x)\,\mathrm{d}x \leqslant C2^{\frac{2\gamma-1}{\gamma}} \int_0^{\frac{1}{2}} x^{-\frac{\alpha+1}{\gamma}+\frac{\theta}{\gamma}} y^2\,\mathrm{d}x + C2^{\frac{2\gamma-1}{\gamma}} \int_{\frac{1}{2}}^1 (1-x)^{-\frac{\beta+1}{\gamma}+\frac{\theta}{\gamma}} y^2\,\mathrm{d}x$$
$$\leqslant C2^{\frac{2\gamma-1}{\gamma}} \left(\int_0^{\frac{1}{2}} x^{-2} y^2\,\mathrm{d}x + \int_{\frac{1}{2}}^1 (1-x)^{-2} y^2\,\mathrm{d}x \right) \leqslant C2^{\frac{4\gamma-1}{\gamma}} \int_0^1 y'^2\,\mathrm{d}x$$

and $m_{\alpha,\beta,\gamma} \leqslant (1 + C2^{(4\gamma-1)/\gamma})\pi^2$. For $\theta = \alpha - 2\gamma + 1$ and $C = (\theta 2^{\theta-1})^{1/\gamma}$ we have $m_{\alpha,\beta,\gamma} \leqslant (1 + (\alpha - 2\gamma + 1)^{1/\gamma} 2^{(\alpha+2\gamma-1)/\gamma})\pi^2$.

2.4) Consider the case $\gamma < 0$, $\alpha \leqslant 2\gamma - 1$ and $\beta < 0$. Consider the functions

$$Q_{\varepsilon,\alpha,\beta,\gamma}(x) = \varepsilon x^{-\alpha/\gamma}(1-x)^{-\beta/\gamma} x^{(\varepsilon^\gamma-1)/\gamma} \quad \text{and} \quad y_1(x) = \begin{cases} x^\theta, & 0 \leqslant x \leqslant \frac{1}{2}; \\ (1-x)^\theta, & \frac{1}{2} < x \leqslant 1, \end{cases}$$

where θ is a real number such that $2\theta - \alpha/\gamma + (\varepsilon^\gamma - 1)/\gamma > -1$, $2\theta > 1$ and $2\theta - \beta/\gamma > -1$. Denote $\int_0^1 y_1'^2\,\mathrm{d}x = C_1$, $\int_0^1 y_1^2\,\mathrm{d}x = C_2$, $\int_0^1 Q_{\varepsilon,\alpha,\beta,\gamma}(x) y_1^2\,\mathrm{d}x = \varepsilon C_3$. Then $R[Q_{\varepsilon,\alpha,\beta,\gamma}, y_1] = (C_1 + \varepsilon C_3)/C_2$ and $m_{\alpha,\beta,\gamma} \leqslant C_1/C_2$. The case $\beta \leqslant 2\gamma - 1$, $\alpha < 0$ is symmetric to the case $\alpha \leqslant 2\gamma - 1$, $\beta < 0$. By substitution $x = 1 - t$, $\alpha \leftrightarrow \beta$ the case $2\gamma - 1 < \alpha < 0$ and $\beta \leqslant 2\gamma - 1$ can be included into the case 2.4).

P r o o f of Theorem 2. 1.1) First we suppose that $\gamma < 0$, $\alpha > 0$, $\beta > 0$. Let us prove that $M_{\alpha,\beta,\gamma} = +\infty$. Assume that $\alpha \geqslant \beta$. Consider the function

$$Q_{\varepsilon,\alpha,\beta,\gamma}(x) = \begin{cases} ((1 - \varepsilon^{2\alpha}(1-\varepsilon)^\alpha)/2\varepsilon)^{1/\gamma} x^{-\alpha/\gamma}(1-x)^{-\beta/\gamma}, & x \in (0,1) \setminus (\varepsilon, 1-\varepsilon); \\ (\varepsilon^{2\alpha}(1-\varepsilon)^\alpha/(1-2\varepsilon))^{1/\gamma} x^{-\alpha/\gamma}(1-x)^{-\beta/\gamma}, & x \in (\varepsilon, 1-\varepsilon), \end{cases}$$

