THE GROWTH OF A CLASS OF RANDOM DIRICHLET SERIES ON THE HORIZONTAL ZONE

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Abstract. In the paper we obtain that, under some condition, the Rademacher-Dirichlet series or the Steinhaus-Dirichlet series on the whole plane and on the horizontal zone almost surely have the same growth.

Keywords: random Dirichlet series, Rademacher-Dirichlet series, Steinhaus-Dirichlet series, growth

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1. Introduction and main result

Let $(\Omega, \mathscr{A}, P)$ be a probability space, where $\Omega = [0,1]$, $\mathscr{A}$ is composed of all Lebesgue measurable sets E on Ω and $P(E)$ is the Lebesgue measure of E. Let $\{\varepsilon_n(\omega)\}$ and $\{\gamma_n(\omega)\}$ $(n = 0, 1, 2, \ldots)$ (see [1], [2]), which can be considered to be random variables sequences in $(\Omega, \mathscr{A}, P)$, be respectively the Rademacher and Steinhaus sequence. $\gamma_n(\omega) = \exp\{2\pi i\theta_n(\omega)\}$, the value of $\theta_n(\omega)$ uniformly distributes on $[0,1]$. $\{\varepsilon_n(\omega)\}$ $(n = 0, 1, 2, \ldots)$ satisfies

$$P[\varepsilon_n(\omega) = 1] = P[\varepsilon_n(\omega) = -1] = \frac{1}{2}.$$

Consider the random Dirichlet series

$$f(s, \omega) = \sum_{n=0}^{+\infty} a_n X_n(\omega) e^{-\lambda_n s},$$

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where $X_n(\omega) = \varepsilon_n(\omega)$ or $\gamma_n(\omega)$, $s = \sigma + \mathrm{i}t$ $(\sigma, t \in \mathbb{R})$ denotes the complex variable, $\{a_n\}_{n=0}^{+\infty}$ is a sequence of complex numbers, $0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n \uparrow +\infty$. Following Bohr (see [3]), we define the quantities

$$\sigma_c(\omega) = \inf\Bigl\{\sigma \in \mathbb{R}\colon \sum a_n X_n(\omega)\mathrm{e}^{-\lambda_n\sigma} \text{ converges}\Bigr\},$$
$$\sigma_a(\omega) = \inf\Bigl\{\sigma \in \mathbb{R}\colon \sum |a_n X_n(\omega)|\mathrm{e}^{-\lambda_n\sigma} \text{ converges}\Bigr\},$$
$$\sigma_u(\omega) = \inf\Bigl\{\sigma \in \mathbb{R}\colon \sum a_n X_n(\omega)\mathrm{e}^{-\lambda_n(\sigma+\mathrm{i}t)} \text{ converges uniformly on } \mathbb{R}\Bigr\}.$$

Paley and Zygmund (see [3], 1932) were the first to study the convergence of $f(s,\omega)$. Yu Jiarong (see [4]), 1978) studied the growth of $f(s,\omega)$ under the conditions

$$\varlimsup_{n\to+\infty} \frac{\ln|a_n|}{\lambda_n} = 0, \quad \text{and} \quad \varlimsup_{n\to+\infty} \frac{n}{\lambda_n} < +\infty,$$

and obtained that $f(s,\omega)$ on the right half plane and on any right horizontal zone a.s. (almost surely) has the same growth. In this paper, we discuss Yu's topic on the whole plane. We obtain that, under some condition, $f(s,\omega)$ on the whole plane and on any horizontal zone a.s. has the same growth.

Let $\sigma_u(\omega)$ be the uniformly convergence abscissa of $f(s,\omega)$. When $\sigma_u(\omega) < +\infty$, for any $\sigma > \sigma_u(\omega)$ let

$$M(\sigma,\omega) = \sup\{|f(s,\omega)|\colon -\infty < t < +\infty\},$$
$$M(\sigma;\alpha,\beta;\omega) = \sup\{|f(s,\omega)|\colon -\infty < \alpha \leqslant t \leqslant \beta < +\infty\}.$$

Put

$$\Delta = \varlimsup_{k\to+\infty} \frac{\ln(p_k+1)}{k\ln k},$$

where p_k is given by $[k, k+1) \cap \{\lambda_n\} = \{\lambda_{n_k}, \lambda_{n_k+1}, \ldots, \lambda_{n_k+p_k}\}$, $k \in \mathbb{N}$. Our main result is

Theorem 1. *Assume that $f(s,\omega)$ satisfies*

$$\sigma_u(\omega) = -\infty \text{ a.s.} \quad \textit{and} \quad \Delta = 0.$$

Then $f(s,\omega)$ a.s. satisfies

$$\varlimsup_{\sigma\to-\infty} \frac{\ln\ln M(\sigma,\omega)}{-\sigma} = \varlimsup_{\sigma\to-\infty} \frac{\ln\ln M(\sigma;\alpha,\beta;\omega)}{-\sigma}.$$

