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Abstract

We consider a simplified model based on the Navier-Stokes-Fourier system coupled to a transport equation and the Maxwell system, proposed to describe radiative flows in stars. We establish global-in-time existence for the associated initial-boundary value problem in the framework of weak solutions.

Key words: Radiation magnetohydrodynamics, Navier-Stokes-Fourier system, weak solution

1 Introduction

There are a number of situations when stars can be described by compressible fluids and their dynamics is controlled by intense magnetic fields coupled with a simplified model of radiation. Following studies by Ducomet, Feireisl and Nečasová [11] and Ducomet and Feireisl [10] we consider a mathematical model of radiative flow where the motion of the fluid is described by the standard Galilean fluid mechanics giving the evolution of *the mass density* $\varrho = \varrho(t, x)$, the *velocity field* $\vec{u} = \vec{u}(t, x)$, and the *absolute temperature* $\vartheta = \vartheta(t, x)$ as functions of the time t and the Eulerian spatial coordinate $x \in \Omega \subset \mathbb{R}^3$. The effect of radiation is incorporated in the *radiative intensity* $I = I(t, x, \vec{\omega}, \nu)$, depending on the director $\vec{\omega} \in \mathcal{S}^2$, where $\mathcal{S}^2 \subset \mathbb{R}^3$ denotes the unit sphere, and the frequency $\nu \geq 0$. This system of equations is coupled to a simplified Maxwell system of electrodynamics where we assume the quasineutrality of the plasma described and neglect the Maxwell displacement current. This system describes the evolution of *the magnetic induction* $\vec{B} = \vec{B}(t,x)$ and *the electric field* $\vec{E} = \vec{E}(t,x)$, resp. *the magnetic field* $\vec{H} = \vec{H}(t,x)$ and *the electric induction* $\vec{D} = \vec{D}(t,x)$. The collective effect of the radiation is then expressed in terms of integral means with respect to the variables $\vec{\omega}$ and ν of quantities depending on I: the radiation energy E_R is given as

$$E_R(t,x) = \frac{1}{c}\int_{\mathcal{S}^2}\int_0^\infty I(t,x,\vec{\omega},\nu)\ \mathrm{d}\vec{\omega}\ \mathrm{d}\nu. \tag{1.1}$$

i0

The time evolution of I is described by a transport equation with source terms $\tilde{S}$ depending on nonnegative quantities of the absolute temperature ϑ and frequency of radiation ν, while the effect of radiation on the macroscopic motion of the fluid is represented by an extra source term of radiative heating/cooling in the energy equation and an extra source term of acceleration/deceleration both evaluated in terms of $\tilde{S}$.

The Maxwell system of classical electrodynamics in our case reduces to *the Faraday's law of induction*

$$\partial_t \vec{B} + \mathbf{curl}_x \vec{E} = 0, \tag{1.2}$$

i6

together with *the Gauss's law for magnetism*

$$\mathrm{div}_x \vec{B} = 0, \tag{1.3}$$

i7

the Ampère's law

$$\vec{J} = \mathbf{curl}_x \vec{H}, \tag{1.4}$$

iA

Coulomb's law

$$\mathrm{div}_x \vec{D} = 0, \tag{1.5}$$

iC

and (nonlinear version of) *Ohm's law*

$$\vec{J} = \sigma(\vec{E} - \vec{B}\times\vec{u}), \tag{1.6}$$

i8

where $\vec{B} = \mu\vec{H}$, $\vec{D} = \tilde{\varepsilon}\vec{E}$, σ is the (nonlinear) electrical conductivity, $\mu = \mu(\left|\vec{H}\right|)$ and $\tilde{\varepsilon}$ is the dielectric permittivity. All the material properties are assumed to be scalars.

This gives us from (1.2)

$$\partial_t \vec{B} + \mathbf{curl}_x(\vec{B}\times\vec{u}) + \mathbf{curl}_x(\frac{1}{\sigma}\mathbf{curl}_x(\frac{1}{\mu}\vec{B})) = 0. \tag{1.7}$$

i6a

Following [2] we will denote

$$\mathcal{M}(s) = \int_0^s \tau\partial_\tau(\tau\mu(\tau))d\tau, \tag{1.8}$$

i6b

and rewrite the equation (1.2) as a version of the Poynting theorem

$$\partial_t\mathcal{M}(|\vec{H}|) + \vec{J}\cdot\vec{E} = \mathrm{div}_x(\vec{H}\times\vec{E}). \tag{1.9}$$

i6c

Together with the principles of continuum mechanics, the magnetofluid [6, 30] with radiation effects [37] problem can be described by the system of equations

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = 0 \text{ in } (0,T)\times\Omega; \tag{1.10}$$

i1

$$\begin{gathered}\partial_t(\varrho\vec{u}) + \mathrm{div}_x(\varrho\vec{u}\otimes\vec{u}) + \nabla_x p(\varrho,\vartheta) = \\ \mathrm{div}_x\mathbb{S} - \vec{S}_F + \varrho\nabla_x\psi + \mu\vec{J}\times\vec{H} \text{ in } (0,T)\times\Omega;\end{gathered} \tag{1.11}$$

i2

$$\begin{gathered}\partial_t\left[\varrho\left(\frac{1}{2}|\vec{u}|^2 + e(\varrho,\vartheta)\right) + \mathcal{M}(|\vec{H}|)\right] + \mathrm{div}_x\left[\varrho\left(\frac{1}{2}|\vec{u}|^2 + e(\varrho,\vartheta) + \frac{p(\varrho,\vartheta)}{\varrho}\right)\vec{u}\right] = \\ \varrho\nabla_x\psi\cdot\vec{u} - \mathrm{div}_x\left(\vec{q} - \mathbb{S}\vec{u} + \frac{\mu}{\tilde{\varepsilon}}\vec{D}\times\vec{H}\right) - S_E \text{ in } (0,T)\times\Omega;\end{gathered} \tag{1.12}$$

i3

$$\partial_t I + c\vec{\omega}\cdot\nabla_x I = c\tilde{S} \text{ in } (0,T)\times\Omega\times(0,\infty)\times\mathcal{S}^2. \tag{1.13}$$

i4

Note that, contrary to the model studied in [17], a radiation term appears in the momentum equation in spite of this term may be small. The electrical conductivity σ hidden in (1.11) can depend on the density ϱ, on the temperature ϑ and the magnetic field $\vec{H}$ (cf. (2.8)) of the magnetofluid.

The symbol $p = p(\varrho,\vartheta)$ denotes the (equilibrium) thermodynamic pressure and $e = e(\varrho,\vartheta)$ is the specific internal energy, interrelated through *Maxwell's relation*

$$\frac{\partial e}{\partial\varrho} = \frac{1}{\varrho^2}\left(p(\varrho,\vartheta) - \vartheta\frac{\partial p}{\partial\vartheta}\right). \tag{1.14}$$

i5

Observe that both the pressure and the internal energy involve both a radiation and a thermal term. The meaning of this splitting is that there is a part of the photon gas in the equilibrium with plasma whereas another part is not. The latter is described by the transport equation (1.13) and is caused mainly by inverse Compton scattering and synchrotrone radiation [29]. Naturally, our description is somehow "mixed" since we use classical thermodynamics and classical electrodynamics for the description of matter while the radiation is described by geometrical optics.

Furthermore, $\mathbb{S}$ is the viscous part of the stress tensor determined by *Newton's rheological law*

$$\mathbb{S} = \lambda(\vartheta, \left|\vec{H}\right|)\left(\nabla_x\vec{u} + \nabla_x^T\vec{u} - \frac{2}{3}\mathrm{div}_x\vec{u}\,\mathbb{I}\right) + \eta(\vartheta, \left|\vec{H}\right|)\mathrm{div}_x\vec{u}\;\mathbb{I}, \tag{1.15}$$

i9

where the shear viscosity coefficient $\lambda > 0$ and the bulk viscosity coefficient $\eta \geq 0$ are effective functions of the absolute temperature and the magnitude of the magnetic field. Once again, we tacitly assume isotropy of the considered medium (without the presence of a magnetic field). Similarly, $\vec{q}$ is the heat flux given by *Fourier's law*

$$\vec{q} = -\left(\kappa_R\vartheta^3 + \kappa_M\left(\varrho,\vartheta,\left|\vec{H}\right|\right)\right)\nabla_x\vartheta, \tag{1.16}$$

i10

with the constant radiative heat conductivity coefficient $\kappa_R > 0$ and with a molecular heat conductivity coefficient $\kappa_M > 0$.

Further the source term of radiation is due to absorption/emission and scattering of light

$$\tilde{S} = S_{a,e} + S_s, \tag{1.17}$$ i11

where

$$S_{a,e}(t,x,\vec{\omega},\nu) = \sigma_a(\nu,\vartheta)\Big(\mathfrak{B}(\nu,\vartheta) - I(t,x,\vec{\omega},\nu)\Big), \tag{1.18}$$ i12

$$S_s(t,x,\vec{\omega},\nu) = \sigma_s(\nu,\vartheta)\left(\frac{1}{4\pi}\int_{\mathcal{S}^2} I(t,x,\vec{\omega},\nu)\,\mathrm{d}\vec{\omega} - I(t,x,\vec{\omega},\nu)\right), \tag{1.19}$$ i13

$$S_E(t,x) = \int_{\mathcal{S}^2}\int_0^\infty \tilde{S}(t,x,\vec{\omega},\nu)\,\mathrm{d}\vec{\omega}\,\mathrm{d}\nu\ , \tag{1.20}$$ i14

and

$$\vec{S}_F(t,x) = c^{-1}\int_{\mathcal{S}^2}\int_0^\infty \vec{\omega}\tilde{S}(t,x,\vec{\omega},\nu)\,\mathrm{d}\vec{\omega}\,\mathrm{d}\nu, \tag{1.21}$$ i15

with the absorption coefficient $\sigma_a = \sigma_a(\nu,\vartheta) \geq 0$, and the scattering coefficient $\sigma_s = \sigma_s(\nu,\vartheta) \geq 0$. Here $\mathfrak{B}(\nu,\vartheta)$ denotes (equilibrial) black body radiation. According to Planck's law we recall

$$\mathfrak{B}(\nu,\vartheta) = \frac{2h\vartheta^3 c^{-2}}{e^{\frac{h\nu}{k_B\vartheta}} - 1}. \tag{1.22}$$ i16

More restrictions on the structural properties of constitutive relations will be imposed in Section 2 below.

