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# A remark on polyconvex functions with symmetry

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**Abstract** For a given polyconvex function $W$, among all associated convex functions $g$ of minors there exists the largest one; this function inherits all symmetry properties of $W$. For a given associated (not necessarily the largest) function $g$, one can still find an associated (possibly not the largest) function with the symmetry of $W$. This function is constructed by averaging of symmetry conjugated functions over the symmetry group of $W$ using Haar's measure. It follows that if a symmetric polyconvex function $W$ has class $k = 0, \ldots, \infty$ associated function, then the averaging produces a symmetric associated function that is class $k$ as well.

**Keywords** Polyconvexity; symmetry; convexity; elasticity

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## 1 Introduction

Let $\mathrm{Lin}$ be the set of all linear transformations $F$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ or the corresponding matrices, $n = 2, 3$. Let $\mathrm{Lin}^+$ be the set of all $F \in \mathrm{Lin}$ with $\det F > 0$. The function $W : \mathrm{Lin}^+ \to \bar{\mathbb{R}}$ is said to be polyconvex if there exists a convex function $g$ such that

$$\left.\begin{array}{ll} \text{if } n = 2: & \left\{\begin{array}{ll} g : \mathrm{Lin} \times (0,\infty) \to \bar{\mathbb{R}} & \text{and} \\ W(F) = g(F, \det F) & \text{for every } F \in \mathrm{Lin}^+, \end{array}\right. \\ \\ \text{if } n = 3: & \left\{\begin{array}{ll} g : \mathrm{Lin} \times \mathrm{Lin} \times (0,\infty) \to \bar{\mathbb{R}} & \text{and} \\ W(F) = g(F, \mathrm{cof}\, F, \det F) & \text{for every } F \in \mathrm{Lin}^+; \end{array}\right. \end{array}\right\} \quad (1.1)$$

here $\mathrm{cof}\, F = (\det F) F^{-\mathrm{T}}$ if $F \in \mathrm{Lin}^+$.

The notion is due to Morrey [5; Theorem 4.4.10], but the terminology is due to Ball [1], who applied the polyconvexity to prove the existence theorems in nonlinear elasticity under realistic assumptions. In particular, he showed that polyconvexity is consistent with the principle of objectivity, the optional isotropy of the body, and the injectivity requirement, i.e., respectively,

$$\left.\begin{aligned} W(QF) &= W(F), \quad F \in \mathrm{Lin}^+,\ Q \in \mathrm{SO}(n),\\ W(FR^{\mathrm{T}}) &= W(F), \quad F \in \mathrm{Lin}^+,\ R \in \mathrm{SO}(n),\\ W(F) &\to \infty \quad \text{if} \quad \det F \to 0. \end{aligned}\right\} \qquad (1.2)$$

For a given $W$, the convex function $g$ occurring in (1.1), which is called the associated convex function in this note, is highly nonunique.

The purpose of this note is to show that if the material satisfies any of the two symmetry requirements $(1.2)_{1,\,2}$ then the associated convex function $g$ can be chosen to satisfy, respectively,

$$\left.\begin{aligned} &\text{if } n = 2: \quad \begin{cases} g(QF, \delta) = g(F, \delta)\\ \text{or, optionally, for an isotropic body,}\\ g(FQ, \delta) = g(F, \delta)\\ \text{for every } (F, \delta) \in \mathrm{Lin} \times (0, \infty),\ Q \in \mathrm{SO}(2), \end{cases}\\ &\text{if } n = 3: \quad \begin{cases} g(QF, QG, \delta) = g(F, H, \delta)\\ \text{or, optionally, for an isotropic body,}\\ g(FQ, GQ, \delta) = g(F, H, \delta)\\ \text{for every } (F, G, \delta) \in \mathrm{Lin} \times \mathrm{Lin} \times (0, \infty),\ Q \in \mathrm{SO}(3). \end{cases} \end{aligned}\right\} \qquad (1.3)$$

In fact, in the treatment below, we replace $\mathrm{SO}(n)$ by arbitrary subgroups $\mathscr{G}_{\mathrm{left}}$ and $\mathscr{G}_{\mathrm{right}}$ of $\mathrm{Lin}^+$, defined by

$$\mathscr{G}_{\mathrm{left}} = \{L \in \mathrm{Lin}^+ : W(LF) = W(F) \ \text{ for all } F \in \mathrm{Lin}^+\},$$

$$\mathscr{G}_{\mathrm{right}} = \{M \in \mathrm{Lin}^+ : W(FM^{-1}) = W(F) \ \text{ for all } F \in \mathrm{Lin}^+\}.$$