2. Lemmas

To give our lemmas, we define some symbols by the method of Knoop-Kojima (see [5]). Consider

$$f(s)=\sum_{n=0}^{+\infty} a_n \mathrm{e}^{-\lambda_n s}.$$

For each $k\in\mathbb{N}$, when

$$[k,k+1)\cap\{\lambda_n\}=\{\lambda_{n_k},\lambda_{n_k+1},\dots,\lambda_{n_k+p_k}\}\neq\emptyset, \tag{1}$$

put

$$\overline{A}_k=\sup_{0\leqslant p\leqslant p_k,t\in\mathbb{R}}\left|\sum_{j=0}^{p} a_{n_k+j}\mathrm{e}^{-\mathrm{i}t\lambda_{n_k+j}}\right|,\quad A_k^*=\sum_{j=0}^{p_k}|a_{n_k+j}|;$$

when $[k,k+1)\cap\{\lambda_n\}=\emptyset$, put $\ln\overline{A}_k=\ln A_k^*=-\infty$. Then we have the formulas (see [5], [6]) for the abscissas σ_u, σ_a in terms of $\overline{A}_k$, A_k^*,

$$\sigma_u=\varlimsup_{k\to+\infty}\frac{\ln\overline{A}_k}{k};\quad \sigma_a=\varlimsup_{k\to+\infty}\frac{\ln A_k^*}{k}.$$

When $\sigma_u<+\infty$, for any $\sigma>\sigma_u$ put

$$\overline{M}_u(\sigma)=\sup\left\{\left|\sum_{j=0}^{n} a_j\mathrm{e}^{-\lambda_j(\sigma+\mathrm{i}t)}\right|:\ n\in\mathbb{N},\ t\in\mathbb{R}\right\};\quad \overline{m}(\sigma)=\max\{\overline{A}_k\mathrm{e}^{-k\sigma}:\ k\in\mathbb{N}\}.$$

Consider $f(s,\omega)$. Put

$$\overline{A}_k(\omega)=\sup_{0\leqslant p\leqslant p_k,t\in\mathbb{R}}\left|\sum_{j=0}^{p} a_{n_k+j}X_j(\omega)\mathrm{e}^{-\mathrm{i}t\lambda_{n_k+j}}\right|;$$
$$\overline{M}_u(\sigma,\omega)=\sup\left\{\left|\sum_{j=0}^{n} a_jX_j(\omega)\mathrm{e}^{-\lambda_j(\sigma+\mathrm{i}t)}\right|:\ n\in\mathbb{N},\ -\infty<t<+\infty\right\}.$$

Chuji Tanaka (see [7]) studied the relation between $\overline{m}(\sigma)$ and $\overline{M}_u(\sigma)$. Now we improve his results.

Lemma 1. *If $f(s)$ is uniformly convergent on the whole plane, then*

(I) $\dfrac{\overline{m}(\sigma)}{4\mathrm{e}^{|\sigma|}} \leqslant \overline{M}_u(\sigma) \leqslant \dfrac{2\mathrm{e}^{|\sigma|}\overline{m}(\sigma-\varepsilon)}{1-\mathrm{e}^{-\varepsilon}}$ *for any* $\varepsilon > 0$,

(II) $\overline{\lim\limits_{\sigma\to-\infty}} \dfrac{\ln\ln \overline{M}_u(\sigma)}{-\sigma} = \overline{\lim\limits_{\sigma\to-\infty}} \dfrac{\ln\ln \overline{m}(\sigma)}{-\sigma}$.

Proof. Take $p \in \mathbb{N}$, such that $n_k + p < n_{k+1}$, where n_k is defined by (1). Using Abel's transformation, we arrive at

$$\begin{aligned}\sum_{j=n_k}^{n_k+p} a_j \mathrm{e}^{-\mathrm{i}t\lambda_j} &= \sum_{j=n_k}^{n_k+p} a_j \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_j}\mathrm{e}^{\sigma\lambda_j}\\ &= \sum_{j=n_k}^{n_k+p-1}\Bigg(\sum_{q=n_k}^{j} a_q \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_q}\Bigg)(\mathrm{e}^{\sigma\lambda_j} - \mathrm{e}^{\sigma\lambda_{j+1}})\\ &\quad + \Bigg(\sum_{q=n_k}^{n_k+p} a_q \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_q}\Bigg)\mathrm{e}^{\sigma\lambda_{n_k+p}}.\end{aligned}$$

