Stability issues concerning measure-valued solutions in fluid mechanics

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Very (very) weak solutions

Basic question

What is an ideal class of weak solutions to an initial value problem?

Universality

Any approximation scheme including the numerical ones should give rise to a weak solution in this class. Easy "existence" proofs

Weak-strong uniqueness

Any weak solution coincides with the strong solution originating from the same initial data as long as the latter exists

Compressible Navier-Stokes system

Field equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \mathbf{u}) = 0$$

$$\partial_t(\varrho \mathbf{u}) + \mathrm{div}_x(\varrho \mathbf{u} \otimes \mathbf{u}) + \nabla_x p(\varrho) = \mathrm{div}_x \mathbb{S}(\nabla_x \mathbf{u})$$

Newton's rheological law

$$\mathbb{S}(\nabla_x \mathbf{u}) = \mu \left(\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u} - \frac{2}{3} \mathrm{div}_x \mathbf{u} \mathbb{I} \right) + \eta \mathrm{div}_x \mathbf{u} \mathbb{I}, \ \ \mu > 0, \ \ \eta \geq 0$$

No-slip boundary conditions

$$\mathbf{u}|_{\partial\Omega} = 0$$

Thermodynamics stability hypothesis

Pressure potential

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2} \, \mathrm{d}z$$

Pressure-density state equation

$$p \in C[0,\infty) \cap C^2(0,\infty), \;\; p(0) = 0$$

$$p'(\varrho) > 0 \text{ for } \varrho > 0, \;\; \liminf_{\varrho \to \infty} p'(\varrho) > 0$$

$$\liminf_{\varrho \to \infty} \frac{P(\varrho)}{p(\varrho)} > 0$$

Isentropic pressure-density state equation

$$p(\varrho) = a\varrho^\gamma, \;\; a > 0, \;\; \gamma \geq 1$$

Hierarchy of solutions

Classical solutions

Solutions are (sufficiently) smooth satisfying the equations point-wise, determined uniquely by the data. Requires strong *a priori* bounds usually not available

Weak solutions

Equations satisfied in the sense of distributions. Requires *a priori* bounds to ensure equi-integrability of nonlinearities + compactness

Measure-valued solutions

Equations satisfied in the sense of distributions, nonlinearities replaced by Young measures (weak limits) $f(u)(t,x) \approx \langle \nu_{t,x}; f(\mathbf{v}) \rangle$. Requires *a priori* bounds to ensure equi-integrability of nonlinearities.

Measure-valued solutions with concentration measure

Measure-valued solutions + concentration defects. Requires *a priori* bounds to ensure integrability of nonlinearities.

Do we need (total) energy balance?

A disturbing example, Chiodaroli, EF

For any smooth (C^2) initial data the *compressible Euler system* admits infinitely many global in time *weak* solutions. Apparently "most" of them do not satisfy any kind of energy balance.

Dissipative solutions

Energy (entropy) inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \left(\frac{1}{2}\varrho|\mathbf{u}|^2 + P(\varrho)\right)\ \mathrm{d}x + \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u}\ \mathrm{d}x \boxed{\leq} 0$$

$$P(\varrho) = \varrho \int_1^\varrho \frac{p(z)}{z^2}\ \mathrm{d}z$$

Known results

- **Local strong solution for any data and global weak solutions for small data.** Matsumura and Nishida [1983], Valli and Zajaczkowski [1986], among others
- **Global-in-time weak solutions.** $p(\varrho) = \varrho^\gamma$, $\gamma \geq 9/5, N = 3$, $\gamma \geq 3/2, N = 2$ P.L. Lions [1998], $\gamma > 3/2$, $N = 3$, $\gamma > 1$, $N = 2$ EF, Novotný, Petzeltová [2000], $\gamma = 1$, $N = 2$ Plotnikov and Vaigant [2014]
- **Measure-valued solutions.** Neustupa [1993], related results Málek, Nečas, Rokyta, Růžička, Nečasová - Novotný

Bounded sequences of integrable functions

Boundedness

$$\mathbf{v}_n \to \mathbf{v} \text{ weakly in } L^1(Q; R^M)$$

$$\|F(\mathbf{v}_n)\|_{L^1(Q)} \leq C \;\Rightarrow\; F(\mathbf{v}_n) \to \overline{F(\mathbf{v})} \not\equiv F(\mathbf{v}) \text{ weakly-(*) in } \mathcal{M}(\overline{Q})$$

Biting limit - parameterized Young measure

$$\langle \nu_{t,x}; F_k(\mathbf{v}) \rangle = \overline{F_k(\mathbf{v})}(t,x),\; F_k \in BC(R^M)$$

$$\langle \nu_{t,x}; F(\mathbf{v}) \rangle = \lim_{k \to \infty} \overline{F_k(\mathbf{v})}(t,x),\; F_k \nearrow F,\; \|F(\mathbf{v}_n)\|_{L^1(Q)} \leq C$$

