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Infinite Games and σ-porosity

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Outline

1. Porosity-like relation.
2. Known results.
3. Characterization of σ-P-porosity.
4. Inscribing theorems.

Definition of Porosity

Definition

Let (X, d) be a metric space. Let $A \subseteq X$, $x \in X$ and $R > 0$. Denote

$$\gamma(x, R, A) = \sup \big\{ r > 0 \colon \text{there exists } z \in X \text{ such that } d(x, z) < R \text{ and } B(z, r) \cap A = \emptyset \big\},$$

$$p(x, A) = \limsup_{R \to 0+} \frac{\gamma(x, R, A)}{R}.$$

A set $A \subseteq X$ is said to be porous at $x \in X$ if $p(x, A) > 0$.
A set $A \subseteq X$ is said to be porous if it is porous at every its point.
A set $A \subseteq X$ is said to be σ-porous if it is a countable union of porous sets.

Point-set relation

Let X be a metric space and let $P \subseteq X \times 2^X$ be a relation between points of the space X and subsets of X. Then we say that P is a point-set relation on X.
For any $x \in X$ and $A \subseteq X$, the symbol $P(x, A)$ means $(x, A) \in P$.

Porosity-like relation

A point-set relation P on X is called a porosity-like relation if for every $A \subseteq X$, $B \subseteq X$ and $x \in X$ we have:

(P1) $[A \subseteq B \text{ and } P(x, B)] \implies P(x, A)$,

(P2) $P(x, A) \iff$ there exists $r > 0$ such that $P(x, A \cap B(x, r))$,

(P3) $P(x, A) \iff P\left(x, \overline{A}\right)$.

Whenever P is a porosity-like relation on X, $A \subseteq X$ and $x \in X$, we say that

- the set A is P-porous at x if $P(x, A)$,
- the set A is P-porous if it is P-porous at every its point,
- the set A is σ-P-porous if it is a countable union of P-porous sets.

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Question

Let X be a compact metric space and let $A \subseteq X$ be a Borel (analytic) set which is not σ-porous. Does there exist a compact set $K \subseteq A$ which is not σ-porous?

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Known results

- M. Zelený, J. Pelant, 2004:
  - Let X be a topologically complete metric space and let $A \subseteq X$ be a Suslin set which is not σ-porous. Then there exists a closed set $F \subseteq A$ which is not σ-porous.
- M. Zelený, L. Zajíček, 2005:
  - Let X be a locally compact metric space and let $A \subseteq X$ be an analytic set which is not σ-porous. Then there exists a compact set $K \subseteq A$ which is not σ-porous.
  - Analogous variants of this theorem for g-porosity, symmetrical porosity, ...

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Known results

- I. Farah, J. Zapletal, 2006:
  - Let $A \subseteq 2^\omega$ be a Borel set which is not σ-porous. Then there exists a compact set $K \subseteq A$ which is not σ-porous.
- D. Rojas-Rebolledo, 2007:
  - Let X be a zero-dimensional compact metric space and let $A \subseteq X$ be an analytic set which is not σ-porous. Then there exists a compact set $K \subseteq A$ which is not σ-porous.
  - Analogous variant of this theorem for strong porosity.

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4 Inscribing theorems.

Infinite game $G(A)$

Let (X, d) be a complete metric space, let P be a porosity-like relation on X and $A \subseteq X$. Then we define an infinite game $G(A)$ in this way:

I

II

- B_n is an open ball in X, $n \in \mathbb{N}$,
- $\overline{B_{n+1}} \subseteq B_n$ and diam $B_{n+1} \leq \frac{1}{2}$diam B_n, $n \in \mathbb{N}$,
- S_n^j is an open subset of B_n, $j \in \{1, \dots, n\}$, $n \in \mathbb{N}$.

The uniquely determined point $x \in \bigcap_{n=1}^{\infty} B_n$ is called an *outcome* of a run of the game $G(A)$.

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I	B_1		B_2
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Infinite game $G(A)$

The second player wins if at least one of the following two conditions is satisfied:

(a) $x \notin A$,

(b) there exists $m \in \mathbb{N}$ such that $x \in X \setminus \bigcup_{n=m}^{\infty} S_n^m$ and the set $X \setminus \bigcup_{n=m}^{\infty} S_n^m$ is P-porous at x.

The first player wins in the opposite case.

Characterization of σ-P-porosity

Theorem

Let (X, d) be a complete metric space, let P be a porosity-like relation on X and $A \subseteq X$. Then the first player has a winning strategy in the game $G(A)$ if and only if the set A is σ-P-porous.

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Inscribing theorems

- Let X be a locally compact metric space and let $A \subseteq X$ be a Borel (analytic) set which is not σ-porous. Then there exists a compact set $K \subseteq A$ which is not σ-porous.
- Analogous variants of this theorem for strong, symmetric and strong symmetric porosity.

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Sketch of the proof of the inscribing theorem (in case of ordinary porosity and for Borel set A)

- Finding an infinite game $H(A)$ such that:
  - the second player has a winning strategy in the game $H(A)$ $\Longleftrightarrow$ the set A is σ-porous,
  - the set A is Borel $\Longrightarrow$ the game $H(A)$ is determined,
  - the second player has only a finite number of possible choices in every his move.
- The set A is Borel but not σ-porous $\Longrightarrow$ the first player has a winning strategy in the game $H(A)$.
- There exists a compact set $K \subseteq A$ such that the same strategy is winning for the first player even in the game $H(K)$. Therefore, this set cannot be σ-porous. □

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Another result

- There exists a closed set $F \subseteq [0, 1]$ which is σ-$(1 - \varepsilon)$-symmetrically porous for every $0 < \varepsilon < 1$ but which is not σ-1-symmetrically (i.e. σ-strong symmetrically) porous.

Thank you for your attention!