where $\varepsilon \to +0$. Thus we have

$$\int_\varepsilon^{1-\varepsilon} y^2(x)\,\mathrm{d}x \leqslant \int_\varepsilon^{1-\varepsilon} \frac{x^{-\alpha/\gamma}(1-x)^{-\alpha/\gamma}}{\varepsilon^{-\alpha/\gamma}(1-\varepsilon)^{-\alpha/\gamma}} y^2(x)\,\mathrm{d}x$$
$$\leqslant \int_\varepsilon^{1-\varepsilon} \frac{x^{-\alpha/\gamma}(1-x)^{-\alpha/\gamma}}{\varepsilon^{-\alpha/\gamma}(1-\varepsilon)^{-\alpha/\gamma}} (1-x)^{(\alpha-\beta)/\gamma} y^2(x)\,\mathrm{d}x$$
$$= \frac{\varepsilon^{-\alpha/\gamma}}{(1-2\varepsilon)^{-1/\gamma}} \int_\varepsilon^{1-\varepsilon} Q_{\varepsilon,\alpha,\beta,\gamma}(x) y^2(x)\,\mathrm{d}x.$$

By the Hölder inequality we get

$$\int_0^1 y^2(x)\,\mathrm{d}x \leqslant \frac{\varepsilon^2}{2}\int_0^\varepsilon y'^2(x)\,\mathrm{d}x + \frac{\varepsilon^{-\frac{\alpha}{\gamma}}}{(1-2\varepsilon)^{-\frac{1}{\gamma}}}\int_\varepsilon^{1-\varepsilon} Q_{\varepsilon,\alpha,\beta,\gamma}(x)y^2(x)\,\mathrm{d}x$$
$$+\frac{\varepsilon^2}{2}\int_{1-\varepsilon}^1 y'^2(x)\,\mathrm{d}x \leqslant a(\varepsilon)\biggl(\int_0^1 y'^2(x)\,\mathrm{d}x + \int_0^1 Q_{\varepsilon,\alpha,\beta,\gamma}(x)y^2(x)\,\mathrm{d}x\biggr),$$

where $a(\varepsilon) = \varepsilon^2/2 + \varepsilon^{-\alpha/\gamma}/(1-2\varepsilon)^{-1/\gamma}$. Then $R[Q_{\varepsilon,\alpha,\beta,\gamma}, y] \geqslant 1/a(\varepsilon)$ for all functions $y \in H_Q$. Consequently, $\inf\limits_{y\in H_Q, y\neq 0} R[Q, y] \geqslant 1/a(\varepsilon)$. Taking into account that $a(\varepsilon) \to 0$ as $\varepsilon \to 0$, we obtain $M_{\alpha,\beta,\gamma} = +\infty$.

1.2) Consider the case $\gamma < 0$, $\alpha > 0$, $\beta \leqslant 0$. Let us prove that $M_{\alpha,\beta,\gamma} = +\infty$.

For $\varepsilon \to +0$ consider the function

$$Q_{\varepsilon,\alpha,\beta,\gamma}(x) = \begin{cases} (\alpha+1)^{1/\gamma}\varepsilon^{-(\alpha+1)/\gamma}(1-\varepsilon)^{1/\gamma}(1-x)^{-\beta/\gamma}, & 0 < x < \varepsilon; \\ (\alpha+1)^{1/\gamma}\varepsilon^{1/\gamma}(1-\varepsilon^{\alpha+1})^{-1/\gamma}(1-x)^{-\beta/\gamma}, & \varepsilon < x < 1. \end{cases}$$

As in the previous case $\int_0^1 y^2(x)\,\mathrm{d}x \leqslant a(\varepsilon)(\int_0^1 y'^2(x)\,\mathrm{d}x + \int_0^1 Q_{\varepsilon,\alpha,\beta,\gamma}(x)y^2)$, where $a(\varepsilon) = \varepsilon^2/2 + (1/(\alpha+1))^{1/\gamma}\varepsilon^{-1/\gamma}(1-\varepsilon^{\alpha+1})^{1/\gamma}$, and by the same argument $M_{\alpha,\beta,\gamma} = +\infty$. The case $\gamma < 0$, $\beta > 0$, $\alpha \leqslant 0$ is symmetric to the case $\gamma < 0$, $\alpha > 0$, $\beta \leqslant 0$.

