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Abstract:
We study a dynamic contact problem for a thermoelastic von Kármán plate vibrating against a rigid obstacle. Dynamics is described by a hyperbolic variational inequality for deflections. The plate is subjected to a perpendicular force and to a heat source. The parabolic equation for a thermal strain resultant contains the time derivative of the deflection. We formulate a weak solution of the system and verify its existence using the penalization method.

Keywords: Thermoelastic plate, unilateral dynamic contact, rigid obstacle, penalization

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1 Introduction and notation

The dynamic contact problems are not frequently solved in the framework of variational inequalities. For the elastic problems there is only a very limited amount of results available (cf. [5] and there cited literature). We have solved these problems for geometrically nonlinear plates and shells in [2] and [3] respectively. We concentrate here not only on purely mechanical impact to the plate being under some load and possibly contacting an rigid obstacle, we also take in mind the heat balance of this process. We shall use the model derived in [7] under the assumption of a small change of temperature compared with its reference temperature. In contrast to it the hyperbolic equation for the deflections is substituted here by the variational inequality, involving the geometrical nonlinearities in deflections.

For convenience of readers we describe the genesis of the model solved more in detail. The simplifications made don't allow the model to be fully rate-independent as it is e.g. in [9].

A thin isotropic elastic plate occupies the domain

$$G = \{(x,z) \in \mathbb{R}^3 : x = (x_1,x_2) \in \Omega,\ |x_3| < h/2\}.$$

Its middle plane $\Omega \subset \mathbb{R}^2$ is a bounded domain with a $C^{2,1}$ boundary Γ. Further we set $I \equiv (0,T)$ a bounded time interval, $Q = I \times \Omega,\ S = I \times \Gamma$. The unit outer normal vector is denoted by $\boldsymbol{n} = (n_1,n_2)$. The displacement is denoted by $\boldsymbol{u} \equiv (u_i)$. The strain tensor is defined as

$$\varepsilon_{ij} = \frac{1}{2}(\partial_i u_j + \partial_j u_i + \partial_i u_3 \partial_j u_3) - x_3 \partial_{ij} u_3,\ i,j = 1,2;\ \varepsilon_{i3} \equiv 0,\ i = 1,2,3.$$

The constants $E > 0$ and $\nu \in [0, \frac{1}{2})$ are the Young modulus of elasticity and the Poisson ratio, respectively. We set

$$a = \frac{h^2}{12},\ b = \frac{Eh^2}{12\varrho(1-\nu^2)},$$

where h is the the plate thickness and ϱ is the density of the material. We denote

$$[u,v] \equiv \partial_{11}u\partial_{22}v + \partial_{22}u\partial_{11}v - 2\partial_{12}u\partial_{12}v.$$

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We assume that the plate is thermally isotropic and is is subjected not only to mechanical loads but also to an unknown temperature distribution τ implying a thermal strain. Due to thermal isotropy the thermal strains have the form $\varepsilon^{\tau}_{ij} = \varepsilon^{\tau}\delta_{ij}$. Employing the Einstein summation convention the constitutional law has then the form

$$\sigma_{ij} = \frac{E}{1-\nu^2}\big((1-\nu)\varepsilon_{ij} + \nu\varepsilon_{kk}\delta_{ij} - (1+\nu)\varepsilon^{\tau}\delta_{ij}\big), \quad i,j = 1,2; \; \sigma_{ij} = 0, \; i = 1,2,3.$$

With respect to a heat conduction we introduce following constants. The specific heat of the body $c > 0$, the coefficients of thermal conductivity $\lambda > 0$. Further we set α the coefficient of thermal expanding and $\Upsilon > 0$ the reference temperature of the plate. The key role in deriving the linear equation for a temperature plays the hypotheses $|\frac{\tau}{\Upsilon}| \ll 1$ i.e. the change τ of the temperature is small compared to the reference temperature Υ of the plate and then $\varepsilon^{\tau} = \alpha\tau$. The thermal entropy of the plate can expressed due to an elastic and thermal isotropy in a linearized form ([8] Chapter 1)

$$S = \frac{E\alpha}{1-2\nu}\varepsilon_{kk} + \frac{\rho c}{\Upsilon}\tau.$$

In order to eliminate the x_3 variable from a temperature equation we introduce the thermal strain resultant function θ by

$$\theta(t,x) = \frac{12\alpha}{h^3}\int_{-h/2}^{h/2} \tau \, dx.$$

After formulation of the original problem in the next chapter we formulate and solve the penalized initial-boundary value problem first. Using the *a priori* estimates and fine interpolation and imbedding technique we achieve the sequence converging to a weak solution of the original problem.