System (1.10) – (1.22) is supplemented with the boundary conditions modelling the mechanical and heat isolation combined with no-slip and transparency (radiation does not reflect back to the domain Ω) at the boundary:

$$\vec{u}|_{\partial\Omega} = \vec{0},\ \vec{q}\cdot\vec{n}|_{\partial\Omega} = 0; \tag{1.23}$$ i17

$$I(t,x,\vec{\omega},\nu) = 0 \text{ for } x \in \partial\Omega,\ \vec{\omega}\cdot\vec{n} \leq 0, \tag{1.24}$$ i18

where $\vec{n}$ denotes the outer normal vector to $\partial\Omega$.

For the electromagnetic fields we adopt boundary conditions of a perfect conductor (assuming outside Ω there are zero fields and using the continuity of the following components of the fields across $\partial\Omega$)

$$\vec{E}\times\vec{n}|_{\partial\Omega} = \vec{0},\ \vec{B}\cdot\vec{n}|_{\partial\Omega} = 0. \tag{1.25}$$ i19

It remains to complement the system with the Poisson equation for the self-gravitational potential ψ from the right-hand side of (1.11)

$$-\Delta\psi = 4\pi G\varrho, \tag{1.26}$$ i20

where G is Newton's gravitational constant.

System (1.10) – (1.26) can be viewed as a simplified model in radiation hydrodynamics, the physical foundations of which were described by Pomraning [37] and Mihalas and Weibel-Mihalas [35] in the framework of the theory of special relativity. Similar systems have been investigated more recently in astrophysics and laser applications (in the relativistic and inviscid case) by Lowrie, Morel and Hittinger [33], Buet and Després [4], with a special attention to asymptotic regimes, see also Dubroca and Feugeas [8], Lin [31] and Lin, Coulombel and Goudon [32] for related numerical issues.

The *existence* of local-in-time solutions and sufficient conditions for blow up of classical solutions in the non-relativistic inviscid case were obtained by Zhong and Jiang [38], see also the recent papers by Jiang and Wang [27, 28] for related one-dimensional "Euler-Boltzmann" type models. Moreover, a simplified version of the system has been investigated by Golse and Perthame [24], where global existence was proved by means of the theory of nonlinear semigroups.

Concerning *viscous* fluids, a number of similar results have been considered in the recent past in the one-dimensional geometry [1, 13, 14, 15, 16] and a global existence result has also recently been proved in the 3D setting in [11] under some hypotheses on transport coefficients, for the "complete system" (when a radiative source appears only in the right-hand side of (1.11)).

Our goal in the present paper is to show that the existence theory developed in [11] and [10] relying on previous works [21], [19] and [20, Chapter 3], can be adapted to the problem (1.10) – (1.26).

As stressed in [11], a complete proof of existence is now well understood (see [20, Chapter 3]) therefore we focus as in [11] on the property of *weak sequential stability* for problem (1.10) – (1.26) in the framework of the weak solutions introduced in [10]. More specifically, we introduce a concept of finite energy weak solution in the spirit of [10] and show that any sequence $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, \vec{H}_\varepsilon, I_\varepsilon\}_{\varepsilon>0}$ of solutions to problem (1.10) – (1.26), bounded in the natural energy norm, possesses a subsequence converging to (another) weak solution of the same problem Such a property highlights the essential ingredients involved in the "complete" proof of existence that may be carried over by means of the arguments delineated in [20, Chapter 3].

The essential contribution to the proof comes from the entropy inequality. Due to a relevant "radiative" contribution one faces a similar situation encountered in [11], namely that the total entropy production has not a "definite sign" and, accordingly, we can establish the strong convergence of the radiative contribution with the help of regularity of velocity averages. This is also connected to the fact that we do not introduce radiation entropy in the total entropy inequality.

The paper is organized as follows. In Section 2, we list the principal hypotheses imposed on constitutive relations, introduce the concept of weak solution to problem (1.10) – (1.26), and state the main result. Uniform bounds imposed on weak solutions by the data are derived in Section 3.1. The property of *weak sequential stability* of a bounded sequence of weak solutions is established in Section 3.2. Finally, we introduce a suitable approximation scheme and discuss the main steps of the proof of existence in Section 3.3.

2 Hypotheses and main results

m

Hypotheses imposed on constitutive relations and transport coefficients are motivated by the general *existence theory* for the Navier-Stokes-Fourier system developed in [20, Chapter 3] and reasonable physical assumptions [37].

Firstly, we consider the pressure in the form

$$p(\varrho,\vartheta) = \vartheta^{5/2} P\left(\frac{\varrho}{\vartheta^{3/2}}\right) + \frac{a}{3}\vartheta^4, \ a > 0, \tag{2.1}$$ m1

where $P : [0,\infty) \to [0,\infty)$ is a given function with the following properties:

$$P \in C^1[0,\infty), \ P(0) = 0, \ P'(Z) > 0 \text{ for all } Z \geq 0, \tag{2.2}$$ m2

$$0 < \frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z} < c \text{ for all } Z \geq 0, \tag{2.3}$$ m3

$$\lim_{Z\to\infty} \frac{P(Z)}{Z^{5/3}} = p_\infty > 0. \tag{2.4}$$ m4

The component $\frac{a}{3}\vartheta^4$ represents the effect of "equilibrium" radiation pressure (see [11] for motivations and [20] for details). Essentially, these hypotheses are implications of general principles of thermodynamical stability and the assumption that there is at least one component in the plasmatic mixture behaving in the degenerate regime as a Fermi gas (we may think of it in most cases as electron gas). The constant a is the Stefan-Boltzmann constant.

According to Maxwell's relation (1.14) and statistical kinetic theory, the internal energy density e is

$$e(\varrho,\vartheta) = \frac{3}{2}\varrho^{-1}\vartheta^{5/2}P\left(\varrho\vartheta^{-3/2}\right) + a\vartheta^4\varrho^{-1}, \tag{2.5}$$ m5

and the associated specific entropy reads

$$s(\varrho,\vartheta) = M\left(\varrho\vartheta^{-3/2}\right) + \frac{4a}{3}\vartheta^3\varrho^{-1}, \tag{2.6}$$ m6

with a function M satisfying by (2.3)

$$M'(Z) = -\frac{3}{2}\frac{\frac{5}{3}P(Z) - P'(Z)Z}{Z^2} < 0.$$

Additional entropy for the photon gas out of equilibrium is not introduced.

The transport coefficients μ, η, and κ_M are continuously differentiable functions of their respective variables admitting a common temperature scaling such that there exist $c_1, c_2, c_3, c > 0$

$$\lambda'_\vartheta(\vartheta, \left|\vec{H}\right|) < c_3, \tag{2.7}$$ m7

$$c_1(1+\vartheta) \le \eta(\vartheta, \left|\vec{H}\right|), \sigma^{-1}(\varrho, \vartheta, \vec{B}), \lambda(\vartheta, \left|\vec{H}\right|) \le c_2(1+\vartheta), \tag{2.8}$$

m8

$$\kappa_M(\varrho, \vartheta, \left|\vec{H}\right|) \le c(1+\vartheta^3), \tag{2.9}$$

m8a

for any $\vartheta \ge 0$. We consider the magnetic permeability μ satisfying the following property

$$\underline{c_k} s(1+s)^{-k} \le \partial_s^k (s\mu(s)) \le \overline{c}_k s(1+s)^{-k}, \tag{2.10}$$

m8b

for any $s \ge 0$ and for $k = 0, 1$ with $\underline{c}_k, \overline{c}_k > 0$. Moreover, we assume that σ_a, σ_s are continuous functions of ν, ϑ such that there exist $c_4, c_5, c_6 > 0$ and $h \in L^1(0,\infty)$ and it holds

$$0 \le \sigma_a(\nu, \vartheta), \sigma_s(\nu, \vartheta) \le c_4, \;\; 0 \le \sigma_a(\nu, \vartheta)\mathfrak{B}(\nu, \vartheta) \le c_5, \tag{2.11}$$

m9

$$\sigma_a(\nu, \vartheta), \sigma_s(\nu, \vartheta), \sigma_a(\nu, \vartheta)\mathfrak{B}(\nu, \vartheta) \le h(\nu), \tag{2.12}$$

m10

$$\sigma_a(\nu, \vartheta), \sigma_s(\nu, \vartheta) \le c_6 \vartheta, \tag{2.13}$$

m11

for all $\nu \ge 0$, $\vartheta \ge 0$. Relations (2.11) – (2.13) represent "cut-off" hypotheses neglecting the effect of radiation at large frequencies ν and small temperatures ϑ.

We just recall the definitions introduced in [11]. In the weak formulation of the Navier-Stokes-Fourier system the equation of continuity (1.10) is replaced by its *renormalized* version introduced in [7] represented by the family of integral identities

$$\int_0^T \int_\Omega \Big[\Big(\varrho + b(\varrho)\Big)\partial_t \varphi + \Big(\varrho + b(\varrho)\Big)\vec{u} \cdot \nabla_x \varphi + \Big(b(\varrho) - b'(\varrho)\varrho\Big) \mathrm{div}_x \vec{u}\, \varphi \Big] \ \mathrm{d}x \ \mathrm{d}t$$
$$= - \int_\Omega \Big(\varrho_0 + b(\varrho_0)\Big)\varphi(0,\cdot) \ \mathrm{d}x, \tag{2.14}$$

m12

to be satisfied for any $\varphi \in C_c^\infty([0,\infty) \times \overline{\Omega})$, and any $b \in C^\infty[0,\infty)$, $b' \in C_c^\infty[0,\infty)$, where (2.14) implicitly includes the initial condition

$$\varrho(0,\cdot) = \varrho_0.$$

Similarly, the momentum equation (1.11) is replaced by its weak version

$$\int_0^T \int_\Omega (\varrho\vec{u} \cdot \partial_t \vec{\varphi} + \varrho\vec{u} \otimes \vec{u} : \nabla_x \vec{\varphi} + p \, \mathrm{div}_x \vec{\varphi}) \ \mathrm{d}x \ \mathrm{d}t \tag{2.15}$$

m13

$$= \int_0^T \int_\Omega \mathbb{S} : \nabla_x \vec{\varphi} - \varrho \nabla_x \psi \cdot \vec{\varphi} + \vec{S}_F \cdot \vec{\varphi} - \mu(\vec{J} \times \vec{H}) \cdot \vec{\varphi} \ \mathrm{d}x \ \mathrm{d}t$$
$$- \int_\Omega (\varrho\vec{u})_0 \cdot \vec{\varphi}(0,\cdot) \ \mathrm{d}x,$$

for any $\vec{\varphi} \in C_c^\infty([0,T) \times \Omega; \mathbb{R}^3)$. For (2.15) to make sense, especially the term $\int_0^T \int_\Omega \mathbb{S} : \nabla_x \varphi \, \mathrm{d}x \, \mathrm{d}t$, the field $\vec{u}$ must belong to a certain Bochner space with Sobolev space with respect to the spatial variable and we require that

$$\vec{u} \in L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^3)), \tag{2.16}$$

m14

where (2.16) already includes the no-slip boundary condition $(1.23)_1$. Gravitational potential ψ is given by (1.26) considered on the whole space $\mathbb{R}^3$, where ϱ was extended to be zero outside Ω.

