CLASSIFYING TREES WITH EDGE-DELETED CENTRAL APPENDAGE NUMBER 2

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Abstract. The eccentricity of a vertex v of a connected graph G is the distance from v to a vertex farthest from v in G. The center of G is the subgraph of G induced by the vertices having minimum eccentricity. For a vertex v in a 2-edge-connected graph G, the edge-deleted eccentricity of v is defined to be the maximum eccentricity of v in $G-e$ over all edges e of G. The edge-deleted center of G is the subgraph induced by those vertices of G having minimum edge-deleted eccentricity. The edge-deleted central appendage number of a graph G is the minimum difference $|V(H)|-|V(G)|$ over all graphs H where the edge-deleted center of H is isomorphic to G. In this paper, we determine the edge-deleted central appendage number of all trees.

Keywords: graphs, trees, central appendage number

MSC 2000: 05C05

1. Introduction

The *distance* $d(u,v)$ between two vertices u and v in a connected graph G is the length of a shortest u-v path in G. The *eccentricity* $e(v)$ of a vertex v in a connected graph G is the distance between v and a vertex farthest from v in G. The minimum eccentricity among the vertices of G is called the *radius* $\mathrm{rad}(G)$ of G, while the maximum eccentricity is the *diameter* $\mathrm{diam}(G)$ of G. A vertex v is called a *central vertex* if $e(v)=\mathrm{rad}(G)$ and called a *peripheral vertex* if $e(v)=\mathrm{diam}(G)$. The *center* $C(G)$ of G is the subgraph induced by the central vertices of G while the *periphery* $P(G)$ of G is the subgraph induced by the peripheral vertices of G.

A graph G is *2-edge-connected* if the removal of any edge of G never results in a disconnected graph. For a vertex v in a 2-edge-connected graph G, the *edge-deleted eccentricity* $g(v)$ of v is defined to be the maximum eccentricity of v in $G-e$

over all edges e of G. The vertices of G with minimum edge-deleted eccentricity are called *edge-deleted central vertices* while the vertices of maximum edge-deleted eccentricity are called *edge-deleted peripheral vertices.* The subgraph induced by the edge-deleted central vertices of G is called the *edge-deleted center* $\mathrm{EDC}(G)$ of G and the subgraph induced by the edge-deleted peripheral vertices $\mathrm{EDP}(G)$ is called the *edge-deleted periphery.* Properties about the edge-deleted eccentricity of vertices and the edge-deleted center of 2-edge-connected graphs were given in [3].

The *central appendage number* of a graph G is the minimum difference $|V(H)| - |V(G)|$ over all graphs H with $C(H) \cong G$. Buckley, Miller, and Slater [2] characterized trees with central appendage number 2 and showed that there are no trees with central appendage number 3. The papers [1] and [5] also study this question. The *edge-deleted central appendage number* $A(G)$ of a graph G is the minimum difference $|V(H)| - |V(G)|$ over all graphs H with $\mathrm{EDC}(H) \cong G$. The edge-deleted central appendage number of several classes of graphs was studied in [4]. In particular, the edge-deleted central appendage number of trees was shown to be 2 or 3. In this paper, we give necessary and sufficient conditions for a tree to have edge-deleted central appendage number 2.

2. Results

Throughout the paper, let T be a tree with $A(T) = 2$ and let H be a graph with $V(H) = V(T) \cup \{x, y\}$ and $\mathrm{EDC}(H) = T$. Since x and y are the only edge-deleted peripheral vertices in H, let $g(x) = g(y) = k$ with $e \in E(H)$ such that $d_{H-e}(x, y) = k$. Let D be the set of peripheral vertices of T and define a *branch* of T as a component of $T - V(C(T))$.

Lemma 1. *Suppose that T is a tree with $A(T) = 2$. Then $g_H(u) = k - 1$ for all $u \in V(T)$.*

Proof. We know that $g_H(x) = g_H(y) = k$ and that there exists a fixed n, $2 \leqslant n \leqslant k - 1$, such that $g_H(u) = n$ for every $u \in V(T)$. Thus it will suffice to show that $g_H(u) = k - 1$ for some $u \in V(T)$.