We shall employ the following notations for space and time derivatives

$$\frac{\partial}{\partial s} \equiv \partial_s, \; \frac{\partial^2}{\partial s \partial r} \equiv \partial_{sr}, \; \partial_i = \partial_{x_i}, \; i = 1,2; \; \dot{v} = \frac{\partial v}{\partial t}, \; \ddot{v} = \frac{\partial^2 v}{\partial t^2}, \; v : Q \mapsto \mathbb{R}.$$

For a domain or an appropriate manifold M and $p \geq 1$ we define the Banach space $L_p(M)$ of real valued measurable functions with integrable power of p. The space $L_{\infty}(M)$ is the Banach space of essentially bounded functions. By $H^k(M) \subset L_2(M)$ with $k \geq 0$ we denote the Hilbert–type Sobolev (for a noninteger k the Sobolev-Slobodetskii) spaces of functions defined on M. For the anisotropic spaces $H^k(M)$, $k = (k_1, k_2) \in \mathbb{R}^2_+$, k_1 is related with the time while k_2 with the space variables provided M is a time-space domain.

By $\mathring{H}^1(\Omega)$ we denote the subspace of functions from $H^1(\Omega)$ with zero traces on Γ and set $V = H^2(\Omega) \cap \mathring{H}^1(\Omega)$. Further we introduce the spaces $\mathring{\mathscr{H}} = L_{\infty}(I; \mathring{H}^1(\Omega))$ and $\mathscr{V} = L_{\infty}(I; V)$. The dual to $\mathring{H}^1(\Omega)$ is denoted by $H^{-1}(\Omega)$ with $\langle\cdot,\cdot\rangle$ the duality pairing between $H^{-1}(\Omega)$ and $\mathring{H}^1(\Omega)$.

2 Formulation of the problem

A triple $\{u, g, \theta\}$ expresses an unknown deflection of the middle plane, an unknown contact force between the plate and the rigid obstacle and an unknown thermal strain resultant. We shall use the thermal constants

$$\kappa = \frac{\lambda}{\rho c} > 0, \; d = \frac{\kappa 12}{h^2} > 0, \; e = \frac{\kappa\alpha^2\Upsilon E}{\lambda(1-2\nu)} > 0.$$

Classical formulation for the plate simply supported, with the zero change of the temperature on the boundary and acting under a perpendicular load f and the heat source p is then composed of the system

$$\left.\begin{aligned}&\ddot u - a\Delta\ddot u + b(\triangle^2 u + \tfrac{1+\nu}{2}\triangle\theta) - [u,v] = f+g,\\ &u\ge 0,\ g\ge 0,\ ug=0,\\ &\triangle^2 v + E[u,u] = 0,\\ &\dot\theta - \kappa\triangle\theta + d\theta - e\triangle\dot u = p\end{aligned}\right\}\ \text{on } Q, \tag{1}$$

the boundary conditions

$$u=w,\ v=\partial_n v=\theta=M(u)=0,\ u\ge 0,\ g\ge 0,\ ug=0\ \text{ on } S, \tag{2}$$

$$M(u)\equiv b\big(\triangle u + (1-\nu)(2n_1n_2\partial_{12}u - n_1^2\partial_{22}u - n_2^2\partial_{11}u)\big)$$

and the initial conditions

$$u(0,\cdot)=u_0,\ \dot u(0,\cdot)=v_0,\ \theta(0,\cdot)=\theta_0 \text{ on } \Omega. \tag{3}$$

For $u,y\in L_2(I;H^2(\Omega))$ we define the following bilinear form

$$A:(u,y)\mapsto b\big(\partial_{11}u\partial_{11}y + \partial_{22}u\partial_{22}y + \nu(\partial_{11}u\partial_{22}y + \partial_{22}u\partial_{11}y) + 2(1-\nu)\partial_{12}u\partial_{12}y\big) \tag{4}$$

almost everywhere on Q, introduce for a fixed function $w:\Omega\mapsto\mathbb{R}$ a shifted cone

$$\mathscr{K} := \{y\in w+\mathscr{V};\ y\ge 0 \text{ on } Q\}, \tag{5}$$

and denote by $\langle\langle\cdot,\cdot\rangle\rangle$ the duality between $(L_\infty(I;L_2(\Omega)))^*$ and $L_\infty(I;L_2(\Omega))$. Then the variational formulation of (1–3) has the following form:

Look for $\{u,v,\theta\}\in\mathscr{K}\times L_2(I;\mathring{H}^2(\Omega)))\times(L_\infty(I;L_2(\Omega))\cap L_2(I;\mathring{H}^1(\Omega))$ *such that* $\ddot u\in(L_\infty(I;L_2(\Omega)))^*$, $\dot\theta\in L_2(I;H^{-1}(\Omega))$, *the relations*

$$\begin{aligned}\langle\langle\ddot u, y-u-a\triangle(y-u)\rangle\rangle + \int_Q \big(A(u,y-u) - b\tfrac{1+\nu}{2}\nabla\theta\cdot\nabla(y-u) - [u,v](y-u)\big)\,dx\,dt&\\ \ge \int_Q f(y-u)\,dx\,dt,&\end{aligned} \tag{6}$$

$$\int_\Omega \big(\triangle v\triangle\varphi + E[u,u]\varphi\big)\,dx = 0, \tag{7}$$

$$\int_I\langle\dot\theta,z\rangle\,dt + \int_Q (d\theta z + \kappa\nabla\theta\cdot\nabla z + e\nabla\dot u\cdot\nabla z)\,dx\,dt = \int_Q pz\,dx\,dt \tag{8}$$

hold for any $\{y,\varphi,z\}\in\mathscr{K}\times L_2(I;\mathring{H}^2(\Omega))\times L_2(I;\mathring{H}^1(\Omega))$ and the initial conditions (3) are fulfilled.

In order to express the Airy stress function v in (6), (7) we define the bilinear operator $\Phi: H^2(\Omega)^2\to\mathring{H}^2(\Omega)$ by means of the variational equation

$$\int_\Omega \triangle\Phi(u,v)\triangle\varphi\,dx = \int_\Omega [u,v]\varphi\,dx,\ \forall\,\varphi\in\mathring{H}^2(\Omega). \tag{9}$$

The equation (9) has a unique solution, because $[u,v]\in L_1(\Omega)\hookrightarrow H^2(\Omega)^*$. The well-defined operator Φ is evidently compact and symmetric. The domain Ω fulfils the assumptions enabling to apply Lemma 1 from [6] due to which $\Phi: H^2(\Omega)^2\to W_p^2(\Omega)$, $2<p<\infty$ and

$$\|\Phi(u,v)\|_{W_p^2(\Omega)} \le c\|u\|_{H^2(\Omega)}\|v\|_{W_p^1(\Omega)}\ \forall u\in H^2(\Omega),\ v\in W_p^1(\Omega). \tag{10}$$

With its help we reformulate the system (6-8) into the following problem:

Problem $\mathscr{P}$. *Look for* $\{u,\theta\} \in \mathscr{K} \times (L_\infty(I;L_2(\Omega)) \cap L_2(I;\mathring{H}^1(\Omega))$ *such that* $\ddot{u} \in (L_\infty(I;L_2(\Omega)))^*,\ \dot{\theta} \in L_2(I;H^{-1}(\Omega))$, *the relations*

$$\begin{aligned}\langle\langle \ddot{u}, y-u-a\triangle(y-u)\rangle\rangle &+ \int_Q \big((A(u,y-u) - b\tfrac{1+\nu}{2}\nabla\theta\cdot\nabla(y-u)\\ &- E[u,\Phi(u,u)](y-u)\big)\,dx\,dt \ge \int_Q f(y-u)\,dx\,dt,\end{aligned} \tag{11}$$

$$\int_I \langle\dot{\theta},z\rangle\,dt + \int_Q (d\theta z + \kappa\nabla\theta\cdot\nabla z + e\nabla\dot{u}\cdot\nabla z)\,dx\,dt = \int_Q pz\,dx\,dt \tag{12}$$

hold for any $\{y,z\} \in \mathscr{K} \times L_2(I;\mathring{H}^1(\Omega))$ *and the initial conditions* (3) *are fulfilled.*

Problem $\mathscr{P}$ will be solved under the following assumptions

$$\begin{aligned}&w \in H^2(\Omega),\ w \ge w_0 > 0 \text{ on } \Omega;\ w_{|\Gamma} = u_{0|\Gamma},\\ &u_0 \in H^2(\Omega),\ u_0 \ge 0 \text{ on } \Omega;\ v_0 \in \mathring{H}^1(\Omega),\ \theta_0 \in L_2(\Omega),\ \{f,p\} \in L_2(Q)^2,\end{aligned} \tag{13}$$

where w_0 is a given constant.