Concentration part - defect measure

$$\overline{F(\mathbf{v})}(t,x) = \underbrace{\langle \nu_{t,x}; F(\mathbf{v}) \rangle}_{integrable} + \underbrace{\left[\overline{F(\mathbf{v})}(t,x) - \langle \nu_{t,x}; F(\mathbf{v}) \rangle \right]}_{concentration\ \ defect}$$

Measure-valued solutions

Parameterized (Young) measure

$$\nu_{t,x} \in L^\infty_{\rm weak}((0,T)\times\Omega; \mathcal{P}([0,\infty)\times R^N)),\ \ [s,\mathbf{v}] \in [0,\infty)\times R^N$$

$$\varrho(t,x) = \langle \nu_{t,x}; s\rangle\ ,\ \ \mathbf{u} = \langle \nu_{t,x}; \mathbf{v}\rangle \in L^2(0,T; W^{1,2}_0(\Omega; R^N))$$

Field equations revisited

$$\int_0^T \int_\Omega \langle \nu_{t,x}; s\rangle \, \partial_t \varphi + \langle \nu_{t,x}; s\mathbf{v}\rangle \cdot \nabla_x \varphi \ {\rm d}x\ {\rm d}t = \langle R_1; \nabla_x \varphi\rangle$$

$$\int_0^T \int_\Omega \langle \nu_{t,x}; s\mathbf{v}\rangle \cdot \partial_t \varphi + \langle \nu_{t,x}; s\mathbf{v}\otimes\mathbf{v}\rangle \cdot \nabla_x \varphi + \langle \nu_{t,x}; p(s)\rangle \, {\rm div}_x \varphi \ {\rm d}x\ {\rm d}t$$

$$= -\int_0^T \int_\Omega \langle \nu_{t,x}; \mathbf{v}\rangle \cdot {\rm div}_x \mathbb{S}(\nabla_x \varphi)\ {\rm d}x\ {\rm d}t + \langle R_2; \nabla_x \varphi\rangle$$

Dissipativity

Energy inequality

$$\int_\Omega \left\langle \nu_{\tau,x}; \left(\frac{1}{2} s |\mathbf{v}|^2 + P(s) \right) \right\rangle \, \mathrm{d}x + \int_0^\tau \int_\Omega \mathbb{S}(\nabla_x \mathbf{u}) : \nabla_x \mathbf{u} \, \mathrm{d}x \, \mathrm{d}t + \boxed{\mathcal{D}(\tau)}$$

$$\leq \int_\Omega \left\langle \nu_0; \left(\frac{1}{2} s |\mathbf{v}|^2 + P(s) \right) \right\rangle \, \mathrm{d}x$$

Compatibility

$$\left| R_1[0,\tau] \times \overline{\Omega} \right| + \left| R_2[0,\tau] \times \overline{\Omega} \right| \leq \int_0^\tau \xi(t) \mathcal{D}(t) \, \mathrm{d}t, \ \xi \in L^1(0,T)$$

$$\int_0^\tau \int_\Omega \left\langle \nu_{t,x}; |\mathbf{v} - \mathbf{u}|^2 \right\rangle \, \mathrm{d}x \, \mathrm{d}t \leq c_P \mathcal{D}(\tau)$$

Truly measure-valued solutions

Truly measure-valued solutions for the Euler system (with E.Chiodaroli, O.Kreml, E. Wiedemann)

There is a measure-valued solution to the compressible Euler system (without viscosity) that *is not* a limit of bounded L^p weak solutions to the Euler system.

Do we need measure valued solutions?

Limits of problems with higher order viscosities

Multipolar fluids with complex rheologies (Nečas - Šilhavý)

$$\mathbb{T}(\mathbf{u}, \nabla_x \mathbf{u}, \ \nabla_x^2 \mathbf{u}, \dots)$$

$$= \mathbb{S}(\nabla_x \mathbf{u}) + \delta \sum_{j=1}^{k-1} \left((-1)^j \mu_j \Delta^j (\nabla_x \mathbf{u} + \nabla_x^t \mathbf{u}) + \lambda_j \Delta^j \mathrm{div}_x \mathbf{u} \ \mathbb{I} \right)$$

$$+ \text{ non-linear terms}$$

Limit for $\delta \to 0$

Limits of numerical solutions

Numerical solutions resulting from Karlsen-Karper and other schemes

Sub-critical parameters

$p(\varrho) = a\varrho^\gamma$, $\gamma < \gamma_{\mathrm{critical}}$

Weak (mv) - strong uniqueness

Theorem - EF, P.Gwiazda, A.Świerczewska-Gwiazda, E. Wiedemann [2015]

A measure valued and a strong solution emanating from the same initial data coincide as long as the latter exists