Let $x, u_1, u_2, \ldots, u_{k-1}, y$ be a shortest x-y path in $H - e$. Clearly $u_i \in V(T)$ for each i, $1 \leqslant i \leqslant k - 1$. Since the distance between u_1 and y is at least $k - 1$ in $H - e$, $g_H(u_1) = k - 1$. Therefore $g(u_1) = k - 1$. □

Lemma 2. *Suppose that T is a tree with $A(T) = 2$. If e is an edge of H with $d_{H-e}(x, y) = k$, then $e \notin E(T)$.*

P r o o f. If $xy \in H$, then the result is obvious. Suppose that $xy \notin E(H)$ and that $e = uu' \in E(T)$. Let $x, u_1, u_2, \ldots, u_m, u$ be a shortest x-u path in $H - e$ and $y, u'_1, u'_2, \ldots, u'_r, u'$ be a shortest $y - u'$ path in $H - e$.

Now, $y \neq u_i$ for $1 \leqslant i \leqslant m$ because if so, then $d_{H-e}(x, u) > d_{H-e}(x, y) = k$, which is a contradiction. Similarly, $x \neq u'_i$ for $1 \leqslant i \leqslant r$. Consider a shortest $u_1 - u'_1$ path in $H - e$. This path must contain either x or y. If not, this path, $u_1 - u$ path, $u' - u'_1$ path, along with the edge uu' would produce a cycle in T. Suppose that the path contains x. Then $k - 1 \geqslant d_{H-e}(u_1, u'_1) \geqslant d_{H-e}(x, u'_1) + 1 \geqslant k$, a contradiction. Switching the roles of x and y in the previous sentence completes the proof. □

Lemma 3. *Suppose that T is a tree with $A(T) = 2$. If $u, v \in V(T)$ such that ux and vy are edges in $H - e$, then*
(1) *a shortest $u - v$ path in $H - e$ lies entirely in T*
(2) *$d_{H-e}(u, v) = k - 1$ or $k - 2$*
(3) *$e_{H-e}(u) = e_{H-e}(v) = k - 1$.*

P r o o f. If (1) is false, then a shortest $u - v$ path contains x or y. Without loss of generality, assume that it contains x. Then $k = d_{H-e}(x, y) = d_{H-e}(x, v) + 1 = d_{H-e}(u, v) = k - 1$, a contradiction.

Now Lemma 1 implies that $d_{H-e}(u, v) \leqslant k - 1$, and $d_{H-e}(x, y) = k$ implies that $d_{H-e}(u, v) \geqslant k - 2$; which proves (2).

Finally, $d_{H-e}(x, v) = k - 1 = d_{H-e}(y, u)$ gives $e_{H-e}(u) \geqslant k - 1$ and $e_{H-e}(v) \geqslant k - 1$. But $g_{H-e}(u) = g_{H-e}(v) = k - 1$ implies $e_{H-e}(u) = e_{H-e}(v) \leqslant k - 1$. Thus, (3) holds. □

Lemma 4. *Let T be a tree with $A(T) = 2$. Let u and v be peripheral vertices with $ux, vy \in E(H - e)$. Then $d_{H-e}(u, v) = k - 2 = \operatorname{diam}(T)$.*

P r o o f. *Let if possible $d_{H-e}(u, v) < \operatorname{diam}(T)$.* If $C(T) = \langle\{w\}\rangle$, then u and v must be end-vertices on the same branch of w. If $C(T) = \langle\{w, w'\}\rangle$, then without loss of generality, u and v must be end-vertices on the branches of w (either on the same branch or two separate branches of w). Let $u' \in D$, with $d_T(u, u') = \operatorname{diam}(T)$ (note that in the case where $C(T) = \langle\{w, w'\}\rangle$, u' must be an end-vertex on the branch of w', if u is on the branch of w). If $C(T) = \langle\{w\}\rangle$, or if $C(T) = \langle\{w, w'\}\rangle$ and u and v are on the same branch of w or $d_{H-e}(u, v) = k - 1$, then either $d_{H-e}(u, u')$ or $d_{H-e}(u', v)$ is greater than $k - 1$.