3 Penalized problem

For any $\eta > 0$ we formulate the *penalized problem*

$$\left.\begin{aligned}&\ddot{u} - a\triangle\ddot{u} + b(\triangle^2 u + \tfrac{1+\nu}{2}\triangle\theta) - [u,v] = f + \eta^{-1}u^-,\\ &\triangle^2 v + E[u,u] = 0,\\ &\dot{\theta} - \kappa\triangle\theta + d\theta - e\triangle\dot{u} = p\end{aligned}\right\}\ \text{on } Q, \tag{14}$$

$$u = w,\ v = \partial_n = \theta = M(u) = 0 \text{ on } S \tag{15}$$

and the initial conditions (3) *hold.*

It has the following variational formulation after applying the bilinear operator Φ in the same way as above.

Problem $\mathscr{P}_\eta$. *Look for* $\{u,\theta\} \in (w+\mathscr{V})\times L_2(I;\mathring{H}^1(\Omega))$ *such that* $\{\dot{u},\dot{\theta}\} \in \mathring{\mathscr{H}} \times L_2(I;H^{-1}(\Omega))$, $\ddot{u} \in L_2(Q)$, *the equations*

$$\begin{aligned}\int_Q &\big(\ddot{u}(y-a\triangle y) + A(u,y) - b\tfrac{1+\nu}{2}\nabla\theta\cdot\nabla y + E[u,\Phi(u,u)]y - \eta^{-1}u^-y\big)\,dx\,dt\\ &= \int_Q fy\,dx\,dt,\end{aligned} \tag{16}$$

$$\int_I \langle\dot{\theta},z\rangle\,dt + \int_Q (d\theta z + \kappa\nabla\theta\cdot\nabla z + e\nabla\dot{u}\cdot\nabla z)\,dx\,dt = \int_Q pz\,dx\,dt \tag{17}$$

hold for any $\{y,z\} \in L_2(I;V) \times L_2(I;\mathring{H}^1(\Omega))$ *and the initial conditions* (3) *remain.*

We shall verify the existence of a solution to the penalized problem.

Theorem 3.1 *For every* $\eta > 0$ *there exists a solution* $\{u,\theta\}$ *of the problem* $\mathscr{P}_\eta$.

Proof. Let us denote by $\{v_i \in V;\ i \in \mathbb{N}\}$ a basis of V orthonormal with respect to the inner product

$$(u,v)_a = \int_\Omega (uv + a\nabla u\cdot\nabla v)\,dx,\ u,\ v \in \mathring{H}^1(\Omega)$$

and by $\{w_i \in \mathring{H}^1(\Omega);\ i \in \mathbb{N}\}$ an orthonormal in $L_2(\Omega)$ basis of $\mathring{H}^1(\Omega)$.

We construct the Galerkin approximation $\{u_m, \theta_m\}$ of a solution in the form

$$u_m(t) = w + \sum_{j=1}^{m} \alpha_j(t) v_j, \;\; \theta_m(t) = \sum_{j=1}^{m} \beta_j(t) w_j; \;\; \{\alpha_j(t), \beta_j(t)\} \in \mathbb{R}^2, \;\; t \in I, \quad j = 1, ..., m;$$

to satisfy the following system of equations

$$\begin{aligned} &\int_{\Omega} \left(\ddot{u}_m v_i + a \nabla \ddot{u}_m \cdot \nabla v_i + A(u_m, v_i) + E[u_m, v_i] \Phi(u_m, u_m) - b \tfrac{1+\nu}{2} \nabla \theta_m \cdot \nabla v_i \right) dx \\ &\quad = \int_{\Omega} (\eta^{-1} u_m^- + f) \, v_i \, dx, \end{aligned} \tag{18}$$

$$\int_{\Omega} \left(\dot{\theta}_m w_i + \kappa \nabla \theta_m \cdot \nabla w_i + d \theta_m w_i + e \nabla \dot{u}_m \cdot \nabla w_i \right) dx = \int_{\Omega} p w_i \, dx, \;\; i = 1, ..., m; \tag{19}$$

and the initial conditions

$$\begin{aligned} &u_m(0) = u_{0m}, \;\; u_{0m} \to u_0 \text{ in } H^2(\Omega); \;\; \dot{u}_m(0) = v_{0m}, \;\; v_{0m} \to v_0 \text{ in } \mathring{H}^1(\Omega); \\ &\theta_m(0) = \theta_{0m}, \;\; \theta_{0m} \to \theta_0 \text{ in } L_2(\Omega). \end{aligned} \tag{20}$$

The initial value problem (18–20) fulfils the conditions for the local existence of solution $\{u_m, \theta_m\}$ on some interval $I_m \equiv [0, t_m], \; 0 < t_m < T$.