To verify that $\mathscr{G}_{\mathrm{left}}$ and $\mathscr{G}_{\mathrm{right}}$ are groups, one uses the multiplicativity of cof, i.e.,

$$\mathrm{cof}(AB) = \mathrm{cof}\,A\,\mathrm{cof}\,B, \quad \mathrm{cof}(A^{-1}) = (\mathrm{cof}\,A)^{-1} =: \mathrm{cof}\,A^{-1} \qquad (1.4)$$

$A, B \in \mathrm{Lin}^+$, formulas to be frequently employed below.

It will be shown that the associated convex functions $g$ can be chosen to satisfy the symmetry requirements governed by $\mathscr{G}_{\mathrm{left}}$ and $\mathscr{G}_{\mathrm{right}}$. Two elementary constructions of $g$ will be given. The first one is based on the existence of the largest associated convex function, a consequence of the fact that the pointwise supremum of any family of convex functions is convex. The second construction holds only if $\mathscr{G}_{\mathrm{left}}$ and $\mathscr{G}_{\mathrm{right}}$ are compact, and uses the averaging of symmetry conjugated associated convex functions with respect to Haar's measures on $\mathscr{G}_{\mathrm{left}}$ and/or $\mathscr{G}_{\mathrm{right}}$, a procedure frequently used in the group theory to construct invariant objects.

## 2 Dimensions 2 and 3

Throughout this section, let $n = 2$ or $n = 3$. The following fact, although elementary, forms the crux of the polyconvexity notion.

**Remark 2.1** ([1; Section 4] and [2; Theorem 5.6, Part 2]). *For a given polyconvex function* $W : \mathrm{Lin}^+ \to \bar{\mathbb{R}}$ *the set of all associated convex functions h contains the largest one, g, such that* $g \ge h$ *over* $\mathrm{Lin} \times (0,\infty)$ *or* $\mathrm{Lin} \times \mathrm{Lin} \times (0,\infty)$. *This function is given by*

$$g(F,\delta) = \sup\{h(F,\delta) : h \text{ an associated convex function}\},$$
$$g(F,G,\delta) = \sup\{h(F,G,\delta) : h \text{ an associated convex function}\}$$

*throughout their domains. Alternatively, g is constructed as the convexification of the function* $g_0$ *given by*

$$\text{if } n=2: \qquad g_0(F,\delta) = \begin{cases} W(F) & \text{if } \delta = \det F, \\ \infty & \text{else,} \end{cases}$$

$$\text{if } n=3: \quad g_0(F,G,\delta) = \begin{cases} W(F) & \text{if } (G,\delta) = (\operatorname{cof} F, \det F) \\ \infty & \text{else.} \end{cases}$$

*The convexification is defined by*

$$g = \sup\{h : h \le g_0,\ h \text{ convex}\}. \tag{2.1}$$

To avoid cumbersome formulas, let for each associated convex function $g$ and each $L \in \mathscr{G}_{\mathrm{left}}$, $M \in \mathscr{G}_{\mathrm{right}}$, the function $g_{L,M}$ be defined by

$$\left.\begin{aligned} &\text{for } n=2: \quad g_{L,M}(F,\delta) := g(LFM^{-1}, \det L \det M^{-1}\delta), \\ &\text{for } n=3: \quad g_{L.M}(F,G,\delta) = g(LFM^{-1}, (\operatorname{cof} L) G \operatorname{cof} M^{-1}, \det L \det M^{-1}\delta) \end{aligned}\right\} \tag{2.2}$$

for every $(F,\delta)$ or every $(F,G,\delta)$ from the corresponding domains. If $\mathscr{G}_{\mathrm{left}} = \mathscr{G}_{\mathrm{right}} = \mathrm{SO}(n)$ and $Q, R \in \mathrm{SO}(n)$ then

$$\begin{aligned} &\text{if } n=2: \quad g_{Q,R}(F,\delta) := g(QFR^{\mathrm{T}}, \delta), \\ &\text{if } n=3: \quad g_{Q,R}(F,G,\delta) = g(QFR^{\mathrm{T}}, QGR^{\mathrm{T}}, \delta), \end{aligned}$$

thus recovering the particular case from the introduction.