We may assume that $C(T) = \langle\{w, w'\}\rangle$ and u and v are on two separate branches of w. If there is no vertex on a branch of w' which is adjacent to x, then $d_{H-e}(u', x)$ is at least k, which contradicts $g(u') < k$. Similarly, if there is no vertex on a branch of w' adjacent to y, then $d_{H-e}(u', y) \geqslant k$. We may assume that there are vertices

z and z' on branches of w' with zx and $z'y \in E(H-e)$. Notice that one of these vertices may be u'. Since $d_{H-e}(x,y) = k$, we must have $d_{H-e}(z,z') \geqslant k-2$, and necessarily z and z' are both end-vertices.

If u' is not adjacent to either x or y in $H-e$, then $e = u'x$ or $u'y$. But then $d_{H-ww'}(w,w') = k$. We may assume without loss of generality that $u' = z$.

The edge e is incident with at least one of x and y. If $e = xy$ or if e joins either x or y to an end-vertex of T, then $d_{H-ww'}(w,w') = k$. We may assume that e joins x or y to a vertex of T that is not an end-vertex of T. Without loss of generality, suppose e joins x to a vertex on a branch of w. Then $d_{H-yz'}(y,z') \geqslant k$ which contradicts $g(z') = k-1$.

Therefore $d_{H-e}(u,v) = \mathrm{diam}(T)$.

Let if possible now $d_{H-e}(u,v) = \mathrm{diam}(T) = k-1$. Note that $d_{H-e}(x,y) = k$ and $g(s) = k-1$ for all $s \in V(T)$. Therefore for all $s \in V(T)$ with $sx \in E(H-e)$, we must have $d_{H-e}(s,y) \geqslant k-1$ and in particular $d_{H-e}(u,y) = k-1$. Therefore there exists an $s \in V(T) - D$ such that $sy \in E(H-e)$. Using Lemma 3 and the fact that $s \notin D$, we get $d_{H-e}(u,s) = k-2$. Note that s must be an end-vertex. Otherwise consider an end-vertex on the branch of s, say s', then $d_{H-e}(s',x) > k-1$ which is a contradiction to the fact that $e_{H-e}(s') \leqslant k-1$. By a similar argument we can find an end-vertex $z \notin D$ with $d_{H-e}(v,z) = k-2$ and $zx \in E(H-e)$.

Claim: $d_{H-e}(s,z) = \mathrm{diam}(T)$.

In $H-e$, let a shortest $u-v$ path be $u, u_1, u_2, \ldots, u_r, u_{r+1}, \ldots, u_{r+m}, \ldots, u_{k-2}, v$, shortest $u-s$ path be $u, u_1, u_2, \ldots, u_r, u'_{r+1}, \ldots, u'_{r+m}, \ldots, u'_{k-3}, s$, shortest $v-z$ path be $v, u_{k-2}, u_{k-3}, \ldots, u_{r+m}, v_{r+m-1}, \ldots, v_2, z$. See Figure 1.