1.3) Now suppose that $\gamma < 0$, $\alpha \leqslant 0$, $\beta \leqslant 0$. Let us prove that $M_{\alpha,\beta,\gamma} = +\infty$.

Consider the function

$$Q_{\varepsilon,\alpha,\beta,\gamma}(x) = \begin{cases} (1-\varepsilon)^{1/\gamma}\varepsilon^{-1/\gamma}x^{-\alpha/\gamma}(1-x)^{-\beta/\gamma}, & 0 < x < \varepsilon; \\ (1-\varepsilon)^{-1/\gamma}\varepsilon^{1/\gamma}x^{-\alpha/\gamma}(1-x)^{-\beta/\gamma}, & \varepsilon < x < 1, \end{cases}$$

where $\varepsilon \to +0$. By the same argument $M_{\alpha,\beta,\gamma} = +\infty$.

2.1) Consider the case $0 < \gamma < 1$, $\alpha \geqslant 0$, $\beta \geqslant 0$. Divide the segment $[0,1]$ by points $0 = \varepsilon_0 < \varepsilon_1 < \ldots < \varepsilon_n = 1$ to equal segments of length ε. Consider the function $Q_\varepsilon(x)$ on the segment [0,1] defined on each interval $[\varepsilon_{i-1}, \varepsilon_i)$ $(1 \leqslant i \leqslant n)$ as follows:

$$Q_\varepsilon(x) = \begin{cases} \varepsilon^{-\mu}x^{-\alpha/\gamma}(1-x)^{-\beta/\gamma}, & \varepsilon_{i-1} \leqslant x < \varepsilon_{i-1} + \varepsilon^\varrho; \\ 0, \quad \varepsilon_{i-1} + \varepsilon^\varrho \leqslant x < \varepsilon_i, \end{cases}$$

where $\varepsilon \to +0$, $\varrho = (1+\gamma)/(1-\gamma)$, $\mu = 2/(1-\gamma)$. Then there is $\theta_i \in [\varepsilon_{i-1}, \varepsilon_{i-1} + \varepsilon^\varrho)$ such that

$$\int_{\varepsilon_{i-1}}^{\varepsilon_{i-1}+\varepsilon^\varrho} Q_\varepsilon(x)y^2\,\mathrm{d}x = \varepsilon^{-\mu}\varepsilon^\varrho\theta_i^{-\alpha/\gamma}(1-\theta_i)^{-\beta/\gamma}y^2(\theta_i) = \varepsilon^{-1}\theta_i^{-\alpha/\gamma}(1-\theta_i)^{-\beta/\gamma}y^2(\theta_i).$$

Since $y(x) = y(\theta_i) + \int_{\theta_i}^{x} y'(x)\,\mathrm{d}x$, we have by the Hölder inequality

$$\int_{\varepsilon_{i-1}}^{\varepsilon_i} y^2\,\mathrm{d}x = \int_{\varepsilon_{i-1}}^{\varepsilon_i} \left(y(\theta_i) + \int_{\theta_i}^{x} y'(x)\,\mathrm{d}x\right)^2 \mathrm{d}x \leqslant 2\varepsilon y^2(\theta_i) + 2\varepsilon^2 \int_{\varepsilon_{i-1}}^{\varepsilon_i} y'^2\,\mathrm{d}x$$
$$= 2\varepsilon^2\left(\theta_i^{\alpha/\gamma}(1-\theta_i)^{\beta/\gamma}\int_{\varepsilon_{i-1}}^{\varepsilon_i} Q_\varepsilon(x)y^2\,\mathrm{d}x + \int_{\varepsilon_{i-1}}^{\varepsilon_i} y'^2\,\mathrm{d}x\right)$$
$$< 2\varepsilon^2\left(\int_{\varepsilon_{i-1}}^{\varepsilon_i} Q_\varepsilon(x)y^2\,\mathrm{d}x + \int_{\varepsilon_{i-1}}^{\varepsilon_i} y'^2\,\mathrm{d}x\right)$$

and $\int_0^1 y^2\,\mathrm{d}x < 2\varepsilon^2(\int_0^1 Q_\varepsilon y^2\,\mathrm{d}x + \int_0^1 y'^2\,\mathrm{d}x)$. Hence, $M_{\alpha,\beta,\gamma} = +\infty$.