**Remark 2.2.** *Assume that W satisfies the injectivity requirement* $(1.2)_3$. *Then*

$$\mathscr{G}_{\mathrm{left}} \subset \mathrm{SL}(n), \quad \mathscr{G}_{\mathrm{right}} \subset \mathrm{SL}(n)$$

*where* $\mathrm{SL}(n) := \{L \in \mathrm{Lin}^+ : \det L = 1\}$. *Equations* (2.2) *simplify accordingly.*

***Proof*** Suppose that $\mathscr{G}_{\mathrm{left}}$ contains an element $L$ with $\det L \ne 1$; by passing from $L$ to $L^{-1}$ if necessary, we can assume $\det L < 1$. Picking any $F \in \mathrm{Lin}^+$ and iterating $W(LF) = W(F)$ we obtain

$$W(L^p F) = W(F) = \text{const} \tag{2.3}$$

for any positive integer $p$, but $\det(L^p F) \to 0$ and thus (2.3) contradicts $(1.2)_3$. □

Recall that for each compact group $\mathscr{G}$ there exists a unique nonnegative regular Borel measure $m$ on $\mathscr{G}$, called Haar's measure, such that $m(\mathscr{G}) = 1$ and $m$ is left and right invariant, i.e.,

$$\int_{\mathscr{G}} f(LM^{-1})\, dm(L) = \int_{\mathscr{G}} f(ML)\, dm(L) = \int_{\mathscr{G}} f(L)\, dm(L)$$

for every $m$ integrable function $f : \mathscr{G} \to \mathbb{R}$ and every $M \in \mathscr{G}$, [6; Theorem 5.14].

**Proposition 2.3.**

(i) *The largest convex function g associated with a polyconvex function W is invariant, i.e.,*

$$g_{L,\,M} = g \tag{2.4}$$

*for every* $L \in \mathcal{G}_{\text{left}}$, $M \in \mathcal{G}_{\text{right}}$ *throughout the domain of* $g$.

(ii) *If* $\mathcal{G}_{\text{left}}$ *and* $\mathcal{G}_{\text{right}}$ *are compact with Haar's measures* $m_{\text{left}}$ *and* $m_{\text{right}}$, *respectively, if* $h$ *is any convex function associated with* $W$ *and if* $g$ *is defined by* (cf. [7])

$$\left.\begin{aligned} &\text{for } n=2\text{:}\quad g(F,\delta) = \int_{\mathcal{G}_{\text{left}}\times\mathcal{G}_{\text{right}}} h_{L,\,M}(F,\delta)\,dm_{\text{left}}(L)dm_{\text{right}}(M) \\ &\text{for } n=3\text{:}\quad g(F,G,\delta) = \int_{\mathcal{G}_{\text{left}}\times\mathcal{G}_{\text{right}}} h_{L,\,M}(F,G,\delta)\,dm_{\text{left}}(L)dm_{\text{right}}(M) \end{aligned}\right\} \tag{2.5}$$

*throughout the domain of* $g$, *then* $g$ *is an associated convex function that satisfies* (2.4). *If* $h$ *is of class* $k = 0, \ldots, \infty$ *then* $g$ *is of class* $k$ *also.*

We assume that the integrand in (2.5) is measurable in $L$, $M$ with respect to $m_{\text{left}} \otimes m_{\text{right}}$ for every fixed $(F,\delta)$ or $(F,G,\delta)$ (for example, let $h$ be finite valued and hence continuous).

***Proof*** Only the case $n = 3$ will be proved; $n = 2$ is similar.

(i): Clearly, for any $L \in \mathcal{G}_{\text{left}}$, $M \in \mathcal{G}_{\text{right}}$ and any associated convex function (not necessarily the largest one), also $g_{L,\,M}$ is an associated convex function. Thus if $g$ is the largest associated convex function, we have $g_{L,\,M} \le g$, i.e.,

$$g(LFM^{-1}, (\operatorname{cof} L)G\operatorname{cof} M^{-1}, \det L\det M^{-1}\delta) \le g(F,G,\delta)$$

for all arguments occurring there. Replacing $L \to L^{-1}$, $M^{-1} \to M$, $F \to LFM^{-1}$, $G \to (\operatorname{cof} L)G\operatorname{cof} M^{-1}$, $\delta \to \det L\det M^{-1}\delta$, we obtain the opposite inequality and hence (2.4).