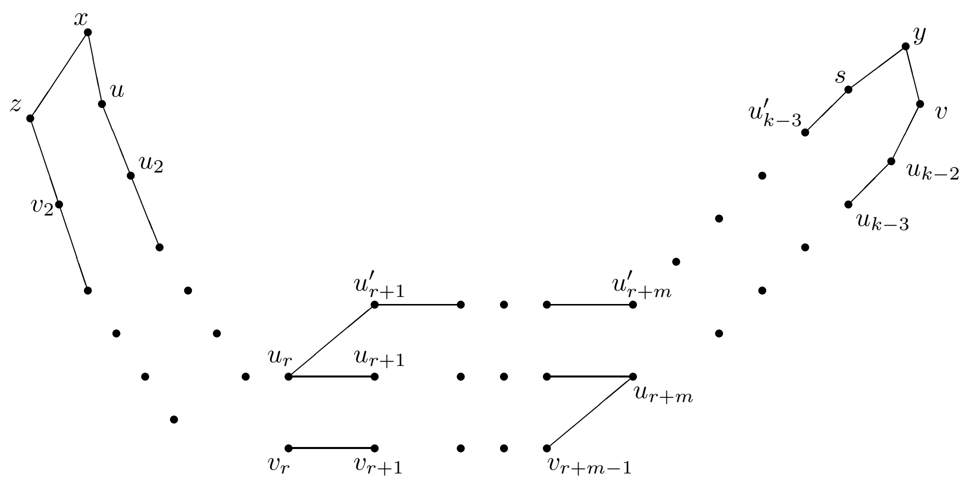

Figure 1

Let $d(u_r, s) = a$, $d(u_{r+m}, z) = b$, $d(u_{r+m}, v) = c$. Then $r + m + c = k - 1$, $r + a = k - 2$, $c + b = k - 2$ and $b + m + a = d_{H-e}(s, z) = k - 2$ as $s, z \notin D$ and by Lemma 3. Solving these equations we would get $2m = 1$ which is not possible since m is a whole number. Therefore our assumption is false. $\square$

Lemma 5. *Let T be a tree with $A(T) = 2$. Then D must contain two vertices u and $v \in V(H - e)$, such that $ux, vy \in E(H - e)$.*

P r o o f. Since all end-vertices of T must be adjacent to x or y in H, in $H - e$ all but possibly one of the end-vertices must be adjacent to either x or y. Let if possible D not contain two vertices u and v in $V(H - e)$, such that $ux, vy \in E(H - e)$. Then without loss of generality, we can assume that in $H - e$ all vertices in D are adjacent to x only (except possibly one). Consider a vertex $u \in D$ with $ux \in E(H - e)$. Note that in $H - e$, $d_{H-e}(x, y) = k$ and therefore all x-y paths must be of length greater than or equal to k. Since $g(u) = k - 1$, there must exist a $s \in V(T) - D$, with $sy \in E(H - e)$ and $d_{H-e}(u, s) = k - 2$.

C a s e 1. Let if possible u and s be on the same branch of T.

Consider a vertex $u' \in D$ with $d_T(u, u') = \operatorname{diam}(T)$. Now, $d_T(u', s) \geqslant d_T(u, s) + 2 = k$. Since $g(u') = k - 1$, there must be a shorter $u' - s$ path in $H - e$. If this path goes through x, then $d_{H-e}(u', s) \geqslant d_{H-e}(x, s) + 1 = k$ which is not possible. The shortest $u' - s$ path must go through y, so $d_{H-e}(u', y) \leqslant k - 2$. Thus, u' cannot be adjacent to x. Thus u' is the unique vertex at distance $\operatorname{diam}(T)$ from u in T. Since $g(u') = k - 1$, we have $d_{H-e}(u', x) \leqslant k - 1$. On a shortest $u' - x$ path, let x' be the vertex adjacent to x. On a shortest $u' - y$ path, let y' be the vertex adjacent to y. We may assume without loss of generality that $y' \notin D$. The portion of the $u' - x'$ and $u' - y'$ paths moving towards $C(T)$ must be the same. This common portion is more than half of the $u' - y'$ path and at least half of the $u' - x'$ path, so $d_{H-e}(x', y') = \lceil \frac{k-3}{2} \rceil - 1 + \lceil \frac{k-2}{2} \rceil = k - 3$. But then $d_{H-e}(x, y) \leqslant k - 1$, a contradiction.

C a s e 2. Let if possible s belong to $C(T)$.

Note that if $C(T)$ has one vertex w, then $\operatorname{rad}(T) = k - 2$ and wy must be an edge in $H - e$. If $C(T)$ has two central vertices w and w', then both of them must be adjacent to y and $\operatorname{rad}(T) - 1 = k - 2$. Note that in both these cases, only vertices in D can be adjacent to x in $H - e$, otherwise x-y paths of length less than k would exist in $H - e$. To make the argument easier to understand we will show later that when s is a central vertex, all end-vertices of T must belong to D. Using some of the similar arguments we can also show that no other vertices of T besides the central vertices of T can be adjacent to y in $H - e$. Therefore, if we assume that all end-vertices of T are in D, $e = xy$ or yz for some z in $V(T)$ in order for H to be 2-connected. (Note that in the case when there are two central vertices we also have to consider

$e = xz$ for some z in $V(T)$.) We will now show that e cannot equal xy in the case $|C(T)| = 1$. The case $|C(T)| = 2$ is similar.