Noting that

$$\begin{aligned}\Bigg|\sum_{q=n_k}^{j} a_q \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_q}\Bigg| &= \Bigg|\sum_{q=0}^{j} a_q \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_q} - \sum_{q=0}^{n_k-1} a_q \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_q}\Bigg|\\ &\leqslant \Bigg|\sum_{q=0}^{j} a_q \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_q}\Bigg| + \Bigg|\sum_{q=0}^{n_k-1} a_q \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_q}\Bigg| \leqslant 2\overline{M}_u(\sigma),\end{aligned}$$

we obtain

$$\begin{aligned}\overline{A}_k &\leqslant 2\overline{M}_u(\sigma)|\mathrm{e}^{\sigma\lambda_{n_k}} - \mathrm{e}^{\sigma\lambda_{n_k+p}}| + \mathrm{e}^{\sigma\lambda_{n_k+p}} 2\overline{M}_u(\sigma)\\ &\leqslant 4\overline{M}_u(\sigma)\mathrm{e}^{(k+\operatorname{sgn}\sigma)\sigma} = 4\mathrm{e}^{|\sigma|}\overline{M}_u(\sigma)\mathrm{e}^{k\sigma}.\end{aligned}$$

By the definition of $\overline{m}(\sigma)$,

$$\frac{\overline{m}(\sigma)}{4\mathrm{e}^{|\sigma|}} \leqslant \overline{M}_u(\sigma). \tag{2}$$

Suppose $n_k + p < n_{k+1}$. Abel's transformation yields

$$\begin{aligned}\sum_{j=n_k}^{n_k+p} a_j \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_j} &= \sum_{j=n_k}^{n_k+p} a_j \mathrm{e}^{-\mathrm{i}t\lambda_j}\mathrm{e}^{-\sigma\lambda_j}\\ &= \sum_{j=n_k}^{n_k+p-1}\Bigg(\sum_{q=n_k}^{j} a_q \mathrm{e}^{-\mathrm{i}t\lambda_q}\Bigg)(\mathrm{e}^{-\sigma\lambda_j} - \mathrm{e}^{-\sigma\lambda_{j+1}})\\ &\quad + \Bigg(\sum_{q=n_k}^{n_k+p} a_q \mathrm{e}^{-\mathrm{i}t\lambda_q}\Bigg)\mathrm{e}^{-\sigma\lambda_{n_k+p}}.\end{aligned}$$

When $\sigma > 0$,

$$\left|\sum_{j=n_k}^{n_k+p} a_j \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_j}\right| \leqslant \overline{A}_k \sum_{j=n_k}^{n_k+p-1} (\mathrm{e}^{-\sigma\lambda_j} - \mathrm{e}^{-\sigma\lambda_{j+1}}) + \overline{A}_k \mathrm{e}^{-\sigma\lambda_{n_k+p}}$$
$$\leqslant \overline{A}_k \mathrm{e}^{-\sigma\lambda_{n_k}} \leqslant \overline{A}_k \mathrm{e}^{-\sigma k}.$$

When $\sigma \leqslant 0$,

$$\left|\sum_{j=n_k}^{n_k+p} a_j \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_j}\right| \leqslant \overline{A}_k \sum_{j=n_k}^{n_k+p-1} (\mathrm{e}^{-\sigma\lambda_{j+1}} - \mathrm{e}^{-\sigma\lambda_j}) + \overline{A}_k \mathrm{e}^{-\sigma\lambda_{n_k+p}} \leqslant 2\overline{A}_k \mathrm{e}^{-\sigma(k+1)}.$$

Then for any $\sigma \in \mathbb{R}$ we have

$$\left|\sum_{j=n_k}^{n_k+p} a_j \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_j}\right| \leqslant 2\overline{A}_k \mathrm{e}^{|\sigma|} \mathrm{e}^{-k\sigma}.$$

Therefore for any $\varepsilon > 0$

$$\left|\sum_{j=0}^{n_k+p} a_j \mathrm{e}^{-(\sigma+\mathrm{i}t)\lambda_j}\right| \leqslant 2\mathrm{e}^{|\sigma|} \sum_{j=0}^{k} \overline{A}_j \mathrm{e}^{-(\sigma-\varepsilon)j} \mathrm{e}^{-j\varepsilon}$$
$$\leqslant 2\mathrm{e}^{|\sigma|} \overline{m}(\sigma-\varepsilon) \sum_{j=0}^{+\infty} \mathrm{e}^{-j\varepsilon} = \frac{2\mathrm{e}^{|\sigma|}\overline{m}(\sigma-\varepsilon)}{1-\mathrm{e}^{-\varepsilon}}.$$

By the definition of $\overline{M}_u(\sigma)$,

$$\overline{M}_u(\sigma) \leqslant \frac{2\mathrm{e}^{|\sigma|}\overline{m}(\sigma-\varepsilon)}{1-\mathrm{e}^{-\varepsilon}}. \tag{3}$$

Combining (2) and (3), we prove (I).