Let us set $\gamma = b \frac{1+\nu}{2e}$. To derive the *a priori* estimates for solutions of (18)-(20) we multiply the equations (18) by $\dot{\alpha}_i(t)$ and (19) by $\gamma \beta_i(t)$ respectively, add with respect to i and integrate on $[0, t_m]$. Taking in mind

$$\int_{\Omega} [u, v] y \, dx = \int_{\Omega} [u, y] v \, dx \tag{21}$$

if at least one element of $\{u, v, y\}$ belongs to $\mathring{H}^2(\Omega)$ (cf. [4], Lemma 2.2.2, Chapter 2), we obtain after integrating for $Q_m := I_m \times \Omega$ the relation

$$\begin{aligned} &\int_{Q_m} \left[\frac{1}{2} \partial_t \left(\dot{u}_m^2 + a |\nabla \dot{u}_m|^2 + A(u_m, u_m) + \frac{E}{2} (\triangle \Phi(u_m, u_m))^2 + \gamma \theta_m^2 + \eta^{-1} (u_m^-)^2 \right) \right. \\ &\quad \left. + \gamma (\kappa |\nabla \theta_m|^2 + d \theta_m^2) \right] dx \, dt = \int_{Q_m} (f \dot{u}_m + \gamma \, p \, \theta_m) \, dx \, dt \end{aligned}$$

which leads to the estimate

$$\begin{aligned} &\|\dot{u}_m\|^2_{L_\infty(I; \mathring{H}^1(\Omega))} + \|u_m\|^2_{L_\infty(I; V)} + \|\Phi(u_m, u_m)\|^2_{L_\infty(I; H^2(\Omega))} + \eta^{-1} \|u_m^-\|^2_{L_\infty(I; L_2(\Omega))} \\ &\quad + \|\theta_m\|^2_{L_\infty(I; L_2(\Omega))} + \|\theta_m\|^2_{L_2(I; \mathring{H}^1(\Omega))} \leq C_1 \equiv C_1(f, p, u_0, v_0, \theta_0). \end{aligned} \tag{22}$$

The prolongation to the whole interval I is due to the original estimate for I_m not depending on m. Moreover the estimate (10) implies

$$\|\Phi(u_m, u_m)\|_{L_\infty(I; W_p^2(\Omega))} \leq c_p \equiv c_p(f, u_0, u_1) \, \forall \, p > 2. \tag{23}$$

The estimate (23) further implies

$$\begin{aligned} &[u_m, \Phi(u_m, u_m)] \in L_2(I; L_r(\Omega)), \; r = \frac{2p}{p+2}, \\ &\big\| [u_m, \Phi(u_m, u_m)] \big\|_{L_2(I; L_r(\Omega))} \leq c_r \equiv c_r(f, u_0, u_1). \end{aligned} \tag{24}$$

From the equation (19) we obtain straightforwardly the estimate

$$\|\dot{\theta}_m\|_{L_2(I; W_m{}^*)} \leq C_2(f, p, u_0, v_0, \theta_0), \;\; m \in \mathbb{N}, \tag{25}$$

where $W_m \subset \mathring{H}^1(\Omega)$ is the linear hull of $\{w_i\}_{i=1}^m$.

From (18) we obtain

$$\|\ddot{u}_m - a\triangle\ddot{u}_m\|^2_{L_2(I;V_m{}^*)} \le C_3(\eta),\ m \in \mathbb{N}, \tag{26}$$

where $V_m \subset H^2(\Omega)$ is the linear hull of $\{v_i\}_{i=1}^m$.

We proceed with the convergence of the Galerkin approximation. Applying the estimate (22), the compact imbedding theorem and interpolation in Sobolev spaces we obtain subsequences of $\{u_m\}$, $\{\theta_m\}$ (denoted again by $\{u_m\}$, $\{\theta_m\}$), and functions u, θ with the convergences

$$\begin{aligned} &u_m \rightharpoonup^* u \ \text{ in } L_\infty(I;V),\\ &\dot{u}_m \rightharpoonup^* \dot{u} \ \text{ in } L_\infty(I;\mathring{H}^1(\Omega)),\\ &\theta_m \rightharpoonup^* \theta \ \text{ in } L_\infty(I;L_2(\Omega)),\\ &\theta_m \rightharpoonup \theta \ \text{ in } L_2(I;\mathring{H}^1(\Omega)), \end{aligned} \tag{27}$$

