Measure-valued solutions to compressible fluid flows revisited

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Compressible Navier-Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x (\varrho \mathbf{u}) = 0$$

$$\partial_t (\varrho \mathbf{u}) + \mathrm{div}_x (\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Newton's rheological law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \mu > 0, \ \eta \geq 0$$

No-slip boundary conditions

$$\mathbf{u}|_{\partial \Omega} = 0$$

Thermodynamics stability hypothesis

Pressure potential

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2}\,\mathrm{d}z$$

Pressure-density state equation

$$p \in C[0,\infty) \cap C^2(0,\infty),\ \ p(0) = 0$$

$$p'(\varrho) > 0 \text{ for } \varrho > 0,\ \ \liminf_{\varrho\to\infty} p'(\varrho) > 0$$

$$\liminf_{\varrho\to\infty} \frac{P(\varrho)}{p(\varrho)} > 0$$

Isentropic pressure-density state equation

$$p(\varrho) = a\varrho^\gamma,\ \ a > 0,\ \ \gamma \geq 1$$

Hierarchy of solutions

Classical solutions

Solutions are (sufficiently) smooth satisfying the equations point-wise, determined uniquely by the data. Requires strong *a priori* bounds usually not available

Weak solutions

Equations satisfied in the sense of distributions. Requires *a priori* bounds to ensure equi-integrability of nonlinearities + compactness

Measure-valued solutions

Equations satisfied in the sense of distributions, nonlinearities replaced by Young measures (weak limits) $f(u)(t,x) \approx \langle \nu_{t,x}; f(\mathbf{v}) \rangle$. Requires *a priori* bounds to ensure equi-integrability of nonlinearities.

Measure-valued solutions with concentration measure

Measure-valued solutions + concentration defects. Requires *a priori* bounds to ensure integrability of nonlinearities.

Dissipative solutions

Energy (entropy) inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)\ \mathrm{d}x + \int_{\Omega}\mathbb{S}(\nabla_x\mathbf{u}) : \nabla_x\mathbf{u}\ \mathrm{d}x \boxed{\leq} 0$$

$$P(\varrho) = \varrho\int_1^{\varrho}\frac{p(z)}{z^2}\ \mathrm{d}z$$

Known results

- **Local strong solution for any data and global weak solutions for small data.** Matsumura and Nishida [1983], Valli and Zajaczkowski [1986], among others
- **Global-in-time weak solutions.** $p(\varrho) = \varrho^{\gamma}$, $\gamma \geq 9/5, N = 3$, $\gamma \geq 3/2, N = 2$ P.L. Lions [1998], $\gamma > 3/2$, $N = 3$, $\gamma > 1$, $N = 2$ EF, Novotný, Petzeltová [2000], $\gamma = 1$, $N = 2$ Plotnikov and Vaigant [2014]
- **Measure-valued solutions.** Neustupa [1993], related results Málek, Nečas, Rokyta, Růžička, Nečasová - Novotný

Bounded sequences of integrable functions

Boundedness

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^1(Q; R^M)$$

$$\|F(\mathbf{v}_n)\|_{L^1(Q)} \le C \;\Rightarrow\; F(\mathbf{v}_n) \to \overline{F(\mathbf{v})} \not\equiv F(\mathbf{v}) \text{ weakly-(*) in } \mathcal{M}(\overline{Q})$$

Biting limit - parameterized Young measure

$$\langle \nu_{t,x}; F_k(\mathbf{v})\rangle = \overline{F_k(\mathbf{v})}(t,x),\; F_k \in BC(R^M)$$

$$\langle \nu_{t,x}; F(\mathbf{v})\rangle = \lim_{k\to\infty} \overline{F_k(\mathbf{v})}(t,x),\; F_k \nearrow F,\; \|F(\mathbf{v}_n)\|_{L^1(Q)} \le C$$

Concentration part - defect measure

$$\overline{F(\mathbf{v})}(t,x) = \underbrace{\langle \nu_{t,x}; F(\mathbf{v})\rangle}_{integrable} + \underbrace{\left[\overline{F(\mathbf{v})}(t,x) - \langle \nu_{t,x}; F(\mathbf{v})\rangle\right]}_{concentration\ \ defect}$$