Note that

$$\ln^+ \ln^+ \overline{m}(\sigma) \leqslant \ln^+ \ln^+ \frac{1}{4}\mathrm{e}^{-|\sigma|}\overline{m}(\sigma) + \ln^+ \ln^+ 4\mathrm{e}^{|\sigma|} + \ln 2, \text{ and } \overline{\lim_{\sigma\to-\infty}} \ln \overline{m}(\sigma) = +\infty.$$

Hence

$$\overline{\lim_{\sigma\to-\infty}} \frac{\ln\ln\overline{m}(\sigma)}{-\sigma} = \overline{\lim_{\sigma\to-\infty}} \frac{\ln^+\ln^+\overline{m}(\sigma)}{-\sigma} \leqslant \overline{\lim_{\sigma\to-\infty}} \frac{\ln^+\ln^+\frac{1}{4}\mathrm{e}^{-|\sigma|}\overline{m}(\sigma)}{-\sigma}.$$

Similarly,

$$\begin{aligned}\varlimsup_{\sigma\to-\infty} &\frac{\ln^+\ln^+ 2\mathrm{e}^{|\sigma|\overline{m}(\sigma-\varepsilon)}/(1-\mathrm{e}^{-\varepsilon})}{-\sigma}\\ &\leqslant \varlimsup_{\sigma\to-\infty}\frac{\ln^+\ln^+\overline{m}(\sigma-\varepsilon)}{-\sigma}+\varlimsup_{\sigma\to-\infty}\frac{\ln^+\ln^+ 2\mathrm{e}^{|\sigma|}/(1-\mathrm{e}^{-\varepsilon})}{-\sigma}\\ &=\varlimsup_{\sigma\to-\infty}\frac{\ln^+\ln^+\overline{m}(\sigma)}{-\sigma}.\end{aligned}$$

Thus (II) is proved.

Lemma 2. *Suppose that $M_x < \mathrm{e}^{-c\sigma} + x\sigma$ holds for any $\sigma < \sigma_\varepsilon$ and any $x \in \mathbb{R}$, $x > 0$, where c is a positive real number, σ_ε is a given real number, M_x is a real number depending on x. Then there exists $x_\varepsilon \in \mathbb{R}$, $x_\varepsilon > 0$ such that for any $x > x_\varepsilon$,*

$$M_x < \min\{\mathrm{e}^{-c\sigma} + x\sigma\colon\ \sigma \in \mathbb{R}\} = \frac{x}{c}(\ln ce - \ln x).$$

P r o o f. Let $\psi(\sigma) = \mathrm{e}^{-c\sigma} + x\sigma$ $(-\infty < \sigma < +\infty)$. Then there exists $\sigma_x = (\ln c - \ln x)/c$ such that $\min\{\psi(\sigma)\colon\ \sigma \in \mathbb{R}\} = \psi(\sigma_x) = xc^{-1}(\ln c\mathrm{e} - \ln x)$. Note that σ_x is a monotonic decreasing function of x, and $\sigma_x \to -\infty \Leftrightarrow x \to +\infty$. Then there exists $x_\varepsilon \in \mathbb{R}$ such that $\sigma_{x_\varepsilon} < \sigma_\varepsilon$. Hence, for any $x > x_\varepsilon$, we have $\sigma_x < \sigma_\varepsilon$. Therefore $M_x < \psi(\sigma_x) = xc^{-1}(\ln c\mathrm{e} - \ln x)$, $x > x_\varepsilon$.

Lemma 3. *If $f(s)$ is uniformly convergent on the whole plane, then*

$$(4)\qquad \varlimsup_{\sigma\to-\infty}\frac{\ln\ln\overline{M}_u(\sigma)}{-\sigma} = \varrho_u \Leftrightarrow \varlimsup_{k\to+\infty}\frac{\ln\overline{A}_k}{k\ln k} = \begin{cases} -\infty, & \varrho = 0;\\ -\dfrac{1}{\varrho_u}, & 0 < \varrho < +\infty;\\ 0, & \varrho = +\infty.\end{cases}$$