The estimates (25), (26) imply the convergence

$$\dot{\theta}_m \rightharpoonup \dot{\theta} \quad \text{in } L_2(I;W^*), \tag{28}$$

$$(\ddot{u}_m - a\triangle\ddot{u}_m) \rightharpoonup (\ddot{u} - a\triangle\ddot{u}) \ \text{ in } L_2(I;Y^*), \tag{29}$$

where $W = \bigcup_{m\in\mathbb{N}} W_m,\ \overline{W} = \mathring{H}^1(\Omega)$ and $Y = \bigcup_{m\in\mathbb{N}} V_m,\ \overline{Y} = V$. The convergences (28), (29) imply

$$\|\dot{\theta}_m\|_{L_2(I;H^{-1}(\Omega))} \le C_2(f,p,u_0,v_0,\theta_0),\ m\in\mathbb{N}, \tag{30}$$

$$\dot{\theta}_m \rightharpoonup \dot{\theta} \ \text{ in } L_2(I;H^{-1}(\Omega)), \tag{31}$$

$$\|\ddot{u}_m - a\triangle\ddot{u}_m\|^2_{L_2(I;V^*)} \le C_3(\eta),\ m\in\mathbb{N}. \tag{32}$$

Moreover we obtain from (32) a better acceleration estimate

$$\|\ddot{u}_m\|_{L_2(Q)} \le C_4(\eta) \tag{33}$$

and the convergence

$$\ddot{u}_m \rightharpoonup \ddot{u} \text{ in } L_2(Q) \tag{34}$$

for a chosen subsequence denoted again by $\{\ddot{u}_m\}$. We have applied also the surjectivity of the elliptic operator $v \mapsto v - a\triangle v,\ v \in V$; in the same way as in [1] setting

$$\|\ddot{u}_m\|_{L_2(Q)} = \sup_{\|\psi\|_{L_2(Q)}\le 1} \left|\int_Q \ddot{u}_m\,\psi\,dx\,dt\right| \le c \sup_{\|v\|_{L_2(I;V)}\le 1} \left|\int_Q \ddot{u}_m(v - a\triangle v)\,dx\,dt\right| \le C_4(\eta).$$

The estimates (30), (33) imply after considering the convergences (27) the uniform convergences

$$\begin{aligned} &u_m \to u \ \text{ in } C(\bar{I};H^{2-\varepsilon}(\Omega)),\\ &\dot{u}_m \to \dot{u} \ \text{ in } C(\bar{I};H^{1-\varepsilon}(\Omega)),\\ &\theta_m \to \theta \ \text{ in } C(\bar{I};H^{1-\varepsilon}(\Omega)) \ \text{ for any } \varepsilon > 0 \end{aligned} \tag{35}$$

and

$$\begin{aligned} &\Phi(u_m,u_m) \to \Phi(u,u) \ \text{ in } L_2(I;H^2(\Omega)),\\ &\Phi(u_m,u_m) \rightharpoonup^* \Phi(u,u) \text{ in } L_\infty(I;W_p^2(\Omega)). \end{aligned} \tag{36}$$

Let $\mu \in \mathbb{N},\ y_\mu = \sum_{i=1}^{\mu} \phi_i(t) v_i,\ z_\mu = \sum_{i=1}^{\mu} \phi_i(t) w_i,\ \phi_i \in \mathscr{D}(0,T),\ i = 1, ..., \mu$. We have for arbitrary $t \in I$ the relations

$$\int_\Omega \big(\ddot{u}_m (y_\mu - a \triangle y_\mu) + A(u_m, y_\mu) + E[u_m, y_\mu] \Phi(u_m, u_m) - b \tfrac{1+\nu}{2} \nabla \theta_m \cdot \nabla y_\mu - \eta^{-1} u_m^- y_\mu) \big)\, dx$$
$$= \int_\Omega f y_\mu \, dx,$$
$$\int_\Omega \big(\dot{\theta}_m z_\mu + \kappa \nabla \theta_m \cdot \nabla z_\mu + d \theta_m z_\mu + e \nabla \dot{u}_m \cdot \nabla z_\mu \big)\, dx = \int_\Omega p z_\mu \, dx,\ \forall\ m \geq \mu,\ t \in I.$$