Claim: e cannot equal xy.

Proof of Claim: Let if possible $e = xy$. Let u_1 be a vertex of T adjacent to w in $H - e$. Then $d_{H-wu_1}(w, u_1) > k - 1$. See Figure 2.

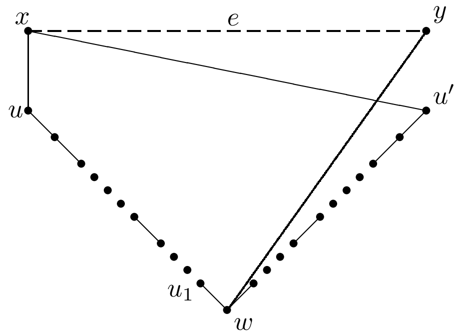

Figure 2

Claim: e cannot equal yz, for some vertex z of $V(T)$.

Proof of Claim: If $e = yz$ for some vertex z of $V(T)$, then consider a vertex u_1 of T adjacent to w in $H - e$ and belonging to a branch of T not containing z. Then $d_{H-wu_1}(w, u_1) > k - 1$. See Figure 3.

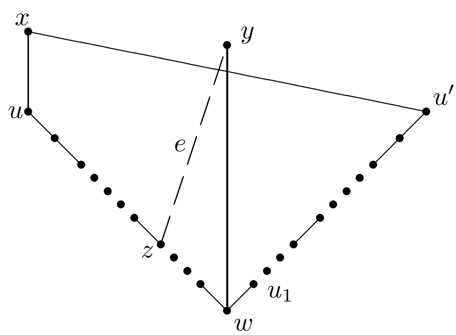

Figure 3

Therefore it is clear that s does not belong to $C(T)$.

(Note that in the case when $|C(T)| = 2$, we would also have to consider that $e = xz$ for z in $V(T)$. The proof to show that e cannot equal xz for some vertex z of $V(T)$ is identical to the proof when we show e cannot equal yz for some vertex z of $V(T)$. Also remember to insert $\mathrm{rad}(T) - 1$ in place of $\mathrm{rad}(T)$ in the above proof.)

After proving the fact that all the end-vertices are in D, then we will know that s cannot be in $C(T)$.

Now we prove the fact that all end-vertices of T must belong to D. We will prove the result for the case when $C(T)$ has only one vertex.

We know that u is a peripheral vertex and s is the central vertex, and $d_{H-e}(u,s) = k-2$. Thus, any vertex in T that is adjacent to x in $H-e$ must be a peripheral vertex of T. Suppose there is a branch of T so that no vertex on that branch is adjacent to x in $H-e$. Then for any vertex u' on that branch, the shortest x-u' path in $H-e$ must go through either w or y, and so have length at least k. This is a contradiction; we can assume without loss of generality that every branch of T contains some peripheral vertex that is adjacent to x in $H-e$.

Let if possible there exist at least one end-vertex, say z, in $V(T)$ that is not in D. Then z is an end-vertex on a branch of T containing at least one end-vertex in D. Assume that at least one of the peripheral end-vertices on the branch containing z is adjacent to x. If zy is an edge in $H-e$, then let u' be the vertex on the branch of z adjacent to x and belonging to D. Let u be an end-vertex in D with $d_T(u,u') = \operatorname{diam}(T)$. Then the shortest $z-u$ path must be either a combination of a shortest $z-u'$ path (which clearly must be of length $k-2$ or more in $H-e$) along with the edges $u'x$ and xu, or a combination of a shortest $z-w$ path (which must be of length 2 or more) along with the shortest $u-w$ path. This would imply that $d(z,u) > k-1$.

If zy is not an edge in $H-e$, without loss of generality we can assume that there are no end-vertices on the branch containing z that are not in D and are adjacent to y. Let u' be one of the end-vertices on the branch of z, adjacent to x and in D. Let u be a vertex in D with $d(u,u') = \operatorname{diam}(T)$ in T. Clearly u is on another branch of T. Let the root of this branch be u_1. Let the shortest u_1-u path be $u_1, u_2, u_3, \ldots, u_{k-2}$ where $u_{k-2} = u$. Let $d(z,w) = n$. Then $d(z, u_{k-n}) > k-1$.

