Perfect cliques with respect to infinitely many relations

Martin Doležal
joint work with Wiesław Kubiś

Institute of Mathematics AS CR

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Cantor-Bendixson Derivative

Let X be a topological space, and let $A \subseteq X$.

The Cantor-Bendixson derivative of A is the set

$$A' = \{x \in A\colon x \text{ is a limit point of } A\}$$

The iterated Cantor-Bendixson derivatives A^γ, $\gamma \in \mathrm{ORD}$, are defined by

$$\begin{aligned} A^0 &= A \\ A^{\gamma+1} &= (A^\gamma)' \\ A^\gamma &= \bigcap_{\alpha<\gamma} A^\alpha, \text{ if } \gamma \text{ is limit} \end{aligned}$$

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Cantor-Bendixson Rank

Let X be a topological space, and let $A \subseteq X$.

The Cantor-Bendixson rank of A (denoted by $\operatorname{rank}(A)$) is the least $\gamma \in \mathrm{ORD}$ such that $A^\gamma = \emptyset$.
If such γ does not exist then the Cantor-Bendixson rank of A is $+\infty$.

Observation 1

$\operatorname{rank}(A) = +\infty \iff$ *A contains a dense in itself subset*

Observation 2

X second countable and $\operatorname{rank}(A) < \omega_1 \implies$ *A is at most countable*

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Perfect Cliques

Let X be a topological space, and let $R \subseteq X^n$ be a relation on X.

A set $S \subseteq X$ is called an R-clique if $(s_1, \ldots, s_n) \in R$ whenever $s_1, \ldots, s_n \in S$ are pairwise distinct.

If $\mathcal{R}$ is a family of relations on X then a set $S \subseteq X$ is called an $\mathcal{R}$-clique if it is an R-clique for every $R \in \mathcal{R}$.

A perfect $\mathcal{R}$-clique is an $\mathcal{R}$-clique which is a perfect set (i.e. completely metrizable without isolated points).

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The Existence of Perfect Cliques

Question

Let X be a topological space, and let $\mathcal{R}$ be a family of relations on X. When does there exist a perfect $\mathcal{R}$-clique?

Similar questions were already studied by J. Mycielski (for comeager relations), Q. Feng (for one binary relation), W. Kubiś (for one symmetric relation)...

Our main theorem is a variant of the previous results.

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Main Result

Theorem

Let X be a completely metrizable space of weight $\kappa \geq \omega_0$, and let $\mathcal{R}$ be a countable family of G_δ relations on X. Then exactly one of the following two statements holds:

(S) *There exists an ordinal $\gamma < \kappa^+$ such that every $\mathcal{R}$-clique has Cantor-Bendixson rank $< \gamma$.*

(P) *There exists a perfect $\mathcal{R}$-clique.*

Note: This theorem fails if we replace 'G_δ relations' by 'F_σ relations' (even for one F_σ relation).
This was proved by S. Shelah, and a concrete example was found by W. Kubiś and B. Vejnar.

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Corollary

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Let X be an analytic space, and let $\mathcal{R}$ be a countable family of G_δ relations on X. If there exists an uncountable $\mathcal{R}$-clique then there exists a perfect $\mathcal{R}$-clique.

Proof: There is a continuous surjection $f\colon Y \to X$ where Y is a completely metrizable space of weight ω_0.
For $R \in \mathcal{R}$, let $\tilde{R} = \{(y_1, \ldots, y_n) \in Y^n\colon (f(y_1), \ldots, f(y_n)) \in R\}$.
Let $\tilde{\mathcal{R}} = \{\tilde{R}\colon R \in \mathcal{R}\}$.
Then exactly one holds:

(S) There exists an ordinal $\gamma < \omega_1$ such that every $\tilde{\mathcal{R}}$-clique has rank $< \gamma \implies$ all $\mathcal{R}$-cliques are at most countable.

(P) There exists a perfect $\tilde{\mathcal{R}}$-clique $\implies$ there exists a perfect $\mathcal{R}$-clique.

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Theorem (Souslin)

Let X be an analytic space. Then either X is at most countable, or else X contains a perfect subset.

Proof:

Let $R = X^2$, and let $\mathcal{R} = \{R\}$.
Then every subset of X is an $\mathcal{R}$-clique.
So if X has an uncountable subset then X has a perfect subset.

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(This proof was already known earlier, using a theorem by Q. Feng instead of our result.)

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S. Głąb, F. Strobin (2015):

Let G_n, $n \in \mathbb{N}$, be countable groups, and let $G = \Pi_{n\in\mathbb{N}} G_n$. Then either all free subgroups of G are countable, or else G contains a free subgroup generated by a set of cardinality $\mathfrak{c}$.

Question:

Does this hold for other groups G as well?

Answer:

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Let G be a Polish group. Then either all free subgroups of G are countable, or else G contains a free subgroup generated by a perfect set.

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For each nonempty word $w(x_1, \ldots, x_n)$ on G, let $R_w = \{(x_1, \ldots, x_n) \in G^n \colon w(x_1, \ldots, x_n) \neq 0\}$.
Let $\mathcal{R} = \{R_w \colon w \text{ is a nonempty word on } G\}$.
Then a subset of G generates a free group $\iff$ it is an $\mathcal{R}$-clique.
So if G has an uncountably generated free subgroup then it has a perfectly generated free subgroup.

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