(ii): Clearly, $g$, being essentially a convex combination of convex functions $h_{L,\,M}$, is convex. Further, the invariant character of $m_{\text{left}}$ and $m_{\text{right}}$ implies that $g$ satisfies (2.4). Finally, if $F \in \mathrm{Lin}^+$ then

$$\begin{aligned} h_{L,\,M}(F,\operatorname{cof} F,\det F) &= h(LFM^{-1}, \operatorname{cof}(LFM^{-1}), \det(LFM^{-1})) \\ &= W(LFM^{-1}) = W(F) \end{aligned}$$

and thus integrating over $\mathcal{G}_{\text{left}} \times \mathcal{G}_{\text{right}}$, we obtain

$$g(F,\operatorname{cof} F,\det F) = W(F).$$

Hence $g$ is an associated convex function. The differentiability follows from the theorem on the differentiation under the integral sign. □

**Remarks 2.4.**

(i) One can replace $\mathcal{G}_{\text{left}}$ and $\mathcal{G}_{\text{right}}$ by arbitrary subgroups of $\mathcal{G}_{\text{left}}$ and $\mathcal{G}_{\text{right}}$.

(ii) Let $\Omega \subset \mathbb{R}^n$ be the reference region of the body and suppose that $W$ depends parametrically on $x \in \Omega$. Fixing $x$, we can apply Proposition 3 to each $W(x,\cdot)$ to obtain $g(x,\cdot)$ on $\mathrm{Lin} \times (0,\infty)$ or $\mathrm{Lin} \times \mathrm{Lin} \times (0,\infty)$. However, the measurability/continuity character of $W$ apparently does not reproduce. Suppose, for example, that $W$ is a Carathéodory function [3; Section IV.1.2], i.e., for almost every

$x \in \Omega$, $W(x, \cdot)$ is continuous and for every $F \in \mathrm{Lin}$, $W(\cdot, F)$ is measurable. There seems to be no guarantee the largest associated convex function $g$ is a Carathéodory function, since the supremum in (2.1) generally does not preserve continuity and measurability. On the positive side, [4; Proposition 6.43] shows that $g$ is a normal integrand, a property weaker than the Carathéodory property of the integrand, but still with many virtues.

(iii) The following fact was pointed to the autor by S. J. Spector [8]: *if the hypotheses of Item* (ii) *of Proposition* 3 *are satisfied and h is a Carathéodory integrand* (*respectively, a normal integrand that is bounded from below*) *then g*, *given by* (2.5), *is an integrand of the same type.* This follows by a straightforward application of Lebesgue's dominated convergence theorem and Fatou's lemma, respectively. The details are left to the reader.

## 3 The general case

This section considers the general case, i.e., a function $W$ of an argument $F$ that is a rectangular matrix of arbitrary dimension, defined on a domain $E$.

Thus let $m$, $n$ be positive integers and let $\mathrm{Lin}(n,m)$ denote the set of all linear transformations from $\mathbb{R}^n$ to $\mathbb{R}^m$ or the corresponding matrices. Let $\mathrm{Lin}^+(n,n)$ be the set of all elements of $\mathrm{Lin}(n,n)$ of positive determinant. Let $\mathbb{I}_r^n$ be the set of all multiindices of order $r$, $0 \le r \le n$, consisting of all $r$-tuples $I = (i_1, \ldots, i_r)$ with $1 \le i_1 < i_2 < \ldots < i_r \le n$. If $1 \le r \le q := \min\{m, n\}$, let $\mathrm{Lin}(\mathbb{I}_r^n, \mathbb{I}_r^m)$ be the set of all $\mathbb{I}_r^m \times \mathbb{I}_r^n$ matrices, i.e., collections $\xi = [\xi_{IJ}]_{I \in \mathbb{I}_r^m,\ J \in \mathbb{I}_r^n}$ of real numbers $\xi_{IJ}$. For each $F \in \mathrm{Lin}(n,m)$, let $F^{(r)} \in \mathrm{Lin}(\mathbb{I}_r^n, \mathbb{I}_r^m)$ be the $\mathbb{I}_r^m \times \mathbb{I}_r^n$ matrix of minors of $F$ of order $r$. Thus if $I = (i_1, \ldots, i_r) \in \mathbb{I}_r^m$, $J = (j_1, \ldots, j_r) \in \mathbb{I}_r^n$ are two multiindices then