When $C(T)$ has two vertices consider u_1 be a vertex adjacent to the other central vertex and replace $k-n$ with $k-1-n$. Therefore all end-vertices must belong to D.

Case 3. Let u and s belong to different branches of T.

When there are two central vertices, note that u and s must belong to branches of different central vertices. Let u' be an end-vertex on the branch of s farthest away from $C(T)$. Note that none of end-vertices on the branch of T containing s could be adjacent to x, otherwise there would exist an x-y path of length less than $k-1$ in $H-e$. Therefore $d_{H-e}(u',x) > k-1$.

(In Case 3, if there are two central vertices w and w' such that one of the branches of w' contains s, then none of the end-vertices of all the branches of w' can be adjacent to x in $H-e$.)

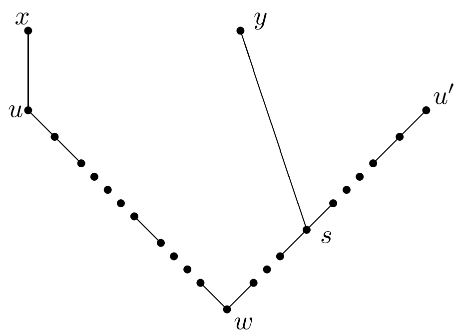

Figure 4

Therefore $d(u,s) = 2\,\mathrm{rad}(T)$ when there is one central vertex, and $d(u,w) = 2\,\mathrm{rad}(T) - 1$ when there are two central vertices. And hence there must be at least two vertices u, s in D with ux and sy as edges in $H - e$. $\square$

Theorem 1. *Let T be a tree with $A(T) = 2$. Then all the end-vertices are equidistant from the center.*

P r o o f. In order to show that all end-vertices are equidistant from the center we will show that all end-vertices belong to D. Note that $|D| \geqslant 2$ for a tree. By Lemma 5, there exist vertices $u, v \in D$ with $ux, vy \in E(H - e)$. By Lemma 4, $d_{H-e}(u,v) = k - 2 = \mathrm{diam}(T)$. Therefore all end-vertices adjacent to x or y must be in D (otherwise there will exist an x-y path of length less than k). Suppose there exists an end-vertex z, such that $z \notin D$. This would imply that $e = xz$ or yz. Without loss of generality assume that $e = xz$. In this case let z_1 be a vertex of T adjacent to z. Then $d_{H-zz_1}(z,y) > k - 1$ and therefore $g(z) \neq k - 1$. Therefore all end-vertices must belong to D. $\square$

Lemma 6. *Let T be a tree with $A(T) = 2$. Let u_n and z_n be end-vertices of the same branch of T. If $u_n x \in E(H - e)$, then $z_n y \notin E(H - e)$ (in other words the end-vertices of the same branch of T cannot be adjacent to x and y in $H - e$).*

P r o o f. Clearly from Lemma 4 if u_n and z_n are end-vertices with $u_n x, z_n y \in E(H - e)$, then $d_{H-e}(u_n, z_n) = k - 2 = \mathrm{diam}(T)$. This implies that u_n and z_n cannot be the end-vertices of the same branch (otherwise $d_{H-e}(u_n, z_n) < \mathrm{diam}(T)$ a contradiction to Lemma 4). $\square$

N o t e 1. In $H - e$ only vertices in D can be adjacent to an x or y (by Lemma 4 and Lemma 5). Also note that it is clear that $e \neq xz$ for any z in D, otherwise $d_{H-zz_1}(z,y) > k - 1$ where z_1 is a vertex adjacent to z. A symmetric argument shows that $e \neq yz$ for any z in D.

N o t e 2. By Lemma 4, Theorem 1 and Lemma 6, for a tree T with $A(T) = 2$, it follows that $k = 2\operatorname{rad}(T) + 2$ when $C(T) = \langle\{w\}\rangle$ and $k = 2\operatorname{rad}(T) + 1$ when $C(T) = \langle\{w, w'\}\rangle$.