As the term $\mathbb{S}\vec{u}$ in the total energy balance (1.12) is not controlled on the (hypothetical) vacuum zones of vanishing density, we replace (1.12) by the internal energy equation as in [20]

$$\partial_t(\varrho e) + \mathrm{div}_x(\varrho e \vec{u}) + \mathrm{div}_x \vec{q} = \mathbb{S} : \nabla_x \vec{u} - p \, \mathrm{div}_x \vec{u} - S_E + \vec{u} \cdot \vec{S}_F + \frac{1}{\sigma} \left| \mathbf{curl}_x \vec{H} \right|^2. \tag{2.17}$$

m15

Furthermore, dividing (2.17) by ϑ and using Maxwell's relation (1.14), we may rewrite (2.17) as the entropy equation

$$\partial_t (\varrho s) + \mathrm{div}_x (\varrho s \vec{u}) + \mathrm{div}_x \left(\frac{\vec{q}}{\vartheta} \right) = r, \tag{2.18}$$

m16

where the entropy production rate r is

$$r = \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} + \frac{1}{\sigma} \left| \mathbf{curl}_x \vec{H} \right|^2 \right) + \frac{\vec{u} \cdot \vec{S}_F - S_E}{\vartheta}, \tag{2.19}$$

iprod

where the first term $r_m := \frac{1}{\vartheta} \left(\mathbb{S} : \nabla_x \vec{u} - \frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} + \frac{1}{\sigma} \left| \mathbf{curl}_x \vec{H} \right|^2 \right)$ is the (nonnegative) matter entropy production by virtue of the constitutive laws (1.15) and (1.16). The second term on the right-hand side of (2.19) is due to radiative entropy rate which does not have a definite sign since it corresponds to radiative heating/cooling.

For the smooth fields we can get an evolution equation for the sum of the density of the kinetic energy $\frac{1}{2} \varrho |\vec{u}|^2$ and of the magnetic energy $\mathcal{M}(\vec{H})$ subtracting (2.17) from (1.12). However, generally for weak solutions we cannot exclude that the entropy dissipation rate due to heat exchange, internal viscous friction and Foucault eddy currents is larger than r in compliance with the Second law of Thermodynamics and equation (2.18) has to be replaced in the weak formulation by the inequality

$$\int_0^T \int_\Omega \left(\varrho s \partial_t \varphi + \varrho s \vec{u} \cdot \nabla_x \varphi + \frac{\vec{q}}{\vartheta} \cdot \nabla_x \varphi \right) \mathrm{d}x \, \mathrm{d}t \tag{2.20}$$

m17

$$\leq - \int_\Omega (\varrho s)_0 \varphi(0, \cdot) \, \mathrm{d}x + \int_0^T \int_\Omega \frac{1}{\vartheta} \left(\frac{\vec{q} \cdot \nabla_x \vartheta}{\vartheta} - \mathbb{S} : \nabla_x \vec{u} - \frac{1}{\sigma} \left| \mathbf{curl}_x \vec{H} \right|^2 - \vec{u} \cdot \vec{S}_F + S_E \right) \varphi \, dx \, \mathrm{d}t,$$

for any $\varphi \in C_c^\infty([0,T) \times \overline{\Omega})$, $\varphi \geq 0$.

Since replacing *equation* (1.12) by *inequality* (2.20) would certainly result in a formally underdetermined problem, system (2.14), (2.15), (2.20) must be supplemented with the total energy balance

$$\frac{d}{dt} \int_\Omega \left(\frac{1}{2} \varrho(t,x) \, |\vec{u}(t,x)|^2 + \varrho e(\varrho(t,x), \vartheta(t,x)) + c^{-1} \int_{\mathcal{S}^2} \int_0^\infty I(t,x,\vec{\omega},\nu) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu \right.$$

$$\left. + \mathcal{M}(|\vec{H}|) - \frac{1}{2} \varrho(t,x) \psi(t,x) \right) dx = \int_\Omega \int_{\mathcal{S}^2} \int_0^\infty \mathrm{div}_x \left(\vec{\omega} I(t,x,\vec{\omega},\nu) \right) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu \, \mathrm{d}x, \tag{2.21}$$

m18

which can be rephrased as follows

$$\int_\Omega \left(\frac{1}{2} \varrho |\vec{u}|^2 + \varrho e(\varrho, \vartheta) + E_R + \mathcal{M}(|\vec{H}|) - \frac{1}{8\pi G} \, |\nabla \psi|^2 \right) (\tau, \cdot) \, \mathrm{d}x \tag{2.22}$$

m19

$$+ \int_0^\tau \iint_{\substack{\partial\Omega \times \mathcal{S}^2 \\ \vec{\omega}\cdot\vec{n} \geq 0}} \int_0^\infty I(t,x,\vec{\omega},\nu) \, \vec{\omega} \cdot \vec{n} \, \mathrm{d}\nu \, \mathrm{d}\vec{\omega} \, \mathrm{d}S_x \, \mathrm{d}t$$

$$= \int_\Omega \left(\frac{1}{2\varrho_0} |(\varrho \vec{u})_0|^2 + (\varrho e)_0 + E_{R,0} + \mathcal{M}(|\vec{H}|)(0,\cdot) - \frac{1}{8\pi G} \, |\nabla \psi_0|^2 \right) \, \mathrm{d}x,$$

$$\text{for a. a. } \tau \in (0,T),$$

where E_R is given by (1.1), and

$$E_{R,0} = c^{-1} \int_{\mathcal{S}^2} \int_0^\infty I_0(\cdot, \vec{\omega}, \nu) \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu.$$

The transport equation (1.13) can be extended to the whole physical space $\mathbb{R}^3$ provided we set

$$\sigma_a(x,\nu,\vartheta) = 1_\Omega \sigma_a(\nu,\vartheta), \;\; \sigma_s(x,\nu,\vartheta) = 1_\Omega \sigma_s(\nu,\vartheta),$$

and take the initial distribution $I_0(x,\vec{\omega},\nu)$ to be zero for $x \in \mathbb{R}^3 \setminus \Omega$. Accordingly, for any fixed $\vec{\omega} \in \mathcal{S}^2$, equation (1.13) can be considered a linear transport equation defined in $(0,T) \times \mathbb{R}^3$, with a right-hand side $c\tilde{S}$. With the above mentioned convention, extending $\vec{u}$ to be zero outside Ω, we may therefore assume that both ϱ and I are defined on the whole physical space $\mathbb{R}^3$. Then the gravitational potential ψ is defined on the whole $\mathbb{R}^3$ by the Newtonian potential.

Finally, we need to state the weak formulation of the reduced Maxwell system (1.2), (1.6)

$$\int_0^T \int_\Omega \left(\vec{B} \cdot \partial_t \vec{\varphi} - \left(\vec{B} \times \vec{u} + \frac{1}{\sigma} \mathbf{curl}_x \vec{H} \right) \cdot \mathbf{curl}_x \vec{\varphi} \right) \, \mathrm{d}x \, \mathrm{d}t = \tag{2.23}$$

m20

$$- \int_\Omega \vec{B}_0 \cdot \vec{\varphi}(0,\cdot) \, \mathrm{d}x,$$

holds for any $\vec{\varphi} \in \mathcal{D}([0,T) \times \mathbb{R}^3, \mathbb{R}^3)$.

In accordance with the boundary conditions (1.24), (1.25) one also take

$$\vec{B}_0 \in L^2(\Omega), \mathrm{div}_x \vec{B}_0 = 0 \text{ in } D'(\Omega), \vec{B}_0 \cdot n|_{\partial\Omega)} = 0. \tag{2.24}$$ m20a

Definition 2.1 *We say that $(\varrho, \vec{u}, \vartheta, \vec{B}, I)$ is a weak solution of problem (1.10) – (1.26) iff*

$$\varrho_0 \geq 0 \ \ a.e. \ \ in \ \Omega, \ \ \varrho_0 \in L^{\frac{5}{3}}(\Omega), \ \ \frac{(\varrho\vec{u})_0}{\sqrt{\rho_0}} \in L^2(\Omega, \mathbb{R}^3),$$
$$(\varrho e(\varrho, \vartheta))_0 = \varrho_0 e(\varrho_0, \vartheta_0) \in L^1(\Omega), \ \ \vartheta_0 > 0 \ \ a.e. \ \ in \ \Omega, \ \ \ \vartheta_0 \in L^\infty(\Omega),$$
$$\psi_0 = G(-\Delta)^{-1}\chi_\Omega \varrho_0, \ \ \vec{B}_0 \in L^2(\Omega, \mathbb{R}^3), \ \ \mathrm{div}_x \vec{B}_0 = 0 \quad \text{in} \quad \mathcal{D}'(\Omega), \ \ \vec{B}_0 \cdot \vec{n}\Big|_{\partial\Omega} = 0,$$
$$I_0 \geq 0 \ \ a.e. \ \ in \ \Omega \times \mathcal{S}^2 \times (0,\infty),$$
$$I_0 \in L^1(\mathbb{R}^3 \times \mathcal{S}^2 \times (0,\infty)) \cap L^\infty(\mathbb{R}^3 \times \mathcal{S}^2 \times (0,\infty)),$$
$$(\varrho s(\varrho, \vartheta))_0 = \varrho_0 s(\varrho_0, \vartheta_0) \in L^1_{\mathrm{loc}}(\Omega),$$

$$\varrho \geq 0, \ \ \vartheta > 0 \ for \ a.a. \ (t,x) \times \Omega, \ \ I \geq 0 \ \ a.a. \ \ in \ (0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty),$$
$$\varrho \in L^\infty(0,T; L^{5/3}(\Omega)), \ \ \vartheta \in L^\infty(0,T; L^4(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)),$$
$$\vec{u} \in L^2(0,T; W_0^{1,2}(\Omega; \mathbb{R}^3)), \ \ \vec{B} \in L^\infty(0,T; L^2(\Omega)) \cap L^2(0,T; W^{1,2}(\Omega)),$$
$$\mathcal{M}\left(\left|\vec{H}\right|\right) \in L^\infty(0,T; L^1(\Omega)),$$
$$\mathrm{div}_x \vec{B}(t) = 0, \ \ \vec{B}(t) \cdot n|_{\partial\Omega} = 0, \ \ t \in (0,T),$$
$$I \in L^\infty((0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty)), \ \ I \in L^\infty(0,T; L^1(\Omega \times \mathcal{S}^2 \times (0,\infty)),$$

and (ϱ, $\vec{u}$, ϑ, $\vec{B}$, I) satisfy the integral identities (2.14), (2.15), (2.20), (2.22), and (2.23) together with the transport equation (1.13) and boundary conditions (1.23) – (1.25) at least in the sense of traces.

