Relative energies and problems of stability in fluid dynamics

Eduard Feireisl

Institute of Mathematics, Academy of Sciences of the Czech Republic, Prague

MoMatFlu 2016, WIAS Berlin, 22–26 February 2016

The research leading to these results has received funding from the European Research Council under the European Union's Seventh Framework Programme (FP7/2007-2013)/ ERC Grant Agreement 320078

Compressible barotropic fluid flow

Equation of continuity - mass conservation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

Momentum equation - Newton's second law

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}$$

$$\mathbf{u}|_{\partial\Omega} = 0$$

Energy inequality

$$E(\tau) + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t \boxed{\leq} E(0)$$

$$E = \int_\Omega \frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \, \mathrm{d}x, \;\; P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \, \mathrm{d}z$$

Relative energy/entropy

Lyapunov function

$$E = \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - 0|^2 + P(\varrho) - P'(\overline{\varrho})(\varrho - \overline{\varrho}) - P(\overline{\varrho}) \right] \, \mathrm{d}x$$

Coercivity of the pressure potential

$$\varrho \mapsto p(\varrho) \text{ non-decreasing } \Rightarrow \ \varrho \mapsto P(\varrho) \text{ convex}$$

Relative energy (relative entropy Dafermos [1979])

$$\mathcal{E}\left(\varrho, \mathbf{u} \ \middle| \ r, \mathbf{U} \right)$$

$$= \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + P(\varrho) - P'(r)(\varrho - r) - P(r) \right] \, \mathrm{d}x$$

Relative energy inequality

Weak formulation

Determine $\left[\mathcal{E} \left(\varrho, \mathbf{u} \ \Big| \ r, \mathbf{U} \right) \right]_{t=0}^{t=\tau}$ in terms of the weak formulation!

$$\mathcal{E} \left(\varrho, \mathbf{u} \ \Big| \ r, \mathbf{U} \right)$$

$$= \int_\Omega \left[\frac{1}{2} \varrho |\mathbf{u}|^2 + P(\varrho) \right] \ \mathrm{d}x - \int_\Omega \varrho \mathbf{u} \cdot \mathbf{U} \ \mathrm{d}x + \int_\Omega \frac{1}{2} \varrho |\mathbf{U}|^2 \ \mathrm{d}x$$

$$- \int_\Omega P'(r) \varrho \ \mathrm{d}x + \int_\Omega \left[P'(r) r - P(r) \right] \ \mathrm{d}x$$

Constraints imposed on the test functions

$$r > 0$$

$$\mathbf{U}|_{\partial \Omega} = 0$$

Dissipative solutions

Relative energy inequality

$$\left[\mathcal{E} \left(\varrho, \mathbf{u} \;\middle|\; r, \mathbf{U} \right) \right]_{t=0}^{t=\tau}$$

$$+ \int_0^\tau \int_\Omega (\mathbb{S}(\nabla_x \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U})) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t$$

$$\leq \int_0^\tau \mathcal{R} \left(\varrho, \mathbf{u} \;\middle|\; r, \mathbf{U} \right) \ \mathrm{d}t$$

Test functions

$$r > 0, \ \mathbf{U}|_{\partial\Omega} = 0 \text{ (or other relevant b.c.)}$$

Remainder

$$\int_0^\tau \mathcal{R}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) \, \mathrm{d}t$$

$$\int_\Omega \varrho \left(\partial_t \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U}\right) \cdot (\mathbf{U} - \mathbf{u}) \, \mathrm{d}x$$

$$+ \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \, \mathrm{d}x + \int_\Omega (p(r) - p(\varrho)) \, \mathrm{div}_x \mathbf{U} \, \mathrm{d}x$$

$$+ \int_\Omega \left[(r - \varrho) \partial_t P'(r) + \nabla_x P'(r) \cdot (r\mathbf{U} - \varrho \mathbf{u}) \right] \, \mathrm{d}x$$

Relative energy for complete fluids

Relative energy

$$\mathcal{E}\left(\varrho, \vartheta, \mathbf{u} \;\middle|\; r, \Theta, \mathbf{U}\right)$$

$$= \int_{\Omega} \left[\frac{1}{2} \varrho |\mathbf{u} - \mathbf{U}|^2 + H_{\Theta}(\varrho, \vartheta) - \frac{\partial H_{\Theta}(r, \Theta)}{\partial \varrho} (\varrho - r) - H_{\Theta}(r, \Theta) \right] \, \mathrm{d}x$$

