Inviscid primitive equations and weak oscillatory solutions

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What are primitive equations 1

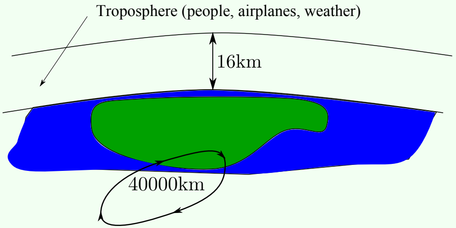

Figure: Scales in global weather models

Figure: Vilhelm Bjerknes (1862-1951)

What are primitive equations 2

Cauchy problem: to find u, v, w, p, $\theta \colon [0, T) \times \Omega \to \mathbb{R}$ satisfying

$$
\begin{aligned}
u_x + v_y + w_z &= 0, \\
u_t + uu_x + vu_y + wu_z + p_x &= \mu_1(u_{xx} + u_{yy}) + \mu_2 u_{zz}, \\
v_t + uv_x + vv_y + wv_z + p_y &= \mu_1(v_{xx} + v_{yy}) + \mu_2 v_{zz}, \\
p_z &= -\theta, \\
\theta_t + u\theta_x + v\theta_y + w\theta_z &= \lambda_1(\theta_{xx} + \theta_{yy}) + \lambda_2 \theta_{zz}
\end{aligned}
$$

with initial conditions $u(0) = u_0$, $v(0) = v_0$ and $\theta(0) = \theta_0$.
When $\mu_1 = \mu_2 = 0$ we call the problem *inviscid primitive equations*.
A special feature: non-deterministic role of w.

Why are they interesting? (for mathematicians)

- Global in time weak solutions existence for the **viscid case**[3],
- global in time regularity of solutions for smooth initial conditions in the **viscid case**[4],
- finite time blow-up of solutions emanating from special smooth initial data in the **inviscid case**[5].

[3] Jacques-Louis Lions, Roger Temam, and Shou Hong Wang. "On the equations of the large-scale ocean". In: *Nonlinearity* 5.5 (1992), pp. 1007–1053.

[4] Chongsheng Cao and Edriss S. Titi. "Global well-posedness of the three-dimensional viscous primitive equations of large scale ocean and atmosphere dynamics". In: *Ann. of Math. (2)* 166.1 (2007).

[5] Chongsheng Cao et al. "Finite-time blowup for the inviscid primitive equations of oceanic and atmospheric dynamics". In: *Comm. Math. Phys.* 337.2 (2015), pp. 473–482.

What about existence of solutions in the inviscid case?

- The term *primitive* might be a bit misleading.
- Blow-up for some smooth initial data.
- If we erase the diffusion in the heat equation, the system is not hyperbolic.
- There are (to the best knowledge of the author) *no* a priori estimates for velocities and temperature. (Hence, it is not surprising that there are no result on existence of local in time smooth solutions for general smooth initial data).

Existence of weak solutions[6]

Theorem

Assume that $T > 0$. Let u_0, v_0, w_0, θ_0 be initial data in $\mathcal{C}^1(\bar{\Omega})$. Then there exist infinitely many weak solutions $u, v, w \in \mathcal{C}([0,T]; L^2_{weak}(\Omega))$, $u, v, w, p \in L^\infty((0,T) \times \Omega)$, $\theta \in W^{1,p}((0,T); L^p(\Omega)) \cap L^p((0,T); W^{2,p}(\Omega))$ to the primitive equations emanating from the initial conditions u_0, v_0, w_0, θ_0.

- Canonically, there will be a jump of the kinetic energy at time $t = 0$. If we denote

$$E(t) = \int_\Omega \frac{1}{2}|u(t,x)|^2 + |v(t,x)|^2 + |w(t,x)|^2 \, \mathrm{d}x$$

then

$$\liminf_{t \to 0^+} E(t) > E(0).$$

[6] E. Chiodaroli, E. Feireisl, and M. Michálek. "Existence of global weak solutions for inviscid primitive equations". In: *preparation* (2016).

Existence of dissipative weak solutions[7]

Definition

We call solutions dissipative if $E(t) \leq E(s)$ whenever $0 \leq s \leq t$.

Theorem

Assume that $T > 0$. *There exist* $u_0, v_0 \in L^\infty(\Omega)$ *and* $\theta_0 \in \mathcal{C}^1(\overline{\Omega})$ *for which we can find infinitely many weak dissipative solutions*
$u, v, w \in \mathcal{C}([0, T]; L^2(\Omega))$, $u, v, w, p \in L^\infty((0, T) \times \Omega)$,
$\theta \in W^{1,p}((0, T); L^p(\Omega)) \cap L^p((0, T); W^{2,p}(\Omega))$ *of the primitive equations emanating from the initial data* u_0, v_0, w_0, θ_0.