The main result of the present paper can be stated as follows. Weak limits are generally denoted with an overbar.

Tm1 **Theorem 2.1** *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain with boundary $\partial\Omega$ of class $C^{2+\alpha}$ for an $\alpha > 0$. Assume that the thermodynamic functions p, e, s satisfy hypotheses (2.1) – (2.6), and that the transport coefficients η, κ_M, λ, σ, μ, σ_a, and σ_s comply with (2.7) – (2.13).*

Let $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, \vec{B}_\varepsilon, I_\varepsilon\}_{\varepsilon>0}$ be a family of weak solutions to problem (1.10) – (1.26) in the sense of Definition 2.1 such that

$$\varrho_\varepsilon(0,\cdot) \equiv \varrho_{\varepsilon,0} \to \varrho_0 \ \ in \ L^{5/3}(\Omega), \tag{2.25}$$ m21

$$\int_\Omega \left(\frac{1}{2}\varrho_\varepsilon |\vec{u}_\varepsilon|^2 + \varrho_\varepsilon e(\varrho_\varepsilon, \vartheta_\varepsilon) + E_{R,\varepsilon} + \mathcal{M}\left(\left|\vec{H}\right|\right) - \frac{1}{2}\varrho\psi \right)(0,\cdot) \ \mathrm{d}x \tag{2.26}$$ m22

$$\equiv \int_\Omega \left(\frac{1}{2\varrho_{0,\varepsilon}} |(\varrho\vec{u})_{0,\varepsilon}|^2 + (\varrho e)_{0,\varepsilon} + E_{R,0,\varepsilon} + \mathcal{M}\left(\left|\vec{H}\right|\right)_{0,\varepsilon} - \frac{1}{2}(\varrho\psi)_{0,\varepsilon} \right) \ {\rm d}x \leq E_0,$$

$$\int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon)(0,\cdot) \ {\rm d}x \equiv \int_\Omega (\varrho s)_{0,\varepsilon} \ {\rm d}x \geq S_0,$$

and

$$0 \leq I_\varepsilon(0,\cdot) \equiv I_{0,\varepsilon}(\cdot) \leq I_0, \ |I_{0,\varepsilon}(x,\vec{\omega},\nu)| \leq h(\nu) \ \text{for a certain } h \in L^1(0,\infty). \tag{2.27}$$

m23

Then

$$\varrho_\varepsilon \to \varrho \ \text{in } C_{\rm weak}([0,T]; L^{5/3}(\Omega)),$$

$$\vec{u}_\varepsilon \to \vec{u} \ \text{weakly in } L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)),$$

$$\vartheta_\varepsilon \to \vartheta \ \text{weakly in } L^2(0,T; W^{1,2}(\Omega)),$$

$$\vec{B}_\varepsilon \to \vec{B} \ \text{weakly in } L^2(0,T; W^{1,2}(\Omega)) \cap L^\infty(0,T; L^2(\Omega)),$$

and

$$I_\varepsilon \to I \ \text{weakly-(*) in } L^\infty((0,T) \times \Omega \times \mathcal{S}^2 \times (0,\infty)),$$

at least for suitable subsequences, where $\{\varrho, \vec{u}, \vartheta, \vec{B}, I\}$ *is a weak solution of problem (1.10) – (1.26).*

Note that *strong convergence* is required only for the initial distribution of the densities $\{\varrho_{\varepsilon,0}\}_{\varepsilon>0}$ (see (2.25)), while the remaining initial data are only bounded in suitable norms. This is due to the fact that the evolution of the density is governed by continuity equation (1.10) having hyperbolic character without any smoothing effect incorporated.

3 Proof of Theorem 2.1

Following [11], the proof consists of three steps. We establish uniform estimates on the family $\{\varrho_\varepsilon, \vec{u}_\varepsilon, \vartheta_\varepsilon, \vec{B}_\varepsilon, I_\varepsilon\}_{\varepsilon>0}$ independent of $\varepsilon \to 0+$ first. Secondly, we observe that the extra forcing terms in (2.15), (2.20) due to radiation are bounded in suitable Lebesgue norms. In particular, the analysis of the macroscopic variables ϱ_ε, $\vec{u}_\varepsilon$, ϑ_ε is essentially the same as in the case of the Navier-Stokes-Fourier system presented in [20]. Consequently as in [11] the proof of Theorem 2.1 reduces to the study of the transport equation (1.13) governing the time evolution of the radiation intensity I_ε and Maxwell's system (1.2) and (1.6). In the last step we introduce an approximation scheme similar to that used in [20, Chapter 3] and sketch the main ideas of a complete proof of the existence of global-in-time weak solutions to problem (1.10) – (1.26).

3.1 Uniform bounds

a

Uniform (*a priori*) bounds follow immediately from the total energy balance and the entropy production equation.

The total energy balance (2.21), combined with hypotheses of Theorem 2.1 give

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\sqrt{\varrho_\varepsilon}\vec{u}_\varepsilon\|_{L^2(\Omega;\mathbb{R}^3)} \leq c, \tag{3.1}$$ a1

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\varrho_\varepsilon e(\varrho_\varepsilon,\vartheta_\varepsilon)\|_{L^1(\Omega)} \leq c, \tag{3.2}$$ a2

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|E_{R,\varepsilon}\|_{L^1(\Omega)} \leq c, \tag{3.3}$$ a3

$$\operatorname*{ess\ sup}_{t\in(0,T)} \left\|\mathcal{M}\left(\left|\vec{H}_\varepsilon\right|\right)\right\|_{L^1(\Omega)} \leq c, \tag{3.4}$$ a3a

and

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\vec{B}_\varepsilon\|_{L^2(\Omega)} \leq c. \tag{3.5}$$ a40

Thus, as the internal energy contains the radiation component proportional to ϑ^4 (cf. (2.1)), we deduce from (3.2) that

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\vartheta_\varepsilon\|_{L^4(\Omega)} \leq c, \tag{3.6}$$ a4

and, by virtue of hypotheses (2.1) – (2.4),

$$\operatorname*{ess\ sup}_{t\in(0,T)} \|\varrho_\varepsilon\|_{L^{5/3}(\Omega)} \leq c. \tag{3.7}$$ a5

This crucial uniform estimate we get "for free" by the proportionality of pressure and internal energy density of the fluid part of the internal energy and by the assumption that we deal with a component behaving like a Fermi gas (cf. [20, Chapter 2])

$$\varrho e \geq a\vartheta^4 + \frac{3}{2}p_\infty \varrho^{\frac{5}{3}}. \tag{3.8}$$ a50

Since the quantity I_ε is non-negative, we have from (1.13)

$$\frac{1}{c}\partial_t I_\varepsilon + \vec{\omega}\cdot\nabla_x I_\varepsilon \leq \sigma_a(\nu,\vartheta_\varepsilon)\mathfrak{B}(\nu,\vartheta_\varepsilon) + \sigma_s(\nu,\vartheta_\varepsilon)\frac{1}{4\pi}\int_{\mathcal{S}^2} I_\varepsilon(\cdot,\vec{\omega},\cdot)\ \mathrm{d}\vec{\omega} \leq \tag{3.9}$$ a6

$$c_5 + c_4\int_{\mathcal{S}^2} I_\varepsilon(\cdot,\vec{\omega},\cdot)\ \mathrm{d}\vec{\omega},$$

as the coefficients σ_s, σ_a are non-negative and bounded by (2.11). Thus we deduce a uniform bound

$$0 \leq I_\varepsilon(t,x,\nu,\vec{\omega}) \leq c(T)(1 + \sup_{x\in\Omega,\ \nu\geq 0,\vec{\omega}\in\mathcal{S}^2} I_{0,\varepsilon}) \leq c(T)(1+I_0) \text{ for any } t\in[0,T] \tag{3.10}$$ a7

by (2.27) with a certain non-negative function $c(t)$.

Cut-off hypothesis (2.12) together with (3.10) yield

$$\|S_{E,\varepsilon}\|_{L^\infty((0,T)\times\Omega)} + \|\vec{S}_{F,\varepsilon}\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \leq c. \qquad (3.11) \quad \boxed{\texttt{a16}}$$

Moreover, due to (2.13) it holds

$$\left\|\frac{S_{E,\varepsilon}}{\vartheta_\varepsilon}\right\|_{L^\infty((0,T)\times\Omega)} + \left\|\frac{\vec{S}_{F,\varepsilon}}{\vartheta_\varepsilon}\right\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \leq c. \qquad (3.12) \quad \boxed{\texttt{a17}}$$

As the viscosity coefficients satisfy (2.7) – (2.8), we get

$$\|\varrho_\varepsilon\vec{u}_\varepsilon\|^2_{L^2((0,T),L^1(\Omega))} + \int_0^T\int_\Omega \frac{1}{\vartheta_\varepsilon}\mathbb{S}_\varepsilon : \nabla_x\vec{u}_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \geq$$

$$c_1\left\|\nabla_x\vec{u}_\varepsilon + \nabla_x^T\vec{u}_\varepsilon - \frac{2}{3}\mathrm{div}_x\vec{u}_\varepsilon\mathbb{I}\right\|^2_{L^2((0,T)\times\Omega;\mathbb{R}^{3\times3})} + \|\varrho_\varepsilon\vec{u}_\varepsilon\|^2_{L^2((0,T),L^1(\Omega))}$$

$$\geq c_7\|\vec{u}_\varepsilon\|^2_{L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))},$$

where we have used a variant of the Korn-Poincaré inequality (see [20, Chapter 2, Proposition 2.1]) and c_7 depends only on the uniform bounds of ϱ and c_1.