Ballistic free energy

$$H_{\Theta}(\varrho, \vartheta) = \varrho \left(e(\varrho, \vartheta) - \Theta s(\varrho, \vartheta) \right)$$

Hypothesis of thermodynamic stability

$$\frac{\partial p(\varrho, \vartheta)}{\partial \varrho} > 0, \quad \frac{\partial e(\varrho, \vartheta)}{\partial \vartheta} > 0$$

Weak-strong uniqueness, conditional regularity

Theorem (EF, B.J. Jin, A.Novotný, Y.Sun [2014])

- *Weak and strong solutions of the barotropic Navier-Stokes system emanating from the same initial data coincide as long as the latter exists*
- *Weak solution with bounded density component emanating from smooth initial data is smooth*

Singular limits

Rotating fluids

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \frac{1}{\varepsilon} \varrho \mathbf{b} \times \mathbf{u} + \frac{1}{\varepsilon^{2M}} \nabla_x p(\varrho)$$

$$= \varepsilon^R \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u}) + \frac{1}{\varepsilon^{2F}} \nabla_x G$$

Path dependent singular limit

$$\varepsilon \to 0, \text{ certain relation between } M, R, F > 0$$

- low Mach $\Rightarrow$ compressible $\to$ incompressible
- high Rossby $\Rightarrow$ 3D $\to$ 2D
- high Reynolds $\Rightarrow$ viscous $\to$ inviscid

Convergence to singular limit system

Target problem - Euler system

$$\partial_t \mathbf{v} + \mathbf{v} \cdot \nabla_x \mathbf{v} + \nabla_x \Pi = 0, \ \mathrm{div}_x \mathbf{v} = 0, \ x \in R^2$$

Convergence results (joint work with A.Novotný, Y.Lu [2014])

- **Spatial geometry - infinite strip:**

$$\Omega = R^2 \times (0, \pi)$$

- **Complete slip (Navier) boundary conditions:**

$$\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega} = 0, \ (\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}) \times \mathbf{n}|_{\partial\Omega} = 0$$

Limits on domains with variable geometry

Channel like domains

$$\Omega_\varepsilon = \left\{ (\mathbf{x}, z) \ \Big| \ z \in (0,1), \ |\mathbf{x} - \varepsilon \mathbf{X}(z)|^2 < \varepsilon^2 R^2(z) \right\}, \ |\mathbf{X}(z)| < R(z)$$

Boundary conditions

$$\mathbf{u} \cdot \mathbf{n}|_\Sigma = 0, \ (\mathbb{S}(\nabla_x \mathbf{u}) \cdot \mathbf{n}) \times \mathbf{n}|_\Sigma = 0$$

$$\Sigma = \partial\Omega \cap \{ z \in (0,1) \}$$

$$\mathbf{u}|_{z=0,1} = 0$$

Target systems

Inviscid limit

$$\partial_t(\varrho_E A) + \partial_z(\varrho_E u_E A) = 0$$

$$\partial_t(\varrho_E u_E A) + \partial_z(\varrho_E u_E^2 A) + A\partial_z p(\varrho_E) = 0$$

Viscous limit

$$\partial_t(\varrho_{NS} A) + \partial_z(\varrho_{NS} u_{NS} A) = 0$$

$$\partial_t(\varrho_{NS} u_{NS} A) + \partial_z(\varrho_{NS} u_{NS}^2 A) + A\partial_z p(\varrho_{NS})$$

$$= A\nu\partial_z^2 u_{NS} + \nu\partial_z\left(R'(z)/R(z) u_{NS}\right), \ \nu = \frac{4}{3}\mu + \eta > 0$$

$$A = R^2$$

Convergence

Korn-Poincaré inequality

$$\int_{\Omega_\varepsilon} |\mathbf{v}|^2 \, \mathrm{d}x \leq c_{KP} \int_{\Omega_\varepsilon} \left| \nabla_x \mathbf{v} + \nabla_x^t \mathbf{v} \right|^2 \, \mathrm{d}x$$

Convergence (joint work with P.Bella, M.Lewicka, A.Novotný [2015])

- Convergence to the target Euler system with geometric terms in the inviscid limit
- Convergence to the Navier-Stokes system in the viscous limit provided the bulk viscosity in the primitive system is positive