P r o o f. We prove the necessity of the right hand side condition of (4). Consider the case $0 < \varrho_u < \infty$. Using Lemma 1(II), for $\forall\varepsilon > 0$ we have, when $-\sigma$ is large enough,

$$\overline{m}(\sigma) < \exp\{\mathrm{e}^{-(\varrho_u+\varepsilon)\sigma}\}.$$

Then for each $k \in \mathbb{N}$,

$$\ln\overline{A}_k < \mathrm{e}^{-(\varrho_u+\varepsilon)\sigma} + k\sigma.$$

Note that

$$\min\{\mathrm{e}^{-(\varrho_u+\varepsilon)\sigma} + k\sigma\colon\ \sigma \in \mathbb{R}\} = -\frac{k}{\varrho_u+\varepsilon}\ln\frac{k}{\mathrm{e}(\varrho_u+\varepsilon)}.$$

By Lemma 2, when k is sufficiently large,

$$\ln \overline{A}_k < -\frac{k}{\varrho_u+\varepsilon}\ln\frac{k}{\mathrm{e}(\varrho_u+\varepsilon)}.$$

As $\varepsilon \to 0$,

$$\overline{\lim_{k\to+\infty}}\frac{\ln \overline{A}_k}{k\ln k} \leqslant -\frac{1}{\varrho_u}.$$

Suppose $\overline{\lim\limits_{k\to+\infty}}(\ln \overline{A}_k)/k\ln k < -1/\varrho_u$. Then there exists $\varrho_u' \in (0,\varrho_u)$, $c>0$ such that for each $k\in\mathbb{N}$ and for any $\sigma<0$,

$$\overline{A}_k \mathrm{e}^{-k\sigma} < \exp\Big\{-\frac{k\ln k}{\varrho_u'} - k\sigma + c\Big\}.$$

Note that

$$\max\Big\{-\frac{k\ln k}{\varrho_u'} - k\sigma\colon\ k\in\mathbb{N}\Big\} \leqslant \max\Big\{-\frac{k\ln k}{\varrho_u'} - k\sigma\colon\ k>0,\ k\in\mathbb{R}\Big\} = \mathrm{e}^{-\varrho_u'\sigma-\ln(\varrho_u')-1}.$$

Then

$$\overline{m}(\sigma) < \exp\{\mathrm{e}^{-\varrho_u'\sigma-\ln(\varrho_u')-1}+c\}.$$

Applying Lemma 1(II), we conclude

$$\overline{\lim_{\sigma\to-\infty}}\frac{\ln\ln \overline{M}_u(\sigma)}{-\sigma} \leqslant \varrho_u' < \varrho_u,$$

which contradicts the left-hand side of (4). Thus the necessity of the right-hand side of (4) is proved. By the above proof, we can easily prove the sufficiency of the right-hand side of (4).

Using the conclusion of the case $0<\varrho_u<\infty$, we can easily prove the case $\varrho_u=0$, $\varrho_u=+\infty$. Thus the lemma is proved.

Lemma 4. *If $f(s)$ is uniformly convergent on the whole plane and $\Delta=0$, then*

$$(5)\qquad \overline{\lim_{\sigma\to-\infty}}\frac{\ln\ln M(\sigma)}{-\sigma}=\varrho \Leftrightarrow \overline{\lim_{n\to+\infty}}\frac{\ln|a_n|}{\lambda_n\ln\lambda_n} = \begin{cases} -\infty, & \varrho=0;\\ -\dfrac{1}{\varrho}, & 0<\varrho<+\infty;\\ 0, & \varrho=+\infty. \end{cases}$$

Proof. For any $\varepsilon>0$, when $-\sigma$ is large enough,

$$M(\sigma) < \exp\{\mathrm{e}^{-(\varrho+\varepsilon)\sigma}\}.$$

Taking into account Hadamard's theorem (see [8]), $a_n e^{-\lambda_n\sigma} \leqslant M(\sigma)$,

$$|a_n| < \exp\{e^{-(\varrho+\varepsilon)\sigma} + \lambda_n\sigma\}, \quad \forall n \in \mathbb{N}.$$

Note that

$$\min\{e^{-(\varrho+\varepsilon)\sigma} + \lambda_n\sigma\colon\ \sigma \in \mathbb{R}\} = -\frac{\lambda_n}{\varrho+\varepsilon}\ln\frac{\lambda_n}{e(\varrho+\varepsilon)}.$$

By Lemma 2, when λ_n is sufficiently large,

$$|a_n| < \exp\Big\{-\frac{\lambda_n}{\varrho+\varepsilon}\ln\frac{\lambda_n}{e(\varrho+\varepsilon)}\Big\}.$$

Let $\varepsilon \to 0$, then

$$\varlimsup_{n\to+\infty}\frac{\ln|a_n|}{\lambda_n\ln\lambda_n} \leqslant -\frac{1}{\varrho}.$$