On the other hand, in accordance with (3.12) by Hölder's inequality

$$\left|\int_0^T\int_\Omega \frac{1}{\vartheta_\varepsilon}\vec{u}_\varepsilon\cdot\vec{S}_{F,\varepsilon}\,\mathrm{d}x\,\mathrm{d}t\right| \leq c\|\vec{u}_\varepsilon\|_{L^1((0,T)\times\Omega;\mathbb{R}^3)}.$$

Then the entropy inequality (2.20) (with positive production terms, the rest estimated by (3.12)) yields the uniform bounds *for* Ω *bounded*

$$\|\vec{u}_\varepsilon\|_{L^2(0,T;W_0^{1,2}(\Omega;\mathbb{R}^3))} \leq c, \qquad (3.13) \quad \boxed{\texttt{a18}}$$

$$\|\nabla_x\vartheta_\varepsilon\|_{L^2((0,T)\times\Omega)} \leq c, \qquad (3.14) \quad \boxed{\texttt{a19}}$$

$$\left\|\frac{1}{\vartheta_\varepsilon\sigma_\varepsilon}\left|\mathbf{curl}_x\vec{H}_\varepsilon\right|^2\right\|_{L^1((0,T)\times\Omega)} \leq c, \qquad (3.15) \quad \boxed{\texttt{a200}}$$

upon testing with approximations of the test function $\varphi = \chi_\Omega(x)\chi_{[0,T]}(t)$ and passing to the limit. Moreover we have

$$\left\|\mathbf{curl}_x\vec{H}_\varepsilon\right\|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \leq c. \qquad (3.16) \quad \boxed{\texttt{a230}}$$

Finally we summarize the lemmas from [18, Chapter 7] concerning the (linearized) equation for the evolution of magnetic field $\vec{B}$ (2.23):

- The boundary condition expressing continuity of the tangential component of the electric field $(1.25)_1$ is automatically satisfied by the weak formulation (2.23).

- The boundary condition expressing continuity of the normal component of the magnetic field $(1.25)_2$ is satisfied once we choose $\vec{B}_{\varepsilon,0} \cdot \vec{n} = 0$.
- The same is true for the condition of solenoidality (1.3) once we guarantee $\mathrm{div}_x \vec{B}_{\varepsilon,0} = 0$ in $\mathcal{D}'(\Omega)$.
- We have got a Hodge-type estimate

$$\left\| \vec{B}_\varepsilon \right\|_{W^{1,2}(\Omega,\mathbb{R}^3)} \leq c \left(\left\| \mathbf{curl}_x \vec{B}_\varepsilon \right\|_{L^2(\Omega,\mathbb{R}^3)} + \left\| \mathrm{div}_x \vec{B}_\varepsilon \right\|_{L^2(\Omega)} + \left\| \vec{B}_\varepsilon \cdot \vec{n} \right\|_{W^{\frac{1}{2},2}(\partial\Omega)} \right) \leq c \left\| \mathbf{curl}_x \vec{B}_\varepsilon \right\|_{L^2(\Omega,\mathbb{R}^3)} \leq c,$$

meaning that we have got a uniform estimate of the magnetic induction

$$\left\| \vec{B}_\varepsilon \right\|_{L^2(0,T;W^{1,2}(\Omega,\mathbb{R}^3))} \leq c, \tag{3.17}$$ a250

and also for the magnetic field

$$\left\| \vec{H}_\varepsilon \right\|_{L^2(0,T;W^{1,2}(\Omega,\mathbb{R}^3))} \leq c. \tag{3.18}$$ a250a

To estimate the pressure functions $p(\varrho_\varepsilon, \vartheta_\varepsilon)$ globally we start with the observation that estimates (3.7), (3.13) imply that the sequences $\{\varrho_\varepsilon \vec{u}_\varepsilon\}_{\varepsilon>0}$, $\{\varrho_\varepsilon \vec{u}_\varepsilon \otimes \vec{u}_\varepsilon\}_{\varepsilon>0}$ are bounded in $L^p((0,T) \times \Omega)$ for a certain $p > 1$, namely $p = \frac{45}{43}$. Similarly, combining (3.6), (3.13), (3.14), we get

$$\{\mathbb{S}_\varepsilon\}_{\varepsilon>0} \text{ bounded in } L^p((0,T) \times \Omega; \mathbb{R}^{3\times3}) \text{ for a certain } p > 1, \text{namely } p = \frac{34}{23}.$$

Moreover, $\{\varrho_\varepsilon \nabla_x \psi_\varepsilon\}_{\varepsilon>0}$, $\{\vec{J}_\varepsilon \times \vec{B}_\varepsilon\}_{\varepsilon>0}$ are bounded in $L^p((0,T)\times\Omega)$ for a certain $p > 1$. Now, repeating the arguments of [23], we observe that the quantities

$$\varphi(t,x) = \tilde{\psi}(t) \{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha, \ \tilde{\psi} \in \mathcal{D}(0,T) \text{ for sufficiently small parameters } \alpha, \omega > 0$$

may be used as test functions in the momentum equation (2.15), where $\mathcal{B}[v]$ is a suitable branch of solutions to the boundary value problem

$$\mathrm{div}_x \Big(\mathcal{B}[v] \Big) = v - \frac{1}{|\Omega|} \int_\Omega v \, \mathrm{d}x, \ \mathcal{B}[v]|_{\partial\Omega} = 0. \tag{3.19}$$ p27

Here $\mathcal{B}$ is the Bogovskii operator and α denotes a convolution parameter in time since we need to test with a continuously differentiable function.

Next we get estimates of $\{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha$ in $L^q(0,T; W^{1,p}(\Omega, \mathbb{R}^3))$ for all $q \in [1,\infty]$ and $p \in (1,\infty)$ by $\{\varrho_\varepsilon^\omega\}^\alpha$ in $L^q(0,T; L^p(\Omega,\mathbb{R}^3))$ and of $\{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha$ setting the renormalization function in (2.14) $b(\varrho_\varepsilon) := \{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha$. This leads to an estimate of the term $\int_0^T \tilde{\psi}(t) \int_\Omega p(\varrho_\varepsilon, \vartheta_\varepsilon) \{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha \, \mathrm{d}x \, \mathrm{d}t$ by eight integrals resulting from the momentum equation (2.15). We omit most details which can the reader find in [20, Chapter 2]. Let us just note that the "worst" term arises from the time derivative which we a priori do not control in a Lebesgue space and therefore we partially integrate in time and estimate it as follows

$$
\begin{gathered}
\left|-\int_0^T \tilde{\psi}(t)\int_\Omega \varrho_\varepsilon \vec{u}_\varepsilon \cdot \partial_t\{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha \ \mathrm{d}x \ \mathrm{d}t\right| \leq \left\|\tilde{\psi}\right\|_{C(0,T)} \|\varrho_\varepsilon \vec{u}_\varepsilon\|_{L^\infty(0,T;L^{\frac{5}{4}}(\Omega,\mathbb{R}^3))} \times \\
\|\partial_t\{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha\|_{L^1(0,T;L^5(\Omega,\mathbb{R}^3))} \leq c \left\|\tilde{\psi}\right\|_{C(0,T)} \|\varrho_\varepsilon \vec{u}_\varepsilon\|_{L^\infty(0,T;L^{\frac{5}{4}}(\Omega,\mathbb{R}^3))} \times \\
\Big\{\|\{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha \vec{u}_\varepsilon\|_{L^1(0,T;L^5(\Omega,\mathbb{R}^3))} + \\
\left\|\left[\varrho_\varepsilon\left(\{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha\right)' - \{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha\right] \mathrm{div}_x \vec{u}_\varepsilon\right\|_{L^1(0,T;L^{\frac{15}{8}}(\Omega,\mathbb{R}^3))}\Big\} \leq c \quad \text{for } \ \omega \leq \frac{11}{18},
\end{gathered} \tag{3.20}
$$

p280

by (3.7), (3.1) and (3.13) and repeated use of Hölder's inequality and Sobolev embedding. "New terms" in comparison to [20, Chapter 2] are estimated as follows

$$
\begin{gathered}
\left|-\int_0^T \tilde{\psi}(t)\int_\Omega \varrho_\varepsilon \nabla_x \psi_\varepsilon \cdot \{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha \ \mathrm{d}x \ \mathrm{d}t\right| \leq c\left\|\tilde{\psi}\right\|_{C(0,T)} \|\nabla\psi_\varepsilon\|_{L^\infty(0,T;L^{\frac{15}{4}}(\Omega))} \times \\
\|\varrho_\varepsilon^\omega\|_{L^1(0,T;L^{\frac{15}{7}}(\Omega))} \|\varrho_\varepsilon\|_{L^\infty(0,T;L^{\frac{5}{3}}(\Omega))} \leq c \quad \text{for } \ \omega \leq \frac{7}{9},
\end{gathered} \tag{3.21}
$$

p290

$$
\begin{gathered}
\left|\int_0^T \tilde{\psi}(t)\int_\Omega \vec{S}_{F,\varepsilon} \cdot \{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha \ \mathrm{d}x \ \mathrm{d}t\right| \leq c\left\|\tilde{\psi}\right\|_{C(0,T)} \left\|\vec{S}_{F,\varepsilon}\right\|_{L^\infty((0,T)\times\Omega;\mathbb{R}^3)} \times \\
\|\varrho_\varepsilon^\omega\|_{L^1(0,T;L^{1+\delta}(\Omega))} \leq c \quad \text{for small } \delta > 0 \text{ and } \omega < \frac{5}{3}
\end{gathered} \tag{3.22}
$$

p300

and

$$
\begin{gathered}
\left|-\int_0^T \tilde{\psi}(t)\int_\Omega \sigma(\varrho_\varepsilon,\vartheta_\varepsilon,\vec{B}_\varepsilon)\left(\vec{E}_\varepsilon + \vec{u}_\varepsilon \times \vec{B}_\varepsilon\right) \times \vec{B}_\varepsilon \cdot \{\mathcal{B}[\varrho_\varepsilon^\omega]\}^\alpha \ \mathrm{d}x \ dt\right| \leq c\times \\
\left\|\tilde{\psi}\right\|_{C(0,T)} \left\|\mathbf{curl}_x \vec{H}_\varepsilon\right\|_{L^2((0,T)\times\Omega;\mathbb{R}^3)} \|\varrho_\varepsilon^\omega\|_{L^{\frac{15}{8}}(0,T;L^{\frac{15}{8}}(\Omega))} \left\|\vec{B}_\varepsilon\right\|_{L^{\frac{10}{3}}((0,T)\times\Omega;\mathbb{R}^3)} \leq c \\
\text{for } \ \omega \leq \frac{8}{9}.
\end{gathered} \tag{3.23}
$$

p310

The resulting (uniform in ε) estimate reads

$$
\int_0^T \int_\Omega p(\varrho_\varepsilon,\vartheta_\varepsilon)\varrho_\varepsilon^\omega \ \mathrm{d}x \ \mathrm{d}t < c, \tag{3.24}
$$

p28

where $\omega \leq \min\left\{\frac{5}{3}, \frac{8}{9}, \frac{7}{9}, \frac{55}{102}, \frac{11}{18}, \frac{2}{27}\right\} = \frac{2}{27}$, in particular, we can arrive at the following regularity by upper bounds for pressure in the non-degenerate region (for small Zs in the sense of (2.2)) and in the degenerate region, respectively and homogeneous regularity of temperature by (3.6) $\|\vartheta_\varepsilon\|_{L^{\frac{17}{3}}((0,T)\times\Omega)} \le c$

$$\{p(\varrho_\varepsilon, \vartheta_\varepsilon)\}_{\varepsilon>0} \text{ is bounded in } L^p((0,T)\times\Omega) \text{ for a } p>1, \text{namely } p = \frac{47}{45}. \quad (3.25)$$ p29

3.2 Weak sequential stability

w

We sketch the principal part of the proofs and focus mainly on the issues related to weak sequential stability of quantities related to radiation and magnetic field that require new ideas. In particular, we examine in details the extra terms in the entropy balance equation (2.20).