Measure-valued solutions

Parameterized (Young) measure

$$\nu_{t,x} \in L^\infty_{\rm weak}((0,T) \times \Omega; \mathcal{P}([0,\infty) \times R^N)), \ [s, \mathbf{v}] \in [0,\infty) \times R^N$$

$$\varrho(t,x) = \langle \nu_{t,x}; s \rangle, \ \mathbf{u} = \langle \nu_{t,x}; \mathbf{v} \rangle \in L^2(0,T; W^{1,2}_0(\Omega; R^N))$$

Field equations revisited

$$\int_0^T \int_\Omega \langle \nu_{t,x}; s \rangle \partial_t \varphi + \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \nabla_x \varphi \ {\rm d}x \ {\rm d}t = \langle R_1; \nabla_x \varphi \rangle$$

$$\int_0^T \int_\Omega \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \partial_t \varphi + \langle \nu_{t,x}; s\mathbf{v} \otimes \mathbf{v} \rangle \cdot \nabla_x \varphi + \langle \nu_{t,x}; p(s) \rangle {\rm div}_x \varphi \ {\rm d}x \ {\rm d}t$$

$$= \int_0^T \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \varphi \ {\rm d}x \ {\rm d}t + \langle R_2; \nabla_x \varphi \rangle$$

Dissipativity

Energy inequality

$$\int_\Omega \left\langle \nu_{\tau,x}; \left(\frac{1}{2}s|\mathbf{v}|^2 + P(s)\right)\right\rangle \,\mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \,\mathrm{d}x\,\mathrm{d}t + \boxed{\mathcal{D}(\tau)}$$

$$\leq \int_\Omega \left\langle \nu_0; \left(\frac{1}{2}s|\mathbf{v}|^2 + P(s)\right)\right\rangle \,\mathrm{d}x$$

Compatibility

$$\left|R_1[0,\tau] \times \overline{\Omega}\right| + \left|R_2[0,\tau] \times \overline{\Omega}\right| \leq \xi(\tau)\mathcal{D}(\tau),\ \xi \in L^1(0,T)$$

$$\int_0^\tau \int_\Omega \left\langle \nu_{t,x}; |\mathbf{v} - \mathbf{u}|^2 \right\rangle \,\mathrm{d}x\,\mathrm{d}t \leq c_P \mathcal{D}(\tau)$$

Truly measure-valued solutions

Truly measure-valued solutions for the Euler system (with E.Chiodaroli, O.Kreml, E. Wiedemann)

There is a measure-valued solution to the compressible Euler system (without viscosity) that *is not* a limit of bounded L^p weak solutions to the Euler system.

Do we need measure valued solutions?

Limits of problems with higher order viscosities

Multipolar fluids with complex rheologies (Nečas - Šilhavý)

$$\mathbb{T}(\mathbf{u}, \nabla_x \mathbf{u},\ \nabla_x^2 \mathbf{u}, \dots)$$

$$= \mathbb{S}(\nabla_x \mathbf{u}) + \delta \sum_{j=1}^{k-1} \left((-1)^j \mu_j \Delta^j (\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}) + \lambda_j \Delta^j \mathrm{div}_x \mathbf{u}\ \mathbb{I} \right)$$

$$+ \text{ non-linear terms}$$

Limit for $\delta \to 0$

Limits of numerical solutions

Numerical solutions resulting from Karlsen-Karper and other schemes

Sub-critical parameters

$p(\varrho) = a\varrho^\gamma,\ \gamma < \gamma_{\rm critical}$

Weak (mv) - strong uniqueness

Theorem - EF, P.Gwiazda, A.Świerczewska-Gwiazda, E. Wiedemann [2015]

A measure valued and a strong solution emanating from the same initial data coincide as long as the latter exists

Relative energy (entropy)