Assume that $\varlimsup\limits_{n\to+\infty}(\ln|a_n|)/\lambda_n\ln\lambda_n < -1/\varrho$. Then there exists $0 < \varrho' < \varrho$ such that for sufficiently large $n \in \mathbb{N}$

$$|a_n| < \exp\Big\{-\frac{1}{\varrho'}\lambda_n\ln\lambda_n\Big\}.$$

By the definition of $\overline{A}_k$, for sufficiently large k,

$$\overline{A}_k < \sum_{j=n_k}^{n_k+p_k}\exp\Big\{-\frac{\lambda_j}{\varrho'}\ln\lambda_j\Big\} \leqslant \exp\Big\{-\frac{k}{\varrho'}\ln k + \ln(p_k+1)\Big\}.$$

Note that $\Delta = 0$. Then

$$\varlimsup_{k\to+\infty}\frac{\ln\overline{A}_k}{k\ln k} \leqslant -\frac{1}{\varrho'} < -\frac{1}{\varrho}.$$

By Lemma 3,

$$\varlimsup_{\sigma\to-\infty}\frac{\ln\ln\overline{M}(\sigma)}{-\sigma} < \varrho.$$

Note $M(\sigma) \leqslant \overline{M}(\sigma)$. Then

$$\varlimsup_{\sigma\to-\infty}\frac{\ln\ln M(\sigma)}{-\sigma} < \varrho,$$

which contradicts the left-hand side of (5). Thus the necessity of the right-hand side of (5) is proved. By the above proof, we can easily prove the sufficiency of the right-hand side of (5). Thus we have proved the case $0 < \varrho < +\infty$.

By the case $0 < \varrho < +\infty$, we can easily prove the case $\varrho = 0$, $\varrho = +\infty$. Thus the lemma is proved.

Lemma 5. *If* $f(s,\omega)$ *satisfies* $\sigma_u(\omega) = -\infty$ a.s. *and* $\Delta = 0$, *then*

$$\varlimsup_{\sigma\to-\infty} \frac{\ln\ln M(\sigma,\omega)}{-\sigma} = \varrho \text{ a.s.} \quad \Leftrightarrow \quad \varlimsup_{n\to+\infty} \frac{\ln|a_n|}{\lambda_n \ln\lambda_n} = \begin{cases} -\infty, & \varrho = 0; \\ -\dfrac{1}{\varrho}, & 0 < \varrho < +\infty; \\ 0, & \varrho = +\infty. \end{cases}$$

P r o o f. Note that $|a_n X_n(\omega)| = |a_n|$ a.s. Then

$$\varlimsup_{n\to+\infty} \frac{\ln|a_n X_n(\omega)|}{\lambda_n \ln\lambda_n} = \varlimsup_{n\to+\infty} \frac{\ln|a_n|}{\lambda_n \ln\lambda_n} \text{ a.s.}$$

By Lemma 4, the lemma is proved.

Lemma P.Z (see [2])**.** *For every* $E \subseteq \Omega$ *satisfying* $P(E) > 0$, *there exists a positive integer* $N = N(E)$ *such that*

$$\int_E \Big|\sum_{n=N}^{N'} c_n\varepsilon_n(\omega)\Big|^2 \,\mathrm{d}\omega \geqslant \frac{1}{2}P(E)\sum_{n=N}^{N'}|c_n|^2,$$
$$\int_E \Big|\sum_{n=N}^{N'} c_n\gamma_n(\omega)\Big|^2 \,\mathrm{d}\omega \geqslant \frac{1}{2}P(E)\sum_{n=N}^{N'}|c_n|^2,$$

where $\{c_n\}_{n=1}^{+\infty}$ *is an arbitrary sequence of complex numbers.*

3. Proofs of main result

P r o o f of Theorem 1. By Lemma 5, there exists $\varrho \geqslant 0$ such that

$$\varrho = \varlimsup_{\sigma\to-\infty} \frac{\ln\ln M(\sigma,\omega)}{-\sigma} \quad \text{a.s.} \tag{6}$$

C a s e I: $\varrho = 0$. The theorem holds obviously.
C a s e II: $0 < \varrho < \infty$. Evidently,

$$\varlimsup_{\sigma\to-\infty} \frac{\ln\ln M(\sigma;\alpha,\beta;\omega)}{-\sigma} \leqslant \varrho \quad \text{a.s.}$$

Assume that $\varlimsup\limits_{\sigma\to-\infty} -\sigma^{-1}\ln\ln M(\sigma;\alpha,\beta;\omega) < \varrho$ a.s. Then there exists $E \subset \Omega$ satisfying $P(E) > 0$ and $0 < \varrho' < \varrho$ and α_0, β_0, $\alpha_0 < \beta_0$ such that

$$\varlimsup_{\sigma\to-\infty} \frac{\ln\ln M(\sigma;\alpha_0,\beta_0;\omega)}{-\sigma} < \varrho', \quad \omega \in E.$$