3.2.1 Weak sequential stability of macroscopic thermodynamic quantities

After the uniform estimates on the radiation forcing terms $S_{E,\varepsilon}$ and $\vec{S}_{F,\varepsilon}$ in (3.11), strong (pointwise) convergence of the macroscopic thermodynamic quantities $\{\varrho_\varepsilon\}_{\varepsilon>0}$, $\{\vartheta_\varepsilon\}_{\varepsilon>0}$ can be shown exactly as in [10].

To begin, using the uniform bounds established in Section 3.1 we observe that

$$\varrho_\varepsilon \to \varrho \text{ in } C_{\text{weak}}([0,T]; L^{\frac{5}{3}}(\Omega)), \quad (3.26)$$ w1

$$\vartheta_\varepsilon \to \vartheta \text{ weakly in } L^2(0,T; W^{1,2}(\Omega)), \quad (3.27)$$ w2

and

$$\log(\vartheta_\varepsilon) \to \overline{\log(\vartheta)} \text{ weakly in } L^2((0,T)\times\Omega), \quad (3.28)$$ w3

and

$$\vec{u}_\varepsilon \to \vec{u} \text{ weakly in } L^2(0,T; W_0^{1,2}(\Omega;\mathbb{R}^3)), \quad (3.29)$$ w4

possibly passing to suitable subsequences. Moreover, since the (weak) time derivative $\partial_t(\varrho_\varepsilon \vec{u}_\varepsilon)$ of momenta can be expressed by means of momentum balance (2.15) (Lorentz force density bounded as in (3.23)), we have got

$$\varrho_\varepsilon \vec{u}_\varepsilon \to \varrho\vec{u} \text{ in } C_{\text{weak}}([0,T]; L^{\frac{5}{4}}(\Omega;\mathbb{R}^3)). \quad (3.30)$$ w4a

Since our system contains quantities depending on ϱ and ϑ in a general nonlinear way, pointwise (resp. a. e.) convergence of $\{\varrho_\varepsilon\}_{\varepsilon>0}$, $\{\vartheta_\varepsilon\}_{\varepsilon>0}$ must be established in order to perform the limit $\varepsilon \to 0+$. This step is apparently easier to carry out for the temperature because of the uniform bounds available for $\nabla_x \vartheta_\varepsilon$.

3.2.2 Pointwise convergence of temperature

A. e. convergence of the sequence $\{\vartheta_\varepsilon\}_{\varepsilon>0}$ can be established essentially by the monotonicity arguments. The main problem are possible uncontrollable time oscillations in hypothetical zones of vacuum, here eliminated by the presence of radiation component in the entropy inequality. More specifically, our goal is to show that

$$\int_0^T \int_\Omega \Big(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) - \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta) \Big) \Big(\vartheta_\varepsilon - \vartheta \Big) \ \mathrm{d}x \ \mathrm{d}t \to 0 \text{ as } \varepsilon \to 0+, \tag{3.31}$$

np1

which, in accordance with hypothesis (2.6), implies the desired conclusion

$$\vartheta_\varepsilon \to \vartheta \text{ in } L^4((0,T)\times\Omega), \text{ in particular, } \vartheta_{\varepsilon_k} \to \vartheta \text{ a. e. in } (0,T)\times\Omega. \tag{3.32}$$

np2

In order to see (3.31), we first observe that

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon) \Big(\vartheta_\varepsilon - \vartheta \Big) \ \mathrm{d}x \ \mathrm{d}t \to 0 \text{ as } \varepsilon \to 0+ .$$

Indeed this follows by means of the Aubin-Lions compactness lemma as

$$\vartheta_\varepsilon - \vartheta \to 0 \text{ weakly in } L^2(0,T; W^{1,2}(\Omega)),$$

and the (weak) time derivative $\partial_t(\varrho_\varepsilon s(\varrho_\varepsilon, \vartheta_\varepsilon))$ can be expressed by means of the entropy inequality (2.20).

Consequently, it remains to show that

$$\int_0^T \int_\Omega \varrho_\varepsilon s(\varrho_\varepsilon, \vartheta) \Big(\vartheta_\varepsilon - \vartheta \Big) \ \mathrm{d}x \ \mathrm{d}t \to 0 \text{ as } \varepsilon \to 0+ . \tag{3.33}$$

np3

To see (3.33), we combine the bounds imposed on $\partial_t b(\varrho_\varepsilon)$ by the renormalized equation (2.14), with the estimates on the temperature gradient (3.14), to deduce that

$$\nu_{t,x}[\varrho_\varepsilon, \vartheta_\varepsilon] = \nu_{t,x}[\varrho_\varepsilon] \otimes \nu_{t,x}[\vartheta_\varepsilon] \ \text{ a. e. in } \ (0,T)\times\Omega, \tag{3.34}$$

np4

where the symbol $\nu[\varrho_\varepsilon, \vartheta_\varepsilon]$ denotes a Young measure associated to the family $\{\varrho_\varepsilon, \vartheta_\varepsilon\}_{\varepsilon>0}$, while $\nu[\varrho_\varepsilon]$, $\nu[\vartheta_\varepsilon]$ stand for Young measures generated by $\{\varrho_\varepsilon\}_{\varepsilon>0}$, $\{\vartheta_\varepsilon\}_{\varepsilon>0}$, respectively. In order to conclude, we use the following result frequently called *Fundamental theorem of the theory of Young measures* (see Pedregal [36, Chapter 6, Theorem 6.2]):

TYM **Theorem 3.1** *Let $\{\vec{v}_n\}_{n=1}^\infty$, $\vec{v}_n : Q \subset \mathbb{R}^N \to \mathbb{R}^M$ be a sequence of functions bounded in $L^1(Q;\mathbb{R}^M)$, where Q is a domain in $\mathbb{R}^N$.*

Then there exist a subsequence (not relabeled) and a parametrized family $\{\nu_y\}_{y\in Q}$ of probability measures on $\mathbb{R}^M$ depending measurably on $y \in Q$ with the following property:

For any Caratheodory function $\Phi = \Phi(y,z)$, $y \in Q$, $z \in \mathbb{R}^M$ such that

$$\Phi(\cdot, \vec{v}_n) \to \overline{\Phi} \text{ weakly in } L^1(Q),$$

we have

$$\overline{\Phi}(y) = \int_{R^M} \Phi(y,z) \ \mathrm{d}\nu_y(z) \text{ for a.a. } y \in Q.$$

In virtue of Theorem 3.1, relation (3.34) implies (3.33). We have proved the almost everywhere convergence of the temperature claimed in (3.32). Note that this step leans heavily on the presence of the *radiative entropy flux*.

3.2.3 Pointwise convergence of density

Although the pointwise convergence of the family of densities $\{\varrho_\varepsilon\}_{\varepsilon>0}$ represents one of the most delicate questions of the existence theory for the compressible Navier-Stokes system, this step is nowadays well understood. The idea is to use the quantities

$$\varphi(t,x) = \psi(t)\phi(x)\nabla_x\Delta^{-1}[\chi_\Omega T_k(\varrho_\varepsilon)]$$

as test functions in the weak formulation of momentum equation (2.15). Similarly, we can let $\varepsilon \to 0+$ in (2.15) and test the resulting expression on

$$\varphi(t,x) = \psi(t)\phi(x)\nabla_x\Delta^{-1}[\chi_\Omega \overline{T_k(\varrho_\varepsilon)}],$$

where $\psi \in C_c^\infty(0,T)$, $\phi \in C_c^\infty(\Omega)$, and T_k is a cut-off function,

$$T_k(z) = \min\{z, k\}.$$

In the limit for $\varepsilon \to 0+$, this procedure yields the celebrated relation for the *effective viscous pressure* discovered by Lions [34], relating last two expressions whose weak limits have not been identified yet, which reads in the present setting as

$$\int_0^T \int_\Omega \psi\phi \left(\overline{p(\varrho,\vartheta)T_k(\varrho)} - \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)} \right) \ \mathrm{d}x\ \mathrm{d}t \tag{3.35}$$

evp

$$= \int_0^T \int_\Omega \psi\phi \left(\overline{\mathbb{S} : \mathcal{R}[\chi_\Omega T_k(\varrho)]} - \mathbb{S} : \mathcal{R}[\chi_\Omega \overline{T_k(\varrho)}] \right) \ \mathrm{d}x\ \mathrm{d}t,$$

where the ovebars denote weak limits of the composed quantities (in the appropriate Lebesgue spaces $L^p((0,T)\times\Omega)$, for $p = \frac{47}{45}$ and $p = \frac{34}{23}$, respectively, thus in $L^1((0,T)\times\Omega)$ as well) and where $\mathcal{R} = \mathcal{R}_{i,j} = \partial_{x_i}\Delta^{-1}\partial_{x_j}$ is a pseudodifferential operator with its symbol

$$R = \frac{\xi \otimes \xi}{|\xi|^2}$$

(see [20, Section 3.6]). Note that the presence of the extra term $\vec{S}_F$ in (2.15) does not present any additional difficulty as

$$\int_0^T \int_\Omega \psi \vec{S}_{F,\varepsilon} \cdot \phi\nabla_x\Delta^{-1}[T_k(\varrho_\varepsilon)] \ \mathrm{d}x\ \mathrm{d}t \to \int_0^T \int_\Omega \psi \vec{S}_F \cdot \phi\nabla_x\Delta^{-1}[\overline{T_k(\varrho)}] \ \mathrm{d}x\ \mathrm{d}t.$$

The same applies to the Lorentz force density as we can apply the Aubin-Lions lemma due to (3.17), $\partial_t\vec{B}_\varepsilon \in L^2(0,T;[W^{1,4}(\Omega)]^*)$ by (3.5), (3.13), (2.8) and (3.15) to obtain

$$\vec{B}_\varepsilon \to \vec{B} \quad \text{in } L^2\left((0,T)\times\Omega,\mathbb{R}^3\right) \tag{3.36}$$

evp0

which together with once again (3.15) leads to the identification of the weak limit in $L^{\frac{5}{4}}\left((0,T)\times\Omega\right)$ of the term involving magnetic induction $\vec{B}$ in (2.15).