Lemma 7. *If T is a tree with $C(T) = \langle\{w, w'\}\rangle$, then $A(T) \neq 2$.*

P r o o f. Let if possible $A(T) = 2$. By the note above we know that $k = 2\operatorname{rad}(T) + 1$. Therefore all end-vertices of w are adjacent to x and that of w' to y. In order for H to be 2-connected, $e = xy$. Then $d_{H-ww'}(w, w') > k - 1$ which is a contradiction to the fact that $g(w) = k - 1$. $\square$

Lemma 8. *Let T be a tree with $A(T) = 2$ and $C(T) = \langle\{w\}\rangle$. Then $e = xy$.*

P r o o f. From Lemma 5 it is clear that $d_{H-e}(x, y) = k = \operatorname{diam}(T) + 2$. Note 1 gives us that $e \neq xz$ for any $z \in D$. Let if possible $e = xz$ for $z \in V(T) - D$. For cases 1 through 3, let $z \in V(T) - (D \cup \{w\})$.

C a s e 1. *Let z belong to a branch of w where all end-vertices are adjacent to x.* In this case in order for H to be 2-edge-connected, we must have $\deg(w) \geqslant 4$, at least two of the branches must have all their end-vertices adjacent to x, and at least two of the branches must have all their end-vertices adjacent to y. Let $u_1 \in V(T)$ be a vertex on a branch whose end-vertices are adjacent to x, with $u_1w \in E(T)$. Then $d_{H-u_1w}(u_1, y) > k - 1$ which is a contradiction to the fact that $g_H(u_1) = k - 1$.

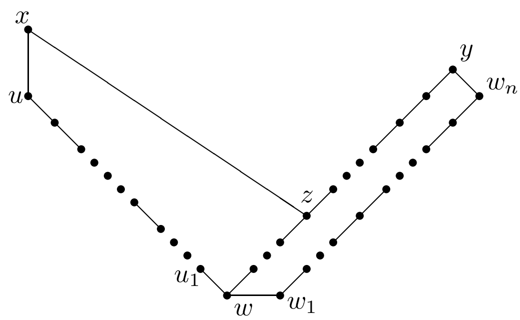

Figure 5

C a s e 2. *Let z belong to a branch of w where all end-vertices are adjacent to y and there is more than one branch of w whose end-vertices are adjacent to y.* Let $u_1 \in V(T)$ be a vertex on a branch whose end-vertices are adjacent to x with $u_1w \in E(T)$. Consider a branch of w not containing z whose end-vertices are adjacent to y. Let w_1 be a vertex on this branch adjacent to w. Then $d_{H-w_1w}(u_1, w_1) > k-1$ which is a contradiction to the fact that $g_H(u_1) = k - 1$. See Figure 5.

C a s e 3. *Let z belong to a branch of w where all the end-vertices are adjacent to y and there is only one branch of w whose end-vertices are adjacent to y.* For H to remain 2-edge-connected, the degree of y must be 2 or more and the degree of at least one of the vertices z' on the branch containing z with $d_T(z', w) \leqslant d_T(z, w)$ must be at least 3. Notice that $e_{H-e}(z) < k - 1$ for all edges $e \in E(H)$. Therefore $g(z) < k - 1$ which is a contradiction. See Figure 6.

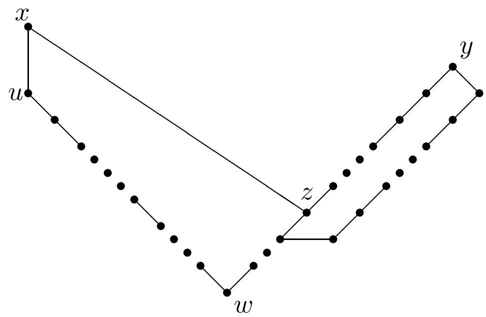

Figure 6

C a s e 4. *Let $z = w$.* Without loss of generality we can assume that $e = xw$. Consider a vertex u_1 on a branch of w where end-vertices are adjacent to x and $u_1w \in E(T)$. Then $d_{H-u_1w}(u_1, y) > k - 1$ which is a contradiction. When $e = yw$ a similar proof can be given.