Then, when $-\sigma$ is large enough, $\alpha_0 \leqslant t \leqslant \beta_0$, we have

$$\Big|\sum_{n=0}^{+\infty} a_n X_n(\omega)\mathrm{e}^{-\lambda_n(\sigma+\mathrm{i}t)}\Big| < \exp\{\mathrm{e}^{-\varrho'\sigma}\}, \quad \omega \in E,$$

where $N = N(E)$ is determined by Lemma P.Z for E. By Lemma P.Z, when $-\sigma$ is large enough,

$$\frac{1}{2}P(E)\sum_{n=N}^{N'} |a_n|^2\mathrm{e}^{-2\lambda_n\sigma} \leqslant \int_E \Big|\sum_{n=N}^{N'} a_n X_n(\omega)\mathrm{e}^{-\lambda_n(\sigma+\mathrm{i}t)}\Big|^2 \mathrm{d}\omega, \quad N' > N,\ \omega \in E.$$

Since

$$\Big|\sum_{n=N}^{N'} a_n X_n(\omega)\mathrm{e}^{-\lambda_n(\sigma+\mathrm{i}t)}\Big| \leqslant \sum_{n=0}^{+\infty} |a_n|\mathrm{e}^{-\lambda_n\sigma} < +\infty, \quad -\infty < \sigma < 0,\ \omega \in E,$$

we have

$$\sum_{n=N}^{\infty} |a_n|^2\mathrm{e}^{-2\lambda_n\sigma} \leqslant \frac{2}{P(E)}\int_E \Big|\sum_{n=N}^{+\infty} a_n X_n(\omega)\mathrm{e}^{-\lambda_n(\sigma+\mathrm{i}t)}\Big|^2 \mathrm{d}\omega \leqslant 8\exp\{2\mathrm{e}^{-\varrho'\sigma}\}.$$

Therefore, when $-\sigma$ is large enough, $n \geqslant N$, then

$$|a_n|\mathrm{e}^{-\lambda_n\sigma} < 3\exp\{\mathrm{e}^{-\varrho'\sigma}\}.$$

Note that

$$\min\{\mathrm{e}^{-\varrho'\sigma} + \lambda_n\sigma\colon\ \sigma \in \mathbb{R}\} = -\frac{\lambda_n}{\varrho'}\ln\frac{\lambda_n}{\mathrm{e}(\varrho')}.$$

By Lemma 2, when n is sufficiently large,

$$|a_n| < 3\exp\Big\{-\frac{\lambda_n}{\varrho'}\ln\frac{\lambda_n}{\mathrm{e}(\varrho')}\Big\}.$$

Then

$$\varlimsup_{n\to+\infty}\frac{\ln|a_n|}{\lambda_n\ln\lambda_n} \leqslant -\frac{1}{\varrho'} < -\frac{1}{\varrho}.$$

By Lemma 5,

$$\varlimsup_{\sigma\to-\infty}\frac{\ln\ln M(\sigma,\omega)}{-\sigma} < \varrho \quad \text{a.s.},$$

which contradicts (6).

C a s e III: $\varrho = +\infty$. We must have

$$\overline{\lim_{\sigma \to -\infty}} \frac{\ln \ln M(\sigma; \alpha, \beta; \omega)}{-\sigma} = +\infty \quad \text{a.s.}$$

Otherwise, assume that $\overline{\lim}_{\sigma \to -\infty} -\sigma^{-1} \ln \ln M(\sigma; \alpha, \beta; \omega) < +\infty$ a.s. Then by the proof of cases I and II,

$$\overline{\lim_{\sigma \to -\infty}} \frac{\ln \ln M(\sigma, \omega)}{-\sigma} < +\infty \quad \text{a.s.}$$

which contradicts the given condition $\overline{\lim}_{\sigma \to -\infty} -\sigma^{-1} \ln \ln M(\sigma, \omega) = +\infty$ a.s. □

4. Corollary and example

Corollary 1. *Consider $f(s, \omega)$ as in the introduction. If*

$$\overline{\lim_{n \to +\infty}} \frac{\ln |a_n|}{\lambda_n} = -\infty \text{ and } \overline{\lim_{n \to +\infty}} \frac{\ln n}{\lambda_n} = D < +\infty,$$

then $f(s, \omega)$ a.s. (almost surely) satisfies

$$\overline{\lim_{\sigma \to -\infty}} \frac{\ln \ln M(\sigma, \omega)}{-\sigma} = \overline{\lim_{\sigma \to -\infty}} \frac{\ln \ln M(\sigma; \alpha, \beta; \omega)}{-\sigma}. \tag{7}$$