2.2) If $0<\gamma<1$, $\alpha<0$, $\beta\geqslant 0$, then divide the segment $[0,1]$ in a way similar to the previous case and define the function Q_ε on each interval $[\varepsilon_{i-1},\varepsilon_i)$ $(1\leqslant i\leqslant n)$ as follows:

$$Q_\varepsilon(x) = \begin{cases} 0, \ \varepsilon_{i-1}\leqslant x<\varepsilon_{i-1}+\frac12\varepsilon-\frac12\varepsilon^\varrho \quad\text{or}\quad \varepsilon_{i-1}+\frac12\varepsilon+\frac12\varepsilon^\varrho\leqslant x<\varepsilon_i;\\ \varepsilon^{-\mu}x^{-\alpha/\gamma}(1-x)^{-\beta/\gamma}, \quad \varepsilon_{i-1}+\frac12\varepsilon-\frac12\varepsilon^\varrho\leqslant x<\varepsilon_{i-1}+\frac12\varepsilon+\frac12\varepsilon^\varrho,\end{cases}$$

where $\varepsilon\to+0$, $\varrho=(1+\gamma-\alpha)/(1-\gamma)$, $\mu=(2-\alpha/\gamma)/(1-\gamma)$. By the same argument as for the case $\alpha\geqslant 0$, $\beta\geqslant 0$ we have

$$\int_0^1 y^2\,\mathrm{d}x \leqslant 2\varepsilon^2\left(\max_i(\theta_i^{\alpha/\gamma})\varepsilon^{-\alpha/\gamma}\int_0^1 Q_\varepsilon y^2\,\mathrm{d}x + \int_0^1 y'^2\,\mathrm{d}x\right)$$
$$= 2\varepsilon^2\left(\theta_1^{\alpha/\gamma}\varepsilon^{-\alpha/\gamma}\int_0^1 Q_\varepsilon y^2\,\mathrm{d}x + \int_0^1 y'^2\,\mathrm{d}x\right)$$
$$< 2^{1-\alpha/\gamma}\varepsilon^2(1-\varepsilon^{\varrho-1})^{\alpha/\gamma}\left(\int_0^1 Q_\varepsilon y^2\,\mathrm{d}x + \int_0^1 y'^2\,\mathrm{d}x\right).$$

Taking a sufficiently small ε we get $R[Q_\varepsilon,y]\geqslant(\frac12)^{-\alpha/\gamma}/2^{1-\alpha/\gamma}\varepsilon^2$. Therefore, $M_{\alpha,\beta,\gamma}=+\infty$. Note that the case $0<\gamma<1$, $\beta<0$, $\alpha\geqslant 0$ is symmetric to the case $0<\gamma<1$, $\alpha<0$, $\beta\geqslant 0$.

2.3) Consider the case $0<\gamma<1$, $\alpha<0$, $\beta<0$. If for example $\beta>\alpha$, then divide the segment $[0,1]$ in a way similar to the previous cases and define the function Q_ε on each interval $[\varepsilon_{i-1},\varepsilon_i)$ $(1\leqslant i\leqslant n)$ as follows:

$$Q_\varepsilon(x) = \begin{cases} 0, \quad \varepsilon_{i-1}\leqslant x<\varepsilon_{i-1}+\frac12\varepsilon-\frac12\varepsilon^\varrho \text{ or } \varepsilon_{i-1}+\frac12\varepsilon+\frac12\varepsilon^\varrho\leqslant x<\varepsilon_i;\\ \varepsilon^{-\mu}x^{-\alpha/\gamma}(1-x)^{-\beta/\gamma}, \quad \varepsilon_{i-1}+\frac12\varepsilon-\frac12\varepsilon^\varrho\leqslant x<\varepsilon_{i-1}+\frac12\varepsilon+\frac12\varepsilon^\varrho,\end{cases}$$

where $\varepsilon\to+0$, $\varrho=(1+\gamma-\alpha)/(1-\gamma)$, $\mu=(2-\alpha/\gamma)/(1-\gamma)$. The proof in this case is similar to the proof of 2.2) and also $M_{\alpha,\beta,\gamma}=+\infty$.