Relative energy functional

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right)(\tau)$$

$$= \int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2} s|\mathbf{v} - \mathbf{U}|^2 + P(s) - P'(r)(s-r) - P(r) \right\rangle \, \mathrm{d}x$$

$$= \int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2} s|\mathbf{v}|^2 + P(s) \right\rangle \, \mathrm{d}x - \int_\Omega \langle \nu_{\tau,x}; s\mathbf{v} \rangle \cdot \mathbf{U} \, \mathrm{d}x$$

$$+ \int_\Omega \frac{1}{2} \langle \nu_{\tau,x}; s \rangle \, |\mathbf{U}|^2 \, \mathrm{d}x$$

$$- \int_\Omega \langle \nu_{\tau,x}; s \rangle \, P'(r) \, \mathrm{d}x + \int_\Omega p(r) \, \mathrm{d}x$$

Relative energy (entropy) inequality

Relative energy inequality

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) + \int_0^\tau \mathbb{S}(\nabla_x \mathbf{u}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t + \mathcal{D}(\tau)$$

$$\leq \int_\Omega \left\langle \nu_{0,x}; \frac{1}{2} s |\mathbf{v} - \mathbf{U}_0|^2 + P(s) - P'(r_0)(s - r_0) - P(r_0) \right\rangle \ \mathrm{d}x$$

$$+ \int_0^\tau \mathcal{R}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) \ \mathrm{d}t$$

Remainder

$$\mathcal{R}\left(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\right)$$

$$= - \int_0^\tau \int_\Omega \langle \nu_{t,x}, s\mathbf{v} \rangle \cdot \partial_t \mathbf{U} \ \mathrm{d}x \ \mathrm{d}t$$

$$- \int_0^\tau \int_{\overline{\Omega}} \left[\langle \nu_{t,x}; s\mathbf{v} \otimes \mathbf{v} \rangle : \nabla_x \mathbf{U} + \langle \nu_{t,x}; p(s) \rangle \, \mathrm{div}_x \mathbf{U} \right] \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \left[\langle \nu_{t,x}; s \rangle \, \mathbf{U} \cdot \partial_t \mathbf{U} + \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \mathbf{U} \cdot \nabla_x \mathbf{U} \right] \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \int_\Omega \left[\left\langle \nu_{t,x}; \left(1 - \frac{s}{r} \right) \right\rangle p'(r) \partial_t r - \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \frac{p'(r)}{r} \nabla_x r \right] \ \mathrm{d}x \ \mathrm{d}t$$

$$+ \int_0^\tau \left\langle R_1; \frac{1}{2} \nabla_x \left(|\mathbf{U}|^2 - P'(r) \right) \right\rangle \ \mathrm{d}t - \int_0^\tau \langle R_2; \nabla_x \mathbf{U} \rangle \, \mathrm{d}t$$

Regularity

Theorem - EF, P.Gwiazda, A. Świerczewska-Gwiazda, E. Wiedemann

Suppose that the initial data are smooth and satisfy the relevant compatibility conditions. Let $\nu_{t,x}$ be a measure-valued solution to the compressible Navier-Stokes system with a dissipation defect $\mathcal{D}$ such that

$$\mathrm{supp}\ \nu_{t,x} \subset \left\{ (s, \mathbf{v}) \ \Big| \ 0 \leq s \leq \overline{\varrho},\ \mathbf{v} \in R^N \right\}$$

for a.a. $(t,x) \in (0,T) \times \Omega$.
Then $\mathcal{D} = 0$ and

$$\nu_{t,x} = \delta_{\varrho(t,x), \mathbf{u}(t,x)}$$

where ϱ, $\mathbf{u}$ is a smooth solution.

Sketch of the proof

- The Navier-Stokes system admits a local-in-time smooth solution
- The measure-valued solution coincides with the smooth solution on its life-span
- The smooth solution density component remains bounded by $\overline{\varrho}$ as long as the solution exists
- Y. Sun, C. Wang, and Z. Zhang [2011]: The strong solution can be extended as long as the density component remains bounded

Corollary

Convergence of numerical solutions

Bounded numerical solutions emanating from smooth data that converge to a measure-valued solution converge, in fact, unconditionally to the unique strong solution