P r o o f. Note that $\overline{\lim}_{n \to +\infty} \lambda_n^{-1} \ln |a_n X_n(\omega)| = \overline{\lim}_{n \to +\infty} \lambda_n^{-1} \ln |a_n|$ a.s. Then, by G. Valiron's formula (see [6], [9]),

$$\overline{\lim_{n \to +\infty}} \frac{\ln |a_n|}{\lambda_n} \leqslant \sigma_c(\omega) \leqslant \sigma_u(\omega) \leqslant \sigma_a(\omega) \leqslant \overline{\lim_{n \to +\infty}} \frac{\ln |a_n|}{\lambda_n} + \overline{\lim_{n \to +\infty}} \frac{\ln n}{\lambda_n} = -\infty \quad \text{a.s.,}$$

which gives $\sigma_u(\omega) = -\infty$ a.s.

By virtue of $\overline{\lim}_{n \to +\infty} \lambda_n^{-1} \ln n = D < +\infty$, for any $\varepsilon > 0$, when $n \in \mathbb{N}$ is large enough,

$$n < \mathrm{e}^{\lambda_n (D + \varepsilon)}.$$

Hence, when k is large enough,

$$p_k + 1 \leqslant n_{k+1} < \mathrm{e}^{\lambda_{n_{k+1}} (D + \varepsilon)} \leqslant \mathrm{e}^{(k+2)(D + \varepsilon)}.$$

Therefore

$$0 \leqslant \overline{\lim_{k\to+\infty}} \frac{\ln(p_k+1)}{k\ln k} \leqslant \overline{\lim_{k\to+\infty}} \frac{(k+2)(D+\varepsilon)}{k\ln k} = 0.$$

Thus $\Delta = 0$. By Theorem 1, (7) holds.

Now we give an example to show that $\Delta = 0$ is less restricting than

$$\overline{\lim_{n\to+\infty}} \frac{\ln n}{\lambda_n} < +\infty.$$

E x a m p l e 1. Consider $f(s,\omega)$. For any $i \in \mathbb{N}$, put $[\ln i] = \max\{z\colon z \in \mathbb{N}, z \leqslant \ln i\}$. For any $k \in \mathbb{N}$, take $n_k = 1 + \sum\limits_{i=1}^{k-1} ([\ln i])^i$, $p_k = ([\ln k])^k - 1$, $0 \leqslant p \leqslant p_k$, $\lambda_{n_k+p} = k + p/p_k$, $a_{n_k+p} = ([\ln k])^{-2k}$. Prove that $\sigma_u(\omega) = -\infty$ a.s., $\Delta = 0$, $\overline{\lim\limits_{n\to+\infty}} \lambda_n^{-1}\ln n = +\infty$.

P r o o f. By the definition of $\overline{A}_k(\omega)$,

$$\begin{aligned}\overline{A}_k(\omega) &= \sup_{0\leqslant p\leqslant p_k, t\in\mathbb{R}} \left| \sum_{j=0}^{p} a_{n_k+j} X_{n_k+j}(\omega) \mathrm{e}^{-\mathrm{i}t\lambda_{n_k+j}} \right| \\ &\leqslant \sum_{j=0}^{p_k} |a_{n_k+j}| = ([\ln k])^k ([\ln k])^{-2k} = ([\ln k])^{-k}, \text{ a.s.}\end{aligned}$$

Then by the formulas for $\sigma_u(\omega)$ in terms of $\overline{A}_k(\omega)$ in Section 2,

$$\sigma_u(\omega) = \overline{\lim_{k\to+\infty}} \frac{\ln \overline{A}_k(\omega)}{k} \leqslant \overline{\lim_{k\to+\infty}} \frac{\ln([\ln k])^{-k}}{k} \leqslant \overline{\lim_{k\to+\infty}} \frac{\ln(\ln k - 1)^{-k}}{k} = -\infty \text{ a.s.}$$

By the definition of Δ,

$$\Delta = \overline{\lim_{k\to+\infty}} \frac{\ln(p_k+1)}{k\ln k} = \overline{\lim_{k\to+\infty}} \frac{\ln([\ln k])^k}{k\ln k} \leqslant \overline{\lim_{k\to+\infty}} \frac{\ln(\ln k)^k}{k\ln k} = 0$$

but

$$\overline{\lim_{n\to+\infty}} \frac{\ln n}{\lambda_n} \geqslant \overline{\lim_{n\to+\infty}} \frac{\ln(p_k+1)}{k+1} = \overline{\lim_{k\to+\infty}} \frac{\ln([\ln k])^k}{k+1} \geqslant \overline{\lim_{k\to+\infty}} \frac{\ln(\ln k - 1)^k}{k+1} = +\infty.$$

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