The following *commutator lemma* is in the spirit of Coifman and Meyer [5]:

LLL1 **Lemma 3.1** *Let $w \in W^{1,2}(\mathbb{R}^3)$ and $\vec{Z} \in L^p(\mathbb{R}^3;\mathbb{R}^3)$ be given, with $6/5 < p < \infty$.*

Then, for any $1 < s < 6p/(6+p)$,

$$\left\| \mathcal{R}[w\vec{Z}] - w\mathcal{R}[\vec{Z}] \right\|_{W^{\beta,s}(\mathbb{R}^3;\mathbb{R}^3)} \leq c\|w\|_{W^{1,2}(\mathbb{R}^3)} \|\vec{Z}\|_{L^p(\mathbb{R}^3;\mathbb{R}^3)},$$

where $0 < \beta = \frac{3}{s} - \frac{6+p}{6p}$, and $c = c(p)$ are positive constants.

Applying Lemma 3.1 to the expression on the right-hand side of (3.35) and using (weak) compactness in time of $T_k(\varrho_\varepsilon)$ following[1] from the renormalized equation (2.14), we obtain

$$\overline{p(\varrho,\vartheta)T_k(\varrho)} - \left(\frac{4}{3}\lambda\left(\vartheta, \left|\vec{H}\right|\right) + \eta\left(\vartheta, \left|\vec{H}\right|\right)\right) \overline{T_k(\varrho)\mathrm{div}_x\vec{u}} \tag{3.37}$$ evp1

$$= \overline{p(\varrho,\vartheta)}\ \overline{T_k(\varrho)} - \left(\frac{4}{3}\lambda\left(\vartheta, \left|\vec{H}\right|\right) + \eta\left(\vartheta, \left|\vec{H}\right|\right)\right) \overline{T_k(\varrho)}\mathrm{div}_x\vec{u},$$

cf. [20, Section 3.7.4] with help of (3.32) and (3.36).

Now, introducing the functions

$$L_k(\varrho) = \int_1^\varrho \frac{T_k(z)}{z^2}\ \mathrm{d}z,$$

we deduce from renormalized equation (2.14) that

$$\int_0^T \int_\Omega \left(\overline{\varrho L_k(\varrho)}\partial_t\varphi + \overline{\varrho L_k(\varrho)}\vec{u}\cdot\nabla_x\varphi - \overline{T_k(\varrho)\mathrm{div}_x\vec{u}}\varphi \right)\ \mathrm{d}x\ \mathrm{d}t$$

$$= -\int_\Omega \varrho_0 L_k(\varrho_0)\varphi(0,\cdot)\ \mathrm{d}x$$

for any $\varphi \in C_c^\infty([0,T)\times\overline{\Omega})$. It follows from (3.37) that

$$\mathbf{osc}_q[\varrho_\varepsilon \to \varrho]\left((0,T)\times\Omega\right) \tag{3.38}$$ odm

$$\equiv \sup_{k\geq 1}\left(\limsup_{\varepsilon\to 0} \int_0^T \int_\Omega |T_k(\varrho_\varepsilon) - T_k(\varrho)|^q\ \mathrm{d}x\ \mathrm{d}t \right) < \infty,\ \forall q \in \left(2, \frac{8}{3}\right),$$

where **osc** is the oscillation defect measure introduced in [22]. In particular, relation (3.38) implies that the limit functions ϱ, $\vec{u}$ satisfy renormalized equation (2.14) (see [20, Lemma 3.8]); whence

$$\int_\Omega \left(\overline{\varrho L_k(\varrho)} - \varrho L_k(\varrho)\right)(\tau)\ \mathrm{d}x + \int_0^\tau \int_\Omega \left(\overline{T_k(\varrho)\mathrm{div}_x\vec{u}} - \overline{T_k(\varrho)}\mathrm{div}_x\vec{u}\right)\ \mathrm{d}x \tag{3.39}$$ for

$$= \int_0^\tau \int_\Omega \left(T_k(\varrho)\mathrm{div}_x\vec{u} - \overline{T_k(\varrho)}\mathrm{div}_x\vec{u}\right)\ \mathrm{d}x\ \mathrm{d}t \text{ for any } \tau \in [0,T].$$

[1] Naturally the T_k function has to be approximated by differentiable functions first.

Using once more (3.38), we can let $k \to \infty$ in (3.39) to obtain the desired conclusion

$$\overline{\varrho \log(\varrho)} = \varrho \log(\varrho),$$

particularly,

$$\varrho_\varepsilon \to \varrho \text{ in } L^1((0,T) \times \Omega), \tag{3.40}$$ convd

see [20, Section 3.7.4] for details.

Relations (3.26) – (3.29), (3.32), (3.40), and by means of interpolation from (3.36) and (3.17)

$$\vec{B}_\varepsilon \to \vec{B} \text{ in } L^2((0,T); L^q(\Omega, \mathbb{R}^3)), \quad 1 \le q < 6. \tag{3.41}$$ convb

together with the previous uniform bounds allow us to pass to the limit in the weak formulation of the Navier-Stokes-Fourier system and the simplified Maxwell's system, as soon as we show convergence of the sequence $\{I_\varepsilon\}_{\varepsilon>0}$. This will be accomplished in the forthcoming section.

3.2.4 Convergence of radiation intensity

Our ultimate goal is to establish convergence of the quantities arising in the entropy production rate by radiation

$$\frac{1}{\vartheta_\varepsilon} S_{E,\varepsilon} = \frac{1}{\vartheta_\varepsilon} \int_0^\infty \sigma_a(\nu, \vartheta_\varepsilon) \left[\int_{\mathcal{S}^2} \left(\mathfrak{B}(\nu, \vartheta_\varepsilon) - I_\varepsilon \right) \, \mathrm{d}\vec{\omega} \right] \, \mathrm{d}\nu +$$

$$\frac{1}{\vartheta_\varepsilon} \int_0^\infty \sigma_s(\nu, \vartheta_\varepsilon) \int_{\mathcal{S}^2} \left[\frac{1}{4\pi} \int_{\mathcal{S}^2} I_\varepsilon(t,x,\vec{\omega},\nu) \, \mathrm{d}\vec{\omega} - I_\varepsilon(t,x,\vec{\omega},\nu) \right] \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu =$$

$$\frac{1}{\vartheta_\varepsilon(t,x)} \int_0^\infty \sigma_a(\nu, \vartheta_\varepsilon(t,x)) \left[\int_{\mathcal{S}^2} \left(\mathfrak{B}(\nu, \vartheta_\varepsilon(t,x)) - I_\varepsilon(t,x,\vec{\omega},\nu) \right) \, \mathrm{d}\vec{\omega} \right] \, \mathrm{d}\nu$$

and

$$\frac{1}{\vartheta_\varepsilon} \vec{S}_{F,\varepsilon} \cdot \vec{u}_\varepsilon =$$

$$\frac{1}{c\vartheta_\varepsilon} \vec{u}_\varepsilon \cdot \int_0^\infty \sigma_s(\nu, \vartheta_\varepsilon) \int_{\mathcal{S}^2} \vec{\omega} \left[\frac{1}{4\pi} \int_{\mathcal{S}^2} I_\varepsilon(t,x,\vec{\omega},\nu) \, \mathrm{d}\vec{\omega} - I_\varepsilon(t,x,\vec{\omega},\nu) \right] \, \mathrm{d}\vec{\omega} \, \mathrm{d}\nu +$$

$$\frac{1}{c\vartheta_\varepsilon(t,x)} \vec{u}_\varepsilon \cdot \int_0^\infty \sigma_a(\nu, \vartheta_\varepsilon(t,x)) \left[\int_{\mathcal{S}^2} \vec{\omega} \left(\mathfrak{B}(\nu, \vartheta_\varepsilon(t,x)) - I_\varepsilon(t,x,\vec{\omega},\nu) \right) \, \mathrm{d}\vec{\omega} \right] \, \mathrm{d}\nu$$

Since $\vartheta_\varepsilon \to \vartheta$ a. e. in $(0,T) \times \Omega$, the desired result follows from compactness of the velocity averages over the sphere $\mathcal{S}^2$ established by Golse et al. [25, 26], see also Bournaveas and Perthame [3], and hypothesis (2.12). Specifically, we use the following result (see [25]):

Ppr1 **Proposition 3.1** *Let $I \in L^q([0,T] \times \mathbb{R}^n \times \mathcal{S}^2 \times \mathbb{R})$, $\partial_t I + c\omega \cdot \nabla_x I \in L^q([0,T] \times \mathbb{R}^n \times \mathcal{S}^2 \times \mathbb{R})$ for a certain $q > 1$. In addition, let $I_0 \equiv I(0,\cdot) \in L^\infty(\mathbb{R}^n \times \mathcal{S}^2 \times \mathbb{R})$.*

Then

$$\tilde{I}(t,x,\nu) \equiv \int_{\mathcal{S}^2} I(t,x,\vec{\omega},\nu) \, \mathrm{d}\vec{\omega}$$

belongs to the space $W^{s,q}([0,T]\times\mathbb{R}^n\times\mathbb{R})$ *for any* s, $0<s<\inf\{1/q,1-1/q\}$, *and*

$$\|\tilde I\|_{W^{s,q}}\le c(I_0)(\|I\|_{L^q}+\|\partial_t I+c\omega\cdot\nabla I\|_{L^q}).$$

As the radiation intensity I_ε satisfies the transport equation (1.13), by virtue of hypotheses (2.9) and (2.10) where $\tilde S$ is bounded in $L^q\cap L^\infty([0,T)\times\Omega\times\mathcal{S}^2\times\mathbb{R})$, a direct application of Proposition 3.1 yields the desired conclusion

$$\int_{\mathcal{S}^2} I_\varepsilon(t,x,\vec\omega,\nu)\,\mathrm{d}\vec\omega\to\int_{\mathcal{S}^2} I(t,x,\vec\omega,\nu)\,\mathrm{d}\vec\omega \text{ in } L^2((0,T)\times\Omega),$$

and

$$\int_{\mathcal{S}^2} \vec\omega I_\varepsilon(t,x,\vec\omega,\nu)\,\mathrm{d}\vec\omega\to\int_{\mathcal{S}^2} \vec\omega I(t,x,\vec\omega,\nu)\,\mathrm{d}\vec\omega \text{ in } L^2((0,T)\times\Omega),$$

for any fixed $\nu>0$. Consequently

$$\frac{1}{\vartheta_\varepsilon}S_{E,\varepsilon}\to\frac{1}{\vartheta}S_E, \tag{3.42}$$ convSE

and

$$\frac{1}{\vartheta_\varepsilon}\vec S_{F,\varepsilon}\cdot\vec u_\varepsilon\to\frac{1}{\vartheta}\vec S_F\cdot\vec u \tag{3.43}$$ convSF

as required, and Theorem 2.1 is proved by convergences (3.40), (3.29), (3.32), (3.41), (3.42) and (3.43). The entropy inequality (2.20) needs additionally convergence of the initial total entropy of approximations to the initial entropy of its limit and weak upper semicontinuity of the right-hand side of (2.20) due to (1.16), (1.15), (3.29), (3.27). Moreover we need the positivity of the absolute temperature ϑ. This is due to the convergences (3.28) and (3.32).