The convergences (27), (31), (34) imply that functions u, θ fulfil

$$\begin{aligned} &\int_\Omega \big(\ddot{u} (y_\mu - a \triangle y_\mu) + A(u, y_\mu) + E[u_m, \Phi(u_m, u_m)] y_\mu - b \tfrac{1+\nu}{2} \nabla \theta \cdot \nabla y_\mu - \eta^{-1} u^- y_\mu) \big)\, dx \\ &\quad = \int_\Omega f y_\mu \, dx, \end{aligned} \tag{37}$$

$$\int_\Omega \big(\dot{\theta} z_\mu + \kappa \nabla \theta \cdot \nabla z_\mu + d \theta z_\mu + e \nabla \dot{u} \cdot \nabla z_\mu \big)\, dx = \int_\Omega p z_\mu \, dx. \tag{38}$$

Functions $\{y_\mu\},\ \{z_\mu\}$ form a dense subsets of the spaces $L_2(I; V)$ and $L_2(I; \mathring{H}^1(\Omega))$ respectively. Then we obtain from (37), (38) the relations (16), (17). which together with properties (20) imply the initial conditions (3) and the proof of the existence of a solution is complete.

4 Solvability of the original problem

The estimates (22), (30) imply the following η independent estimates :

$$\begin{aligned} &\|\dot{u}_\eta\|^2_{L_\infty(I;\mathring{H}^1(\Omega))} + \|u_\eta\|^2_{L_\infty(I;V)} + \|\theta_\eta\|^2_{L_\infty(I;L_2(\Omega))} + \|\theta_\eta\|^2_{L_2(I;\mathring{H}^1(\Omega))} + \|\dot{\theta}_\eta\|^2_{L_2(I;H^{-1}(\Omega))} \\ &\quad + \eta^{-1} \|u_\eta^-\|^2_{L_\infty(I;L_2(\Omega))} \leq C_5 \equiv C_5(f, p, u_0, v_0, \theta_0). \end{aligned} \tag{39}$$

for a solution $\{u_\eta, \theta_\eta\},\ \eta > 0$ of the penalized problem. The acceleration term $\ddot{u}_\eta$ does not appear in (39). It is then suitable to transform the penalized relation (16) using the integration by parts with respect to t and the Green formula with respect to x. We obtain the system

$$\begin{aligned} &\int_Q \big(A(u_\eta, y) - \dot{u}_\eta \dot{y} - a \nabla \dot{u}_\eta \cdot \nabla \dot{y} - b \tfrac{1+\nu}{2} \nabla \theta_\eta \cdot \nabla y \big)\, dx\, dt \\ &\quad + \int_\Omega \left(\dot{u}_\eta y + a \nabla \dot{u}_\eta \cdot \nabla y \right)(T, \cdot)\, dx \\ &\quad = \int_\Omega \left(v_0 y(0, \cdot) + a \nabla v_0 \cdot \nabla y(0, \cdot) \right)\, dx + \int_Q (f + \eta^{-1} u_\eta^-) y \, dx\, dt, \end{aligned} \tag{40}$$

$$\int_Q \big(\dot{\theta}_\eta z + \kappa \nabla \theta_\eta \cdot \nabla z + d \theta_\eta z + e \nabla \dot{u}_\eta \cdot \nabla z \big)\, dx\, dt = \int_Q p y \, dx\, dt \tag{41}$$

holding for any $\{y, z\} \in L_2(I; V) \times L_2(I; \mathring{H}^1(\Omega))$ with $\dot{y} \in L_2(I; \mathring{H}^1(\Omega))$.

We derive an $\eta-$independent estimate of the penalty term $\eta^{-1} u_\eta^-$. Applying the assumptions (13) and the definition of u_η^- we obtain

$$0 \leq w_0 \int_Q \eta^{-1} u_\eta^- \, dx\, dt \leq \int_Q \eta^{-1} u_\eta^- w \, dx\, dt \leq \int_Q \eta^{-1} u_\eta^- (w - u_\eta)\, dx\, dt.$$

After inserting $y = w - u_\eta$ in (40) we achieve using the estimates (39) the crucial estimate

$$\|\eta^{-1} u_\eta^-\|_{L_1(Q)} \leq C_6 \equiv C_6(f, p, u_0, v_0, \theta_0). \tag{42}$$

In order to achieve the L_1 estimate of the acceleration terms $\{\ddot{u}_\eta\}$ we express the identity (40) in a form

$$\int_Q (\ddot{u}_\eta(y - a\triangle y)\,dx\,dt = \int_Q \left[-A(u_\eta, y) + b\tfrac{1+\nu}{2}\nabla\theta_\eta \cdot \nabla y + (\eta^{-1}u_\eta^- + f)y\right]\,dx\,dt. \tag{43}$$