Therefore $e = xy$. □

Lemma 9. *Let T be a tree with $A(T) = 2$ and $C(T) = \langle\{w\}\rangle$. Then $\deg(w) \geqslant 4$.*

P r o o f. Let if possible $\deg(w) < 4$. Clearly $\deg(w) \geqslant 2$, therefore without loss of generality let us assume that only one branch of T has end-vertices adjacent to x. Let u_1 be a vertex on this branch with $u_1w \in E(T)$. By Lemma 8, since $e = xy$, $d_{H-u_1w}(u_1, w) > k - 1$. This is a contradiction to the fact that $g(u_1) = k - 1$. □

Theorem 2. *Let T be a tree with $C(T) = \langle\{w\}\rangle$. Then $A(T) = 2$ if and only if the following are satisfied:*

(a) *All end-vertices are equidistant from the center.*

(b) $\deg(w) \geqslant 4$, *and*

(c) *for $z \in V(T)$, if $1 \leqslant d_T(z, w) < n-1$, then $\deg_T(z) = 2$, and if $d_T(w, z) = n-1$, then $\deg_T(z) \geqslant 2$.*

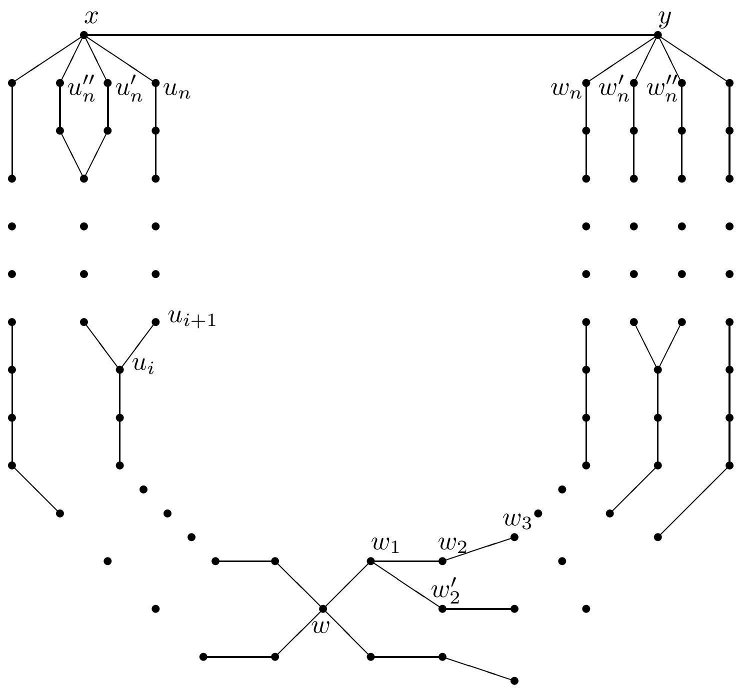

Figure 7

P r o o f. From [4], we have that a), b) and c) imply $A(T) = 2$. To see this, construct a graph H from the tree T by adding two new vertices x and y to T, joining x to all end-vertices of T in two branches of w, joining y to the remaining end-vertices of T, and adding the edge xy. In the graph H, we calculate $g(z) = 2n+1$ for $z \in V(T)$ and $g(x) = g(y) = 2n+2$.

If $A(T) = 2$, then there is a graph H with $V(H) = V(T) \cup \{x, y\}$ with $\mathrm{EDC}(H) = T$. It follows that all end-vertices are equidistant from the center by Theorem 1 and $\deg(w) \geqslant 4$ by Lemma 9. Let $u_i \in V(T)$ such that $d(u_i, w) < n-1$ and $\deg(w) > 2$. Let u_1 be a vertex on this branch adjacent to w and without loss of generality, assume that all end-vertices of this branch are adjacent to x. Also assume that $u_i, u_{i+1}, \ldots, u_n$ and $u_i, u'_{i+1}, \ldots, u'_n$, are at least two of the sub-branches of this vertex. If $i \neq 1$, then $g(u_{i+1}) < k-1$ and if $i = 1$, then $g(u_3) < k-1$, which are both contradictions. See Figure 7. □

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