3.1) Consider the case $\gamma = 1$, $0 \leqslant \alpha \leqslant 1$, $\beta < 0$. Since $y^2(x) \leqslant x\int_0^1 y'^2\,\mathrm{d}t$ for all $x \in (0,1)$, we have

$$\int_0^1 Qy^2(x)\,\mathrm{d}x \leqslant \sup_{[0,1]}\frac{y^2}{x^\alpha}\int_0^1 Qx^\alpha(1-x)^\beta\,\mathrm{d}x \leqslant \sup_{[0,1]}\frac{y^2}{x} \leqslant \int_0^1 y'^2(x)\,\mathrm{d}x.$$

Therefore, $M_{\alpha,\beta,\gamma} \leqslant 2\pi^2$. Note that the case $\gamma = 1$, $0 \leqslant \beta \leqslant 1$, $\alpha < 0$ is symmetric to the case $\gamma = 1$, $0 \leqslant \alpha \leqslant 1$, $\beta < 0$.

3.2) Consider the case $\gamma = 1$, $0 \leqslant \alpha \leqslant 1$, $0 \leqslant \beta \leqslant 1$. We have $M_{\alpha,\beta,\gamma} \leqslant 3\pi^2$, because

$$\int_0^1 Qy^2(x)\,\mathrm{d}x \leqslant \sup_{[0,1]}\frac{y^2}{x^\alpha(1-x)^\beta}\int_0^1 Qx^\alpha(1-x)^\beta\,\mathrm{d}x \leqslant \sup_{[0,1]}\frac{y^2}{x} + \sup_{[0,1]}\frac{y^2}{1-x}.$$

3.3) Now suppose that $\gamma = 1$, $\alpha < 0$, $\beta < 0$.

One can show [1], [2] that for all $y \in H_Q$ the following inequality holds: $\sup\limits_{[0,1]} y^2 \leqslant \frac{1}{4}\int_0^1 y'^2(x)\,\mathrm{d}x$. Then

$$\int_0^1 Qy^2(x)\,\mathrm{d}x \leqslant \sup_{[0,1]}\frac{y^2}{x^\alpha}\int_0^1 Qx^\alpha(1-x)^\beta\,\mathrm{d}x \leqslant \sup_{[0,1]}\frac{y^2}{x^\alpha} \leqslant \sup_{[0,1]} y^2 \leqslant \frac{1}{4}\int_0^1 y'^2(x)\,\mathrm{d}x.$$

Hence, $M_{\alpha,\beta,\gamma} \leqslant \frac{5}{4}\pi^2$.

3.4) Now we consider the case $\gamma > 1$, $0 \leqslant \alpha \leqslant 2\gamma - 1$, $\beta < 0$. By the Hölder inequality we have

$$\int_0^1 Qy^2(x)\,\mathrm{d}x \leqslant \left(\int_0^1 |y|^{\frac{2\gamma}{\gamma-1}} x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}\,\mathrm{d}x\right)^{\frac{\gamma-1}{\gamma}} \leqslant \left(\int_0^1 |y|^{\frac{2\gamma}{\gamma-1}} x^{-\frac{2\gamma-1}{\gamma-1}}\,\mathrm{d}x\right)^{\frac{\gamma-1}{\gamma}}.$$

By the generalized Hardy inequality [3]

$$\left(\int_0^1 |y|^{\frac{2\gamma}{\gamma-1}} x^{-\frac{2\gamma-1}{\gamma-1}}\,\mathrm{d}x\right)^{\frac{\gamma-1}{2\gamma}} \leqslant \left(\frac{2\gamma-1}{\gamma}\right)^{\frac{2\gamma-1}{2\gamma}}\left(\int_0^1 y'^2(x)\,\mathrm{d}x\right)^{\frac{1}{2}}$$

we have $\int_0^1 Qy^2(x)\,\mathrm{d}x \leqslant ((2\gamma-1)/\gamma)^{(2\gamma-1)/\gamma}\int_0^1 y'^2(x)\,\mathrm{d}x$ and $M_{\alpha,\beta,\gamma} \leqslant (1 + ((2\gamma-1)/\gamma)^{(2\gamma-1)/\gamma})\pi^2$. The case $\gamma > 1$, $0 \leqslant \beta \leqslant 2\gamma - 1$, $\alpha < 0$ is symmetric to the case $\gamma > 1$, $0 \leqslant \alpha \leqslant 2\gamma - 1$, $\beta < 0$.