$$(F^{(r)})_{IJ} := \det[F_{i_\alpha j_\beta}]_{1 \le \alpha,\ \beta \le r},$$

where $F_{ij}$ are the matrix elements of $F$. The matrices $F^{(r)}$ can be interpreted as exterior powers of $F$ and the notation has been chosen to emphasize this fact. The multiplicativity (1.4) of cof now generalizes as follows: If $A \in \mathrm{Lin}^+(m,m)$, $F \in \mathrm{Lin}(n,m)$ and $B \in \mathrm{Lin}^+(n,n)$ then

$$(AFB)^{(r)} = A^{(r)} F^{(r)} B^{(r)}, \quad (A^{-1})^{(r)} = (A^{(r)})^{-1} =: A^{(r)-1}.$$

Let

$$\mathbf{M}(n,m) = \bigoplus_{r=1}^{q} \mathrm{Lin}(\mathbb{I}_r^n, \mathbb{I}_r^m);$$

the elements $\eta$ of $\mathbf{M}(n,m)$ are $q$-tuples

$$\eta = (\eta_1, \ldots, \eta_q) \quad \text{where} \quad \eta_r \in \mathrm{Lin}(\mathbb{I}_r^n, \mathbb{I}_r^m), \quad 1 \le r \le q.$$

Let $E \subset \mathrm{Lin}(n,m)$. A function $W : E \to \bar{\mathbb{R}}$ can be extended to $\mathrm{Lin}(n,m)$ by setting $W = \infty$ on $\mathrm{Lin}(n,m) \sim E$. In this way it suffices to consider only functions $W : \mathrm{Lin}(n,m) \to \bar{\mathbb{R}}$. Such a function $W$ is said to be polyconvex if there exists a convex function $g : \mathbf{M}(n,m) \to \bar{\mathbb{R}}$ such that

$$W(F) = g(F^{(1)}, \ldots, F^{(q)}),$$

$F \in \mathrm{Lin}(n,m)$. We put

$$\mathscr{G}_{\mathrm{left}} = \{L \in \mathrm{Lin}^+ : W(LF) = W(F) \text{ for all } F \in \mathrm{Lin}^+(m,m)\},$$
$$\mathscr{G}_{\mathrm{right}} = \{M \in \mathrm{Lin}^+ : W(FM^{-1}) = W(F) \text{ for all } F \in \mathrm{Lin}^+(n,n)\}.$$

We have the following general version of Proposition 3, with identical proof.

**Proposition 3.1.**

(i) *The largest convex function g associated with a polyconvex function W is invariant, i.e.,*

$$g(L^{(1)}\eta_1 M^{(1)-1}, L^{(q)}\eta_q M^{(q)-1}) = g(\eta_1, \dots, \eta_q) \tag{3.1}$$

*for every* $(\eta_1, \dots, \eta_q) \in \mathbf{M}(n,m)$ *and* $L \in \mathscr{G}_{\mathrm{left}}$, $M \in \mathscr{G}_{\mathrm{right}}$.

(ii) *If* $\mathscr{G}_{\mathrm{left}}$ *and* $\mathscr{G}_{\mathrm{right}}$ *are compact with Haar's measures* $m_{\mathrm{left}}$ *and* $m_{\mathrm{right}}$, *respectively, if h is any convex function associated with W and if* $g : \mathbf{M}(n,m) \to \bar{\mathbb{R}}$ *is defined by*

$$g(\eta_1, \dots, \eta_q) = \int_{\mathscr{G}_{\mathrm{left}} \times \mathscr{G}_{\mathrm{right}}} h(L^{(1)}\eta_1 M^{(1)-1}, L^{(q)}\eta_q M^{(q)-1})\, dm_{\mathrm{left}}(L) dm_{\mathrm{right}}(M)$$

$(\eta_1, \dots, \eta_q) \in \mathbf{M}(n,m)$, *then g is an associated convex function that satisfies* (3.1).

We again assume that the integrand is measurable in $L, M$ with respect to $m_{\mathrm{left}} \otimes m_{\mathrm{right}}$ for every fixed $(\eta_1, \dots, \eta_q)$.

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