Equations driven by stochastic forces

Navier–Stokes system with stochastic forcing

$$\mathrm{d}\varrho + \mathrm{div}_x(\varrho \mathbf{u})\ \mathrm{d}t = 0$$

$$\mathrm{d}(\varrho \mathbf{u}) + \left[\mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) \right]\ \mathrm{d}t = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})\ \mathrm{d}t + \mathbb{G}(\varrho, \varrho \mathbf{u}) \mathrm{d}W,$$

White–noise forcing

$$\mathbb{G}(\varrho, \varrho \mathbf{u})\ \mathrm{d}W = \sum_{k \geq 1} \mathbf{G}_k(\varrho, \varrho \mathbf{u})\ \mathrm{d}W_k.$$

Relative energy inequality

Relative energy inequality - joint result with D.Breit and M.Hofmanová [2015]

$$- \int_0^T \partial_t \psi \ \mathcal{E} \left(\varrho, \mathbf{u} \Big| r, \mathbf{U} \right) \ \mathrm{d}t$$

$$+ \int_0^T \psi \int_\Omega \left(\mathbb{S}(\nabla \mathbf{u}) - \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \right) \mathrm{d}x \ \mathrm{d}t$$

$$\leq \psi(0) \mathcal{E} \left(\varrho, \mathbf{u} \ \Big| r, \mathbf{U} \right) (0) + \int_0^T \psi \mathrm{d} M_{RE} + \int_0^T \psi \mathcal{R} \left(\varrho, \mathbf{u} \Big| r, \mathbf{U} \right) \mathrm{d}t$$

$$\psi \in C_c^\infty [0, T) \ \text{(deterministic)}, \ \psi \geq 0.$$

Test functions

$$\mathrm{d} \ r = D_t^d r \ \mathrm{d}t + D_t^s r \ \mathrm{d}W, \ \mathrm{d}\mathbf{U} = D_t^d \mathbf{U} \ \mathrm{d}t + D_t^s \mathbf{U} \ \mathrm{d}W$$

Stochastic remainder

Remainder

$$\mathcal{R}\left(\varrho, \mathbf{u} \middle| r, \mathbf{U}\right)$$

$$= \int_\Omega \mathbb{S}(\nabla_x \mathbf{U}) : (\nabla_x \mathbf{U} - \nabla_x \mathbf{u}) \ \mathrm{d}x + \int_\Omega \varrho \Big(D_t^d \mathbf{U} + \mathbf{u} \cdot \nabla_x \mathbf{U} \Big) (\mathbf{U} - \mathbf{u}) \ \mathrm{d}x$$

$$+ \int_\Omega \Big((r - \varrho) H''(r) D_t^d r + \nabla_x H'(r)(r\mathbf{U} - \varrho \mathbf{u}) \Big) \ \mathrm{d}x$$

$$- \int_\Omega \mathrm{div}_x \mathbf{U} (p(\varrho) - p(r)) \ \mathrm{d}x + \frac{1}{2} \sum_{k \geq 1} \int_\Omega \varrho \left| \frac{\mathbf{G}_k(\varrho, \varrho \mathbf{u})}{\varrho} - D_t^s \mathbf{U}_k \right|^2 \ \mathrm{d}x$$

$$+ \frac{1}{2} \sum_{k \geq 1} \int_\Omega \varrho H'''(r) |D_t^s r_k|^2 \ \mathrm{d}x + \frac{1}{2} \sum_{k \geq 1} \int_\Omega p''(r) |D_t^s r_k|^2 \ \mathrm{d}x$$

Results for stochastic Navier-Stokes system

Weak–strong uniqueness (with D.Breit, M.Hofmanová [2015])

- Pathwise weak-strong uniqueness
- Weak-strong uniqueness in law

Inviscid–incompressible limit in the stochastic setting (with D.Breit, M.Hofmanová [2015])

Convergence to the limit stochastic Euler system for vanishing viscosity and the Mach number. Results for well-prepared data.

Possible extensions

Numerical analysis (T. Gallouet, R. Herbin, D. Maltese, A.Novotný [2014])

Relative energy inequality for the numerical scheme proposed by K.Karlsen and T. Karper. Error estimates.
(cf. also the talk and results by J.Fischer at this workshop)

Measure–valued solutions

Weak–strong uniqueness for measure-valued solutions - joint work with P. Gwiazda, A. Swierczewska-Gwiazda, E.Wiedemann