Using the estimates (39) and (42) and the imbedding $L_1(Q) \hookrightarrow L_1(I;V^*) \subset [L_\infty(I;V)]^*$ we obtain

$$\|\ddot{u}_\eta - a\triangle\ddot{u}_\eta\|_{L_1(I;V)} \le C_7 \equiv C_7(f,p,u_0,v_0,\theta_0).$$

Applying the relations

$$\|\ddot{u}_\eta\|_{L_1(I;L_2(\Omega))} = \sup_{\|\psi\|_{L_\infty(I;L_2(\Omega))}\le 1} \left|\int_Q \ddot{u}_\eta\,\psi\,dx\,dt\right| \le c \sup_{\|v\|_{L_\infty(I;V)}\le 1} \left|\int_Q \ddot{u}_\eta(v - a\triangle v)\,dx\,dt\right|$$

in the same way as above we obtain the η-independent estimate

$$\|\ddot{u}_\eta\|_{L_1(I;L_2(\Omega))} \le C_8 \equiv C_8(f,p,u_0,v_0,\theta_0). \tag{44}$$

Hence there exists a sequence $\eta_k \searrow 0$, a couple of functions $\{u,\theta\}$ and a functional g such that for $\{u_k,\eta_k\} \equiv \{u_{\eta_k},\theta_{\eta_k}\}$ the following convergences hold:

$$\begin{aligned}
u_k &\rightharpoonup^* u \ \text{ in } L_\infty(I;V),\\
\dot{u}_k &\rightharpoonup^* \dot{u} \ \text{ in } L_\infty(I;\mathring{H}^1(\Omega)),\\
\ddot{u}_k &\rightharpoonup^* \ddot{u} \ \text{ in } (L_\infty(I;L_2(\Omega)))^*,\\
u_k &\to u \ \text{ in } C(I;H^{2-\varepsilon}(\Omega)) \text{ for any } \varepsilon > 0,\\
\eta^{-1}u_k^- &\rightharpoonup^* g \ \text{ in } (L_\infty(Q))^*,\\
\theta_k &\rightharpoonup^* \theta \ \text{ in } L_\infty(I;L_2(\Omega)) \cap L_2(I;\mathring{H}^1(\Omega)),\\
\dot{\theta}_k &\rightharpoonup \dot{\theta} \ \text{ in } L_2(I;H^{-1}(\Omega)),\\
\theta_m &\to \theta \ \text{ in } C(\bar{I};H^{1-\varepsilon}(\Omega)) \ \text{ for any } \varepsilon > 0.
\end{aligned} \tag{45}$$

The above convergences together with the appropriate convergencies of the type (36), which obviously remain true also for this sequence, prove the relation (12) and the initial conditions $u(0,\cdot) = u_0,\ \theta(0,\cdot) = \theta_0$. The initial condition $\dot{u}(0,\cdot) = v_0$ is fulfilled in a weak sense due to the estimates of the accelarations. The convergence in (45) together with (43) imply further

$$\langle\langle \ddot{u}, y - a\triangle y\rangle\rangle + \int_Q \left(A(u,y) - b\tfrac{1+\nu}{2}\nabla\theta \cdot \nabla y\right)\,dx\,dt = \int_Q fy\,dx\,dt + \langle\langle g,y\rangle\rangle_Q$$

for any $y \in L_\infty(I;V)$, where $\langle\langle\cdot,\cdot\rangle\rangle_Q$ is the duality pairing between $(L_\infty(Q))^*$ and $L_\infty(Q)$.

We have the orthogonality

$$\langle\langle g,u\rangle\rangle = 0$$

due to the relations

$$\langle\langle g,u\rangle\rangle = \lim_{k\to\infty} \eta_k^{-1}\|u_k^-\|^2_{L_2(Q)} = 0.$$

The relations

$$\langle\langle g,y\rangle\rangle = \lim_{k\to\infty} \int_Q \eta_k^{-1}u_k^- y\,dx\,dt \ge 0 \ \forall\, y \in \mathscr{K}$$

imply together with the orthogonality proved above that the variational inequality (11) is fulfilled and we have verified the existence theorem:

Theorem 4.1 *Let the assumptions* (13) *hold. Then there exists a solution of Problem* $\mathscr{P}$*.*

Remark 4.2 *Boundary conditions* (2) *for simply supported plate and zero boundary thermal stress resultant enable in an easy way to get the a priori estimates in the previous and this chapter. It is possible to consider also another types of boundary conditions with a bit more complicated deriving of a priori estimates inevitable for the convergence process.*

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