3.5) Now consider the case $\gamma > 1$, $0 \leqslant \alpha \leqslant 2\gamma - 1$, $0 \leqslant \beta \leqslant 2\gamma - 1$. Since

$$\int_0^1 Qy^2(x)\,\mathrm{d}x \leqslant \left(\int_0^1 Q^\gamma x^\alpha(1-x)^\beta\,\mathrm{d}x\right)^{\frac{1}{\gamma}}\left(\int_0^1 |y|^{\frac{2\gamma}{\gamma-1}} x^{\frac{\alpha}{1-\gamma}}(1-x)^{\frac{\beta}{1-\gamma}}\,\mathrm{d}x\right)^{\frac{\gamma-1}{\gamma}},$$

we have by the generalized Hardy inequality

$$\int_0^1 |y|^{\frac{2\gamma}{\gamma-1}} x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}}\,\mathrm{d}x \leqslant 2C\Big(\frac{2\gamma-1}{\gamma}\Big)^{\frac{2\gamma-1}{\gamma-1}} \Big(\int_0^1 y'^2(x)\,\mathrm{d}x\Big)^{\frac{\gamma}{\gamma-1}},$$

where $C = 2^{(2\gamma-1)/(\gamma-1)}$ and $M_{\alpha,\beta,\gamma} \leqslant (1 + 2^{(3\gamma-2)/\gamma}((2\gamma-1)/\gamma)^{(2\gamma-1)/\gamma})\pi^2$.

3.6) Suppose that $\gamma > 1$, $\alpha < 0$, $\beta < 0$. It follows from

$$\int_0^1 Qy^2(x)\,\mathrm{d}x \leqslant \Big(\int_0^1 |y|^{\frac{2\gamma}{\gamma-1}} x^{\frac{\alpha}{1-\gamma}} (1-x)^{\frac{\beta}{1-\gamma}}\,\mathrm{d}x\Big)^{\frac{\gamma-1}{\gamma}} \leqslant \Big(\int_0^1 |y|^{\frac{2\gamma}{\gamma-1}}\,\mathrm{d}x\Big)^{\frac{\gamma-1}{\gamma}} \leqslant \int_0^1 y'^2(x)\,\mathrm{d}x$$

that $M_{\alpha,\beta,\gamma} \leqslant 2\pi^2$.

3.7) Consider the case $\gamma \geqslant 1$, $\alpha > 2\gamma - 1$, $\beta < 0$. Let $y_1 = x^{\alpha/(2\gamma)} \sin \pi x$ and $\int_0^1 y_1'^2\,\mathrm{d}x = C_1$, $\int_0^1 y_1^2\,\mathrm{d}x = C_2$. Then we have that $M_{\alpha,\beta,\gamma} \leqslant (C_1+1)/C_2$, because

$$R[Q,y_1] \leqslant \frac{C_1 + \int_0^1 Q(x)x^{\alpha/\gamma}\,\mathrm{d}x}{C_2} \leqslant \frac{C_1 + \big(\int_0^1 Q^\gamma(x)x^\alpha(1-x)^\beta\,\mathrm{d}x\big)^{1/\gamma}}{C_2} = \frac{C_1+1}{C_2}.$$

The case $\gamma \geqslant 1$, $\beta > 2\gamma-1$, $\alpha < 0$ is symmetric to the case $\gamma \geqslant 1$, $\alpha > 2\gamma - 1$, $\beta < 0$.

3.8) Finally, let $\gamma \geqslant 1$, $\alpha > 2\gamma - 1$, $\beta \geqslant 0$. Taking $y_1 = x^{\alpha/(2\gamma)}(1-x)^{\beta/(2\gamma)} \sin \pi x$ the proof in this case is similar to the proof 3.7) and $M_{\alpha,\beta,\gamma} \leqslant (C_1+1)/C_2$. Note that by substitution $x = 1 - t$, $\alpha \leftrightarrow \beta$ the case $\gamma = 1$, $0 \leqslant \alpha \leqslant 1$, $\beta > 1$ can be included into the case 3.8).

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