Relative energy (entropy)

Relative energy functional

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\right)(\tau)$$

$$= \int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2} s|\mathbf{v} - \mathbf{U}|^2 + P(s) - P'(r)(s-r) - P(r) \right\rangle \,\mathrm{d}x$$

$$= \int_\Omega \left\langle \nu_{\tau,x}; \frac{1}{2} s|\mathbf{v}|^2 + P(s) \right\rangle \,\mathrm{d}x - \int_\Omega \langle \nu_{\tau,x}; s\mathbf{v} \rangle \cdot \mathbf{U} \,\mathrm{d}x$$

$$+ \int_\Omega \frac{1}{2} \langle \nu_{\tau,x}; s \rangle |\mathbf{U}|^2 \,\mathrm{d}x$$

$$- \int_\Omega \langle \nu_{\tau,x}; s \rangle P'(r) \,\mathrm{d}x + \int_\Omega p(r) \,\mathrm{d}x$$

Relative energy (entropy) inequality

Relative energy inequality

$$\mathcal{E}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) + \int_0^\tau \mathbb{S}(\nabla_x \mathbf{u}) : (\nabla_x \mathbf{u} - \nabla_x \mathbf{U}) \ \mathrm{d}x \ \mathrm{d}t + \mathcal{D}(\tau)$$

$$\leq \int_\Omega \left\langle \nu_{0,x}; \frac{1}{2} s |\mathbf{v} - \mathbf{U}_0|^2 + P(s) - P'(r_0)(s - r_0) - P(r_0) \right\rangle \ \mathrm{d}x$$

$$+ \int_0^\tau \mathcal{R}\left(\varrho, \mathbf{u} \,\middle|\, r, \mathbf{U}\right) \ \mathrm{d}t$$

Remainder

$$
\begin{aligned}
&\mathcal{R}\left(\varrho, \mathbf{u} \,\Big|\, r, \mathbf{U}\right) \\
&= - \int_0^\tau \int_\Omega \langle \nu_{t,x}, s\mathbf{v} \rangle \cdot \partial_t \mathbf{U} \, \mathrm{d}x \, \mathrm{d}t \\
&- \int_0^\tau \int_{\overline{\Omega}} \left[\langle \nu_{t,x}; s\mathbf{v} \otimes \mathbf{v} \rangle : \nabla_x \mathbf{U} + \langle \nu_{t,x}; p(s) \rangle \, \mathrm{div}_x \mathbf{U} \right] \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\langle \nu_{t,x}; s \rangle \, \mathbf{U} \cdot \partial_t \mathbf{U} + \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \mathbf{U} \cdot \nabla_x \mathbf{U} \right] \, \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \int_\Omega \left[\left\langle \nu_{t,x}; \left(1 - \frac{s}{r} \right) \right\rangle p'(r) \partial_t r - \langle \nu_{t,x}; s\mathbf{v} \rangle \cdot \frac{p'(r)}{r} \nabla_x r \right] \, \mathrm{d}x \, \mathrm{d}t \\
&+ \int_0^\tau \left\langle R_1; \frac{1}{2} \nabla_x \left(|\mathbf{U}|^2 - P'(r) \right) \right\rangle \, \mathrm{d}t - \int_0^\tau \langle R_2; \nabla_x \mathbf{U} \rangle \, \mathrm{d}t
\end{aligned}
$$

Regularity

Theorem - EF, P.Gwiazda, A. Świerczewska-Gwiazda, E. Wiedemann

Suppose that the initial data are smooth and satisfy the relevant compatibility conditions. Let $\nu_{t,x}$ be a measure-valued solution to the compressible Navier-Stokes system with a dissipation defect $\mathcal{D}$ such that

$$\mathrm{supp}\ \nu_{t,x} \subset \left\{ (s, \mathbf{v}) \ \middle| \ 0 \leq s \leq \overline{\varrho},\ \mathbf{v} \in R^N \right\}$$

for a.a. $(t, x) \in (0, T) \times \Omega$.
Then $\mathcal{D} = 0$ and

$$\nu_{t,x} = \delta_{\varrho(t,x),\mathbf{u}(t,x)}$$

where ϱ, $\mathbf{u}$ is a smooth solution.

Sketch of the proof

- The Navier-Stokes system admits a local-in-time smooth solution
- The measure-valued solution coincides with the smooth solution on its life-span
- The smooth solution density component remains bounded by $\overline{\varrho}$ as long as the solution exists
- Y. Sun, C. Wang, and Z. Zhang [2011]: The strong solution can be extended as long as the density component remains bounded

Corollary

Convergence of numerical solutions

Bounded numerical solutions emanating from smooth data that converge to a measure-valued solution converge, in fact, unconditionally to the unique strong solution

Vše nejlepší k narozeninám
milý Reinharde!