3.2.5 The Maxwell equation

The Maxwell system is represented by weak formulation (2.23).

We have the following convergences:

- $\vec B_\varepsilon\to\vec B$ weakly in $L^2(0,T,W^{1,2}(\Omega))$,
- $\vec B_\varepsilon\to\vec B$ strongly in $L^2(0,T,L^2(\Omega))$,
- $\vec J_\varepsilon\times\vec B_\varepsilon\to\vec J\times\vec B$ weakly in $L^p((0,T)\times\Omega;\mathbb{R}^3)\quad\forall p\in\left[1,\frac{5}{4}\right]$.

Here we factor the only nonlinear term as follows to use the uniform bound (3.15)

$$\frac{1}{\sigma}\mathbf{curl}_x\vec H_\varepsilon=\sqrt{\vartheta_\varepsilon\sigma^{-1}(\varrho_\varepsilon,\vartheta_\varepsilon,\vec B_\varepsilon)}\sqrt{\frac{1}{\vartheta_\varepsilon\sigma(\varrho_\varepsilon,\vartheta_\varepsilon,\vec B_\varepsilon)}}\mathbf{curl}_x\vec H_\varepsilon, \tag{3.44}$$ magdiff

and get the uniform bound in a reflexive Banach space

$$\left\|\sigma^{-1}(\varrho_\varepsilon,\vartheta_\varepsilon,\vec B_\varepsilon)\mathbf{curl}_x\vec H_\varepsilon\right\|_{L^{\frac{34}{23}}((0,T)\times\Omega)}\le c. \tag{3.45}$$ magdiffest

This suffices to pass to the limit in (2.23).

3.3 Approximating scheme and global-in-time existence

exi

We conclude the paper by proposing an approximation scheme to be used to prove existence of global-in-time weak solutions to problem (1.10) – (1.26). The scheme is essentially the same as in [20, Chapter 3], the extra terms are put in { }. The dependence of approximate solutions on the parameters of approximation δ and d has been in notation suppressed.

- The continuity equation (1.10) is replaced by an "artificial viscosity" approximation

$$\partial_t \varrho + \mathrm{div}_x(\varrho \vec{u}) = \{d\Delta \varrho\}, \ d > 0, \tag{3.46}$$

ap1

to be satisfied on $(0,T)\times\Omega$, and supplemented by the homogeneous Neumann boundary conditions

$$\nabla_x \varrho \cdot \vec{n}|_{\partial\Omega} = 0. \tag{3.47}$$

ap2

The initial distribution of the approximate densities is given through

$$\varrho(0,\cdot) = \varrho_{0,\delta}, \tag{3.48}$$

ap3

where

$$\varrho_{0,\delta} \in C^{2,\nu}(\overline{\Omega}), \ \nabla_x \varrho_{0,\delta}\cdot\vec{n}|_{\partial\Omega} = 0, \ \inf_{x\in\Omega} \varrho_{0,\delta}(x) > 0, \tag{3.49}$$

ap4

with a positive parameter $\delta > 0$. We recall the requirement of the strong convergence of initial approximations of density (2.25) supplemented additionally with the condition

$$|\{\varrho_{0,\delta} < \varrho_0\}| \to 0+ \quad \text{as } \delta \to 0+\,. \tag{3.50}$$

ap5

- The momentum equation is replaced by a Faedo-Galerkin approximation:

$$\int_0^T\int_\Omega \Big(\varrho\vec{u}\cdot\partial_t\vec{\varphi} + \varrho\vec{u}\otimes\vec{u} : \nabla_x\vec{\varphi} + (p + \{\delta(\varrho^\Gamma + \varrho^2)\})\mathrm{div}_x\vec{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}t =$$

$$\int_0^T\int_\Omega \Big(\{d(\nabla_x\varrho\nabla_x\vec{u})\}\cdot\vec{\varphi} + \mathbb{S}_\delta : \nabla_x\vec{\varphi} - \varrho\nabla_x\psi\cdot\vec{\varphi} - \vec{S}_F\cdot\vec{\varphi}\Big)\,\mathrm{d}x\,\mathrm{d}t$$

$$-\int_\Omega (\varrho\vec{u})_0\cdot\vec{\varphi}\,dx, \tag{3.51}$$

ap6

to be satisfied for any test function $\vec{\varphi}\in C^1_c([0,T),X_n)$, where

$$X_n \subset C^{2,\nu}(\overline{\Omega};\mathbb{R}^3) \subset L^2(\Omega;\mathbb{R}^3) \tag{3.52}$$

ap7

is a finite-dimensional space of functions satisfying the no-slip boundary conditions

$$\vec{\varphi}|_{\partial\Omega} = \vec{0}. \tag{3.53}$$

ap8

The space X_n is endowed with the Hilbert structure induced by the scalar product of the Lebesgue space $L^2(\Omega; R^3)$.

We set

$$\begin{gathered}\mathbb{S}_\delta := \mathbb{S}_\delta\left(\nabla_x \vec{u}, \vartheta, \vec{H}\right) = \left(\lambda\left(\vartheta, \left|\vec{H}\right|\right) + \delta\vartheta\right) \times \\ \left(\nabla_x \vec{u} + \nabla_x^T \vec{u} - \tfrac{2}{3}\mathrm{div}_x \vec{u}\,\mathbb{I}\right) + \left(\eta\left(\vartheta, \left|\vec{H}\right|\right) \mathrm{div}_x \vec{u}\right) \mathbb{I}.\end{gathered} \tag{3.54}$$

- We replace the energy equation (1.12) with a modified internal energy balance

$$\partial_t(\varrho e + \{\delta\varrho\vartheta\}) + \mathrm{div}_x\Big((\varrho e + \{\delta\varrho\vartheta\})\vec{u}\Big) - \tag{3.55}$$

ap11

$$\mathrm{div}_x\left(\kappa_M(\varrho, \vartheta, \left|\vec{B}\right|) + \kappa_R\vartheta^3 + \{\delta\left(\vartheta^\Gamma + \vartheta^{-1}\right)\}\nabla_x\vartheta\right) =$$

$$\mathbb{S}_\delta : \nabla_x \vec{u} + \frac{1}{\sigma}\left|\mathbf{curl}_x \vec{H}\right|^2 + \vec{u}\cdot\vec{S}_F + \left\{ d\delta(\Gamma|\varrho|^{\Gamma-2} + 2)|\nabla_x\varrho|^2 + \right.$$

$$\left. \delta\vartheta^{-2} - d\vartheta^5 + 2\delta\vartheta\left[\left|\frac{\nabla_x\vec{u} + \nabla_x\vec{u}^T}{2}\right|^2 - \frac{1}{3}\left(\mathrm{div}_x\vec{u}\right)^2\right]\right\} - p\;\mathrm{div}_x\vec{u} - S_E,$$

to be satisfied in $(0,T)\times\Omega$, together with no-flux boundary conditions

$$\nabla_x\vartheta\cdot\vec{n}|_{\partial\Omega} = 0. \tag{3.56}$$

ap12

The initial condition reads

$$\varrho(e + \delta\vartheta)(0,\cdot) = \varrho_{0,\delta}(e(\varrho_{0,\delta}, \vartheta_{0,\delta}) + \delta\vartheta_{0,\delta}), \tag{3.57}$$

ap13

where the (approximate) temperature distribution satisfies

$$\vartheta_{0,\delta} \in C^1(\overline{\Omega}),\ \nabla_x\vartheta_{0,\delta}\cdot\vec{n}|_{\partial\Omega} = 0,\ \inf_{x\in\Omega}\vartheta_{0,\delta}(x) > 0. \tag{3.58}$$

ap14

- We add the equation for the radiative transfer

$$\frac{1}{c}\partial_t I + \vec{\omega}\cdot\nabla_x I = \tilde{S} \text{ in } (0,T)\times\Omega\times\mathcal{S}^2\times(0,\infty), \tag{3.59}$$

ap15

together with the transparency condition (1.24).

- Finally we require satisfaction of the unmodified equation for magnetic induction $\vec{B}$ (2.23), solenoidality condition (1.3) with an approximate initial condition

$$\vec{B}(0,\cdot) = \vec{B}_{0,\delta} \qquad (3.60)$$ ap16

with $\vec{B}_{0,\delta} \in \mathcal{D}(\Omega, \mathbb{R}^3)$, $\mathrm{div}_x \vec{B}_{0,\delta} = 0$ and

$$\vec{B}_{0,\delta} \to \vec{B}_0 \quad \text{in } L^2(\Omega, \mathbb{R}^3) \text{ as } \delta \to 0+\,. \qquad (3.61)$$ ap17

Given a family of approximate solutions $\{\varrho_{d,\delta}, \vec{u}_{d,\delta}, \vartheta_{d,\delta}, \vec{B}_{d,\delta}, I_{d,\delta}\}_{d>0,\delta>0}$, we may construct a weak solution of system (1.1) – (1.26) letting successively $d \to 0$, $\delta \to 0$ and using compactness arguments delineated in the previous part of this paper. The reader may consult [20, Chapter 3] for all technical details. The approximate solutions can be constructed by means of a fixed point argument applied to the couple $\vec{u}$, I, similarly to [20, Chapter 3, Section 3.4].

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