[7] E. Chiodaroli, E. Feireisl, and M. Michálek. "Existence of global weak solutions for inviscid primitive equations". In: *preparation* (2016).

Convex integration, oscillatory solutions, etc.

What is convex integration?

A recent trend in analysis of partial differential equations.

A technique, how to find a solution of a system of linear differential equations with nonlinear constitutive relations.

- The technique has its origin in differential geometry:
  - pioneering work of Nash on isometric embeddings,
  - general methods of Gromov (h-principle, etc.) leading to solutions of many partial differential relations.
- It was used to construct surprising results about regularity of weak solutions of Euler-Lagrange equations corresponding to quasiconvex functionals.[8]
- Recently, C. De Lellis and L. Székelyhidi adapted the technique to construct paradoxical solutions of the Euler system.[9]

[8] S. Müller and V. Šverák. "Convex integration for Lipschitz mappings and counterexamples to regularity". In: *Ann. of Math. (2)* 157.3 (2003).

[9] Camillo De Lellis and László Székelyhidi Jr. "On admissibility criteria for weak solutions of the Euler equations". In: *Arch. Ration. Mech. Anal.* 195.1 (2010).

An application on the primitive equations

- We would like to apply the machinery of convex integration - take a motivation from already known approaches[10],[11] for abstract Euler equations:

$$\operatorname{div} \mathbf{u} = 0,$$
$$\partial_t \mathbf{u} + \operatorname{div} (\mathbf{u} \otimes \mathbf{u} - \mathbb{H}[\mathbf{u}]) + \nabla \Pi[\mathbf{u}] = 0.$$

- Problem for the primitive equations - the equation for the third component of the velocity is degenerated.

[10] Elisabetta Chiodaroli, Eduard Feireisl, and Ondřej Kreml. "On the weak solutions to the equations of a compressible heat conducting gas". In: *Ann. Inst. H. Poincaré Anal. Non Linéaire* 32.1 (2015), pp. 225–243.

[11] Donatella Donatelli, Eduard Feireisl, and Pierangelo Marcati. "Well/ill posedness for the Euler-Korteweg-Poisson system and related problems". In: *Comm. Partial Differential Equations* 40.7 (2015), pp. 1314–1335.

Inviscid primitive equations

$$
\begin{aligned}
u_x + v_y + w_z &= 0 \\
u_t + uu_x + vu_y + wu_z + p_x &= 0, \\
v_t + uv_x + vv_y + wv_z + p_y &= 0, \\
p_z &= \theta \\
\theta_t + u\theta_x + v\theta_y + w\theta_z &= \lambda_1(\theta_{xx} + \theta_{yy}) + \lambda_2\theta_{zz}
\end{aligned}
$$

Extended inviscid primitive equations

$$
\begin{aligned}
u_x + v_y + w_z &= 0 \\
u_t + uu_x + vu_y + wu_z + p_x &= 0, \\
v_t + uv_x + vv_y + wv_z + p_y &= 0, \\
w_t + uw_x + vw_y + ww_z + p_z &= 0, \\
p_z &= \theta \\
\theta_t + u\theta_x + v\theta_y + w\theta_z &= \lambda_1(\theta_{xx} + \theta_{yy}) + \lambda_2\theta_{zz}
\end{aligned}
$$

An example using the idea of convex integration

$$\operatorname{div} u = 0 \quad \text{in } \mathbb{T}^3, \tag{1}$$

$$|u| = 1 \quad \text{almost everywhere in } \mathbb{T}^3 \tag{2}$$

- we define a set of subsolutions

$$X_0 = \{u \in \mathcal{C}^\infty(\mathbb{T}^3;\mathbb{R}^3) \colon \text{(1) holds and } |\mathrm{u}|<1\},$$

- we define a functional on X_0 by $I(u) = \int_{\mathbb{T}^3} |u|^2 - 1 \, \mathrm{d}x$.

Lemma (Effective oscillations)

Let $u \in X_0$. Then exists $\{w_n\} \subseteq \mathcal{C}^\infty(\mathbb{T}^3;\mathbb{R}^3)$ such that

- $u + w_n \in X_0$,
- $w_n \to 0$ *weakly in* L^2
- $$\liminf_{n\to\infty} I(u + w_n) \geq I(u) + c\,(I(u))^2 \,.$$